ASYMPTOTIC DESCRIPTION OF
DIRAC MASS FORMATION
IN KINETIC EQUATIONS FOR QUANTUM PARTICLES

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Abstract In this paper the detailed asymptotic behaviour of the solutions of a kinetic equation for quantum particles is studied. It is shown that this behaviour is sensitive not only to the total mass of the initial data but also to its precise behaviour near the origin. Finally, the self-similar way in which a Dirac mass is eventually generated in some cases is described in detail.

1.-INTRODUCTION.

The purpose of this article is to describe the long time asymptotics of the solutions to the equation

$$k^2 \frac{\partial f}{\partial t} = \int_0^\infty k^2 k'^2 b(k,k') \{ f'(1+f)e^{-k} - f(1+f')e^{-k'} \} dk' \tag{1.1}$$

where $f = f(k,t)$, $k \geq 0$, $t \geq 0$, f' stands for $f(k',t)$ and $b > 0$. Equations of the type (1.1) are a typical example of kinetic equations for quantum particles (**c.f. for example [EMV], [Nordheim], [Landau] and references therein**). In particular, this equation describe the approach to thermal equilibrium of a distribution of Bose particles that are in contact with a bath of fermions in thermal equilibrium (A physical example would be the system of photons and electrons in a plasma c.f. [Dr], [LY1], [LY2]).

A very interesting mathematical feature of equation (1.1) is that its solutions may develop Dirac masses at $k = 0$ as $t \to \infty$. One simple heuristic explanation for this behaviour is the following. The distributions of Bose particles in thermal equilibrium (the steady states of the equation) are given by the usual Bose-Einstein distributions ($\mu > 0$) and Planck distribution ($\mu = 0$)

$$f_{BE}(k) = \frac{1}{e^{\mu+k} - 1}. \tag{1.2}$$

It turns out that the family of steady states of (1.1) is wider than (1.2). It can be checked by means of a careful analysis of the stationary solutions of (1.1) or by means of careful study of the statistical physics of bosons, as it was made by Bose and Einstein ([B], [E1], [E2]), that the class of steady distributions and includes also the solutions containing a Dirac mass at the origin:

$$k^2 f_{BEC}(k) = \frac{k^2}{e^k - 1} + \rho\,\delta_0, \quad \rho \geq 0. \tag{1.3}$$

Since the total density of Bose particles,

$$M(t) = \int_0^\infty k^2 f(k,t)dk, \tag{1.4}$$

remains constant under the evolution by (1.1) it is natural to expect for its solutions a long time asymptotics given by the steady states (1.2) or (1.3) with the same total density of Bose particles as the initial data. In particular, if the initial distribution of particles has a density larger than the Planck distribution

$$M(0) > \int_0^\infty \frac{k^2}{e^k - 1}\, dk =: m_0, \tag{1.5}$$

(notice that the mass of the Planck distribution is finite and larger that the mass of any Bose-Einstein distributions) then it would be natural to expect a long time asymptotics given by (1.3) with $\rho > 0$. This means Dirac mass formation as $t \to \infty$.

To our knowledge, the first results indicating Dirac mass formation for the solutions of (1.1) were given in [LY1], [LY2] using suitable approximations of the equation for $k \to 0^+$. Numerical simulations showing Dirac mass formation for solutions of equations related to (1.1) have been obtained in [ST1] and [ST2]. A rigourous proof of the fact that solutions of (1.1) behave as steady states contains Dirac masses at the origin, as $t \to \infty$ if (1.5) holds was given in [EM1] and [EM2]. The proof given in [EM2] works under rather broad assumptions since it is based on general features of kinetic equations. First, notice that equation (1.1) can be estimated in terms of linear equations of the variable f (due to mass conservation). This rules out the possibility of finite time blow up for the solutions. On the other hand equation (1.1) has a natural entropy that increases along the solutions. Using this entropy as a Lyapunov function it can be checked that the solutions of (1.1) approach steady states as $t \to \infty$.

The main goal of this paper is to analyse the precise manner in which the Dirac mass forms as $t \to \infty$. To this end, we study first the case of constant kernel b (and we take $b \equiv 1$ without loss of generality in that case). In this case (1.1) is explicitly solvable using Laplace transforms and this allows to describe in a detailed manner the asymptotics of the solutions of (1.1) as $t \to \infty$. We are then able to describe the formation of a Dirac measure as $t \to \infty$ and also to obtain convergence rates of the solutions to the corresponding steady states. We also show that several of the previous results can be generalised to the case $b \not\equiv 1$ by means of matched asymptotic expansions.

High Sensitivity to the initial data ??

For technical reasons it is more convenient to make the change of variables

$$F(k,t) = k^2 f(k,t) \tag{1.6}$$

that transforms (1.1) into

$$\frac{\partial F}{\partial t} = Q(F,F) = \int_0^\infty b(k,k')\big(F'\,(k^2 + F)\,e^{-k} - F\,(k'^2 + F')\,e^{-k'}\big)\,dk'. \tag{1.7}$$

Given a positive measure g we define

$$M(g) = \int_0^\infty k^2 g(k,t)dk. \tag{1.8}$$

Taking into acount (1.2), it follows that the steady states without mass concentration for (1.7) are

$$g_\mu(k) = \frac{k^2}{e^{k+\mu}-1}, \quad \mu \geq 0 \tag{1.9}$$

Roughly speaking, it is proved in [EM1], [EM2] that, for any initial data $F(k,0) = \varphi + \rho\delta_0$, ($\rho \in \mathbb{R}$ and $(1+k)\varphi \in L^1(0,+\infty)$), if $M[F(k,0)] = m \leq m_0$ then,

$$F(k,t) \to g_\mu, \quad \text{with} \quad M[g_\mu] = m$$

If $M[F(k,0)] = m > m_0$ then,

$$F(t,\cdot) \to (m - m_0)\delta(t) + g_0$$

where the convergences take place in the weak sense (**of measures ?**). Under suitable regularity assumptions on the initial data it is possible to obtain stronger convergence away from the origin.

A question that has not been addressed in the papers [EM1], [EM2] is the study of the detailed asymptotics of the Dirac mass formation as $t \to \infty$ (**and in particular to study the rate of convergence of the solutions to the final steady state as** $t \to \infty$ quitar?). Due to the fact that the process of Dirac mass formation takes place in the region $k \approx 0$, such detailed asymptotics can be expected to be very strongly dependent on the asymptotics of the kernel $b(k,k')$ as $(k,k') \to (0,0)$. In this paper we will restrict our attention to the case of kernels with the asymptotics $b(k,k') \approx C > 0$ as $(k,k') \to (0,0)$. Notice that with a suitable rescaling of the time variable we can assume that $C = 1$.

A large part of this paper is devoted to the analysis of the particular kernel $b(k,k') = 1$. The reason for this is that the kinetic equation (1.1) is then explicitly solvable using suitable changes of variables and Laplace transforms (cf. Section 2). Using this representation formula for the solutions it is possible to describe the asymptotics of the solutions of (1.1) in an explicit manner for a large class of data. It turns out that the asymptotic behaviour of the solutions is extremely sensitive on the behaviour of the initial data $f_{in}(k)$ in the region $k \approx 0$. The high sensitivity of the asymptotics of the solutions of kinetic equations on the initial data has been previously observed in other kinetic equations (cf. [CP], [NP]).

We have restricted our study (**of the asymptotics of Dirac mass formation for the solutions of (1.1)** quitar ?) to initial data behaving as powers as $k \to 0^+$ since this will be enough in order to ascertain the high sensitivity of this asymptotic behaviour with respect to the initial data, but it will be apparent from our arguments that a much larger class of initial data could be considered using the representation formulae derived herein with standard asymptotic methods. In any case, in a very rough manner we could formulate our results stating that solutions that yield condensate as $t \to \infty$ (more precisely $m \geq m_0$) approach to equilibrium at an algebraic rate, and that solutions that do not yield condensate ($m < m_0$) approach to equilibrium exponentially.

Let us briefly describe the asymptotic result that we derive in this paper for the kernel $b = 1$. Although the representation formula for the solutions of (1.1) that we derive is valid for general initial data, we will restrict our attention to initial data of the form

$$F(k,0) = \varphi_{in}(k) + \rho_{in}\delta_0, \quad \text{where } (1+k)\,\varphi_{in} \in L^1(0,+\infty),\ \rho_{in} \in \mathbb{R}.$$

Moreover we also assume that

$$\varphi_{in}(k) \sim Ak^\alpha, \quad \alpha > 0, A > 0 \quad \text{as } k \to 0. \tag{1.10}$$

The detailed asymptotics of the Dirac mass formation as well as the rate of convergence of $F(k,t)$ to the corresponding steady state in the region $k > 0$ depend on

(i) the value of m with respect to m_0,
(ii) the presence or not of condensate at the initial stage, or equivalently if $\rho_{in} = 0$ or $\rho_{in} > 0$,
(iii) the value of α in (1.10)

More precisely, the rate of convergence of F to equilibrium will be exponential if $m < m_0$ (i.e. if there is no condensation as $t \to \infty$). On the other hand the rate of convergence will be algebraic when $m \geq m_0$, but the precise rate of convergence depends on α and ρ_{in}.

¿From now on, we assume that

$$b(k,k') \equiv 1, \quad M(F_{in}) = m \tag{1.11}$$

and we decompose a solution F to equation (1.7) as:

$$F = G + g_\mu, \quad G = g + \rho(t)\delta_0, \quad F_{in} = G_{in} + g_\mu, \quad G_{in} = g_{in} + \rho_{in}\delta_0, \tag{1.12}$$

with $\rho \geq 0$, $g + g_\mu \geq 0$, where g_μ is as in (1.9), and μ is chosen as follows

$$\mu = 0 \quad \text{if } m \geq m_0 \quad \text{or} \quad m = M[g_\mu] \quad \text{if} \quad m < m_0. \tag{1.13}$$

Moreover, we supose that there exists $\alpha > 0$, $A > 0$ and $\Gamma > 0$ such that

$$\begin{cases} |g_{in}(k)| \leq e^{-\Gamma k} \ \forall\, k \geq 1, \quad \lim_{k\to 0} k^{-\alpha}\, g_{in}(k) = A, \\ k^{-\alpha}\, g_{in}(k) \in C^2([0,1]) \quad \text{if} \quad \alpha \neq 1 \\ k^{-\beta}\,(g_{in}(k) - Ak^\alpha) \in C^2([0,1]) \quad \text{for some } \beta > 1, \quad \text{if} \quad \alpha = 1. \end{cases} \tag{1.14}$$

The regularity assumptions in (1.14) are convenient in order to compute in a simple manner the sought-for asymptotics of the solutions. We have not intended to reach the greatest possible generality.
In the rest of the paper, C stands by a generic constant which depends on the initial data and can change from line to line. Most of them can be computed explicitly but are not very illuminating.
Finally, let us define

$$\overline{\alpha} = \min(1, \alpha). \tag{1.15}$$

We first describe the detailed formation of the "final agregate" ρ_∞.

Theorem 1. *Assume (1.11)-(1.14) and $m > m_0$.*

(i) Suppose that $\rho_{in} = 0$, then:

$$g(k,t) = Ct\,\Phi_\alpha(k\,t)\ (1 + o(1)) \quad \text{as } t \to \infty$$

uniformly on $0 < kt \leq L$ for any finite positive constant L where

$$\Phi_\alpha(\xi) = \xi^{\overline{\alpha}} e^{-m\xi}.$$

(ii) Suppose that $\rho_{in} > 0$, then:

$$g(k,t) = \frac{C}{t^{\overline{\alpha}}}\Phi_\alpha(kt)\ (1 + o(1)) \quad \text{as } t \to \infty$$

uniformly on $0 < kt \leq L$ for any finite positive constant L and

$$\lim_{t\to\infty} \rho(t) = m - m_0.$$

Remark 1 . Theorem 1 shows that the precise manner in which the Dirac mass develops depends very sensitively of two features of the initial data, namely α and the presence or absence of condensate at initial time. From [EM2] we already know that if $\rho_{in} = 0$ then $\rho(t) = 0$ for any $t > 0$. On the other hand, if

$\rho_{in} > 0$, $\rho(t)$ absorbs as $t \to \infty$ all the "excess" of mass that the initial data has with respect to the Planck distribution $m - m_0$.

Our next result concerns the rate of convergence as $t \to +\infty$ of the solutions to (1.1) towards the regular part of the corresponding steady state. Using the representation formula that is established in Section 2, it is possible to obtain such a rate for k in any region of the real line. Nevertheless, for the sake of brevity, we only consider the region where k is of order one. This is the region where the steady states g_μ, $\mu \geq 0$, undergo the strongest variation.

Theorem 2. *Assume (1.11)-(1.14). Then, when $t \to \infty$ and uniformly for k on compact sets of $(0, +\infty)$:*

(i) If $m > m_0$:
(a) for $\rho_{in} = 0$:

$$g(k,t) = C\, \frac{g_0(k)}{(1 - e^{-k})}\, \frac{1}{t}(1 + o(1))$$

(b) for $\rho_{in} > 0$:

$$g(k,t) = C\, \frac{g_0(k)}{(1 - e^{-k})}\, \frac{1}{t^{\overline{\alpha}+2}}(1 + o(1))$$

(ii) If $m = m_0$:
(a) for $\rho_{in} = 0$:

$$g(k,t) = C\, \frac{g_0(k)}{(1 - e^{-k})}\, \frac{1}{t^{\overline{\alpha}+2}}(1 + o(1))$$

(b) for $\rho_{in} > 0$:

$$g(k,t) = C \frac{g_0(k)}{1 - e^{-k}} \frac{1}{t}(1 + o(1)), \quad \textit{and } \rho(t) = \frac{C}{t}(1 + o(1))$$

(iii) If $m < m_0$

$$g(k,t) = \mathcal{O}(e^{-\delta t}), \quad \textit{and } \rho(t) = \mathcal{O}(e^{-\delta t})$$

for all $\delta \in (0, e^\mu - 1)$.

We conclude this Section with a sketch of the plan of the paper. In Section 2 we obtain a formal explicit representation of the solution F to (1.7) when $b \equiv 1$. Some properties of the functions that appear in that representation formula, and which will be used later, are described in Section 3. In Section 4 we prove some key asymptotic results (Propositions 2.1-2.3) Theorems 1 and 2 follow from these Propositions as it is shown in Section 2. Finally in an Appendix at the end of the paper we describe how the results derived before can be obtained for more general collision kernels b using formal asymptotics.

2. A REPRESENTATION FORMULAE FOR THE SOLUTIONS IN THE CASE $b \equiv 1$.

In this section we obtain a representation formulae for the solutions of (1.7) under general assumptions on g_{in} and for $b \equiv 1$.

Let F be the solution of equation (1.18) that we decompose as indicated in (1.12). Notice that

$$m = M(F) = M(g)(t) + M[g_\mu] + \rho(t), \quad t > 0 \tag{2.1}$$

The function G then satisfies:

$$\frac{\partial G}{\partial t} = \int_0^\infty \left(F'\,(k^2 + G + g_\mu)\, e^{-k} - (G + g_\mu)\,(k'^2 + G' + g'_\mu)\, e^{-k'} \right) dk'.$$

Taking into account that $(k^2 + g_\mu)\, e^{-k} = e^\mu g_\mu$ and $M(F) = m$ for any $t \geq 0$, we obtain

(2.2)
$$\begin{cases} \dfrac{\partial G}{\partial t} = (G + g_\mu)\,\varepsilon(t) - m\,(e^\mu - e^{-k})\,G \\ \quad G(0) = G_{in} = g_{in} + \rho_{in}\,\delta_0, \end{cases}$$

since, by (1.23) and (1.12)

(2.3)
$$\varepsilon(t) = \int_0^\infty (e^\mu - e^{-k'})\,G'\,dk'.$$

Using the decomposition (1.12) of G in (2.2) we can write (2.2) as:

(2.4)
$$\begin{cases} \dfrac{\partial g}{\partial t} = (g + g_\mu)\,\varepsilon(t) - m\,(e^\mu - e^{-k})\,g, \qquad g(0) = g_{in}, \\ \dfrac{\partial \rho}{\partial t} = [\varepsilon(t) - m(e^\mu - 1)]\,\rho, \qquad \rho(0) = \rho_{in}. \end{cases}$$

The two equations of (2.4) are linear in g and ρ respectively, and they can be reduced to a constant coefficient equation by means of the change of variables

(2.5)
$$h = g\,e^{-\int_0^t \varepsilon(s)\,ds}, \qquad \omega(t) = \rho(t)\,e^{-\int_0^t \varepsilon(s)\,ds}$$

and

(2.6)
$$\lambda(t) = \varepsilon(t)\,e^{-\int_0^t \varepsilon(s)\,ds}.$$

Whence, (h, ω) solve the linear equations:

(2.7)
$$\begin{cases} \dfrac{\partial h}{\partial t} = \lambda(t)\,g_\mu - m\,(e^\mu - e^{-k})\,h, \\ \dfrac{\partial \omega}{\partial t} = -m(e^\mu - 1)\omega. \end{cases}$$

Notice that, in this formulation of the problem, the equation for ω decouples. Therefore,

(2.8)
$$\omega(t) = \rho_{in}\,e^{-m(e^\mu - 1)t}$$

The linear equationin (2.7) can be solved explicitly in terms of the (still unknown) function $\lambda(t)$ using the classical Laplace transform. We define:

(2.9)
$$H(z, k) = \int_0^\infty h(t, k)\,e^{-z\,t}\,dt, \qquad \Lambda(z) = \int_0^\infty \lambda(t)\,e^{-z\,t}\,dt.$$

Standard computations yield
$$z\,H - g_{in} = g_\mu\,\Lambda - m\,(e^\mu - e^{-k})\,H.$$

Thus:

(2.10)
$$H(k, z) = \frac{g_{in}(k)}{z + m\,(e^\mu - e^{-k})} + \frac{g_\mu(k)\,\Lambda(z)}{z + m\,(e^\mu - e^{-k})}$$

where Λ is the only quantity that remains to be computed. To this end we use (2.3), (2.5), (2.6) and (2.9) to obtain:
$$\Lambda(z) = \int_0^\infty (e^\mu - e^{-k'})\,H(k', z)\,dk' + \Omega(z)(e^\mu - 1).$$

where

$$\Omega(z) = \frac{\rho_{in}}{z + m\,(e^{\mu} - 1)}$$

is just the Laplace transform of $\omega(t)$. Whence, using (2.10) and making the change of variables

$$\zeta = \frac{z}{m} \tag{2.11}$$

we have:

$$\Lambda(z) = \Lambda(\zeta) = \frac{\Phi(\zeta)}{S(\zeta)}, \tag{2.12}$$

with

$$\Phi(\zeta) = \frac{1}{m}\int_0^{\infty} \frac{(e^{\mu} - e^{-k})\,G_{in}}{\zeta + e^{\mu} - e^{-k}}\,dk, \qquad S(\zeta) = 1 - \frac{1}{m}\int_0^{\infty} \frac{(e^{\mu} - e^{-k})\,g_{\mu}}{\zeta + e^{\mu} - e^{-k}}\,dk. \tag{2.13}$$

Using the classical inversion formula of the Laplace transform (see for instance [D]) we obtain:

$$\lambda(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{zt}\Lambda(\zeta)\,dz \qquad \text{and} \qquad h(k,t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{zt}H(k,z)\,dz, \tag{2.14}$$

where $\gamma > 0$.

As it is shown in Section 3, the function Λ is analytic in the region $\mathbb{C} \setminus [-e^{\mu}, 1 - e^{\mu}]$. The key point on deriving the long time asymptotics of $\lambda(t)$ and $h(k,t)$ is to study the behaviour of Λ near the point $1 - e^{\mu}$. This analysis is the content of Section 3. We sumarise in the following Lemmas the resulting asymptotics for $\lambda(t)$, $h(k,t)$ as $t \to \infty$.

Proposition 2.1. *Under the assumptions (1.11)–(1.14) the function λ given by (2.14) has the following asymptotic behaviours:*

(i) If $m > m_0$, then:

$$\lambda(t) = \frac{C}{t^{\overline{\alpha}+2}}(1 + o(1)), \quad \text{as } t \to +\infty \tag{2.15}$$

where $\overline{\alpha}$ is as in (1.15).

(ii) If that $m = m_0$, then

(a) for $\rho_{in} = 0$,

$$\lambda(t) = \frac{C}{t^{\overline{\alpha}+2}}(1 + o(1)), \quad \text{as } t \to +\infty \tag{2.16}$$

(b) for $\rho_{in} > 0$

$$\lambda(t) = -\lambda_0 + o(1), \quad \text{as } t \to +\infty \tag{2.17}$$

where λ_0 is a positive constant.

(iii) If $m < m_0$, then:

$$\lambda(t) = O(e^{-\delta t}), \quad \text{as } t \to +\infty \tag{2.18}$$

for any $\delta \in (0, e^{\mu} - 1)$.

Proposition 2.2. *Assume again (1.11)–(1.14). The function $h(k,t)$, given by (2.14), is defined for all $t > 0$, an all $k > 0$ and satisfies uniformly in $0 < k\,t < L$:*

(i) If $m > m_0$, then:

$$h(k,t) = C\,k^{\overline{\alpha}} e^{-mkt}(1 + o(1)), \quad \textit{as } t \to +\infty \tag{2.19}$$

(ii) If that $m = m_0$, then

(a) for $\rho_{in} = 0$,

$$h(k,t) = C\,k^{\overline{\alpha}} e^{-mkt}(1 + o(1)), \quad \textit{as } t \to +\infty \tag{2.20}$$

(b) for $\rho_{in} > 0$

$$h(k,t) = [Ck^{\alpha} e^{-mkt} - \lambda_0(1 - e^{-mkt})](1 + o(1)), \quad \textit{as } t \to +\infty \tag{2.21}$$

(iii) If $m < m_0$, then:

$$h(k,t) = O(k^{\overline{\alpha}} e^{-\delta t}) \textit{ for any } \delta \in (0, e^{\mu} - 1), \quad \textit{as } t \to +\infty. \tag{2.22}$$

Proposition 2.3 . *Assume (1.11)–(1.14). Then the asymptotic behaviour of $h(k,t)$, uniformly for k on compact sets of $(0,\infty)$, is the following.*

(i) If $m > m_0$, then:

$$h(k,t) = C\,\frac{g_0(k)}{(1 - e^{-k})}\,\frac{1}{t^{\overline{\alpha}+2}}(1 + o(1)), \quad \textit{as } t \to +\infty \tag{2.23}$$

(ii) If that $m = m_0$, then

(a) for $\rho_{in} = 0$,

$$h(k,t) = C\,\frac{g_0(k)}{(1 - e^{-k})}\,\frac{1}{t^{\overline{\alpha}+2}}(1 + o(1)), \quad \textit{as } t \to +\infty \tag{2.24}$$

(b) for $\rho_{in} > 0$

$$h(k,t) = -\lambda_0 \frac{g_0(k)}{1 - e^{-k}}(1 + o(1)), \quad \textit{as } t \to +\infty \tag{2.25}$$

(iii) If $m < m_0$, then:

$$h(k,t) = O(e^{-\delta t}) \textit{ for any } \delta \in (0, e^{\mu} - 1), \quad \textit{as } t \to +\infty. \tag{2.26}$$

Remark 2.4. Notice that (2.21) and (2.25) match in the intermediate region $1/t << k << 1$. Seemingly, the same does not happen with (2.19)-(2.23), and (2.20)-(2.24). Nevertheless one may check that matching would occur by means of higher order terms in the expansion, since the leading terms in the region kt of order one is given in these cases by terms that decay exponentially in the region k of order one.

Remark 2.5 . It is possible to compute the precise exponential factor in (2.18), (2.22) and (2.26) using the same techniques.

Proof of Theorem 1 and Theorem 2.

Using the propositions 3.1-3.3, we deduce the precise asymptotics of g, ρ as $t \to \infty$ i.e. Theorem 1 and Theorem 2. Indeed, integrating the equation (2.6), we obtain,

$$\lambda(t) = \varepsilon(t)\, e^{-\int_0^t \varepsilon(s)\,ds} = -\frac{d}{dt} e^{-\int_0^t \varepsilon(s)\,ds}. \tag{2.27}$$

We have to distinguish between two cases, namely $|\int_0^\infty \lambda(s) ds| < \infty$ (c.f. (2.15), (2.16) and (2.18)) and $|\int_0^\infty \lambda(s) ds| = \infty$ (c.f. (2.17)). In the first case we have,

$$\Lambda(0) = \int_0^\infty \lambda(t)\,dt.$$

¿From (2.27), we obtain

$$\int_t^\infty \lambda(s)\,ds = e^{-\int_0^t \varepsilon(s)\,ds} - e^{-\int_0^\infty \varepsilon(s)\,ds} \tag{2.28}$$

and in particular for $t = 0$

$$\Lambda(0) = 1 - e^{-\int_0^\infty \varepsilon(s)\,ds}. \tag{2.29}$$

Using (2.27)-(2.29) we obtain

$$\varepsilon(t) = \frac{\lambda(t)}{1 - \Lambda(0) + \int_t^\infty \lambda(s)\,ds}. \tag{2.30}$$

A straightforward computation gives

$$1 - \Lambda(0) = \begin{cases} \dfrac{\rho_{in}}{m - m_0} & \text{if } m > m_0 \\ \dfrac{M[F_{in}/(e^\mu - e^{-k})]}{M[g_\mu/(e^\mu - e^{-k})]} & \text{if} \quad m < m_0 \quad \text{or} \quad m = m_0 \text{ and } \rho_{in} = 0. \end{cases} \tag{2.31}$$

Using (2.31) as well as the asymptotics of $\lambda(t)$ derived in Proposition 2.1, we compute the following asymptotic of $\varepsilon(t)$. Therefore, using again (2.6) we derive the asymptotics of $\exp(\int_0^t \varepsilon(s) ds)$ as $t \to \infty$. Using then, (2.5), (2.8) and the asymptotics for h in Propositions 2.2-2.3 we obtain the corresponding asymptotics for $\rho(t)$ and $g(k,t)$ in Theorems 1 and 2.

It only remains to study the case $|\int_0^\infty \lambda(s)\,ds| = +\infty$, i.e $m = m_0$ and $\rho_{in} > 0$. In this case, (cf. (2.17))

$$\lambda(t) = -\lambda_0 + o(1) \quad \text{as} \quad t \to \infty,$$

whence integring the identity (2.27) we get

$$\exp(-\int_0^t \varepsilon(s) ds) \;=\; \lambda_0\, t + o(1), \quad \text{as} \quad t \to \infty,$$

and the argument follows as in the previous case. □

3. SOME PROPERTIES OF THE FUNCTION Λ.

This section is devoted to the study of the function Λ. To this end we shall need some elementary complex variable results. The relevant properties if Λ are stated in the two following Propositions.

Proposition 3.1 . *Suppose μ is given by $m = M[g_\mu]$ if $m \le m_0$ and $\mu = 0$ if $m > m_0$ (cf. (1.13)).* Then the function Λ defined in (2.12), (2.13) is analytic on $\mathbb{C} \setminus \{[-e^\mu, -e^\mu + 1]\}$ and

$$\lim_{|\zeta|\to\infty} \Lambda(\zeta) = 1.$$

Proof. The functions Φ and S defined in (2.13) are analytic in the set $\mathbb{C} \setminus \{[-e^\mu, -e^\mu + 1]\}$. Therefore, it only remains to show that S does not vanish there. Since

$$\operatorname{Im}(S(\zeta)) = \frac{\zeta_2}{M_{in}} \int_0^\infty \frac{(e^\mu - e^{-k})\, g_\mu}{(\zeta_1 + e^\mu - e^{-k})^2 + \zeta_2^2}\, dk \neq 0 \quad \text{if} \quad \zeta_2 \neq 0.$$

where $\zeta = \zeta_1 + i\,\zeta_2$, $\zeta_1 \in \mathbb{R}$, $\zeta_2 \in \mathbb{R}$ all the zeros of the function S lie in the real line $\zeta \in \mathbb{R}$. On the other hand,

$$S'(\zeta) = \frac{1}{M_{in}} \int_0^\infty \frac{g_\mu(k)(e^\mu - e^{-k})}{(e^\mu - e^{-k} + \zeta)^2} dk > 0$$

whence,

$$\lim_{\zeta\to-\infty} S(\zeta) = \lim_{\zeta\to+\infty} S(\zeta) = 1$$

$$\lim_{\zeta\to(-e^\mu+1)^+} S(\zeta) = 1 - \frac{m_0}{m}, \qquad \lim_{\zeta\to(-e^\mu)^-} S(\zeta) = +\infty$$

where all the limits are taken along the real line $\zeta \in \mathbb{R}$.

It follows that $S(\zeta)$ does not vanish if $m \ge m_0$. If $m > m_0$, $S(\zeta)$ has only a simple zero at $\zeta = 0$ (cf. (2.13)). On the other hand, for such values of m, we also have, $\Phi(0) = 0$. It then follows that Λ is analytic in a neighbourhood of $\zeta = 0$ and Proposition 3.1 follows.

We now obtain the behaviour of the function Λ near the point $\zeta_0 = -e^\mu + 1$.

Proposition 3.2. *Assume (1.11)-(1.14) hold.*

(i) If $m > m_0$

$$\Lambda(\zeta) = a_1 + a_2\,\zeta + \begin{cases} a_3\,\zeta^{\alpha+1} + o(|\zeta|^{\alpha+1}) & \text{if } \alpha \in (0,1) \\ a_3\,\zeta^2\,\log\zeta + a_4\,\zeta^2 + \mathcal{O}(|\zeta|^{2+\delta}) & \text{if } \alpha = 1, \text{ for some } \delta > 0 \\ a_3\,\zeta^2\,\log\zeta + a_4\,\zeta^2 + o(|\zeta|^2) & \text{if } \alpha > 1 \end{cases} \tag{3.1}$$

(ii) If $m = m_0$,

(a) for $\rho_{in} = 0$

$$\Lambda(\zeta) = a_1 + \begin{cases} a_2\,\zeta^\alpha + o(|\zeta|^\alpha) & \text{if } \alpha \in (0,1) \\ a_2\,\zeta\,\log\zeta + a_3\,\zeta + \mathcal{O}(|\zeta|^{1+\delta}) & \text{if } \alpha = 1, \text{ for some } \delta > 0 \\ a_2\,\zeta\log\zeta + a_3\,\zeta + a_4\,\zeta^\alpha + o(|\zeta|^\alpha) & \text{if } \alpha > 1 \end{cases} \tag{3.2}$$

(b) for $\rho_{in} > 0$

$$\Lambda(\zeta) = \frac{a_1}{\zeta} + a_2\,\log\zeta + o(|\log\zeta|). \tag{3.3}$$

(iii) If $m < m_0$,

$$\Lambda(\zeta) = a_1 + \begin{cases} a_2\,(\zeta-\zeta_0)^\alpha + o(|\zeta-\zeta_0|^\alpha) & \text{if } \alpha \in (0,1) \\ a_2\,(\zeta-\zeta_0)\log(\zeta-\zeta_0) + a_3(\zeta-\zeta_0) + o(|\zeta\log\zeta|) & \text{if } \alpha = 1 \text{ for some } \delta > 0 \\ a_2\,(\zeta-\zeta_0) + a_3\,(\zeta-\zeta_0)^\alpha + o(|\zeta-\zeta_0|^\alpha) & \text{if } \alpha > 1, \end{cases} \tag{3.4}$$

as $|\zeta-\zeta_0| \to 0$ and the convergence is uniform in the domain $\mathbb{C}\setminus[-e^\mu, -e^\mu+1]$. The constants a_1, a_2, a_3, a_4 are real numbers, that depend on the initial data and can change froom line to line.

The rationale behind proposition 3.2 is that it is possible to expand the function Λ as $\zeta \to \zeta_0$ as a series of analytic functions plus terms with a branching point at ζ_0 that proceed from the initial data g_{in} as well as from g_μ. The relative size of the corresponding terms of Λ depend on the relative size of $g_{in}(k)$ compared with $g_\mu(k)$.

Proof of Proposition 3.2. By (2.13) to compute the asymptotics of $\Lambda(\zeta)$ reduces to compute those of the two functions Φ and S. This leads in both cases to obtain the asymptotics of integrals of the following form:

$$K(\zeta) = \int_0^\infty \frac{(e^\mu - e^{-k})\,g(k)}{\zeta + e^\mu - e^{-k}}dk \equiv \int_0^\infty \frac{\psi(k)}{\zeta + e^\mu - e^{-k}}dk \tag{3.5}$$

as $\zeta \to \zeta_0$, and where g wich may be g_{in} or g_μ. The asymptotics of this integral strongly depends on the behaviour of $g(k)$ as $k \to 0$. This yields the multiple cases in Proposition 3.2. Let us suppose for instance that $g(k) \sim k^\alpha$ as $k \to 0$ in the precise way indicated in (1.14), $\alpha < 1$ and suppose that $\mu > 0$. Then, the contribution to $K(\zeta)$ due to the part of the integral where k is away from the origin is analytic. On the other part near the origin, we can approximate the contribution to the integral as,

$$\int_0^1 \frac{(e^\mu - e^{-k})\,g(k)}{\zeta + e^\mu - e^{-k}}dk \sim (e^\mu - 1)\int_0^1 \frac{k^\alpha}{\zeta + e^\mu - 1}dk = a_0 + a_1\zeta^\alpha + o|\zeta|^\alpha \quad \text{as } \zeta \to 0. \tag{3.6}$$

To make this argument precise and to prove that the convergence is uniform in $\mathbb{C}\setminus\{[-e^\mu, -e^\mu+1]\}$ we have to use (1.14) as well as some integral singular methods.

First, to get rid of the constant term in (3.6) it is convenient to use $K'(\zeta)$ instead of $K(\zeta)$.

$$K'(\zeta) = \int_0^\infty (\psi'(k) + \psi(k))\frac{e^k}{(\zeta + e^\mu - e^{-k})}dk. \tag{3.7}$$

Let us call $\Phi(k) = (\psi'(k) + \psi(k))e^k$. Notice that

$$\Phi(k) \sim k^{\alpha-1}, \quad \Phi'(k) \sim k^{\alpha-2} \quad \text{as } k \to 0. \tag{3.8}$$

We split the integrals in (3.7) as,

$$K'(\zeta) = \int_0^1 \frac{\Phi(k)}{(\zeta + e^\mu - e^{-k})}dk + \int_1^\infty \frac{\Phi(k)}{(\zeta + e^\mu - e^{-k})}dk.$$

The second integral is an analytical function. On the other hand,

$$\int_0^1 \frac{\Phi(k)}{(\zeta + e^\mu - e^{-k})}dk = \int_0^1 \frac{\Phi(k) - Ck^{\alpha-1}}{(\zeta + e^\mu - e^{-k})}dk + C\int_0^1 \frac{k^{\alpha-1}}{(\zeta + e^\mu - e^{-k})}dk \tag{3.9}$$

The second term in the right hand side of (3.9) may be written as,

$$C\int_0^1 \frac{k^{\alpha-1}}{(\zeta + e^\mu - e^{-k})}dk = C\int_0^1 k^{\alpha-1}\frac{dk}{(\zeta-\zeta_0) + kh(k)}$$

with $h(k) = (1 - e^{-k})/k$ is analytic near $k = 0$. Therefore,

$$\int_0^1 \frac{k^{\alpha-1}}{(\zeta + e^{\mu} - e^{-k})} dk = C(\zeta - \zeta_0)^{\alpha-1} + o(|\zeta - \zeta_0|^{\alpha-1}), \quad \text{as} |\zeta - \zeta_0| \to 0.$$

We consider now the first term in the right hand side of (3.9). It may be decomposed as follows

$$\int_0^1 \frac{\Phi(k) - Ck^{\alpha-1}}{(\zeta + e^{\mu} - e^{-k})} dk = \int_0^{|\zeta|/2} [\cdots]\, dk + \int_{|\zeta|/2}^{2|\zeta|} [\cdots]\, dk + \int_{2|\zeta|}^1 [\cdots]\, dk \equiv I_1 + I_2 + I_3.$$

The two terms I_1 and I_3 are easily estimated using (3.8) to obtain,

$$I_1 + I_2 = o(|\zeta - \zeta_0|^{\alpha-1}) \quad \text{as } |\zeta - \zeta_0| \to 0.$$

Finally, to estimate I_2 we use the change of variables $1 - e^{-k} = |\zeta|\, x$, $\zeta = |\zeta|\, \theta$ that transforms I_2 in an integral of the form

$$\overline{I}_2 = \int_J \frac{u(x, \zeta)}{x - \theta}\, dx$$

where J is an interval of length of order one and u is such that $|u(x, \zeta)| + |u_x(x, \zeta)| = o(|\zeta - \zeta_0|^{\alpha-1})$ as $|\zeta| \to 0$. The standard regularity results for Cauchy integrals (cf. [M]) yields

$$|\overline{I}_2| = o(|\zeta - \zeta_0|^{\alpha-1}) \quad \text{as } |\zeta - \zeta_0|.$$

This provides the asymptotics,

$$K'(\zeta) = C(\zeta - \zeta_0)^{\alpha-1} + o(|\zeta - \zeta_0|^{\alpha-1}) \quad \text{as } |\zeta - \zeta_0| \to 0$$

and after integration,

$$K(\zeta) = K(0) + C(\zeta - \zeta_0)^{\alpha} + o(|\zeta - \zeta_0|^{\alpha}) \quad \text{as } |\zeta - \zeta_0| \to 0.$$

The rest of the cases can be analysed using similar arguments. We leave the details to the reader. □

Proof of Proposition 2.1. Using (2.14) as well as standard contour deformation we obtain

$$\lambda(t) = \frac{m}{2\pi i} \int_S e^{m\zeta t}\, \Lambda(\zeta)\, d\zeta.$$

where S is any contour in the complex plane $\mathbb{C}$, surrounding the interval $[-e^{\mu}, -e^{\mu}+1]$ in the counterclockwise sense. The asymptotics of $\lambda(t)$ depends on the behaviour of Λ near $\zeta = \zeta_0 = 1 - e^{\mu}$. Suppose first that we are not in the case $m = m_0$, $\rho_{in} > 0$. Then, by Proposition 3.1, the function Λ is bounded near ζ_0 and, making deformation of contours,

$$\lambda(t) = \frac{m}{2\pi i} \int_{\zeta_0 - \delta}^{\zeta_0} [\Lambda(\zeta - i0) - \Lambda(\zeta + i0)]\, e^{m\zeta t}\, d\zeta + \mathcal{O}(e^{(\zeta_0 - \delta)t}) \quad \text{as} \quad t \to \infty \tag{3.10}$$

for any δ, where

$$\Lambda(\zeta \pm i0) = \lim_{\varepsilon \to 0^+} \Lambda(\zeta \pm i\varepsilon).$$

using now the asymptotics of Λ given in Proposition 3.2 (3.1), (3.2) and (C. 10), we deduce, (2.15), (2.16) and (2.18).

Finally, if $m = m_0$, and $\rho_{in} > 0$ we must take into account that Λ has a simple pole at $\zeta = 0$ (cf. (3.3)). Therefore,

$$\lambda(t) = \frac{m}{2\pi i} \int_{\zeta_0 - \delta}^{\zeta_0} e^{m\zeta t}\, (\frac{A}{\zeta} + \mathcal{O}(|\log \zeta|))\, d\zeta + \mathcal{O}(e^{\zeta_1 t}) = m\, A + o(1) \quad \text{as } t \to \infty,$$

and (2.17) follows . □

Proof of Proposition 2.2 and Proposition 2.3. Using the representation formula (2.14),

$$h(k,t)=\frac{1}{2\pi i}\int_S H(k,\zeta)e^{m\zeta t}d\zeta$$

where S is any contour in the complex plane $\mathbb{C}$, surrounding the interval $[-e^{\mu},-e^{\mu}+1]$ in the counterclockwise sense. Using, (2.10)

$$h(k,t)=\frac{1}{2\pi i}\int_S\frac{g_{in}(k)e^{m\zeta t}}{\zeta+(e^{\mu}-e^{-k})}d\zeta+\frac{1}{2\pi i}\int_S\frac{g_{\mu}(k)\,\Lambda(\zeta)e^{m\zeta t}}{\zeta+(e^{\mu}-e^{-k})}d\zeta=h_1(k,t)+h_2(k,t)$$

The integral $h_1(k,t)$ is explicitly computed by elementary computations and gives,

$$h_1(k,t)=g_{in}(k)e^{-m(e^{\mu}-e^{-k})t}. \tag{3.11}$$

We now compute $h_2(k,t)$ for k in any compact set $\mathcal{K}$ de $(0,+\infty)$. Since k is of order one, $1-e^{-k}$ is uniformly bounded below, and then, making contour deformation, there exists $\delta>0$ which depend on $\mathcal{K}$, such that we can write
(3.12)

$$h_2(k,t)=\frac{g_{\mu}(k)}{2\pi i}\int_{\zeta_0-\delta}^{\zeta_0-r}\frac{(\Lambda(\zeta-i0)-\Lambda(\zeta+i0))e^{m\zeta t}}{\zeta+(e^{\mu}-e^{-k})}d\zeta+\frac{g_{\mu}(k)}{2\pi i}\int_{\partial B_r(\zeta_0)}\frac{\Lambda(\zeta)e^{m\zeta t}}{\zeta+(e^{\mu}-e^{-k})}d\zeta+\mathcal{O}(e^{m(\zeta_0-\delta)t})$$

as $t\to\infty$, for a fixed $r\in(0,\delta)$. Suppose first that $m=m_o$ and $\rho_{in}>0$.

$$\begin{aligned}\int_{\partial B_r(\zeta_0)}\frac{\Lambda(\zeta)e^{m\zeta t}}{\zeta+(e^{\mu}-e^{-k})}d\zeta=&a_1\int_{\partial B_r(\zeta_0)}\frac{e^{m\zeta t}}{\zeta(\zeta+(e^{\mu}-e^{-k}))}d\zeta+\int_{\partial B_r(\zeta_0)}(\Lambda(\zeta)-\frac{a_1}{\zeta})\frac{e^{m\zeta t}}{\zeta+(e^{\mu}-e^{-k})}d\zeta\\=&a_1g_{\mu}(k)+o(1),\quad\text{as}\quad t\to\infty.\end{aligned}$$

where a_1 is given in (3.3) of Proposition 3.2. In order to estimate the last integral we have collapsed the domain $\partial B_r(\zeta_0)$ towards the real line and use the local integrability of the function $(\Lambda(\zeta)-\frac{a_1}{\zeta})$ near the origin. On the other hand the first integral approaches exponentially to zero. On the other hand, the first integral in (3.12) decays exponentially fast as $t\to\infty$ in that case.

If $m=m_0$ and $\rho_{in}=0$ or $m\neq m_0$, we can take $r=0$ in (3.12) and using the asymptotics for Λ in Proposition 3.2 (cf. (3.5), (3.6) and (3.8)) we derive the remaining results of Proposition 2.3.

In order to conclude the proof of Proposition 2.2 it only remains to consider the term $h_2(k,t)$ in the region $0<kt<L$ for any $L>0$ fixed. Let us first consider the case where $\Lambda(\zeta)$ is bounded near $\zeta=\zeta_0$ (i.e. when we are not in the case $m=m_0$ and $\rho_{in}>0$). In that case we may write,

$$h_2(k,t)=\frac{g_{\mu}(k)\Lambda(\zeta_0)}{2\pi i}\int_S\frac{e^{m\zeta t}}{\zeta+(e^{\mu}-e^{-k})}d\zeta+\frac{g_{\mu}(k)}{2\pi i}\int_S\frac{(\Lambda(\zeta)-\Lambda(\zeta_0))e^{m\zeta t}}{\zeta+(e^{\mu}-e^{-k})}d\zeta. \tag{3.13}$$

By residues Theorem

$$\frac{g_{\mu}(k)\Lambda(\zeta_0)}{2\pi i}\int_S\frac{e^{m\zeta t}}{\zeta+(e^{\mu}-e^{-k})}d\zeta=g_{\mu}(k)\Lambda(\zeta_0)e^{m\zeta_0 t}e^{-m(1-e^{-k})t}.$$

To estimate the second one we make a deformation of the contour S to a new contour Γ that locally near $\zeta=\zeta_0$ behaves as two straight lines that intersect at the point ζ_0 and make an angle of $\pi/4$ with the negative real axes. Notice that by Proposition 3.2, in this case

$$|\Lambda(\zeta)-\Lambda(\zeta_0)|\leq C|\zeta-\zeta_0|^{\gamma}$$

for some $C > 0$ and $\gamma > 0$.Then

$$|\int_\Gamma \frac{(\Lambda(\zeta)-\Lambda(\zeta_0))e^{m\zeta t}}{\zeta+(e^\mu-e^{-k})}d\zeta| \le Ce^{m\zeta_0 t}\int_\Gamma \frac{|\zeta-\zeta_0|^\gamma e^{\frac{m}{\sqrt{2}}|\zeta-\zeta_0|t}}{\sqrt{|\zeta-\zeta_0|^2+k^2}}ds \le Ce^{m\zeta_0 t}t^{-\gamma} \quad \text{as } t\to\infty$$

and this gives a negligible contribution to $h_2(k,t)$ in the region $kt = \mathcal{O}(1)$. This yields all the asymptotics in Proposition 2.2 for $m \neq m_0$ or $m = m_0$ and $\rho_{in} = 0$.

Finally if $m = m_0$ and $\rho_{in} > 0$, we argue as in the Proof of Proposition 2.3 above, spliting the singular part a_1/ζ and the locally integrable part $\Lambda(\zeta) - a_1/\zeta$ (cf. (3.3)). The contribution from the singular part is explicitely computed by residue Theorem. The contribution from the locally integrable part is estimated using similar arguments as for the case $m \neq m_0$ or $m = m_0$ and $\rho_{in} = 0$. This concludes the proof of Proposition 2.2. □

APPENDIX.

In this Appendix we show by means of formal asymptotic analysis that the results obtained in the previous set of pages for the case b constant actually hold for more general kernels $b(k,k')$. Some of the results of this Appendix are reminiscent from [LY1], [LY2] and provide a more precise description for the asymptotics of the solutions than those provided by some of the computations therein.

We consider kernels b approaching a nonzero constant as $(k',k) \to (0,0)$. Without loss of generality we can therefore assume that

$$b(0,0) = 1.$$

For the sake of brevity we only consider the cases $m > m_0$. Finally, we shall impose on the kernel b the following conditions:

$$(A.1)\qquad \begin{aligned}&\text{there exists}\quad 0 < C_1 < C_2 < \infty \quad \text{such that:}\\ &C_1\int_0^\infty g_0(k')b(k,k')dk' \le b(k,0) \le C_2\int_0^\infty g_0(k')b(k,k')dk'.\end{aligned}$$

Assume that F solves (1.7). We write

$$F = g_0 + g.$$

Then g solves:

$$(A.3)\qquad \frac{\partial g}{\partial t} = \varepsilon(k,t)g_0 + g(e^{-k}-1)\int_0^\infty b(k,k')(g_0'+g')dk' + \varepsilon(k,t)g$$

where,

$$(A.4) = \qquad \varepsilon(k,t) = \int_0^\infty b(k,k')g'(1-e^{-k'})dk'.$$

Function g behaves in a very different manner in two scales of k. It turns out that for $k \sim 1$, g varies in a smooth manner and it contains a small amount of mass. But on the other hand, in the region $k \to 0^+$, g contains a large amount of mass. This suggest to decompose the function g in two pieces

$$(A.5)\qquad g = g_{\text{sing}}(k,t) + g_{\text{reg}}(k,t)$$

where g_{sing} provides the concentration of mass in the region $k \to 0^+$, and g_{reg} contains a small amount of mass and is uniformly distributed in the region $k > 0$.

Since g_{reg} does contain a small amount of mass it is natural to neglect it to the leading order in the integral term in which g_0 also appears in the right hand side of (A.3). We then approximate (A.3) as:

$$(A.6) \qquad \frac{\partial g}{\partial t} = \varepsilon(k,t) g_0 + g(e^{-k} - 1) \int_0^\infty b(k,k')(g_0' + g_{\text{sing}}') dk' + \varepsilon(k,t) g$$

In the outer region, $k \sim 1$, we neglect g compared to g_0. Moreover, assuming by analogy with the case $b \equiv 1$, that g_{reg} behaves algebraically in time as $t \to \infty$ (something that will be checked "a posteriori"), we could neglect the term $\partial g_{\text{reg}}/\partial t$. With these two approximations, it would follow from (A.6) that

$$(A.7) \qquad g_{\text{reg}}(k,t) \sim \frac{\varepsilon(k,t) g_0(k)}{(1-e^{-k}) \int_0^\infty [g_0(k') + g_{\text{sing}(k')}] b(k,k') dk'}$$

On the other hand, we can use the approximation $g_{\text{sing}}(k,t) \sim (m - m_0)\delta(k)$ in (A.7). Then

$$(A.8) \qquad g_{\text{reg}}(k,t) \sim \frac{\varepsilon(k,t) g_0(k)}{(1-e^{-k})[\int_0^\infty g_0(k') b(k,k') dk' + (m-m_0) b(k,0)]}.$$

In order to complete the description of the asymptotics of $g(k,t)$ we need to precise the function $\varepsilon(k,t)$ in (A.8). At a first glance one could expect to obtain the leading order behaviour of $\varepsilon(k,t)$ approximating g by $g_{\text{sing}} \sim (m - m_0)\delta(k)$ in (A.4), since the mass concentrated concentrated in g_{reg} is much smaller that the one in g_{sing}. Notice however that the term $(1 - e^{-k})$ in (A.4) vanishes at $k = 0$. As a consequence the detailed structure of $g_{\text{sing}}(k,t)$ and also $g_{\text{reg}}(k,t)$ play a role to determine $\varepsilon(k,t)$. We then proceed to compute more precisely the asymptotics of $g_{\text{sing}}(k,t)$ as $k \to 0^+$.

We approximate (A.3) as $k \to 0^+$ to the leading order by

$$\frac{\partial g_{\text{sing}}}{\partial t} = \varepsilon(0,t) k - k g_{\text{sing}} \int_0^\infty b(0,k')(g_0' + (m-m_0)\delta(k')) dk' + \varepsilon(0,t) g_{\text{sing}}.$$

Define:

$$h(k,t) = \exp(-\int_0^t \varepsilon(0,s) ds) g_{\text{sing}}(k,t)$$

and

$$\Gamma = \int_0^\infty b(0,k') g_0(k') dk' + (m - m_0), \qquad \lambda(t) = \varepsilon(0,t) e^{-\int_0^t \varepsilon(0,s) ds}$$

we arrive at

$$(A.10) \qquad \frac{\partial h}{\partial t} = \lambda(t) k - \Gamma k h,$$

whose solution is given by:

$$(A.11) \qquad h(k,t) = e^{-\Gamma k t}[h_0(k) + k \int_0^t e^{\Gamma k t (\frac{s}{t})} \lambda(s) ds].$$

For $kt = \mathcal{O}(1)$ and assuming that $\lambda(t) \to 0$ as $t \to \infty$ (to be checked "a posteriori"), it would follow from (A.11) the approximation

$$(A.12) \qquad h(k,t) = e^{-\Gamma k t}[h_0(k) + k \int_0^t \lambda(s) ds].$$

Notice that we could expect $\lambda(t)$ to be integrable since,

$$\int_0^\infty \lambda(t)dt = \int_0^\infty \varepsilon(0,t)e^{-\int_0^t \varepsilon(0,s)ds}dt$$
$$= -\int_0^\infty \frac{d}{dt}(e^{-\int_0^t \varepsilon(0,s)ds})dt = (1 - e^{-\int_0^\infty \varepsilon(0,s)ds}) = 1.$$

The convergence of the integral appearing in the last formula as well as the integrability of $\lambda(t)$ will be checked later "a posteriori". Notice that the required integrability took place in the case $b \equiv 1$.

The precise asymptotics for $h(k,t)$ in (A.12) depends in a very sensitive manner on the asymptotics of $h_0(k)$ as $k \to 0^+$, something that could be expected for a hyperbolic problem. In particular if $h_0(k) << k$ as $k \to 0^+$, the dynamics of $h(k,t)$ is determined by the last term in the sum of (A.12), and for $h_0(k) >> k$ as $k \to 0^+$, the term $h_0(k)$ is the dominant one. Notice that this was exactly the situation in the explicitly solvable case $b \equiv 1$. Of course, if $h_0(k)$ competes with k as $k \to 0^+$ all kinds of intermediate cases could take place.

Approximating g_{sing} by means of (A.9), (A.12) and g_{reg} using (A.8), we would obtain, from (A.4) the following equation for $\varepsilon(k,t)$

$$(A.13) \qquad \varepsilon(k,t) = \int_0^\infty \frac{b(k,k')\varepsilon(k',t)g_0(k')dk' =}{[\int_0^\infty g_0(k'')b(k',k'')dk'' + (m-m_0)b(k',0)]} + \gamma(t)e^{\int_0^t \varepsilon(0,s)ds}b(k,0).$$

$$(A.14) \qquad \gamma(t) = \beta(t) + \frac{2}{\Gamma^3 t^3}\int_0^\infty \lambda(s)ds, \quad \beta(t) = \int_0^\infty e^{-\Gamma k' t}k' h_0(k')dk'.$$

If $h_0(k') << k'$ as $k' \to 0^+$, $\beta(t) << 1/t^3$ and the term $\beta(t)$ is negligible as expected. If on the contrary $h_0(k') >> k'$, $\beta(t)$ is the dominant term in $\gamma(t)$.

The structure of (A.13) suggests to look for solutions of (A.13) in the form= ,

$$(A.15) \qquad \varepsilon(k,t) = \gamma(t)e^{\int_0^t \varepsilon(0,s)ds}H(k),$$

with $H(k)$ solution of

$$(A.16) \qquad H(k) = T[H](k) + b(k,0)$$

$$(A.17) \qquad T[H](k) = \int_0^\infty \frac{b(k,k')g_0(k')H(k')dk'}{[\int_0^\infty g_0(k'')b(k',k'')dk'' + (m-m_0)b(k',0)]}.$$

A formal solution of (A.16) can be obtained by means of the classical Neumann series:

$$(A.18) \qquad H(k) = \sum_{n=0}^\infty T^n[b(\cdot,0)](k).$$

It turns out that the series in (A.18) converges under rather general hypothesis on the kernel $b(k,k')$. Suppose, for instance that b satisfies (A.1) and let us briefly indicate how to use it in order to obtain the convergence of (A.18). Assume that

$$f(k') \le C\int_0^\infty g_0(k'')b(k',k'')dk''.$$

Then using (A.18), we immediately verify that the following inequality holds

$$T[f](k) \leq \frac{C}{(1+(m-m_0)C_1)} \int_0^\infty b(k,k')g_0(k')dk'$$

or, by iteration:

$$T^n[f](k) \leq \frac{C}{(1+(m-m_0)C_1)^n} \int_0^\infty b(k,k')g_0(k')dk'. \tag{A.19}$$

Using then the second inequality in (A.1) we readily obtain convergence of the series in (A.18) uniformly on $[0,+\infty)$ for b bounded.

Having computed $H(k)$ it only remains to compute $\varepsilon(0,t)$ in order to obtain a full description of $\varepsilon(k,t)$. Notice that (A.14), (A.15) give the following equation for $\varepsilon(0,t)$

$$\varepsilon(0,t) = \gamma(t)e^{\int_0^t \varepsilon(0,s)ds}H(0). \tag{A.20}$$

Since $b(0,0) = 1$ and $T^n[b(\cdot,0)](0) > 0$ for all $n \geq 1$, it follows from (A.19) that $H(0) > 1$. Under the assumption $e^{-\int_0^\infty \varepsilon(0,s)ds} = 0$ (that occurs if $b \equiv 1$, and will be checked in the general case "a posteriori") the solution of (A.20) turns out to be= :

$$\varepsilon(0,t) = \frac{\gamma(t)}{\int_t^\infty \gamma(s)ds}. \tag{A.21}$$

Notice that, for $h_0(k)$ behaving algebraically as $k \to 0^+$, the function $\gamma(t)$ also behaves algebraically as $t \to \infty$ (cf. (A.14)) and therefore, by (A.21) $\varepsilon(0,t) \sim 1/t$ as $t \to \infty$. This implies in articular all the integrability assumptions which where made above for $\lambda(t)$ and $\varepsilon(0,t)$ as $t \to \infty$.

We deduce now the asymptotics for g_{sing} and g_{reg}. Suppose first that $h_0(k) << k$ as $k \to 0$. Then, by (A.14), $\gamma(t) \sim 1/t^3$ as $t \to \infty$ and from (A.21),

$$\int_0^t \varepsilon(0,s)ds = \log \frac{\int_0^\infty \gamma(s)}{\int_t^\infty \gamma(s)ds}$$

and

$$e^{\int_0^t \varepsilon(0,s)ds} \sim [\int_0^\infty \gamma(s)ds]^{-1}t^2 \quad \text{as } t \to \infty,$$

from where,

$$g_{\text{sing}}(k,t) \sim \frac{\int_0^\infty \lambda(s)ds}{\int_0^\infty \gamma(s)ds} e^{-\Gamma kt}t^2 k \quad \text{for } kt = 0(1),\ t \to \infty, \tag{A.22}$$

$$g_{\text{reg}}(k,t) \sim \frac{\int_0^\infty \gamma(s)dsH(k)}{t(1-e^{-k})[\int_0^\infty g_0(k')b(k,k')dk' + (m-m_0)b(k,0)]} \quad \text{for } k = 0(1),\ t \to \infty. \tag{A.23}$$

If on the other hand, $h_0(k) \sim k^\alpha$ with $0 < \alpha < 1$ then $\gamma(t) \sim 1/t^{\alpha+2}$. In that case,

$$e^{\int_0^t \varepsilon(0,s)ds} \sim [\int_0^\infty \gamma(s)ds]^{-1}t^{1+\alpha} \quad \text{as } t \to \infty$$

and then

$$g_{\text{sing}}(k,t) \sim \frac{\int_0^\infty \lambda(s)ds}{\int_0^\infty \gamma(s)ds} e^{-\Gamma kt}t^{1+\alpha}k^\alpha \quad \text{for } kt = 0(1),\ t \to \infty \tag{A.24}$$

and

$$(A.25) \qquad g_{\text{reg}}(k,t) \sim \frac{\int_0^\infty \gamma(s)dsH(k)}{t(1-e^{-k})[\int_0^\infty g_0(k')b(k,k')dk' + (m-m_0)b(k,0)]} \quad \text{for } k = 0(1),\ t \to \infty.$$

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