Homogeneous Boltzmann eqeation in quantum relativistic kinetic theory *

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Abstract

Some mathematical questions about Boltzmann equations for quantum particles, relativistic or not, are considered. Relevant particular cases as Bose, Bose-Fermi, photon-electron gases are studied as well as some simplifications (such as the isotropy of the distribution functions) and asymptotic limits (systems where one of the species is at equilibrium) of special interest from a physical point of view and which give rise to interesting mathematical questions. New results about the existence and long time behaviour of the solutions to some of these problems are exposed.

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1 Introduction

When quantum methods are applied to molecular encounters, some divergence from the classical results appear. It is then necessary in some cases to modify the classical theory in order to account for the quantum effects which are present in the collision processes; see [11, Sec. 17], where the domain of applicability of the classical kinetic theory is discussed in detail. In spite of their formal similarity, the equations for classical and quantum kinetic theory display very different features. Surprisingly, the appropriate Boltzmann equations, which account for quantum effects, have received scarce attention in the mathematical literature.

In this work, we consider some mathematical questions about Boltzmann equations for quantum particles, relativistic and not relativistic. The general interest in different models involving that kind of equations has increased recently. This is so because they are supposedly reliable for computing non equilibrium

properties of Bose-Einstein condensates on sufficiently large times and distance scales; see for example [32, 48, 49] and references therein. We study some relevant particular cases (Bose, Bose-Fermi, photon-electron gases), simplifications such as the isotropy of the distribution functions, and asymptotic limits (systems where one of the species is at equilibrium) which are important from a physical point of view and give rise to interesting mathematical questions.

Since quantum and classic or relativistic particles are involved, we are lead to consider such a general type of equations. We first consider the homogeneous Boltzmann equation for a quantum gas constituted by a single specie of particles, bosons or fermions. We solve the entropy maximization problem under the moments constraint in the general quantum relativistic case. The question of the well posedness, i.e. existence, uniqueness, stability of solutions and of the long time behavior of the solutions is also treated in some relevant particular cases. One could also consider other qualitative properties such as regularity, positivity, eternal solution in a purely kinetic perspective or study the relation between the Boltzmann equation and the underlying quantum field theory, or a more phenomenological description, such as the based on hydrodynamics, but we do not go further in these directions.

1.1   The Boltzmann equations

To begin with, we focus our attention on a gas composed of identical and indiscernible particles. When two particles with respective momentum p and p_* in $\mathbb{R}^3$ encounter each other, they collide and we denote p' and p'_* their new momenta after the collision. We assume that the collision is elastic, which means that the total momentum and the total energy of the system constituted by this pair of particles are conserved. More precisely, denoting by $\mathcal{E}(p)$ the energy of one particle with momentum p, we assume that

$$
\begin{aligned}
p' + p'_* &= p + p_* \\
\mathcal{E}(p') + \mathcal{E}(p'_*) &= \mathcal{E}(p) + \mathcal{E}(p_*).
\end{aligned}
\tag{1.1}
$$

We denote $\mathcal{C}$ the set of all 4-tuplets of particles $(p, p_*, p', p'_*) \in \mathbb{R}^{12}$ satisfying (1.1). The expression of the energy $\mathcal{E}(p)$ of a particle in function of its momentum p depends on the type of the particle;

$$
\mathcal{E}(p) = \mathcal{E}_{nr}(p) = \frac{|p|^2}{2\,m} \quad \text{for a non relativistic particle,}
$$

$$
\mathcal{E}(p) = \mathcal{E}_r(p) = \gamma m c^2; \ \gamma = \sqrt{1 + (|p|^2/c^2 m^2)} \quad \text{for a relativistic particle,}
$$

$$
\mathcal{E}(p) = \mathcal{E}_{ph}(p) = c|p| \quad \text{for massless particle such as a photon or neutrino.}
$$
$$
\tag{1.2}
$$

Here, m stands for the mass of the particle and c for the velocity of light. The velocity $v = v(p)$ of a particle with momentum p is defined by $v(p) = \nabla_p \mathcal{E}(p)$,

and therefore

$$v(p) = v_{nr}(p) = \frac{p}{m} \quad \text{for a non relativistic particle,}$$

$$v(p) = v_r(p) = \frac{p}{m\gamma} \quad \text{for a relativistic particle,} \tag{1.3}$$

$$v(p) = v_{ph}(p) = c\frac{p}{|p|} \quad \text{for a photon .}$$

Now we consider a gas constituted by a very large number (of order the Avogadro number $A \sim 10^{23}/\text{mol}$) of a single specie of identical and indiscernible particles. The very large number of particles makes impossible (or irrelevant) the knowledge of the position and momentum (x, p) (with x in a domain $\Omega \subset \mathbb{R}^3$ and $p \in \mathbb{R}^3$) of every particle of the gas. Then, we introduce $f = f(t, x, p) \geq 0$, the gas density distribution of particles which at time $t \geq 0$ have position $x \in \mathbb{R}^3$ and momentum $p \in \mathbb{R}^3$. Under the hypothesis of molecular chaos and of low density of the gas, so that particles collide by pairs (no collision between three or more particles occurs), Boltzmann [5] established that the evolution of a classic (i.e. no quantum nor relativistic) gas density f satisfies

$$\frac{\partial f}{\partial t} + v(p) \cdot \nabla_x f = Q(f(t, x, .))(p)$$
$$f(0, .) = f_{\text{in}}, \tag{1.4}$$

where $f_{\text{in}} \geq 0$ is the initial gas distribution and $Q(f)$ is the so-called Boltzmann collision kernel. It describes the change of the momentum of the particles due to the collisions.

A similar equation was proposed by Nordheim [42] in 1928 and by Uehling & Uhlenbeck [52] in 1933 for the description of a quantum gas, where only the collision term $Q(f)$ had to be changed to take into account the quantum degeneracy of the particles. The relativistic generalization of the Boltzmann equation including the effects of collisions was given by Lichnerowicz and Marrot [38] in 1940.

Although this is by no means a review article we may nevertheless give some references for the interested readers. For the classical Boltzmann equation we refer to Villani's recent review [55] and the rather complete bibliography therein. Concerning the relativistic kinetic theory, we refer to the monograph [51] by J. M. Stewart and the classical expository text [29] by Groot, Van Leeuwen and Van Weert. A mathematical point of view, may be found in the books by Glassey [25] and Cercignani and Kremer [9]. In [31], Jüttner gave the relativistic equilibrium distribution. Then, Ehlers in [17], Tayler & Weinberg in [53] and Chernikov in [12] proved the H-theorem for the relativistic Boltzmann equation. The existence of global classical solutions for data close to equilibrium is shown by R. Glassey and W. Strauss in [27]. The asymptotic stability of the equilibria is studied in [27], and [28]. For these questions see also the book [25]. The global existence of renormalized solutions is proved by Dudynsky and Ekiel Jezewska in [16]. The asymptotic behaviour of the global solutions is also considered in [2].

In all the following we make the assumption that the density f only depends on the momentum. The collision term $Q(f)$ may then be expressed in all the cases described above as

$$Q(f)(p) = \iiint_{\mathbb{R}^9} W(p, p_*, p', p'_*) q(f) \, dp_* dp' dp'_*$$

$$q(f) \equiv q(f)(p, p_*, p', p'_*) = [f' f'_* (1 + \tau f)(1 + \tau f_*) - f f_* (1 + \tau f')(1 + \tau f'_*)]$$

$$\tau \in \{-1, 0, 1\},$$

(1.5)

where as usual, we denote:

$$f = f(p), \quad f_* = f(p_*), \quad f' = f(p'), \quad f'_* = f(p'_*),$$

and W is a non negative measure called transition rate, which may be written in general as:

$$W(p, p_*, p', p'_*) = w(p, p_*, p', p'_*) \delta(p + p_* - p' - p'_*) \delta(\mathcal{E}(p) + \mathcal{E}(p_*) - \mathcal{E}(p') - \mathcal{E}(p'_*))$$

(1.6)

where δ represents the Dirac measure. The quantity $W \, dp' dp'_*$ is the probability for the initial state $|p, p_*\rangle$ to scatter and become a final state of two particles whose momenta lie in a small region $dp' \, dp'_*$.

The character relativistic or not, of the particles is taken into account in the expression of the energy of the particle $\mathcal{E}(p)$ given by (1.2). The effects due to quantum degeneracy are included in the term $q(f)$ when $\tau \neq 0$, and depend on the bosonic or fermionic character of the involved particles. These are associated with the fact that, in quantum mechanics, identical particles cannot be distinguished, not even in principle. For dense gases at low temperature, this kind of terms are crucial. However, for non relativistic dilute gases, quantum degeneracy plays no role and can be safely ignored ($\tau = 0$).

The function w is directly related to the differential cross section σ (see (5.11)), a quantity that is intrinsic to the colliding particles and the kind of interaction between them. The calculation of σ from the underlying interaction potential is a central problem in non relativistic quantum mechanics, and there are a few examples of isotropic interactions (the Coulomb potential, the delta shell,...) which have an exact solution. However, in a complete relativistic setting or when many-body effects due to collective dynamics lead to the screening of interactions, the description of these in terms of a potential is impossible. Then, the complete framework of quantum field theory (relativistic or not) must be used in order to perform perturbative computations of the involved scattering cross section in w. We give some explicit examples in the Appendix 8 but let us only mention here the case $w = 1$ which corresponds to a non relativistic short range interaction (see Appendix 8). Since the particles are indiscernible, the collisions are reversible and the two interacting particles form a closed physical system. We have then:

$$W(p, p_*, p', p'_*) = W(p_*, p, p', p'_*) = W(p', p'_*, p, p_*)$$

$$+ \quad \text{Galilean invariance (in the non relativistic case)} \quad (1.7)$$

$$+ \quad \text{Lorentz invariance (in the relativistic case)}.$$

To give a sense to the expression (1.5) under general assumptions on the distribution f is not a simple question in general. Let us only remark here that $Q(f)$ is well defined as a measure when f and w are assumed to be continuous. But we will see below that this is not always a reasonable assumption. It is one of the purposes of this work to clarify this question in part.

The Boltzmann equation reads then very similar, formally at least, in all the different contexts: classic, quantum and relativistic. In particular some of the fundamental physically relevant properties of the solutions f may be formally established in all the cases in the same way: conservation of the total number of particles, mean impulse and total energy; existence of an "entropy function" which increases along the trajectory (Boltzmann's H-Theorem). For any $\psi = \psi(p)$, the symmetries (1.7), imply the fundamental and elementary identity

$$\int_{\mathbb{R}^3} Q(f)\psi \, dp = \frac{1}{4} \iiint_{\mathbb{R}^{12}} W(p, p_*, p', p'_*) \, q(f)\big(\psi + \psi_* - \psi' - \psi'_*\big) \, dp \, dp_* \, dp' \, dp'_*. \tag{1.8}$$

Taking $\psi(p) = 1$, $\psi(p) = p_i$ and $\psi(p) = \mathcal{E}(p)$ and using the definition of $\mathcal{C}$, we obtain that the particle number, the momentum and the energy of a solution f of the Boltzmann equation (1.5) are conserved along the trajectoires, i.e.

$$\frac{d}{dt} \int_{\mathbb{R}^3} f(t, p) \begin{pmatrix} 1 \\ p \\ \mathcal{E}(p) \end{pmatrix} dp = \int_{\mathbb{R}^3} Q(f) \begin{pmatrix} 1 \\ p \\ \mathcal{E}(p) \end{pmatrix} dp = 0, \tag{1.9}$$

so that

$$\int_{\mathbb{R}^3} f(t, p) \begin{pmatrix} 1 \\ p \\ \mathcal{E}(p) \end{pmatrix} dp = \int_{\mathbb{R}^3} f_{\text{in}}(p) \begin{pmatrix} 1 \\ p \\ \mathcal{E}(p) \end{pmatrix} dp. \tag{1.10}$$

The entropy functional is defined by

$$H(f) := \int_{R^3} h(f(p)) \, dp, \quad h(f) = \tau^{-1}(1 + \tau f)\ln(1 + \tau f) - f \ln f. \tag{1.11}$$

Taking in (1.8) $\psi = h'(f) = \ln(1 + \tau f) - \ln f$, we get

$$\int_{\mathbb{R}^3} Q(f) \, h'(f) \, dp = \frac{1}{4} \, D(f) \tag{1.12}$$

with

$$D(f) = \iiiint_{\mathbb{R}^{12}} W \, e(f) \, dp \, dp_* \, dp' \, dp'_*$$
$$e(f) = j\big(ff_*(1 + \tau f')(1 + \tau f'_*), f'f'_*(1 + \tau f)(1 + \tau f_*)\big) \tag{1.13}$$
$$j(s, t) = (t - s)(\ln t - \ln s) \geq 0.$$

We deduce from the equation that the entropy is increasing along trajectories, i.e.

$$\frac{d}{dt} H(f(t, .)) = \frac{1}{4} D(f) \geq 0. \tag{1.14}$$

The main qualitative characteristics of f are described by these two properties: conservation (1.9) and increasing entropy (1.14). It is therefore natural to expect that as t tends to ∞ the function f converges to a function f_∞ which realizes the maximum of the entropy $H(f)$ under the moments constraint (1.10). A first simple and heuristic remark is that if f_∞ solves the entropy maximization problem with constraints (1.10), there exist Lagrange multipliers $\mu \in \mathbb{R}$, $\beta^0 \in \mathbb{R}$ and $\beta \in \mathbb{R}^3$ such that

$$\langle \nabla H(f_\infty), \varphi \rangle = \int_{\mathbb{R}^3} h'(f_\infty)\varphi \, dp = \langle \beta^0 \mathcal{E}(p) - \beta \cdot p - \mu, \varphi \rangle \quad \forall \varphi,$$

which implies

$$\ln(1 + \tau f_\infty) - \ln f_\infty = \beta^0 \mathcal{E}(p) - \beta \cdot p - \mu$$

and therefore

$$f_\infty(p) = \frac{1}{e^{\nu(p)} - \tau} \quad \text{with } \nu(p) := \beta^0 \mathcal{E}(p) - \beta \cdot p - \mu. \tag{1.15}$$

The function f_∞ is called a Maxwellian when $\tau = 0$, a Bose-Einstein distribution when $\tau > 0$ and a Fermi-Dirac distribution when $\tau < 0$.

1.2 The classical case

Let us consider for a moment the case $\tau = 0$, i.e. the classic Boltzmann equation, which has been widely studied. It is known that for any initial data f_{in} there exists a unique distribution f_∞ of the form (1.15) such that

$$\int_{\mathbb{R}^3} f_\infty(p) \begin{pmatrix} 1 \\ p \\ \mathcal{E}(p) \end{pmatrix} dp = \int_{\mathbb{R}^3} f_{\text{in}}(p) \begin{pmatrix} 1 \\ p \\ \mathcal{E}(p) \end{pmatrix} dp. \tag{1.16}$$

We may briefly recall the main results about the Cauchy problem and the long time behaviour of the solutions which are known up to now. We refer to [8, 39], for a more detailed exposition and their proofs.

Theorem 1.1 (Stationary solutions) *For any measurable function $f \geq 0$ such that*

$$\int_{\mathbb{R}^3} f(1, p, \frac{|p|^2}{2}) \, dp = (N, P, E) \tag{1.17}$$

for some $N, E > 0$, $P \in \mathbb{R}^3$, the following four assertions are equivalent:

(i) f is the Maxwellian

$$\mathcal{M}_{N,P,E} = \mathcal{M}[\rho, u, \Theta] = \frac{\rho}{(2\pi\Theta)^{3/2}} \exp\left(-\frac{|p - u|^2}{2\Theta}\right)$$

where (ρ, u, Θ) is uniquely determined by $N = \rho$, $P = \rho u$, and $E = \frac{\rho}{2}(|u|^2 + 3\Theta)$;

(ii) f is the solution of the maximization problem

$$H(f) = \max\{H(g), \ g \ \text{satisfies the moments equation (1.10)}\},$$

where $H(g) = -\int_{\mathbb{R}^3} g \log g \, dp$ stands for the classical entropy;

(iii) $Q(f) = 0$;

(iv) $D(f) = 0$.

Concerning the evolution problem one can prove.

Theorem 1.2 *Assume that $w = 1$ (for simplicity). For any initial data $f_{\text{in}} \geq 0$ with finite number of particles, energy and entropy, there exists a unique global solution $f \in C([0, \infty); L^1(\mathbb{R}^3))$ which conserves the particle number, energy and momentum. Moreover, when $t \to \infty$, $f(t, .)$ converges to the Maxwellian M with same particle number, momentum an energy (defined by Theorem 1.1) and more precisely, for any $m > 0$ there exists $C_m = C_m(f_0)$ explicitly computable such that*

$$\|f - M\|_{L^1} \leq \frac{C_m}{(1+t)^m}. \tag{1.18}$$

We refer to [3, 17, 41, 40] for existence, conservations and uniqueness and to [4, 55, 56, 8, 54] for convergence to the equilibrium. Also note that Theorem 1.2 can be extended (sometimes only partially) to a large class of cross-section W we refer to [55] for details and references.

Remark 1.3 The proof of the equivalence (i) - (ii) only involves the entropy $H(f)$ and not the collision integral $Q(f)$ itself.

Remark 1.4 To show that (i), (iii) and (iv) are equivalent one has first to define the quantities $Q(f)$ and $D(f)$ for the functions f belonging to the physical functional space. The first difficulty is to define precisely the collision integral $Q(f)$, (see Section 3.2).

1.3   Quantum and/or relativistic gases

The Boltzmann equation looks formally very similar in the different contexts: classic, quantum and relativistic, but it actually presents some very different features in each of these different contexts. The two following remarks give some insight on these differences.

The natural spaces for the density f are the spaces of distributions $f \geq 0$ such that the "physical" quantities are bounded:

$$\int_{\mathbb{R}^3} f(1 + \mathcal{E}(p)) \, dp < \infty \quad \text{and} \quad H(f) < \infty, \tag{1.19}$$

where H is given by (1.11). This provides the following different conditions:

$f \in L_s^1 \cap L \log L$ in the non quantum case, relativistic or not

$f \in L_s^1 \cap L^\infty$ in the Fermi case, relativistic or not

$f \in L_s^1$ in the Bose case, relativistic or not,

where

$$L_s^1 = \{f \in L^1(\mathbb{R}^3); \int_{\mathbb{R}^3} (1 + |p|^s)\, d|f|(p) < \infty\} \qquad (1.20)$$

and $s = 2$ in the non relativistic case, $s = 1$ in the relativistic case.

On the other hand, remember that the density entropy h given by (1.11) is:

$$h(f) = \tau^{-1}(1 + \tau f)\ln(1 + \tau f) - f \ln f.$$

In the Fermi case we have $\tau = -1$ and then $h(f) = +\infty$ whenever $f \notin [0, 1]$. Therefore the estimate $H(f) < \infty$ provides a strong L^∞ bound on f. But, in the Bose case, $\tau = 1$. A simple calculus argument then shows that $h(f) \sim \ln f$ as $f \to \infty$. Therefore the entropy estimate $H(f) < \infty$ does not gives any additional bound on f.

Moreover, and still concerning the Bose case, the following is shown in [7], in the context of the Kompaneets equation (cf. Section 5). Let $a \in \mathbb{R}^3$ be any fixed vector and $(\varphi_n)_{n \in \mathbb{N}}$ an approximation of the identity:

$$(\varphi_n)_{n \in \mathbb{N}}; \quad \varphi_n \to \delta_a.$$

Then for any $f \in L_2^1$, the quantity $H(f + \varphi_n)$ is well defined by (1.11) for all $n \in \mathbb{N}$ and moreover,

$$N(f + \alpha\varphi_n) \to N(f) + \alpha, \quad \text{and} \quad H(f + \varphi_n) \to H(f) \quad \text{as} \quad n \to \infty. \quad (1.21)$$

See Section 2 for the details. This indicates that the expression of H given in (1.11) may be extended to nonnegative measures and that, moreover, the singular part of the measure does not contributes to the entropy. More precisely, for any non negative measure F of the form $F = g\,dp + G$, where $g \geq 0$ is an integrable function and $G \geq 0$ is singular with respect to the Lebesgue measure dp, we define the Bose-Einstein entropy of F by

$$H(F) := H(g) = \int_{\mathbb{R}^3} \big[(1 + g)\ln(1 + g) - g \ln g\big]\, dp. \qquad (1.22)$$

The discussion above shows how different is the quantum from the non quantum case, and even the Bose from the Fermi case. Concerning the Fermi gases, the Cauchy problem has been studied by Dolbeault [15] and Lions [39], under the hypothesis (H1) which includes the hard sphere case $w = 1$. As it is indicated by the remark above, the estimates at our disposal in this case are even better than in the classical case. In particular the collision term $Q(f)$ may be defined in the same way as in the classical case. But as far as we know, no analogue of Theorem 1.1 was known for Fermi gases. The problem for Bose gases is essentially open as we shall see below. Partial results for radially symmetric L^1 distributions have been obtained by Lu [40] under strong cut off assumptions on the function w.

1.3.1 Equilibrium states, Entropy

As it is formally indicated by the identity (1.9), the particle number, momentum and energy of the solutions to the Boltzmann equation are conserved along the trajectories. It is then very natural to consider the following entropy maximization problem: given $N > 0$, $P \in \mathbb{R}^3$ and $E \in \mathbb{R}$, find a distribution f which maximises the entropy H and whose moments are (N, P, E). The solution of this problem is well known in the non quantum non relativistic case (and is recalled in Theorem 1.1 above). In [31], Jüttner in [Ju] gave the relativistic Maxwellians. The question is also treated by Chernikov in [12]. For the complete resolution of the moments equation in the relativistic non quantum case we refer to Glassey [26] and Glassey & W. Strauss [GS]. We solve the quantum relativistic case in [24]. The general result may be stated as follows.

Theorem 1.5 *For every possible choice of (N, P, E) such that the set K defined by*

$$g \in K \quad \text{if and only if} \quad \int_{\mathbb{R}^3} g(1, p, \frac{|p|^2}{2}) \, dp = (N, P, E),$$

is non empty, there exists a unique solution $\overline{f}$ to the entropy maximization problem

$$\overline{f} \in K, \quad H(\overline{f}) = \max\{H(g); \, g \in K\}.$$

Moreover, $\overline{f} = f_\infty$ given by (1.15) for the nonquantum and Fermi case, while for the Bose case $\overline{f} = f_\infty + \alpha \delta_{\overline{p}}$ for some $\overline{p} \in \mathbb{R}^3$.

It was already observed by Bose and Einstein [5, 18, 19] that for systems of bosons in thermal equilibrium a careful analysis of the statistical physics of the problem leads to enlarge the class of steady distributions to include also the solutions containing a Dirac mass. On the other hand, the strong uniform bound introduced by the Fermi entropy over the Fermi distributions leads to include in the family of Fermi steady states the so called degenerate states. We present in Section 2 the detailed mathematical results of these two facts both for relativistic and non relativistic particles. The interested reader may find the detailed proofs in [24].

1.3.2 Collision kernel, Entropy dissipation, Cauchy Problem

Theorem 1.5 is the natural extension to quantum particles of the results for non quantum particles, i.e. points (i) and (ii) of Theorem 1.1. The extension of the points (iii) and (iv), even for the non relativistic case, is more delicate. In the Fermi case it is possible to define the collision integral $Q(f)$ and the entropy dissipation $D(f)$ and to solve the problem under some additional conditions (see Dolbeault [15] and Lions [39]). We consider this problem and related questions in Section 3.2.

In the equation for bosons, the first difficulty is to define the collision integral $Q(f)$ and the entropy dissipation $D(f)$ in a sufficiently general setting. This question was treated by Lu in [40] and solved under the following additional assumptions:

(i) $f \in L^1$ is radially symmetric.

(ii) Strong truncation on w.

These two conditions are introduced in order to give a sense to the collision integral. Unfortunately, the second one is not satisfied by the main physical examples such as $w = 1$ (see Appendix 8). Moreover Theorem 2.3 shows that the natural framework to study the quantum Boltzmann equation, relativistic or not, for Bose gases is the space of non negative measures. This is an additional difficulty with respect to the non quantum or Fermi cases. We partly extend the study of Lu to the case where f is a non negative radially symmetric measure.

1.4 Two species gases, the Compton-Boltzmann equation

Gases composed of two different species of particles, for example bosons and fermions, are interesting by themselves for physical reasons and have thus been considered in the physical literature (see the references below). On the other hand, from a mathematical point of vue, they provide simplified but still interesting versions of Boltzmann equations for quantum particles. Their study may be then a first natural step to understand the behaviour of this type of equations.

Let us then call $F(t, p) \geq 0$ the density of Bose particles and $f(t, p) \geq 0$ that of Fermi particles. Under the low density assumption, the evolution of the gas is now given by the following system of Boltzmann equations (see [11]):

$$
\begin{aligned}
\frac{\partial F}{\partial t} &= Q_{1,1}(F, F) + Q_{1,2}(F, f) \quad F(0, .) = F_{\text{in}} \\
\frac{\partial f}{\partial t} &= Q_{2,1}(f, F) + Q_{2,2}(f, f) \quad f(0, .) = f_{\text{in}}.
\end{aligned}
\tag{1.23}
$$

The collision terms $Q_{1,1}(F, F)$ and $Q_{2,2}(f, f)$ stand for collisions between particles of the same specie and are given by (1.2). The collision terms $Q_{1,2}(F, f)$ and $Q_{2,1}(f, F)$ stands for collisions between particles of two different species:

$$
Q_{1,2}(F, f) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} W_{1,2} q_{1,2} \, dp_* dp' dp'_*,
\tag{1.24}
$$
$$
q_{1,2} = F' f'_*(1 + F)(1 - \tau f_*) - F f_*(1 + F')(1 - \tau f'_*),
$$

and $Q_{2,1}(f, F)$ is given by a similar expression. Note that the measure $W_{1,2} = W_{1,2}(p, p_*, p', p'_*)$ satisfies the micro-reversibility hypothesis

$$
W_{1,2}(p', p'_*, p, p_*) = W_{1,2}(p, p_*, p', p'_*),
\tag{1.25}
$$

but not the indiscernibility hypothesis $W_{1,2}(p_*, p, p', p'_*) = W_{1,2}(p, p_*, p', p'_*)$ as in (1.7) since the two colliding particles belong now to different species.

We consider in Section 4 some mathematical questions related to the systems (1.23)-(1.25). We do not perform in detail the general study of the steady states since that would be mainly a repetition of what is done in Section 3 and in [24].

Here again, to give a sense to the integral collision $Q_{1,2}$ and $Q_{2,1}$ is the first question to be considered. Since the kernels $Q_{1,1}(F, F)$ and $Q_{2,2}(f, f)$ have already been treated in the precedent section, we focus in the collision terms $Q_{1,2}(F, f)$ and $Q_{2,1}(f, F)$. Let us notice that in the Fermi case, $\tau > 0$, the Fermi density f satisfies an a priori bound in L^∞. Nevertheless, even with this extra estimate, the problem of existence of solutions and their asymptotic behaviour for generic interactions, even with strong unphysical truncation kernel and for radially symmetric distributions f, remains an open question.

In order to get some insight on these problems, we consider two simpler situations which are important from a physical point of view and still mathematically interesting since, in particular, they display Bose condensation in infinite time. These are the equations describing boson-fermion interactions with fermions at equilibrium, and photon-electron Compton scattering. Of course the deductions of these two reduced models are well known in the physical literature but we believe nevertheless that it may be interesting to sketch them here. In the first one, still considered in Section 4, we suppose that the Fermi particles are at rest at isothermal equilibrium. This is nothing but to fix the distribution of Fermi particles f in the system to be a Maxwellian or a Fermi state. Without any loss of generality, this may be chosen to be centered at the origin, so that it is radially symmetric. Moreover, the boson-fermion interaction is short range and we may consider the "slow particle interaction" approximation of the differential cross section $w = 1$ (see Appendix 8). The system reduces then to a single equation which moreover is quadratic and not cubic. Namely:

$$\frac{\partial F}{\partial t} = \int_0^\infty S(\varepsilon, \varepsilon') \big[F'(1 + F)\, e^{-\epsilon} - F(1 + F')\, e^{-\epsilon'} \big]\, d\epsilon', \qquad (1.26)$$

for some kernel S (see Section 4).

We prove in Section 4 the following result about existence, uniqueness and asymptotic behaviour of global solutions for the Cauchy problem associated to (1.26) .

Theorem 1.6 *For any initial datum $F_{\text{in}} \in L_1^1(\mathbb{R}_+)$, $F_{\text{in}} \geq 0$, there exists a solution $F \in C([0, \infty), L_{1/2}^1))$ to the equation (1.26) such that*

$$\lim_{t \to 0} \|F(t) - F_{\text{in}}\|_{L^1(\mathbb{R}^+)} = 0.$$

Moreover, if $\overline{f} = \mathcal{B}_N$ is the unique solution to the maximization problem

$$H(\overline{f}) = \max\{ H(g); \int g(\varepsilon)\, \varepsilon^2\, d\varepsilon = \int F_{\text{in}}(\varepsilon)\, \varepsilon^2\, d\varepsilon =: N \},$$

$$H(F) = \int_0^\infty h(f, \varepsilon) \varepsilon^2 d\varepsilon$$

with $h(x, \varepsilon) = (1 + x) \ln(1 + x) - x \ln x - \varepsilon x$,

$$F(t, .) \underset{t \to \infty}{\rightharpoonup} \overline{f} \ weakly \star \ in \ \big(C_c(\mathbb{R}_+) \big)'$$
$$\lim_{t \to \infty} \|F(t, .) - \overline{f}\|_{L^1((k_0, \infty))} = 0 \quad \forall k_0 > 0. \tag{4.51}$$

This result shows that the density of bosons F underlies a Bose condensation asymptotically in infinite time if its initial value is large enough. The phenomena was already predicted by Levich & Yakhot in [36, 37], and was described as condensation driven by the interaction of bosons with a cold bath (of fermions) (see also Semikoz & Tkachev [48, 49]).

1.4.1 Compton scattering

In Section 5 we consider the equation describing the photon-electron interaction by Compton scattering. This equation, that we call Boltzmann-Compton equation, has been extensively studied in the physical and mathematical literature (see in particular the works by Kompaneets [33], Dreicer [15], Weymann [56], Chapline, Cooper and Slutz [10]). We show how it can be derived starting from the system which describes the photon electron interaction via Compton scattering. This interaction is described, in the non relativistic limit, by the Thomson cross section, (see Appendix 8).

It is important in this case to start with the full relativistic quantum formulation since photons are relativistic particles. Even if, later on, the electrons are considered at non relativistic classical equilibrium. Finally, The equation has the same form as in (1.26) where the only difference lies in the kernel S.

The possibility of some kind of "condensation" for this Compton Boltzmann equation was already considered in physical literature by Chapline, Cooper and Slutz in [10], and Caflisch & Levermore [7] for the Kompaneets equation (see Section 5 and also [21]).

We end with the so called Kompaneets equation, which is the limit of the Boltzmann-Compton equation in the range $|p|, |p'| \ll mc^2$.

2 The entropy maximization problem

In this Section we describe the solution to the maximization problem for the entropy function under the moment constraint. This problem may be stated as follows.

> Given any of the entropies H and of the energies $\mathcal{E}$ defined in the introduction, given three quantities $N > 0$, $E > 0$ and $P \in \mathbb{R}^3$, find $F \geq 0$ such that
>
> $$\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ p \\ \mathcal{E}(p) \end{pmatrix} F(p)\, dp = \begin{pmatrix} N \\ P \\ E \end{pmatrix} \qquad (2.1)$$
>
> and
>
> $$H(F) = \max\{H(g) : g \text{ satisfies } (2.1)\}. \qquad (2.2)$$

We consider successively the case of a relativistic non quantum gas, then the case of a Bose-Enstein gas, and last the case of a Fermi-Dirac gas. For each of

these two kind of gases, we first consider in detail the relativistic case, where the energy is given by

$$\mathcal{E}(p) = \sqrt{1 + \frac{|p|^2}{c^2 m^2}}. \tag{2.3}$$

Then we describe the non relativistic case, $\mathcal{E}(p) = |p|^2/2m$, which is simplest since by Galilean invariance it can be reduced to $P = 0$.

In order to avoid lengthy technical details which are unnecessary for our purpose here, we only present the results for each of the different cases. For the detailed proofs the reader is referred to [24].

The relativistic non quantum case was completely solved by Glassey and Strauss in [27], see also Glassey in [Gl, even in the non homogeneous case with periodic spatial dependence. However, the proof that we give in [24] is different, uses in a crucial way the Lorentz invariance and may be adapted to the quantum relativistic case. Finally, notice that we do not consider this entropy problem for a gas of photons ($\mathcal{E}(p) = |p|$ and H the Bose-Einstein entropy) since it would not have physical meaning. In Section 5 we discuss the entropy problem for a gas constituted of electrons and photons.

2.1   Relativistic non quantum gas

In this subsection, we consider the Maxwell-Boltzmann entropy

$$H(g) = -\int_{\mathbb{R}^3} g \ln g \, dp. \tag{2.4}$$

of a non quantum gas. From the heuristic argument presented in the introduction, the solution to (2.1)-(2.2) is expected to be a relativistic Maxwellian distribution:

$$\mathcal{M}(p) = e^{-\beta^0 p^0 + \beta \cdot p - \mu}. \tag{2.5}$$

The result is the following.

Theorem 2.1   *(i) Given $E, N > 0$, $P \in \mathbb{R}^3$, there exists a least one function $g \geq 0$ which solves the moments equation (2.1) if, and only if,*

$$m^2 c^2 \, N^2 + |P|^2 < E^2. \tag{2.6}$$

When (2.6) holds we will say that (N, P, E) is admissible.

(ii) For an admissible (N, P, E) there exists at least one relativistic Maxwellian distribution $\mathcal{M}$ satisfying (2.1).

(iii) Let $\mathcal{M}$ be a relativistic Maxwellian distribution. For any function $g \geq 0$ satisfying

$$\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ p \\ p^0 \end{pmatrix} g(p) \, dp = \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ p \\ p^0 \end{pmatrix} \mathcal{M}(p) \, dp, \tag{2.7}$$

one has

$$H(g) - H(\mathcal{M}) = H(g|\mathcal{M}) := \int_{\mathbb{R}^3} [g \ln \frac{g}{\mathcal{M}} - g + \mathcal{M}] \, dp. \qquad (2.8)$$

Moreover, $H(g|\mathcal{M}) \leq 0$ and vanishes if, and only if, $g = \mathcal{M}$.

(iv) As a conclusion, for an admissible (N, P, E), the relativistic Maxwellian constructed in (ii) is the unique solution to the entropy maximization problem (2.1)-(2.4).

2.2 Bose gas

We consider now a gas of Bose particles. As it has been said before, Bose [5] and Einstein [18, 19], noticed that in this case, the set of steady distributions had to include solutions containing a Dirac mass. It is then necessary to extend the entropy function H defined in (1.11) with $\tau = 1$ to such distributions. The way to do this may be well understood with the following remark from [7].

Let $a \in \mathbb{R}^3$ be any fixed vector and $(\varphi_n)_{n \in \mathbb{N}}$ an approximation of the identity:

$$(\varphi_n)_{n \in \mathbb{N}}; \quad \varphi_n \to \delta_a.$$

For any $f \in L_2^1$ and every $n \in \mathbb{N}$ the quantity $H(f + \varphi_n)$ is well defined by (1.11). Moreover,

$$H(f + \varphi_n) \to H(f) \quad \text{as} \quad n \to \infty.$$

Suppose, for the sake of simplicity that $\varphi_n \equiv 0$ if $|p - a| \geq 2/n$, then

$$H(f + \varphi_n) = \int_{|p-a| \geq 2/n} h(f(p,t), p) \, dp + \int_{|p-a| \leq 2/n} h((f(p,t) + \varphi_n(p), p) \, dp.$$

Let us call, $\sigma(z) = (1 + z) \ln(1 + z) - z \ln z$. Using $|\sigma(z)| \leq c\sqrt{z}$ we obtain

$$\int_{|p-a| \leq 2/n} |\sigma(f(p,t) + \varphi_n(p))| dp \leq c \frac{2}{\sqrt{n^3}} \Big(\int_{|p-a| \leq 2/n} (f(p,t) + \varphi_n(p) dp \Big)^{1/2} \to 0.$$

Then

$$\Big| \int_{\mathbb{R}^3} h(f(p,t), p) dp - \int_{|p-a| \geq 2/n} h(f(p,t), p) dp \Big| \leq \int_{|p-a| \leq 2/n} |h(f(p,t), p)| dp \to 0$$

which completes the proof.

This indicates that the expression of H given in (1.11) may be extended to nonnegative measures and that the singular part of the measure does not contributes to the entropy. More precisely, for any non negative measure F of the form $F = g dp + G$, where $g \geq 0$ is an integrable function and $G \geq 0$ is singular with respect to the Lebesgue measure dp, we define the Bose-Einstein entropy of F by

$$H(F) := H(g) = \int_{\mathbb{R}^3} \big[(1 + g) \ln(1 + g) - g \ln g \big] \, dp. \qquad (2.9)$$

On the other hand, as we have seen in the Introduction, the regular solutions to the entropy maximization problem should be the Bose relativistic distributions

$$b(p) = \frac{1}{e^{\nu(p)} - 1} \quad \text{with} \quad \nu(p) = \beta^0 p^0 - \beta \cdot p + \mu. \tag{2.10}$$

The following result explains where the Dirac masses have now to be placed (see [24] for the proof).

Lemma 2.2 *The Bose relativistic distribution b is non negative and belongs to $L^1(\mathbb{R}^3)$ if, and only if, $\beta^0 > 0$, $|\beta| < \beta^0$ and $\mu \geq \mu_\mathbf{b}$ with $\mu_\mathbf{b} := -mc\mathbf{b}$, $\mathbf{b} > 0$ and $\mathbf{b}^2 = (\beta^0)^2 - |\beta|^2$. In this case, all the moments of b are well defined. Finally,*

$$\nu(p) > \nu(p_{m,c}) \geq 0 \quad \forall p \neq p_{m,c} := \frac{mc\beta}{\sqrt{\beta^{0\,2} - |\beta|^2}}. \tag{2.11}$$

We define now the generalized Bose-Einstein relativistic distribution $\mathcal{B}$ by

$$\mathcal{B}(p) = b + \alpha \delta_{p_{m,c}} = \frac{1}{e^{\nu(p)} - 1} + \alpha \delta_{p_{m,c}} \tag{2.12}$$

with

$$\nu(p) = \beta^0 p_0 - \beta \cdot p + \mu > \nu(p_{m,c}) \geq 0 \quad \forall p \neq p_{m,c}, \tag{2.13}$$

and the condition $\alpha \nu(p_{m,c}) = 0$.

Theorem 2.3 *(i) Given $E, N > 0$, $P \in \mathbb{R}^3$, there exists at least one measure $F \geq 0$ which solves the moments equation (2.1) if, and only if,*

$$m^2 c^2 \, N^2 + |P|^2 \leq E^2. \tag{2.14}$$

When (2.14) holds we will say that (N, P, E) is a admissible.

(ii) For any admissible (N, P, E) there exists at least one relativistic Bose-Einstein distribution $\mathcal{B}$ satisfying (2.1).

(iii) Let $\mathcal{B}$ be a relativistic Bose-Einstein distribution. For any measure $F \geq 0$ satisfying

$$\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ p \\ p^0 \end{pmatrix} dF(p) = \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ p \\ p^0 \end{pmatrix} d\mathcal{B}(p), \tag{2.15}$$

one has

$$H(F) - H(\mathcal{B}) = H_1(g|b) + H_2(G|b) \tag{2.16}$$

where

$$H_1(g|b) := \int_{\mathbb{R}^3} \left((1+g) \ln \frac{1+g}{1+b} - g \ln \frac{g}{b} \right) dp,$$
$$H_2(G|b) := -\int_{\mathbb{R}^3} \nu(p) \, dG(p). \tag{2.17}$$

Moreover, $H(F|\mathcal{B}) \leq 0$ and vanishes if, and only if, $F = \mathcal{B}$. In particular, $H(F) < H(\mathcal{B})$ if $F \neq \mathcal{B}$.

(iv) As a conclusion, for an admissible (N, P, E), the relativistic Bose-Einstein distribution constructed in (ii) is the unique solution to the entropy maximization problem (2.1)-(2.3), (2.9).

2.2.1 Nonrelativistic Bose particles

For non relativistic particles the energy is $\mathcal{E}(p) = |p|^2/2m$. By Galilean invariance the problem (2.1) is then equivalent to the following simpler one: given three quantities $N > 0$, $E > 0$, $P \in \mathbb{R}^3$ find $F(p)$ such that

$$\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ p - \frac{P}{N} \\ |p - \frac{P}{N}|^2/(2m) \end{pmatrix} F(p)\, dp = \begin{pmatrix} N \\ 0 \\ E - (|P|^2/(2mN)) \end{pmatrix}. \tag{2.18}$$

It is rather simple, using elementary calculus, to prove that for any $E, N > 0$, $P \in \mathbb{R}^3$ there exists a distribution of the from

$$F(p) = \frac{1}{e^{a|p - \frac{P}{N}|^2 + b^+} - 1} - b^- \delta_{\frac{P}{N}} \tag{2.19}$$

with $a \in \mathbb{R}$, $b \in \mathbb{R}$, $\nu \in \mathbb{R}$, $b^+ = \max(b, 0)$, $b^- = -\max(-b, 0)$ which satisfies (2.18). Once such a solution (2.18) of (2.19) is obtained, the following Bose-Einstein distribution

$$\mathcal{B}(p) = \frac{1}{e^{\nu(p)} - 1} + \alpha \delta_{p_{m,c}}, \quad \nu(p) = a|p|^2 - \frac{2a}{N} \cdot p + (b^+ + \frac{a|P|^2}{N^2}) \tag{2.20}$$

solves (2.1). This shows that for non relativistic particles Theorem 2.3 remains valid under the unique following change: the statements (i) and (ii) have to be replaced by

(i') For every $E, N > 0$, $P \in \mathbb{R}^3$, there exists one relativistic Bose Einstein distribution defined by (2.19) corresponding to these moments, i.e. satisfying (2.1).

Statements (iii) and (iv) of Theorem 2.3 remain unchanged.

2.3 Fermi-Dirac gas

In this subsection we consider the entropy maximization problem for the Fermi-Dirac entropy

$$H_{FD}(f) := -\int_{\mathbb{R}^3} \big((1-f)\ln(1-f) + f\ln f\big)\, dp. \tag{2.21}$$

In particular, this implies the constraint $0 \leq f \leq 1$ on the density f of the gas.

From the heuristics argument presented in the introduction, we know that the solution $\mathcal{F}$ of (2.1)-(2.3), (2.21) is the Fermi-Dirac distribution

$$\mathcal{F}(p) = \frac{1}{e^{\nu(p)} + 1} \quad \text{with} \quad \nu(p) = \beta^0 p^0 - \beta \cdot p + \mu. \tag{2.22}$$

We also introduce the "saturated" Fermi-Dirac (SFD) density

$$\chi(p) = \chi_{\beta^0,\beta}(p) = \mathbf{1}_{\{\beta^0 p^0 - \beta \cdot p \leq 1\}} = \mathbf{1}_{\mathcal{E}} \text{ with } \mathcal{E} = \{\beta^0 p^0 - \beta \cdot p \leq 1\}, \quad (2.23)$$

with $\beta \in \mathbb{R}^3$ and $\beta^0 > |\beta|$.

Our main result is the following.

Theorem 2.4 *(i) For any P and E such that $|P| < E$ there exists an unique SFD state $\chi = \chi_{P,E}$ such that $P(\chi) = P$ and $E(\chi) = E$. This one realizes the maximum of particle number for given energy E and mean momentum P. More precisely, for any f such that $0 \leq f \leq 1$ one has*

$$P(f) = P, \quad E(f) = E \quad implies \quad N(f) \leq N(\chi_{P,E}). \quad (2.24)$$

As a consequence, given (N, P, E) there exists F satisfying the moments equation (2.1) if, and only if, $E > |P|$ and $0 \leq N \leq N(\chi_{P,E})$. In this case, we say that (N, P, E) is admissible.

(ii) For any (N, P, E) admissible there exists a Fermi-Dirac state $\mathcal{F}$ ("saturated" or not) which solves the moments equation (2.1).

(iii) Let $\mathcal{F}$ be a Fermi-Dirac state. For any f such that $0 \leq f \leq 1$ and

$$\int_{\mathbb{R}^3} f(p) \begin{pmatrix} 1 \\ p \\ p^0 \end{pmatrix} dp = \int_{\mathbb{R}^3} \mathcal{F}(p) \begin{pmatrix} 1 \\ p \\ p^0 \end{pmatrix} dp,$$

one has

$$\begin{aligned} H_{FD}(f) - H_{FD}(\mathcal{F}) &= H_{FD}(f|\mathcal{F}) \\ &:= \int_{\mathbb{R}^3} \left((1-f) \ln \frac{1-f}{1-\mathcal{F}} - f \ln \frac{f}{\mathcal{F}} \right) dp. \end{aligned} \quad (2.25)$$

(iv) As a conclusion, for any admissible (N, P, E), the relativistic Fermi-Dirac distribution constructed in (ii) is the unique solution to the entropy maximization problem (2.1)-(2.3), (2.21).

The new difficulty with respect to the classic or the Bose case is to be managed with the constraint $0 \leq f \leq 1$.

2.3.1 Nonrelativistic Fermi-Dirac particles

Here again, since the energy is $\mathcal{E}(p) = |p^2|/2m$ the problem (2.1) is equivalent to (2.18): given three quantities $N > 0$, $E > 0$, $P \in \mathbb{R}^3$ find $f(p)$ such that $0 \leq f \leq 1$ and satisfying (2.18). One may then check that for non relativistic particles Theorem 2.4 remains valid under the unique following change: the statements (i) and (ii) have to be replaced by

(i') For every $E, N > 0$, $P \in \mathbb{R}^3$, satisfying $5E \geq 3^{5/3}(4\pi)^{2/3} N^{5/3}$ there exists a non relativistic fermi Dirac state, saturated or not, defined by

$$\mathcal{F}(p) = \begin{cases} \frac{1}{e^{\nu(p)}+1}, \text{where} \\ \nu(p) = a|p|^2 - \frac{2a}{N} P \cdot p + (b + \frac{a|P|^2}{N^2}) & \text{if } E > \frac{3^{5/3}(4\pi)^{2/3}}{5} N^{5/3}, \\ \mathbf{1}_{\{|p - \frac{P}{N}| \leq c\}} & \text{if } E = \frac{3^{5/3}(4\pi)^{2/3}}{5} N^{5/3} \end{cases}$$

corresponding to these moments, i.e., satisfying (2.1).

Statements (iii) and (iv) of Theorem 2.4 remain unchanged.

3   The Boltzmann equation for one single specie of quantum particles

We consider now the homogeneous Boltzmann equation for quantum non relativistic particles, and treat both Fermi-Dirac and Bose-Einstein particles. We begin with the Fermi-Dirac Boltzmann equation for which we may slightly improve the existence result of Dolbeault [JD]and Lions [39]. We also state a very simple (and weak) result concerning the long time behavior of solutions. We finally consider the Bose-Einstein Boltzmann equation. We discuss the work of Lu [40] and slightly extend some of its results to the natural framework of measures

Consider the non relativistic quantum Boltzmann equation

$$\frac{\partial f}{\partial t} = Q(f) = \iiint_{\mathbb{R}^9} w \delta_{\mathcal{C}} q(f) \, dv_* dv' dv'_*,$$

$$q(f) = \left(f' f'_*(1 + \tau\, f)(1 + \tau\, f_*) - f f_*(1 + \tau\, f')(1 + \tau\, f'_*) \right)$$

(3.1)

where $\tau = \pm 1$ and $\mathcal{C}$ is the non relativistic collision manifold of $\mathbb{R}^{12}$, $(\mathcal{C})$:

$$v_* + v'_* = v + v_*$$
$$\frac{|v_*|^2}{2} + \frac{|v'_*|^2}{2} = \frac{|v|^2}{2} + \frac{|v_*|^2}{2}.$$

(3.2)

We assume, without any loss of generality in this Section that the mass m of the particles is one.

3.1   The Boltzmann equation for Fermi-Dirac particles

We first want to give a mathematical sense to the collision operator Q in (3.1) under the physical natural bounds on the distribution f. Of course, if f is smooth (say $C_c(\mathbb{R}^3)$) and w is smooth (for instance $w = 1$) the collision term $Q(f)$ is defined in the distributional sense as it has been mentioned in the Introduction. But, as we have already seen, the physical space for the densities of Fermi- Dirac particles is $L_2^1 \cap L^\infty(\mathbb{R}^3)$.

To give a pointwise sense to the formula (3.1), we first recall the following elementary argument from [26]. After integration with respect to the v'_* variable in (3.1) we have

$$Q(f) = \iint_{\mathbb{R}^6} wq(f)\delta_{\left\{\frac{|v'|^2}{2} + \frac{|v+v_*-v'|^2}{2} - \frac{|v|^2}{2} - \frac{|v_*|^2}{2} = 0\right\}} dv_* dv'.$$

By the change of variable $v' \to (r, \omega)$ with $r \in \mathbb{R}$, $\omega \in S^2$, and $v' = v + r\omega$, and using Lemma 3, we obtain

$$\begin{aligned}
Q(f) &= \frac{1}{2} \int_{\mathbb{R}^3} \int_{S^2} \int_0^\infty wq(f)\delta_{\{r(r-(v_*-v)\cdot\omega)=0\}} r^2 \, dr d\sigma dv_* \\
&= \int_{\mathbb{R}^3} \int_{S^2} w \frac{|(v_* - v, \omega)|}{2} q(f) \, d\omega dv_*,
\end{aligned} \tag{3.3}$$

where v' and v'_* are defined by

$$v' = v + (v_* - v, \omega)\omega, \quad v'_* = v_* - (v_* - v, \omega)\omega. \tag{3.4}$$

Formula (3.3) gives a pointwise sense to $Q(f)$, say for $f \in C_c(\mathbb{R}^3)$.

To extend the definition of $Q(f)$ to measurable functions we make the following assumptions on the cross-section:

$$B = \frac{1}{2}w|(v_* - v, \omega)| \text{ is a function of } v_* - v \text{ and } \omega, \tag{3.5}$$

$$B \in L^1(\mathbb{R}^3 \times S^2). \tag{3.6}$$

Though (3.5) is a natural assumption in view of Section 3.1, (3.6) is a strong restriction, in particular it does not hold when $w = 1$. With these assumptions, first introduced in [15], we now explain how to give a sense to the collision term $Q(f)$ when $f \in L^1 \cap L^\infty$. In one hand, since $\Phi : (v, v_*, \omega) \mapsto (v', v'_*, \omega)$ (with v', v'_* given by (3.4)) is a C^1-diffeomorphism on $\mathbb{R}^3 \times \mathbb{R}^3 \times S^2$ with jacobian $\text{Jac}\Phi = 1$, we clearly have that $(v, v_*, \omega) \mapsto f'f'_*$ is a measurable function of $\mathbb{R}^3 \times \mathbb{R}^3 \times S^2$. On the other hand, performing a change of variable, we get

$$\iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} Bf'f'_* \, dv dv_* d\omega = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} ff_* \left(\int_{S^2} B \, d\omega\right)(v_* - v) \, dv dv_*$$

$$\leq \|f\|_{L^1}\|f\|_{L^\infty}\|B\|_{L^1} < \infty,$$

and by the Fubini-Tonelli Theorem,

$$\int_{S^2} Bf'f' \, d\omega \in L^1(\mathbb{R}^3 \times \mathbb{R}^3).$$

That gives a sense to the gain term

$$Q^+(f) = \int_{\mathbb{R}^3} (1-f)(1-f_*)\left(\int_{S^2} Bf'f' \, d\omega\right) dv_*$$

as an L^1 function. The same argument gives a sense to the loss term $Q^-(f)$ as an L^1 function.

Note that, under the assumptions $B \in L^1_{\text{loc}}(\mathbb{R}^3 \times S^2)$ and

$$\frac{1}{1+|z|^2} \iint_{B_R^v \times S^2} B(z+v, \omega) \, d\omega dv \xrightarrow[|z| \to \infty]{} 0 \quad \forall R > 0, \tag{3.7}$$

one may give a sense to $Q^\pm(f)$ as a function of $L^1(B_R)$ $(\forall R > 0)$ for any $f \in L^1_2 \cap L^\infty$, see [39]. In particular, the cross-section B associated to $w = 1$ satisfies (3.7).

Finally, we can make a third assumption on the cross-section, namely

$$0 \le B(z, \omega) \le (1 + |z|^\gamma)\zeta(\theta), \text{ with } \gamma \in (-5, 0), \quad \int_0^{\pi/2} \theta\zeta(\theta) \, d\theta < \infty. \tag{3.8}$$

This assumption allows singular cross-sections, both in the z variable and the θ variable, near the origin. In that case, the collision term may be defined as a distribution as follows:

$$\int_{\mathbb{R}^3} Q(f)\varphi \, dv = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} ff_*(1 - f' - f'_*) \, B \, K_\varphi \, dvdv_*d\omega \tag{3.9}$$

with the notation

$$K_\varphi = \varphi' + \varphi'_* - \varphi - \varphi_*. \tag{3.10}$$

To see that (3.9) is well defined we note that (see e.g. [55]),

$$|K_\varphi| \le C_\varphi |v - v_*|^2 \theta, \tag{3.11}$$

from where,

$$|ff_*(1 - f' - f'_*) \, B \, K_\varphi| \le C_\varphi \theta\zeta(\theta)ff_*(|v-v_*|^2 + |v-v_*|^{\gamma+2}) \in L^1(\mathbb{R}^3 \times \mathbb{R}^3 \times S^2). \tag{3.12}$$

To see this last claim, we just put $g(z) = |z|^a$; if $a \in (-3, 0]$ one has $g \in L^1 + L^\infty$ and therefore $f(f \star g) \in L^1$ when $f \in L^1 \cap L^\infty$, and if $a \in [0, 2]$ then writing $g(v - v_*) \le (1 + |v|^2)(1 + |v_*|^2)$ we see that $f(f \star g) \in L^1$ when $f \in L^1_2$.

Theorem 3.1 *Assume that one of the conditions (3.6), (3.7) or (3.8) holds. Then, for any $f_{\text{in}} \in L^1_2(\mathbb{R}^3)$ such that $0 \le f_{\text{in}} \le 1$ there exists a solution $f \in C([0, +\infty); L^1(\mathbb{R}^3))$ to equation (3.1). Furthermore,*

$$\int_{\mathbb{R}^3} f(t, v) \, dv = \int_{\mathbb{R}^3} f_{\text{in}} \, dv =: N,$$

$$\int_{\mathbb{R}^3} f(t, v)v \, dv = \int_{\mathbb{R}^3} f_{\text{in}}(v)v \, dv =: P, \tag{3.13}$$

$$\int_{\mathbb{R}^3} f(t, v)\frac{|v|^2}{2} \, dv \le \int_{\mathbb{R}^3} f_{\text{in}}(v)\frac{|v|^2}{2} \, dv =: E,$$

and

$$\int_0^\infty \tilde{D}(f)\, dt \le C(f_{\text{in}}), \qquad (3.14)$$

where

$$\tilde{D}(f) := \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} B_2 (f'f'_*(1-f)(1-f_*) - ff_*(1-f')(1-f'_*))^2 \, d\omega dv_* dv. \quad (3.15)$$

Remark 3.2 In the non homogeneous context, the existence of solutions has been proved by Dolbeault [15] under the assumption (3.6) and by Lions [39] under the assumption (3.7). Existence in a classical context under assumption (3.8) (without Grad's cut-off) has been established by Arkeryd [4], Goudon [25] and Villani [55]. Finally, bounds on modified entropy dissipation term of the kind of (3.15) have been introduced by Lu [40] for the Boltzmann-Bose equation.

Concerning the behavior of the solutions we prove the following result.

Theorem 3.3 *For any sequence* (t_n) *such that* $t_n \to +\infty$ *there exists a subsequence* $(t_{n'})$ *and a stationary solution* $\mathcal{S}$ *such that*

$$f(t_{n'} + ., .) \underset{n' \to \infty}{\rightharpoonup} \mathcal{S} \quad in \ C([0, T]; L^1 \cap L^\infty(\mathbb{R}^3) - weak) \quad \forall T > 0 \quad (3.16)$$

and

$$\int_{\mathbb{R}^3} \mathcal{S}\, dv = N, \quad \int_{\mathbb{R}^3} \mathcal{S}v\, dv = P, \quad \int_{\mathbb{R}^3} \mathcal{S}\frac{|v|^2}{2}\, dv \le E. \qquad (3.17)$$

Remark 3.4 By stationary solution we mean a function $\mathcal{S}$ satisfying

$$\mathcal{S}'\mathcal{S}'_*(1-\mathcal{S})(1-\mathcal{S}_*) = \mathcal{S}\mathcal{S}_*(1-\mathcal{S}')(1-\mathcal{S}'_*). \qquad (3.18)$$

Of course, such a function is, formally at least, a Fermi-Dirac distribution, we refer to [15] for the proof of this claim in a particular case. Theorem 3.3 also holds for the non homogeneous Boltzmann-Fermi equation, when, for example, the position space is the torus, and under assumption (3.7) on B (in order to have existence).

Open questions:

1. Is Theorem 3.1 true under assumption (3.8) for all $\gamma \in (-5, -2]$?

2. Is it possible to prove the entropy identity (1.14) instead of the modified entropy dissipation bound (3.14)? Of course (1.14) implies the dissipation entropy bound (3.14) as it will be clear in the proof of Theorem 3.1.

3. Is any function satisfying (3.18) a Fermi-Dirac distribution?

4. Is it true that
$$\sup_{[0,\infty)} \int_{\mathbb{R}^3} f(t, v)|v|^{2+\epsilon}\, dv \le C(f_{\text{in}}, \epsilon)$$

for some $\epsilon > 0$? Notice that with such an estimate one could prove the conservation of the energy (instead of (3.17)). If we could also give a positive answer to the question 3, we could then prove the convergence to the Fermi-Dirac distribution $\mathcal{F}_{N,P,E}$.

5. Finally, is it possible to improve the convergence (3.16), and prove for instance strong L^1 convergence?

Proof of Theorem 3.1 Suppose that B satisfies (3.6), (3.7) or (3.8) and define $B_\epsilon = B\mathbf{1}_{\theta > \epsilon}\mathbf{1}_{\epsilon < |z| < 1/\epsilon}$. Notice that B_ϵ satisfies (3.6), $0 \leq B_\epsilon \leq B$ and $B_\epsilon \to B$ a.e. From [15] there exists a sequence of solutions (f_ϵ) to (3.1) corresponding to (B_ϵ). Moreover, for any $\epsilon > 0$, the solution f_ϵ satisfies (3.13) and (1.14). As it is shown by Lions in [39]:

$$
\begin{aligned}
a_\epsilon f'_\epsilon f'_{\epsilon*}(1 - f_\epsilon - f_{\epsilon*}) &\underset{n' \to \infty}{\rightharpoonup} af'f'_*(1 - f - f_*) \quad L^1 \quad \text{weak}, \\
a_\epsilon f_\epsilon f_{\epsilon*}(1 - f'_\epsilon - f'_{\epsilon*}) &\underset{n' \to \infty}{\rightharpoonup} aff_*(1 - f' - f'_*) \quad L^1 \quad \text{weak},
\end{aligned}
\tag{3.19}
$$

for any sequence (f_ϵ) such that $f_\epsilon \rightharpoonup f$ in $L^1 \cap L^\infty$, and any sequence (a_ϵ) satisfying (3.8) uniformly in ϵ and such that $a_\epsilon \to a$ a.e. In particular, under the assumption (3.8) on B and taking $a_\epsilon = B_\epsilon, a = B$, it is possible to pass to the limit $\epsilon \to 0$ in the equation (3.1). That gives existence of a solution f to the Fermi-Boltzmann equation for B satisfying (3.8).

Now, for B satisfying (3.8), we just write

$$
\int_{\mathbb{R}^3} Q_\epsilon(f_\epsilon)\varphi \, dv
$$

$$
= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} f_\epsilon f_{\epsilon*}(1 - f'_\epsilon - f'_{\epsilon*}) \, B_\epsilon \, K_\varphi \, \mathbf{1}_{\{\theta \geq \delta, |v - v_*| \geq \delta\}} \, dv dv_* d\omega
$$

$$
+ \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} f_\epsilon f_{\epsilon*}(1 - f'_\epsilon - f'_{\epsilon*}) \, B_\epsilon \, K_\varphi \left(\mathbf{1}_{\{\theta \leq \delta, |v - v_*| \geq \delta\}} + \mathbf{1}_{\{|v - v_*| \leq \delta\}}\right) dv dv_* d\omega
$$

$$
\equiv \mathcal{Q}_{\delta,\epsilon} + r_{\delta,\epsilon}.
$$

For the first term $\mathcal{Q}_{\delta,\epsilon}$ we easily pass to the limit $\epsilon \to 0$ using (3.19). For the second term $r_{\delta,\epsilon}$, using (3.11), we have

$$
\lim_{\epsilon \to 0} r_{\delta,\epsilon}
$$

$$
\leq \lim_{\epsilon \to 0} C_\varphi \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\epsilon f_{\epsilon*}(|v - v_*|^2 + |v - v_*|^{\gamma+2})\left(\int_{S^2} \theta\zeta(\theta)\mathbf{1}_{\{\theta \leq \delta\}} \, d\omega\right) dv dv_*
$$

$$
+ \lim_{\epsilon \to 0} C_\varphi \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\epsilon f_{\epsilon*}(|v - v_*|^2 + |v - v_*|^{\gamma+2})\mathbf{1}_{\{|v_* - v| \leq \delta\}}\left(\int_{S^2} \theta\zeta(\theta) \, d\omega\right) dv dv_*
$$

$$
\leq C_{\varphi,f} \int_{S^2} \theta\zeta(\theta)\mathbf{1}_{\{\theta \leq \delta\}} \, d\omega
$$

$$
+ C_{\varphi,\zeta} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\epsilon f_{\epsilon*}(|v - v_*|^2 + |v - v_*|^{\gamma+2})\mathbf{1}_{\{|v_* - v| \leq \delta\}} \, dv dv_*.
$$

Therefore,

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^3} Q_\epsilon(f_\epsilon)\varphi \, dv$$
$$= \int_{\mathbb{R}^3}\!\!\int_{\mathbb{R}^3}\!\!\int_{S^2} f f_*(1 - f' - f'_*) \, B \, K_\varphi \, \mathbf{1}_{\{\theta \geq \delta, |v - v_*| \geq \delta\}} \, dv dv_* d\omega + r_\delta,$$

with $r_\delta \to 0$ when $\delta \to 0$. Since we also have

$$\int_{\mathbb{R}^3} Q(f)\varphi \, dv = \int_{\mathbb{R}^3}\!\!\int_{\mathbb{R}^3}\!\!\int_{S^2} f f_*(1 - f' - f'_*) \, B \, K_\varphi \, \mathbf{1}_{\{\theta \geq \delta, |v - v_*| \geq \delta\}} \, dv dv_* d\omega + \tilde{r}_\delta,$$

with $\tilde{r}_\delta \to 0$ when $\delta \to 0$, we conclude that

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}^3} Q_\epsilon(f_\epsilon)\varphi \, dv = \int_{\mathbb{R}^3} Q(f)\varphi \, dv$$

and f is a solution to (3.1). We refer to [39, 55] for more details. To establish (3.14), we take $a_\epsilon = a = \sqrt{B_\delta}$ in (3.19) and we then have

$$\sqrt{B_\delta}[f'_\epsilon f'_{\epsilon*}(1 - f_\epsilon - f_{\epsilon*}) - f_\epsilon f_{\epsilon*}(1 - f'_\epsilon - f'_{\epsilon*})]$$
$$\xrightarrow[\epsilon \to 0]{} \sqrt{B_\delta}[f' f'_*(1 - f - f_*) - f f_*(1 - f' - f'_*)]. \qquad (3.20)$$

Using the elementary inequality

$$(b - a)^2 \leq j(a, b) \quad \forall \, a, b \in [0, 1],$$

we have for $\epsilon \in (0, \delta]$

$$\int_0^\infty \tilde{D}_\delta(f_\epsilon) \, dt \leq \int_0^\infty \tilde{D}_\epsilon(f_\epsilon) \, dt \leq \int_0^\infty D_\epsilon(f_\epsilon) \, dt \leq C(f_{\text{in}}). \qquad (3.21)$$

Gathering (3.20) and (3.21) we obtain, using the convexity of $s \mapsto s^2$,

$$\int_0^\infty \tilde{D}_\delta(f) \, dt \leq \liminf_{\epsilon \to 0} \int_0^\infty \tilde{D}_\delta(f_\epsilon) \, dt \leq C(f_{\text{in}}),$$

and we recover (3.15) letting $\delta \to 0$. $\qquad \square$

Proof of Theorem 3.3 Let consider $f_n = f(t + t_n, .)$ as in the statement of the Theorem 3.3. We know that there exists n' and $\mathcal{S}$ such that $f_{n'} \rightharpoonup_{n' \to \infty} \mathcal{S}$ weakly in $L^1 \cap L^\infty((0, T) \times \mathbb{R}^3)$ for all $T > 0$, and we only have to identify the limit $\mathcal{S}$. On one hand, $\mathcal{S}$ satisfies the moments equation (3.17). On the other hand by (3.21) and lower semicontinuity we get

$$\int_0^T \tilde{D}_\delta(\mathcal{S}) \, ds \leq \liminf \int_0^T \tilde{D}(f_{n'}) \, ds \leq \liminf \int_{t_{n'}}^{T + t_{n'}} \tilde{D}(f) \, ds = 0$$

for any $\delta > 0$, so that

$$\mathcal{S}'\mathcal{S}'_*(1-\mathcal{S})(1-\mathcal{S}_*)-\mathcal{S}\mathcal{S}_*(1-\mathcal{S}')(1-\mathcal{S}'_*) = \mathcal{S}'\mathcal{S}'_*(1-\mathcal{S}-\mathcal{S}_*)-\mathcal{S}\mathcal{S}_*(1-\mathcal{S}'-\mathcal{S}'_*) = 0$$

for a.e. $(v, v_*, \omega) \in \mathbb{R}^3 \times \mathbb{R}^3 \times S^2$. In particular, $\frac{\partial \mathcal{S}}{\partial t} = Q(\mathcal{S}) = 0$ and $\mathcal{S}$ is a constant function in time. We finally improve the convergence of $f_{n'}$ to $\mathcal{S}$ and establish (3.16). To this end we note that for any $\psi \in C_c(\mathbb{R}^3)$,

$$\frac{d}{dt} \int_{\mathbb{R}^3} f_{n'} \psi\, dv = \int_{\mathbb{R}^3} Q(f_{n'}) \psi\, dv$$

and the right hand side is bounded in $L^1(0, T)$. Therefore, $\int_{\mathbb{R}^3} f_n \psi\, dv$ is bounded in $BV(0, T)$. We can then extract a second subsequence (not relabeled) such that $\left(\int_{\mathbb{R}^3} f_{n'} \psi\, dv \right)$ converges strongly $L^1(0, T)$ and a.e. on $(0, T)$ and the limit is obviously $\int_{\mathbb{R}^3} \mathcal{S} \psi\, dv$. Let $\tau \in (0, T)$ be such that $\left(\int_{\mathbb{R}^3} f_{n'} \psi\, dv \right) \to \int_{\mathbb{R}^3} \mathcal{S} \psi\, dv$. We deduce that

$$\int_{\mathbb{R}^3} f_{n'}(t, .) \psi\, dv = \int_{\mathbb{R}^3} f_{n'}(\tau) \psi\, dv - \int_{\tau}^{t} \int_{\mathbb{R}^3} Q(f_{n'}) \psi\, dvds$$

and using P. L. Lions' result we get

$$\int_0^T \int_{\mathbb{R}^3} Q(f_{n'}) \psi\, dvds \to \int_0^{\tau} \int_{\mathbb{R}^3} Q(\mathcal{S}) \psi\, dvds = 0,$$

and therefore

$$\sup_{[0,T]} \Big| \int_{\mathbb{R}^3} f(t_{n'}) \psi\, dv - \int_{\mathbb{R}^3} \mathcal{S} \psi\, dv \Big| \to 0.$$

$\square$

3.2 Bose-Einstein collision operator for isotropic density

We consider now the Boltzmann equation for Bose-Einstein particles and take $\tau = 1$ in (3.1). We start with an elementary remark. Assume that $w = 1$ and consider a sequence (f_ϵ) defined by $f_\epsilon(v) := \epsilon^{-3} f(v/\epsilon)$ for a given $0 \leq f \in C_c(\mathbb{R}^3)$, $f \not\equiv 0$. An elementary change of variables leads to

$$\|Q^\pm(f_\epsilon)\|_{L^1} = \frac{1}{\epsilon^3} \|Q^\pm(f)\|_{L^1} \to +\infty.$$

Therefore, no a priori estimate of the form

$$\|Q(f)\| \leq \Phi(\|f\|_{L^1}), \text{ with } \Phi \in C(\mathbb{R}_+),$$

can be expected. In particular we will not be able to give a sense to the kernel $Q(f)$ under the only physical bound $f \in M_2^1(\mathbb{R}^3)$ for such a w.

This first remark motivates the two following simplifications, originally performed by Lu [40] to give a sense to $Q(f)$: we assume that the density is isotropic

and we make a strong (and unphysical) truncation assumption on w. We use them here, in a slightly different way that we believe to be simpler.

We then assume, until the end of this Section, that the density f only depends on the quantity $|v|$, and denote $f(v) = f(|v|) = f(r)$ with $r = |v|$. For a given function $q = q(r, r_*; r', r'_*)$ we define

$$\mathcal{Q}[q](v) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} w q \delta_{\mathcal{C}} \, dv_* dv' dv'_*. \tag{3.22}$$

By introducing the new function

$$\hat{w}(r, r_*, r', r'_*) := \iiint_{S^2 \times S^2 \times S^2} w(v, v_*, v', v'_*)\delta_{\{v + v_* = v' + v'_*\}} \, d\sigma_* d\sigma' d\sigma'_* \tag{3.23}$$

and the notation $v_* = r_* \sigma_*$, $v_* = r'\sigma'$, $v'_* = r'_* \sigma'_*$, $d\hat{r}_* = r_*^2 \, dr_*$, $d\hat{r}' = r'^2 \, dr'$, $d\hat{r}'_* = r'^2_* \, dr'_*$,

$$\widehat{\mathcal{C}} = \{(r, r_*, r', r'_*) \in \mathbb{R}_+^4, \ r^2 + r_*^2 = r'^2 + r'^2_*\}, \tag{3.24}$$

we may write $\mathcal{Q}[q]$ in the simple way

$$\mathcal{Q}[q](r) = \iiint_{\mathbb{R}_+^3} \hat{w} \, \delta_{\widehat{\mathcal{C}}} \, q \, d\hat{r}_* d\hat{r}' d\hat{r}'_*. \tag{3.25}$$

Let us emphasize that $\hat{w}$ is indeed a function of $r = |v|$ (and not on the whole variable v) since, for any $R \in SO(3)$, we have

$$\iiint_{S^2 \times S^2 \times S^2} w(Rv, v_*, v', v'_*)\delta_{\{Rv + v_* = v' + v'_*\}} \, d\sigma_* d\sigma' d\sigma'_*$$

$$= \iiint_{S^2 \times S^2 \times S^2} w(Rv, Rv_*, Rv', Rv'_*)\delta_{\{Rv + Rv_* = Rv' + Rv'_*\}} \, d\sigma_* d\sigma' d\sigma'_*$$

$$= \iiint_{S^2 \times S^2 \times S^2} w(v, v_*, v', v'_*)\delta_{\{v + v_* = v' + v'_*\}} \, d\sigma_* d\sigma' d\sigma'_*$$

by the change of variables $(\sigma_*, \sigma', \sigma'_*) \to (R\sigma_*, R\sigma', R\sigma'_*)$ and the rotation invariance of w. Moreover, $\hat{w}$ clearly satisfies the property

$$\hat{w}(r, r_*, r', r'_*) = \hat{w}(r_*, r, r', r'_*) = \hat{w}(r', r'_*, r, r_*). \tag{3.26}$$

Lemma 3.5 *Assume that w is such that B defined by (3.5) satisfies*

$$\sup_{z \in \mathbb{R}^3} \frac{1}{1 + |z|^s} \int_{S^2} B(z, \omega) \, d\omega < \infty \tag{3.27}$$

with $s = 2$ and that $\hat{w}$ defined by (3.26) satisfies

$$\hat{w}(r + r_* + r' + r'_*) \in L^\infty(\mathbb{R}_+^4). \tag{3.28}$$

Then for any isotropic function $f \in L_2^1(\mathbb{R}^3)$ the kernel $Q^\pm(f)$ is well defined and

$$\|Q^\pm(f)\|_{L^1} \leq C_B \|f\|_{L_2^1(\mathbb{R}^3)}^2 + C_{\hat{w}} \|f\|_{L^1(\mathbb{R}^3)}^3. \tag{3.29}$$

A refined version of the bound (3.29) has been used, by Lu [40], in order to prove a global existence result when $s = 0$ in (3.27). For $s = 1$, X. Lu also proves an existence result under the additional assumption that B has the particular form: $B(z, \omega) = |z|^\gamma \zeta(\theta)$ with $\gamma \in [0, 1]$, $\zeta \in L^1$. Here condition (3.28) has to be understood as a truncation assumption near the origin

$$\exists B_0 \in (0, \infty), \quad B(z, \omega) \le B_0 (\cos \theta)^2 \sin \theta |z|^3, \qquad (3.30)$$

introduced in [40]. In order to clarify the assumption (3.28) let us state the following result.

Lemma 3.6   *1. For $w = 1$,*

$$\hat{w} = \frac{4\pi^2}{r\, r_* \, r'\, r'_*} \min(r, r_*, r', r'_*). \qquad (3.31)$$

2. As a consequence, any cross-section w such that

$$\exists w_0 \in (0, \infty), \quad w \le w_0 ((|v' - v||v'_* - v|) \wedge 1) \qquad (3.32)$$

satisfies (3.27)-(3.28).

Note that condition (3.32) is exactly the X. Lu's assumption (3.30) near the origin. This condition kills the interaction between particles with low energy. We emphasize that this assumption is never satisfied by the physically relevant cross-sections.

Proof of Lemma 3.5   We may write

$$f'f'_*(1+f)(1+f_*) - ff_*(1+f')(1+f'_*) = f'f'_*(1+f+f_*) - ff_*(1+f'+f'_*). \quad (3.33)$$

Then, we have to define $\mathcal{Q}[q]$ for two kinds of terms q: for the quadratic terms $q = ff_*$ and $q = f'f'_*$ and for cubic terms $q = f'f'_*f$, $q = f'f'_*f_*$, $q = ff_*f'$ and $q = ff_*f'_*$. The quadratic terms may be defined in the same way that for the Fermi-Dirac Boltzmann equation in Section 3.1 thanks to assumption (3.28) and they are bounded by the first term in the right hand side of estimate (3.29).

We then focus on the cubic terms. We define them by performing one integration more in the representation formula (3.25) (with respect to one of the variables r_*, r', r'_*) but we still use the formula (3.25) in order to preserve the symmetries of $\mathcal{Q}$.

Let us first assume that moreover, $f \in C_c(\mathbb{R}^3)$. Performing in (3.25) the integration in the r_* variable we get, using Lemma 6.1,

$$\mathcal{Q}[q](r) = \iint_{\mathbb{R}^2_+} \hat{w} \frac{r_*}{2} \mathbf{1}_{\{r'^2 + r'^2_* \ge r^2\}}\, q\, d\hat{r}' d\hat{r}'_*. \qquad (3.34)$$

where $r_* = \sqrt{r'^2 + r'^2_* - r^2}$. It is then clear that (3.33) defines $\mathcal{Q}[q]$ as measurable function.

Moreover, we write

$$\int_{\mathbb{R}_+} \mathcal{Q}[q](r)d\hat{r} = \iiiint_{\mathbb{R}_+^4} \hat{w}\, \delta_{\widehat{C}}\, q\, d\hat{r}d\hat{r}_*d\hat{r}'d\hat{r}'_*$$

and first, perform an integration with respect to the variable which does not appear in q (namely, r_* if $q = f'f'_*f$, r if $q = f'f'_*f_*$, and so on) and we get (using Lemma 6.1 again)

$$\int_{\mathbb{R}_+} \mathcal{Q}[q](r)d\hat{r}$$
$$= \iiint_{\mathbb{R}_+^3} \hat{w}(r_1, r_2, r_3, r_4)\frac{r_4}{2}\mathbf{1}_{\{r_1^2+r_2^2\geq r_3^2\}}\, f(r_1)f(r_2)f(r_3)\, d\hat{r}_1 d\hat{r}_2 d\hat{r}_3 \qquad (3.35)$$
$$\leq \|\hat{w}(r + r_* + r' + r'_*)\|_{L^\infty(\mathbb{R}_+^4)}\|f\|_{L^1(\mathbb{R}^3)}^3,$$

where in the first inequality we have used the notation $r_4 = \sqrt{r_1^2 + r_2^2 - r_3^2}$. We finally deduce (3.29) using a density-continuity argument. $\qquad\square$

Proof of Lemma 3.6 We start by proving (3.31). Using that

$$\delta(v+v_*-v'-v'_*) = \int_{\mathbb{R}^3} e^{i(z,v+v_*-v'-v'_*)}\frac{dz}{(2\pi)^3} = \int_0^\infty \int_{S^2} e^{i(z,v+v_*-v'-v'_*)}\, d\sigma\, \frac{r^2\, dr}{(2\pi)^3}$$

where $z = |z|\sigma$ and $\epsilon = |z|$ in polar coordinates, we obtain

$$\int_{S^2}\int_{S^2}\int_{S^2} \delta(v + v_* - v' - v'_*)\, d\sigma_* d\sigma' d\sigma'_*$$
$$= \int_0^\infty \int_{S^2}\int_{S^2}\int_{S^2}\int_{S^2} e^{i(z,v+v_*-v'-v'_*)}\, d\sigma d\sigma_* d\sigma' d\sigma'_* \frac{\epsilon^2\, d\epsilon}{(2\pi)^3}$$
$$= \int_0^\infty \frac{\epsilon^2\, d\epsilon}{(2\pi)^3} \int_{S^2} e^{i(z,v)}\, d\sigma \int_{S^2} e^{i(z,v_*)}\, d\sigma_* \int_{S^2} e^{i(z,v')}\, d\sigma' \int_{S^2} e^{i(z,v'_*)}\, d\sigma'_*$$
$$= \frac{16\pi}{r\, r_*\, r'\, r'_*}\int_0^\infty \frac{d\epsilon}{(2\pi)^3\epsilon^2}\sin(\epsilon r)\sin(\epsilon r_*)\sin(\epsilon r')\sin(\epsilon r'_*).$$

Then (3.31) follows by an lengthy elementary trigonometric computation. It is clear, thanks to (3.25), that (3.32) implies (3.27). We then have to prove that (3.32) implies (3.28). For any r, r_*, r', r'_* given, we set $m_1 = \min(r, r_*, r', r'_*)$, $m_2 = \min(\{r, r_*, r', r'_*\}\setminus\{m_1\})$, $m_3 = \min(\{r, r_*, r', r'_*\}\setminus\{m_1, m_2\})$ and finally, $m_4 = \max(r, r_*, r', r'_*)$. Since $|v - v'| = |v_* - v'_*|$, $|v - v'_*| = |v_* - v'|$ and $\{r, r_*\} \neq \{m_1, m_2\}$, $\{r, r_*\} \neq \{m_3, m_4\}$, the assumption (3.32) leads to

$$w \leq w_0\{[4\min(\max(r, r'), \max(r_*, r'_*))\min(\max(r, r'_*), \max(r_*, r'))] \wedge 1\}$$
$$\leq w_0\{[4\, m_2\, m_3] \wedge 1\}.$$

Gathering (3.31) and this last inequality we deduce

$$\hat{w}(r + r_* + r' + r'_*) \leq \frac{4\pi^2\, m_1}{r\, r_*\, r'\, r'_*}(4\, w_0\, m_2\, m_3)(r + r_* + r' + r'_*) \leq 64\pi^2\, w_0,$$

and that concludes the proof. □

We want now to extend the previous arguments and give a sense to $Q(F)$ for $F \in M^1(\mathbb{R}^3)$. For the quadratic term that problem has been solved by Povzner in [45]. For the cubic term we write for a radial measure $dF = f \, r^2 \, dr$

$$\langle \mathcal{Q}[q], \phi \rangle = \langle F \otimes F \otimes F, B_{\hat{w}}[\psi] \rangle \qquad (3.36)$$

with

$$B_{\hat{w}}[\psi](r_1, r_2, r_3) := \hat{w}(r_1, r_2, r_3, r_4) \frac{r_4}{2} \mathbf{1}_{\{r_1^2 + r_2^2 \geq r_3^2\}} \psi$$

and $r_4 = \sqrt{r_1^2 + r_2^2 - r_3^2}$ and $\psi(r_1, r_2, r_3) = \phi(r_i)$ $i = 1, 2, 3$ or 4 depending on q. Under the condition

$$B_{\hat{w}}[1] \in C(\mathbb{R}_+^3) \qquad (3.37)$$

we clearly have $B_{\hat{w}}[\psi] \in C_c(\mathbb{R}_+^3)$ for any $\phi \in C_c(\mathbb{R}_+)$ and then the right hand side term of (3.36) is well defined. The assumption

$$B \in C(\mathbb{R}^3 \times S^2), \qquad (3.38)$$

which is satisfied by the hard potentials, guaranties that the quadratic term is well defined and that $\hat{w} \in C(E \backslash \{0\})$ where $E = \{(r_1, r_2, r_3, r_4) \in \mathbb{R}_+^4, \, r_1^2 + r_2^2 - r_3^2 - r_4^2 = 0\}$. The condition (3.37) is then satisfied if moreover

$$\lim_{(r_1, r_2, r_3) \to 0} \hat{w}(r_1, r_2, r_3, r_4) \, r_4 = 0 \, .$$

From the computations performed in the proof of Lemma 3.6. it is clear that the last condition holds if we assume, for instance

$$w \leq w_0 \, |v' - v|^\gamma \wedge 1, \quad \gamma > 1. \qquad (3.39)$$

As a conclusion, under assumption (3.27), (3.38), (3.39) we may define the collision kernel for general non negative bounded and isotropic measures. From all the above one easily deduces the following existence result for (3.1) with $\tau = 1$ in the framework of non negative bounded measures.

Theorem 3.7 Assume w satisfies (3.39). Let $F_{\text{in}} \in M_{\text{rad}}^1(\mathbb{R}^3)$, $F_{\text{in}} \geq 0$. Then, there exists a unique global solution $F = g + G \in C([0, \infty), M_{\text{rad}}^1(\mathbb{R}^3))$ to (3.1) with $\tau = 1$.

Remark 3.8 It is straightforward to check that in the radially symmetric case Theorem **??** gives:

Theorem 3.9 Let $0 \leq F \in M_{\text{rad}}^1(\mathbb{R}^3)$ an isotropic measure on $\mathbb{R}^3$ such that

$$\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ |v|^2/2 \end{pmatrix} dF(v) = \begin{pmatrix} N \\ E \end{pmatrix} \quad \text{and of course} \quad \int_{\mathbb{R}^3} v \, dF(v) = 0, \qquad (3.41)$$

with $N > 0$, $E > 0$. Therefore the two following assertions are equivalent:

(i) F is the Bose-Einstein distribution $\mathcal{B}[N, 0, E]$,

(ii) F is the solution of the maximization problem:

$$\mathcal{H}(F) = \max\{\mathcal{H}(F') \text{ with } F' \text{ satisfying (3.41)}.$$

Open questions: In the L^1 setting X. Lu has proved in [40]:

(i) For any (t_n) such that $t_n \to \infty$ there exists $m' \le m$, $E' \le E$ and a subsequence $(t_{n'})$ such that

$$g(t_{n'}) \rightharpoonup b_{m',0,E'} \quad \text{biting } L^1_{\text{rad}} \text{ weak.}$$

(ii) For a given E there exists $N_c = N_c(E)$ such that if $N(f_{\text{in}}) < N_c$ and $E(f_{\text{in}}) = E$ then

$$g(t) \rightharpoonup \mathcal{B}_{N,0,E} = b_{N,0,E} \quad L^1_{\text{rad}} \text{ weak.}$$

Where the distributions b and $\mathcal{B}$ are defined in (2.22) and (2.25). The two following questions are then natural.

1. Is it possible to construct (global?) solutions to (3.1) for $\tau = 1$ without the strong truncation condition (3.39); for instance for $w = 1$? What is the qualitative behavior of such solutions ?

2. Is it possible to prove that, under the strong truncation condition (3.39),

$$F(t) \rightharpoonup \mathcal{B}_{N,0,E} \quad \text{weakly } \sigma(M^1, C_c) \quad \text{and} \quad g \to b_{N,0,E} \quad \text{strongly} L^1(\mathbb{R}^3 \setminus \{0\})$$

as it may be expected from the stationary analysis? If $N(g_{\text{in}}) < N_c$, is it possible to prove strong convergence instead of result (ii) in Theorem 3.5?

Remark 3.10 The Boltzmann-Compton equation, introduced in Section 5 below, is a particular case of (3.1) with $\tau = 1$. It has been proved in [22, 23] that it also has global solutions $F = g + G \in C([0, \infty), M^1_{\text{rad}}(\mathbb{R}^3))$, where g is the regular and G the singular part of the measure F with respect to the Lebesgue measure. Moreover it was proved that the Boltzmann-Compton may be splitted as a system of two coupled equations for the pair (g, G). This allows in particular for a detailed study of the asymptotic behavior of the solutions. Let us briefly show that this is not true for the general isotropic solutions $F = g + G$ of the equation (3.1) unless G is one single Dirac mass.

Let us write $F = g + G$ with g regular with respect to the Lebesgue measure ($g \prec dv$) and G singular with respect to the Lebesgue measure ($G \perp dv$). We have then

$$
\begin{aligned}
F'F'_* &(1 + F)(1 + F_*) - FF_*(1 + F')(1 + F'_*) = \\
&= (1 + g)(1 + g_* + G_*)(g' + G')(g'_* + G'_*) + G(1 + F_*)F'F'_* \\
&\quad - g(g_* + G_*)(1 + g' + G')(1 + g'_* + G'_*) - GF_*(1 + F')(1 + F'_*) \\
&= (1 + g)\big\{(1 + g_*)\, g'\, g'_* + (1 + g_*)\, g'\, G'_* + (1 + g_*)\, G'\, g'_* + (1 + g_*)\, G'\, G'_* \\
&\qquad + G_*\, g'\, g'_* + G_*\, g'\, G'_* + G_*\, G'\, g'_* + G_*\, G'\, G'_*\big\} \\
&\quad - g\big\{g_*(1 + g')(1 + g'_*) + g_*(1 + g')\, G'_* + g_*\, G'(1 + g'_*) + g_*\, G'\, G'_* \\
&\qquad + G_*(1 + g')(1 + g'_*) + G_*(1 + g')\, G'_* + G_*\, G'(1 + g'_*) + G_*\, G'\, G'_*\big\} \\
&\quad + G(1 + F_*)F'F'_* - GF_*(1 + F')(1 + F'_*) \\
&= q(g) + q_1(g, G) + q_2(g, G) + q_3(G, g),
\end{aligned}
$$

with

$$q(g) := (1 + g)(1 + g_*)\, g'\, g'_* - g\, g_*(1 + g')(1 + g'_*),$$
$$q_1(g, G) := G_*[(1 + g)\, g'\, g'_* - g(1 + g')(1 + g'_*)]$$
$$+ G'[(1 + g)(1 + g_*)\, g'_* - g\, g_*(1 + g'_*)$$
$$+ G'_*[(1 + g)(1 + g_*)\, g' - g\, g_*(1 + g'),$$
$$q_2(g, G) := G_*\, G'[(1 + g)\, g'_* - g(1 + g'_*)]$$
$$+ G_*\, G'_*[(1 + g)\, g' - g(1 + g')]$$
$$+ G'\, G'_*[(1 + g)(1 + g_*) - g\, g_*],$$
$$q_3(G) := (1 + g)\, G_*\, G'\, G'_* - g\, G_*\, G'\, G'_*,$$
$$q_4(G, g) := G(1 + F_*)F'F'_* - GF_*(1 + F')(1 + F'_*).$$

Defining

$$Q_i(g, G) = \iiint_{\mathbb{R}^9} w\delta_{\mathcal{C}} q_i(g, G)\, dv_* dv' dv'_*,$$

we may write

$$Q(F) = Q(g) + Q_1(g, G) + Q_2(g, G) + Q_3(G) + Q_4(g, G).$$

The key result in all our analysis is the following result.

Lemma 3.11 *Assume (3.27), (3.38) and (3.39). For any $F \in M^1_{\mathrm{rad}}(\mathbb{R}^3)$, $F_{\mathrm{in}} \geq 0$, we have*

$$Q(g),\ Q_1(g, G),\ Q_2(g, G) \in L^1_{\mathrm{rad}}(\mathbb{R}^3),$$
$$Q_4(g, G) \in M^1_{\mathrm{rad}}(\mathbb{R}^3)\ and\ Q_4(g, G) \perp dv.$$

For any finite sum of Dirac masses G the kernel $Q_3(G)$ is also finite sum of Dirac masses and, moreover, if G is not single Dirac mass then $\mathrm{supp}\, G \backslash \{0\}$ is strictly contained in $\mathrm{supp}\, Q_3(G)$. Finally, there exists G singular such that $Q_3(G)$ is regular.

Proof of Lemma 3.5 We already know from the above that $Q(g) \in L^1_{\mathrm{rad}}(\mathbb{R}^3)$ and $Q_1(g, G), Q_2(g, G), Q_4(g, G), Q_3(G) \in M^1_{\mathrm{rad}}(\mathbb{R}^3)$. Let us denote by q the first term in $q_1(g, G)$ and write, for any ϕ

$$\langle \mathcal{Q}[q], \phi \rangle = \iiiint_{\mathbb{R}^4_+} \hat{w}\, \delta_{\widehat{\mathcal{C}}}\, g'\, g'_*\, \phi\, dG_* d\hat{r}' d\hat{r}'_* = \iint_{\mathbb{R}^2_+} \psi\, g'\, d\hat{r}'\, dG(r_*),$$

with

$$\psi = \int_{\mathbb{R}_+} \hat{w}\, r\, \mathbf{1}_{\{r'^2 + r'^2_* \geq r^2_*\}}\, \phi(r)\, g'_*\, d\hat{r}'_*$$

where in the expression of ψ we have used the notation $r = \sqrt{r'^2 + r'^2_* - r^2_*}$. Observe that, by assumption, $(r_*, r', r'_*) \mapsto \hat{w}\, r\, \mathbf{1}_{\{r'^2 + r'^2_* \geq r^2_*\}}$ is continuous so

that $(r_*, r') \mapsto \psi$ is continuous for any $\phi \in L_{\mathrm{rad}}^\infty(\mathbb{R}^3)$. Moreover, taking $\phi = \mathbf{1}_A$ for any set A with Lebesgue measure equal zero we get $\psi = 0$, so that the Radon-Nykodim Theorem implies that $\mathcal{Q}[q] \in L^1$. By the same way we prove that all the terms in $Q_1(g, G)$ and $Q_2(g, G)$ belongs to $L_{\mathrm{rad}}^1(\mathbb{R}^3)$.

It is clear that taking $q = F'F'_* - F_* - F_*F' - F_*F'_*$ we have $\mathcal{Q}[q] \in C_b(\mathbb{R}^3)$ so that $Q_4(g, G) \perp dv$. For given $R_*, R', R'_* \geq 0$ we set $q = \delta_{R_*}(r_*)\delta_{R'}(r')\delta_{R'_*}(r'_*)$ and we verify

$$\langle \mathcal{Q}[q], \phi \rangle = \phi(R)\, R\hat{w}(R, R_*, R', R'_*)\mathbf{1}_{\{R'^2 + R'^{\,2}_* \geq R_*^2\}}$$

where R is defined by $R = \sqrt{R'^2 + R'^{\,2}_* - R_*^2}$ if $R'^2 + R'^{\,2}_* \geq R_*^2$. In particular, if $G = \alpha\delta_0$ then $Q_3(G) = 0$ and if $G = \alpha\delta_a$ with $a, \alpha \neq 0$ then $Q_3(G) = \beta\delta_a$ with $\beta > 0$. If

$$G = \sum_{a \in E} \alpha_a \delta_a$$

then

$$Q_3(G) = \sum_{b \in E'} \beta_b \delta_b,$$

with $E' = \{b \geq 0, \exists a_*, a', a'_* \in E \, s.t. (b, a_*, a', a'_*) \in \widehat{\mathcal{C}}\}$. That proves the claim of the Lemma since E' strictly contain E if E is not a single point.

Finally, let G be a measure supported by $\sqrt{C}$, where C is a Cantor set, constructed for example as follows. For every $n \in \mathbb{N}$ consider the set

$$C_n := \{x \in [0, 1], \exists k \in \mathbb{N}, 2\, k \leq 3^n\, x \leq 2\, k + 1\},$$

so that $C_n \searrow C$ as $n \to \infty$. Then $g_n := |C_n|^{-1}\mathbf{1}_{C_n} \rightharpoonup H$, a singular non negative measure whose support is C. If we take now $G := H \circ s$ with $s : \mathbb{R} \to \mathbb{R}$, $s(r) = \sqrt{r}$ we have, for any $\phi \in C_{rad,c}(\mathbb{R}^3)$:

$$\langle Q_3(G) = \iiint_{\mathbb{R}^3_+} dG(r_*)\, dG(r')\, dG(r'_*)[\phi(r)\, r\hat{w}\mathbf{1}_{\{r'^2 + r'^{\,2}_* \geq r_*^2\}}].$$

Since

$$\{z \in \mathbb{R}_+;\ \exists \epsilon_*, \epsilon', \epsilon'_* \in C\, z = \epsilon' + \epsilon'_* - \epsilon_*\} = \mathbb{R}_+$$

we see that $\langle Q_3(G); \phi \rangle > 0$ for any non negative and not vanishing ϕ, and thus $Q_3(G)$ has a regular part which is not equal to 0. $\qquad\square$

Remark 3.12 If we assume that for every time $t > 0$, $G(t) \equiv \alpha(t)\,\delta_0$, then the equation (3.1) with $\tau = 1$ may be split into a coupled system of equations for the pair (g, α).

Remark 3.13 The equation (3.1) with $\tau = 1$ has deserved some interest in the recent physical literature in the context of Bose condensation. We particularly refer here to the works by Semikoz & Tkachev [48, 49], and by Josserand & Pomeau [32] due to their strong mathematical point of view (but see also the

references therein). In these two articles the authors present a possible "scenario" to describe the occurrence of Bose condensation in finite time, based on the isotropic version of the equation (3.1) with $\tau = 1$. Their arguments are based on formal asymptotics and, in [48, 49], also supported by numerical simulations.

4   Boltzmann equation for two species

In this Section we consider a system of two coupled homogeneous equations describing a Bose gas interacting with a heat bath, chosen to be a Fermi gas. This is a particular case of a gas composed of two species and has already been considered in the physical literature (see references below). One of the species are Bose particles and the other are either Fermi-Dirac particles or non quantum particles. In this Section we only deal with non relativistic particles. Relativistic particles, in particular photons will be considered in the next Section. In order to avoid lengthy repetitions, we do not specify the energy (relativistic or not relativistic) unless necessary.

From a mathematical point of view, the study of the Boltzmann equation for a gas of Bose particles is rather difficult, as we have already seen it in the preceding Section. But it is possible to derive, from the Boson-Fermion interaction system, a physically relevant model which turns out to be a sort of "linearization" of equation (3.1). This model is then simpler and gives some insight into the behavior of the Boltzman equations for quantum particles. It describes the interaction of Bose particles with isotropic distribution and non quantum Fermi particles at isotropic equilibrium with non truncated cross-section. The equation is now quadratic instead of cubic and its mathematical analysis is easier. We prove that Bose- Einstein condensation takes place in infinite time, in contrast with the finite time condensation which is expected for the Bose-Bose interaction equation. Similar results had previously been obtained in the physical literature, using formal and numerical methods for similar situations (cf. [36, 37, 48, 49]).

We thus consider in what follows a gas composed of two species of particles. The first are Bose particles. The second are either Fermi particles or their non quantum approximation but will always be designed as Fermi particles. We suppose that when a Bose particle of momentum p collides with a Fermi particle of momentum p_* they undergo an elastic collision, so that the total momentum and the total energy of the system constituted by that pair of particles are conserved. More precisely, denoting by $\mathcal{E}_1(p)$ the energy of Bose particles with momentum p and by $\mathcal{E}_2(p_*)$ the energy of Fermi particles with momentum p_* we assume that after collision the particles have momentum p' (for Bose particles) and p'_* (for Fermi particles) which satisfy $\mathcal{C}_{12}$:

$$
\begin{aligned}
p' + p'_* &= p + p_* \\
\mathcal{E}_1(p') + \mathcal{E}_2(p'_*) &= \mathcal{E}_1(p) + \mathcal{E}_2(p_*).
\end{aligned}
\tag{4.1}
$$

The gas is described by the density $F(t, p) \geq 0$ of Bose particles and the

density $f(t, p) \geq 0$ of Fermi particles. We assume that the evolution of the gas is given by the following Boltzmann equation (see for instance [11])

$$
\begin{aligned}
\frac{\partial F}{\partial t} &= Q_{1,1}(F, F) + Q_{1,2}(F, f), \quad F(0, .) = F_{\text{in}}, \\
\frac{\partial f}{\partial t} &= Q_{2,1}(f, F) + Q_{2,2}(f, f), \quad f(0, .) = f_{\text{in}}.
\end{aligned}
\tag{4.2}
$$

The collision terms $Q_{1,1}(F, F)$ and $Q_{2,2}(f, f)$ stand for collisions between particles of the same specie and therefore are given by (1.5). The collision terms $Q_{1,2}(F, f)$ and $Q_{2,1}(f, F)$ stand for collisions between particles of the two different species, they are given by

$$
\begin{aligned}
Q_{1,2}(F, f) &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} w_{1,2} \delta_{\mathcal{C}_{1,2}} q_{1,2} \, dp_* dp' dp'_*, \\
q_{1,2} &= F' f'_* (1 + F)(1 + \tau f_*) - F f_* (1 + F')(1 + \tau f'_*)
\end{aligned}
\tag{4.3}
$$

with $\tau = -1$ when the second specie is composed of true Fermi particles and $\tau = 0$ when the second specie is constituted of non quantum particles. The collision kernel $Q_{2,1}(f, F)$ is given by an obvious similar expression. The measure $w_{1,2} \delta_{\mathcal{C}_{1,2}} = w_{1,2}(p, p_*, p', p'_*) \delta_{\mathcal{C}_{1,2}}$ satisfies the micro-reversibility hypothesis

$$
w_{1,2}(p', p'_*, p, p_*) \delta_{\mathcal{C}_{1,2}} = w_{1,2}(p, p_*, p', p'_*) \delta_{\mathcal{C}_{1,2}},
\tag{4.4}
$$

but not the indiscernibility $w_{1,2}(p_*, p, p', p'_*) \delta_{\mathcal{C}_{1,2}} = w_{1,2}(p, p_*, p', p'_*) \delta_{\mathcal{C}_{1,2}}$ as in (1.7) since the two species are now different. When both energies are non relativistic $w_{1,2}$ is invariant by Galilean transformations, and when both energies are relativistic it is invariant by Lorentz transformations. In the mixed case of one non relativistic specie and one relativistic specie, the situation is a little more complicated and we postpone the analysis to the next Section.

We start with some simple formal properties of the solutions of (4.2). Thanks to symmetry (4.4), performing a change of variables $(p', p'_*, p, p_*) \to (p, p_*, p', p'_*)$ we get the fundamental formula: for any $\psi = \psi(p)$,

$$
\begin{aligned}
\int_{\mathbb{R}^3} Q_{1,2}(F, f) \psi \, dp = \frac{1}{2} \iiiint_{\mathbb{R}^{12}} w_{1,2} \delta_{\mathcal{C}_{1,2}} \big(F' f'_* (1 + F)(1 + \tau f_*) \\
- F f_* (1 + F')(1 + \tau f'_*) \big) \big(\psi - \psi' \big) \, dp \, dp_* dp' \, dp'_* \, .
\end{aligned}
\tag{4.5}
$$

A similar formula holds for $Q_{2,1}(f, F)$.

After integration of the two equations of (4.2) separately, and using (1.7) and (4.5) we formally get the particle number conservation of each specie:

$$
\int_{\mathbb{R}^3} F(t, p) \, dp = \int_{\mathbb{R}^3} F_{\text{in}}(p) \, dp \, , \quad \int_{\mathbb{R}^3} f(t, p) \, dp = \int_{\mathbb{R}^3} f_{\text{in}}(p) \, dp.
\tag{4.6}
$$

Multiplying both equations in (4.2) by $\psi(p) = p$, summing up and using (1.7), (4.5) and $w_{1,2} = w_{2,1}$ we obtain the global momentum conservation

$$
\int_{\mathbb{R}^3} \big(F(t, p) + f(t, p) \big) p \, dp = \int_{\mathbb{R}^3} \big(F_{\text{in}}(p) + f_{\text{in}}(p) \big) p \, dp.
\tag{4.7}
$$

Multiplying the first equation of (4.2) by $\mathcal{E}_1$, the second equation of (4.2) by $\mathcal{E}_2$, using (1.7), (4.5) and the collision invariance (4.1) we get the global energy conservation

$$\int_{\mathbb{R}^3} \big(F(t,p)\mathcal{E}_1(p) + f(t,p)\mathcal{E}_2(p)\big)\, dp = \int_{\mathbb{R}^3} \big(F_{\text{in}}(p)\mathcal{E}_1(p) + f_{\text{in}}(p)\mathcal{E}_2(p)\big)\, dp\,. \quad (4.8)$$

Finally, we define the entropy of the system by

$$H_{\mathcal{S}}(F,f) := H_1(F) + H_\tau(f) \quad (4.9)$$

with H_τ given by (1.1). Multiplying the first equation by $h'_1(F)$ and the second equation by $h'_\tau(f)$ we obtain using (4.5)

$$\frac{d}{dt} H_{\mathcal{S}}(F,f) = D_{\mathcal{S}}(F,f) := \frac{1}{4}D_1(F) + \frac{1}{2}D_{1,2}(F,f) + \frac{1}{4}D_\tau(f) \geq 0, \quad (4.10)$$

where $D_1(F)$ and $D_\tau(f)$ are the usual dissipation entropy production of one specie, and $D_{1,2}(F,f)$ is the mixed dissipation entropy production given by

$$D_{1,2}(F,f) :=$$
$$\iiiint_{\mathbb{R}^{12}} w_{1,2}\delta_{\mathcal{C}_{1,2}} j\big(F'f'_*(1+F)(1+\tau f_*), F'_*(1+F')(1+\tau f'_*)\big)\, dp dp_* dp' dp'_*.$$
$$(4.11)$$

We list now several questions that one may naturally consider about the system (4.2).

1. For the single quantum equation, one may consider the maximization entropy problem under particle number, momentum and energy restriction i.e.: for any given $N_B, N_F, E > 0$, $P \in \mathbb{R}^3$, to find a pair of functions (F, f) such that

$$\int_{\mathbb{R}^3} dF(p) = N_B, \quad \int_{\mathbb{R}^3} f(p)\, dp = N_F$$
$$\int_{\mathbb{R}^3} p(F(p) + f(p))\, dp = P, \quad \int_{\mathbb{R}^3} (F(p)\mathcal{E}1(p) + f(p)\mathcal{E}_2(p))\, dp = E. \quad (4.12)$$

and satisfying

$$H_{\mathcal{S}}(F,f) = \max_{G, g\, s.t.\, (4.12)} H_{\mathcal{S}}(G,g), \quad \text{with } H_{\mathcal{S}}(G,g) = H_{BE}(G) + H_\tau(g). \quad (4.13)$$

We believe that a complete analysis of this problem can be done using the same ideas exposed in the Section 2 and used in [24].

2. One can also address the well posedness of the Cauchy problem for the system (4.2). Of course the situation here is the same as for the Boltzmann Bose equation. The analysis performed in the Section 3.2 may be readily extended to prove the existence of solutions under the assumption of isotropy of the distribution and with Lu's truncation on the cross sections. A simpler question would be to consider the case when $Q_{1,1}(F,F)$ and $Q_{2,2}(f,f)$ vanish and to

address the well posedness of the Cauchy problem in this case. Even when $\tau > 0$ (which gives an L^∞ a priori bound on the Fermi density f) we do not know if it is possible to give a sense to the collision terms $Q_{1,2}(F, f)$ and $Q_{2,1}(f, F)$ without Lu's truncation on the cross sections.

We do not try to go further in any of these two directions and consider instead the following question. In the study of gases formed by Bose and Fermi particles, it is particularly relevant to consider the case where the Fermi particles are at equilibrium and where the collisions between Bose particles can not distort significantly their distribution function (cf. [36, 37]). This moreover constitutes a first important simplification from a mathematical point of view. The system (4.2) reduces then to a unique equation on the Bose distribution F. Moreover this equation is quadratic and not cubic (c.f. sub Section 4.1)

A second simplification arises if we consider non relativistic, isotropic densities and we assume on physical grounds, that $w_{1,2}$ is constant, (c.f below). In that case, we keep the same quadratic structure for the equation on the Bose distribution F, but we obtain an explicit and quite simple cross-section (c.f. sub Section 4.2).

In both situations, our main concern is to understand if it is possible to obtain a global existence result without the Lu's truncation on the cross sections and then to describe the long time asymptotic behaviour of the solutions.

4.1 Second specie at thermodynamical equilibrium

Let us assume that $f = \mathcal{F}$ is at thermodynamical equilibrium, which means that it is a Fermi or a Maxwellian distribution defined by

$$\mathcal{F}(p) = \frac{1}{e^{\nu(p)} - \tau}, \quad \nu(p) = \beta^0 \mathcal{E}_2(p) - \beta \cdot p + \mu. \tag{4.14}$$

This greatly simplifies the situation since the system (4.2) reduces now to a single equation for the Bose distribution F which reads

$$\frac{\partial F}{\partial t} = Q_{BQ}(F) := Q_{1,2}(F, \mathcal{F}) \tag{4.15}$$

with

$$Q_{BQ}(F) = \int_{\mathbb{R}^3} S(p, p') \big(F'(1 + F) \, e^{-\beta^0 \mathcal{E}_1(p)} - F(1 + F') \, e^{-\beta^0 \mathcal{E}_1(p')} \big) \, dp', \tag{4.16}$$

$$S(p, p') = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} w_{1,2} \delta_{\mathcal{C}_{12}} e^{\beta^0 \mathcal{E}_1(p)} \mathcal{F}'_*(1 + \tau \mathcal{F}_*) \, dp_* dp'_*. \tag{4.17}$$

To establish (4.16)-(4.17) we have used the elementary identity

$$e^{\beta^0 \mathcal{E}_1(p)} \mathcal{F}'_*(1 + \tau \mathcal{F}_*) = e^{\beta^0 \mathcal{E}_1(p')} \mathcal{F}_*(1 + \tau \mathcal{F}'_*) \tag{4.18}$$

which holds on $\mathcal{C}_{12}$. Notice that, using the micro-reversibility symmetry (4.4) and the identity (4.18) we have

$$
\begin{aligned}
S(p', p) &= \iint_{\mathbb{R}^3 \times \mathbb{R}^3} w_{1,2}(p', p_*, p, p'_*) \delta_{\mathcal{C}_{12}} e^{\beta^0 \mathcal{E}_1(p')} \mathcal{F}'_*(1 + \tau \mathcal{F}_*) \, dp_* dp'_* \\
&= \iint_{\mathbb{R}^3 \times \mathbb{R}^3} w_{1,2}(p, p_*, p', p'_*) \delta_{\mathcal{C}_{12}} e^{\beta^0 \mathcal{E}_1(p)} \mathcal{F}_*(1 + \tau \mathcal{F}'_*) \, dp_* dp'_* \\
&= S(p, p'),
\end{aligned}
$$

so that S is symmetric.

Using the symmetry above one can observe that, at least formally, a solution F of equation (4.15) still satisfies the qualitative properties

$$
\int_{\mathbb{R}^3} F \, dp = \int_{\mathbb{R}^3} F_{\text{in}} \, dp \tag{4.19}
$$

and

$$
\frac{d}{dt} H_{BQ}(F) = D_{BQ}(F), \tag{4.20}
$$

with

$$
H_{BQ}(F) = \int_{\mathbb{R}^3} \left((1 + F) \ln(1 + F) - F \ln F - F \beta^0 \mathcal{E}_1(p) \right) dp \tag{4.21}
$$

and

$$
D_{BQ}(F) = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} S(p, p') \, j \left(F'(1 + F) \, e^{-\beta^0 \mathcal{E}_1(p)} - F(1 + F') \, e^{-\beta^0 \mathcal{E}_1(p')} \right) dp dp'. \tag{4.22}
$$

In other words, the particle number is preserved along the trajectories and H_{BQ} is a Lyapunov function (the relative entropy H_{BQ} is a decreasing function along the trajectories).

4.1.1 Non relativistic particles, fermions at isotropic Fermi Dirac equilibrium

It is possible, under further simplifications of the model, to obtain more explicit expressions of the cross section S. As a first step in that direction we consider nonrelativistic particles. We also assume, without any loss of generality, that the two particles have the same mass $m = 1$ from where their energies are $\mathcal{E}_i(p) = \mathcal{E}(p) = |p|^2/2$, $i = 1, 2$. We assume moreover that fermions are at isotropic equilibrium (see (2.39)):

$$
\mathcal{F}(p) = \frac{1}{e^{\beta^0 \frac{|p|^2}{2} + b} - \tau} \quad \text{for some } \theta 0, b \geq 0.
$$

We introduce now the Carleman parametrization of the collision manifold (4.1). Starting by performing the dp_* integration one finds

$$
S(p, p') = \int_{\mathbb{R}^3} w_{1,2}(p, p_*, p', p'_*) \delta_{|p'|^2 + |p'_*|^2 - |p|^2 - |p_*|^2 = 0} \, e^{\beta^0 \frac{|p|^2}{2}} \mathcal{F}'_*(1 + \tau \mathcal{F}_*) \, dp'_*, \tag{4.23}
$$

where now p_* is defined by

$$p_* := p' + p'_* - p_*. \tag{4.24}$$

An elementary computation shows that

$$|p'|^2 + |p'_*|^2 - |p|^2 - |p_*|^2 = -2(p'_* - p) \cdot (p' - p),$$

so that in (4.23) p'_* describes all the plane $E_{p,p'}$ orthogonal to $p' - p$ and containing p. Then, using the distributional Lemma 6.1, we get

$$S(p, p') = \int_{E_{p,p'}} \frac{w_{1,2}(p, p_*, p', p'_*)}{|p' - p|} e^{\beta^0 \frac{|p|^2}{2}} \mathcal{F}'_*(1 + \tau \mathcal{F}_*) \, dE(p'_*), \tag{4.25}$$

where $dE(p'_*)$ stands for the Lebesgue measure on $E_{p,p'}$ and p_* is again defined thanks to (4.24).

A further simplification may be performed, noticing that, on physical ground (cf. Appendix 8), the boson-fermion interaction is a short range interaction and that the velocities of the particles undergoing scattering are small . This allows to consider that $w_{1,2}$ is constant, assuming without loss of generality that $w_{1,2} = 1$. We then obtain a more explicit expression of S in (4.25). Namely:

$$S(p, p') = \int_{E_{p,p'}} \frac{e^{\beta^0 \frac{|p|^2}{2}}}{|p' - p|} \frac{1}{e^{\beta^0 \frac{|p'_*|^2}{2} + b} - \tau} \frac{e^{\beta^0 \frac{|p_*|^2}{2} + b}}{e^{\beta^0 \frac{|p_*|^2}{2} + b} - \tau} \, dE(p'_*).$$

Note that

$$\frac{e^b}{1 + e^b} \leq \frac{e^{\beta^0 \frac{|p_*|^2}{2} + b}}{e^{\beta^0 \frac{|p_*|^2}{2} + b} - \tau} < 1$$

and

$$\delta \frac{e^b}{1 + e^b} \int_{E_{p,p'}} \frac{e^{\beta^0 \frac{|p|^2}{2}}}{|p' - p|} e^{-\beta^0 \frac{|p'_*|^2}{2} - b} \, dE(p'_*)$$

$$\leq S(p, p') < \int_{E_{p,p'}} \frac{e^{\beta^0 \frac{|p|^2}{2}}}{|p' - p|} e^{-\beta^0 \frac{|p'_*|^2}{2} - b} \, dE(p'_*).$$

Let us consider then the function

$$S_0(p, p') \equiv \int_{E_{p,p'}} \frac{e^{\beta^0 \frac{|p|^2}{2}}}{|p' - p|} e^{-\beta^0 \frac{|p'_*|^2}{2}} \, dE(p'_*)$$

In the orthonormal basis $(\vec{k}, \vec{i}, \vec{j})$ where $\vec{k} := (p' - p)/|p' - p|$, $p'_\star$ may be written as follows $p'_\star = p + s\vec{i} + t\vec{j}$ with $s, t \in \mathbb{R}$.

$$|p'_\star|^2 = |p|^2 + (s + p \cdot i)^2 - (p \cdot i)^2 + (t + p \cdot j)^2 - (p \cdot j)^2$$

$$= \left(p, \frac{p' - p}{|p' - p|}\right)^2 + (s + p \cdot i)^2 + (t + p \cdot j)^2,$$

and, integrating (4.25), we find

$$S_0(p, p') = \frac{e^{\beta^0 \frac{|p|^2}{2}}}{|p' - p|} \iint_{\mathbb{R}^2} e^{-\beta^0 |p'_*|^2/2} \, ds \, dt$$

$$= \frac{2\pi}{\beta^0 |p' - p|} \exp\Big[\frac{\beta^0}{2}\Big(|p|^2 - (p, \frac{p' - p}{|p' - p|})^2\Big)\Big].$$

This can also be written in the symmetric form

$$S_0(p, p') = \frac{2\pi}{\beta^0 |p' - p|} \exp\Big[\frac{\beta^0}{4}\Big(|p|^2 + |p'|^2 - (p, \frac{p' - p}{|p' - p|})^2 - (p', \frac{p' - p}{|p' - p|})^2\Big)\Big]. \tag{4.26}$$

Remark 4.1 If one considers the system (4.2) when collisions between Boson particles are very weak (so that they may be neglected) and collisions between Fermi particles are very strong (in such a way that the distribution of Fermi particles goes through thermodynamical equilibrium very rapidly) we may consider as relevant the following scaling

$$\frac{\partial F_\epsilon}{\partial t} = \epsilon \, Q_{1,1}(F_\epsilon, F_\epsilon) + Q_{1,2}(F_\epsilon, f_\epsilon)$$

$$\frac{\partial f_\epsilon}{\partial t} = Q_{2,1}(f_\epsilon, F_\epsilon) + \frac{1}{\epsilon} \, Q_{2,2}(f_\epsilon, f_\epsilon), \tag{4.27}$$

in the limit $\epsilon \to 0$. Notice that the particle number conservation (4.6) and energy conservation (4.8) provide the a priori bounds

$$\sup_{\epsilon > 0} \sup_{t \in [0,\infty)} \int_{\mathbb{R}^3} F_\epsilon(t, p)(1 + \mathcal{E}_1(p)) \, dp, \quad \int_{\mathbb{R}^3} f_\epsilon(t, p)(1 + \mathcal{E}_2(p)) \, dp \leq C. \tag{4.28}$$

Moreover, since

$$\frac{d}{dt} H_{\mathcal{S}}(F_\epsilon, f_\epsilon) = \frac{\epsilon}{4} D_1(F_\epsilon) + \frac{1}{2} D_{1,2}(F_\epsilon, f_\epsilon) + \frac{1}{4\epsilon} D_\tau(f_\epsilon),$$

and all the terms at the right hand side are positive, we obtain

$$\int_0^\infty D_\tau(f_\epsilon) \leq \epsilon \, C(F_{\mathrm{in}}, f_{\mathrm{in}}). \tag{4.29}$$

Formally, these bounds imply that, up to the extraction of a subsequence, we have

$$f_\epsilon \to \mathcal{F}, \quad F_\epsilon \to F, \tag{4.30}$$

where $\mathcal{F}$ has same momentum that f_{in} and $D_\tau(\mathcal{F}) = 0$ so that $\mathcal{F}$ is the Fermi or Maxwellian distribution associated to f_{in} given by (4.4) and F solves the Quadratic Bose equation (4.15)-(4.17).

Remark 4.2 It is important to notice that there is no conservation of the energy for equation (4.15)-(4.16). The conserved quantity in the system (4.2) is the total energy of the bosons-fermions gas but not of any of the two species as it is shown by (4.8).

Open questions:

1. Establish rigorously (4.30).

2. Solve the Cauchy problem (4.15)-(4.17) for physically relevant S.

4.2 Isotropic distribution and second specie at the thermodynamical equilibrium

We now consider the Fermi particles at non relativistic isotropic equilibrium and assume that the Bose distributions are also isotropic. In other words we suppose that

$$f(p,t) \equiv f(|p|) = \frac{1}{e^{\beta^0 \frac{|p|^2}{2}+b} - \tau} \quad \text{and} \quad F(p,t) = F(|p|,t). \tag{4.31}$$

The interaction of an isotropic gas of Bose particles and Fermi particles at equilibrium has been considered by Levich and Yakhot in [36] and [37], by means of formal arguments. They were interested in particular in the occurrence of Bose Einstein condensation. Numerical simulations showing Dirac mass formation in infinite time for a related equation have been obtained by Semikoz and Tkachev in [49, 49]. The Fermi distributions considered in that case are quantum, isotropic, saturated Fermi Dirac distributions (SFD in Section 2). This corresponds to the choice

$$f(p) \equiv f(|p|) = \mathbf{1}_{\{0 \le |p| \le \mu\}}$$

for some $\mu > 0$, and gives rise to a slightly different equation than ours.

The Cauchy problem for the resulting quadratic Bose equation is a "linearized model" of the Bose-Bose interaction equation considered in Section 3, where moreover, the function $S(p,p')$ may be calculated explicitly.

Similar equations have been considered in [22, 23]. Nevertheless, the global existence results obtained in these references do not apply to our case, because the collision kernel does not fulfill the required hypothesis. Therefore, we end this Section proving global existence of solutions, with integrable initial data, to our problem. Finally, the long time behaviour of these global solutions may be addressed exactly as in [23] and then, we only state the result for the sake of completeness.

We start with the following:

Proposition 4.3 *If the function F is radially symmetric then so is $Q_{BQ}(F)$. In that case we write $Q_{BQ}(F)(p) = Q_{BQ}(F)(\frac{|p|^2}{2\theta})$ by abuse of notation. Moreover,*

$$Q_{BQ}(F)(\varepsilon) = \int_0^\infty S\left[F'(1+F)e^{-\epsilon} - F(1+F')e^{-\epsilon'}\right] d\epsilon', \tag{4.32}$$

with $S(\epsilon, \epsilon') = \Sigma(\epsilon, \epsilon')/\sqrt{\epsilon}$ and

$$\Sigma(\epsilon, \epsilon') = \int_{\max(0,\epsilon-\epsilon')}^\infty e^{\epsilon-\epsilon'_*} \frac{e^{\varepsilon'+\varepsilon'_*-\varepsilon+b}}{e^{\varepsilon'+\varepsilon'_*-\varepsilon+b} - \tau} \min(\sqrt{\epsilon}, \sqrt{\epsilon'+\epsilon'_*-\epsilon}, \sqrt{\epsilon'}, \sqrt{\epsilon'_*}) \, d\epsilon'_*. \tag{4.33}$$

Proof It follows from (4.17) and (4.31) that

$$S(p, p') = e^{\beta^0 \frac{|p|^2}{2}} \int \int_{\mathbb{R}^3 \times \mathbb{R}^3} \delta_{|p'|^2 + |p'_*|^2 - |p|^2 - |p_*|^2 = 0} \, \delta_{p' + p'_* - p - p_* = 0}$$

$$\times \frac{e^{\beta^0 \frac{|p_*|^2}{2} + b}}{e^{\beta^0 \frac{|p'_*|^2}{2} + b} - \tau} \frac{dp_* dp'_*}{e^{\beta^0 \frac{|p_*|^2}{2} + b} - \tau}$$

$$= \int_0^\infty \int_0^\infty \delta_{|p'|^2 + |p'_*|^2 - |p|^2 - |p_*|^2 = 0} \frac{e^{\beta^0 \frac{|p|^2}{2}}}{e^{\beta^0 \frac{|p'_*|^2}{2} + b} - \tau} \frac{e^{\beta^0 \frac{|p_*|^2}{2} + b}}{e^{\beta^0 \frac{|p_*|^2}{2} + b} - \tau}$$

$$\times \int_{S^2} \int_{S^2} \delta_{p' + p'_* - p - p_* = 0} \, d\omega_* d\omega'_* |p'_*|^2 d|p'_*| |p_*|^2 d|p_*|.$$

Therefore,

$$Q_{BQ}(F) = \int_0^\infty \int_0^\infty \int_0^\infty \delta_{|p'|^2 + |p'_*|^2 - |p|^2 - |p_*|^2 = 0} \frac{e^{\beta^0 \frac{|p|^2}{2}}}{e^{\beta^0 \frac{|p'_*|^2}{2} + b} - \tau} \frac{e^{\beta^0 \frac{|p_*|^2}{2} + b}}{e^{\beta^0 \frac{|p_*|^2}{2} + b} - \tau}$$

$$\times [F'(1 + F)e^{-\beta^0 \frac{|p|^2}{2}} - F(1 + F')e^{-\beta^0 \frac{|p|^2}{2}}].$$

$$\times \int_{S^2} \int_{S^2} \int_{S^2} \delta_{p' + p'_* - p - p_* = 0} \, d\omega_* d\omega'_* d\omega' |p'_*|^2 d|p'_*| |p_*|^2 d|p_*| |p'|^2 d|p'|.$$

As we have already seen in Section 3,

$$\int_{S^2} \int_{S^2} \int_{S^2} \delta_{p' + p'_* - p - p_* = 0} \, d\omega_* d\omega'_* d\omega' = \frac{4\pi^2}{|p||p'||p_*||p'_*|} \min[|p|, |p'|, |p_*|, |p'_*|]$$

from where,

$$Q_{BQ}(F) = \frac{4\pi^2}{|p|} \int_0^\infty [F'(1 + F)e^{-\beta^0 \frac{|p|^2}{2}} - F(1 + F')e^{-\frac{|p'|^2}{2\theta}}] e^{\beta^0 \frac{|p|^2}{2}}$$

$$\left\{ \int_0^\infty \int_0^\infty \delta_{|p'|^2 + |p'_*|^2 - |p|^2 - |p_*|^2 = 0} \frac{e^{\beta^0 \frac{|p|^2}{2}}}{e^{\beta^0 \frac{|p'_*|^2}{2} + b} - \tau} \frac{e^{\beta^0 \frac{|p_*|^2}{2} + b}}{e^{\beta^0 \frac{|p_*|^2}{2} + b} - \tau} \right.$$

$$\left. \min[|p|, |p'|, |p_*|, |p'_*|] \, |p'_*| d|p'_*| |p_*| d|p_*| \right\} |p'| d|p'|$$

$$= \int_0^\infty [F'(1 + F)e^{-\beta^0 \frac{|p|^2}{2}} - F(1 + F')e^{-\beta^0 \frac{|p'|^2}{2}}] \, \overline{S}(|p|, |p'|) |p'| d|p'|,$$

$$\overline{S}(|p|, |p'|) = \frac{4\pi^2}{|p|} \int_0^\infty \int_0^\infty \delta_{|p'|^2 + |p'_*|^2 - |p|^2 - |p_*|^2 = 0} \frac{e^{\beta^0 \frac{|p|^2}{2}}}{e^{\beta^0 \frac{|p'_*|^2}{2} + b} - \tau} \frac{e^{\beta^0 \frac{|p_*|^2}{2} + b}}{e^{\beta^0 \frac{|p_*|^2}{2} + b} - \tau}$$

$$\times \min[|p|, |p'|, |p_*|, |p'_*|] |p'_*| d|p'_*| |p_*| d|p_*|$$

$$= \frac{4\pi^2}{|p|} \int_0^\infty \frac{e^{\beta^0 \frac{|p|^2}{2}}}{e^{\beta^0 \frac{|p'_*|^2}{2} + b}} \frac{e^{\beta^0 \frac{|p'|^2 + |p'_*|^2 - |p|^2}{2} + b}}{e^{\beta^0 \frac{|p'|^2 + |p'_*|^2 - |p|^2}{2} + b} - \tau} \mathbf{1}_{\{|p'|^2 + |p'_*|^2 \ge |p|^2\}}$$

$$\times \min[|p|, |p'|, \sqrt{|p'|^2 + |p'_*|^2 - |p|^2}, |p'_*|] \, |p'_*| d|p'_*|$$

By defining

$$\varepsilon = \beta^0 \frac{|p|^2}{2}, \quad \varepsilon' = \beta^0 \frac{|p'|^2}{2}, \varepsilon_* = \beta^0 \frac{|p_*|^2}{2}, \quad \varepsilon'_* = \beta^0 \frac{|p'_*|^2}{2},$$

we have

$$\overline{S}(\sqrt{2\varepsilon/\beta^0}, \sqrt{2\varepsilon'/\beta^0}) = 4\pi^2 \sqrt{2}(\beta^0)^{-3/2} e^{-b} \int_0^\infty e^{\varepsilon - \varepsilon'_*} \frac{e^{\varepsilon' + \varepsilon'_* - \varepsilon + b}}{e^{\varepsilon' + \varepsilon'_* - \varepsilon + b} - \tau} \mathbf{1}_{\{\epsilon' + \epsilon'_* \geq \epsilon\}}$$

$$\times \frac{\min(\sqrt{\epsilon}, \sqrt{\varepsilon' + \varepsilon'_* - \varepsilon}, \sqrt{\epsilon'}, \sqrt{\epsilon'_*})}{\sqrt{\varepsilon}} d\varepsilon'_* \equiv S(\varepsilon, \varepsilon').$$

Finally

$$Q_{BQ}(F) = \int_0^\infty \left[F'(1 + F) e^{-\epsilon} - F(1 + F') e^{-\epsilon'} \right] S \, d\epsilon'$$

with $S(\epsilon, \epsilon') = \Sigma(\epsilon, \epsilon')/\sqrt{\epsilon}$ and

$$\Sigma(\epsilon, \epsilon') = \int_{\max(0, \epsilon - \epsilon')}^\infty e^{\epsilon - \epsilon'_*} \frac{e^{\varepsilon' + \varepsilon'_* - \varepsilon + b}}{e^{\varepsilon' + \varepsilon'_* - \varepsilon + b} - \tau} \min(\sqrt{\epsilon}, \sqrt{\epsilon' + \epsilon'_* - \epsilon}, \sqrt{\epsilon'}, \sqrt{\epsilon'_*}) \, d\epsilon'_*.$$

By a change of variables in the integral definition of Σ see that it is a symmetric function. We may then write equation (4.15) as

$$\sqrt{\epsilon} \frac{\partial F}{\partial t} = \int_0^\infty \Sigma \left[F'(1 + F) e^{-\epsilon} - F(1 + F') e^{-\epsilon'} \right] d\epsilon'.$$

and perform the usual change of variable

$$\sqrt{\epsilon} F \to F, \quad b(\epsilon, \epsilon') = \frac{\Sigma(\epsilon, \epsilon')}{\sqrt{\epsilon}\sqrt{\epsilon'}}$$

to obtain the more suitable form

$$\frac{\partial F}{\partial t} = \int_0^\infty b(\epsilon, \epsilon') \left[F'(\sqrt{\epsilon} + F) e^{-\epsilon} - F(\sqrt{\epsilon'} + F') e^{-\epsilon'} \right] d\epsilon', \qquad (4.34)$$

Notice that b is singular near the origin and as a consequence we can not apply the results obtained in [22, 23].

We first want to give a precise mathematical sense to the collision term in (4.34). Notice that it can be written t in the following way

$$Q(F) = \int_0^\infty b(\epsilon, \epsilon') \sqrt{\epsilon} \, e^{-\epsilon} F' \, d\epsilon' - F \int_0^\infty b(\epsilon, \epsilon') \sqrt{\epsilon'} \, e^{-\epsilon'} \, d\epsilon'$$

$$+ \int_0^\infty b(\epsilon, \epsilon')(e^{-\epsilon} - e^{-\epsilon'}) \, FF' \, d\epsilon'.$$

Lemma 4.4 *The function*

$$\ell(\epsilon) := \int_0^\infty b(\epsilon, \epsilon') \sqrt{\epsilon'} \, e^{-\epsilon'} \, d\epsilon'. \qquad (4.37)$$

satisfies $\ell \in C([0,\infty))$, $\ell \leq C_1(1 + \sqrt{\varepsilon})$ for some positive constant C_2, $\ell \to C_2(b,\tau)$ when $\epsilon \to 0$ and $\ell \sim \gamma\sqrt{\epsilon}$ when $\epsilon \to \infty$, with $\gamma > 0$. Moreover, we have

$$\chi(\epsilon, \epsilon') := (e^{-\epsilon} - e^{-\epsilon'})b(\varepsilon, \varepsilon') \in C_b([0,\infty) \times [0,\infty)). \tag{4.38}$$

As a conclusion, for any F such that $(1 + \sqrt{\epsilon})F \in L^1$, $Q(F)$ belongs to L^1 and the map $F \mapsto Q(F)$ is continuous from $L^1_{1/2}$ to L^1.

Proof of Lemma 4.4 In order to prove the properties of ℓ we write

$$\ell(\epsilon) = \frac{1}{\sqrt{\epsilon}} \int_0^\epsilon \zeta_b(\epsilon')\, e^{-\epsilon'}\, d\epsilon' + \frac{1}{\sqrt{\epsilon}} \int_0^\epsilon \xi_b(\epsilon, \varepsilon')\, e^{-\epsilon'}\, d\epsilon'$$

where,

$$\zeta_b(z) = \int_0^\infty e^{z-k} \frac{e^{k+b}}{e^{k+b} - \tau} \min(\sqrt{z}, \sqrt{k})\, dk$$

$$\xi_b(z, y) = \int_0^\infty e^{z-k} \frac{e^{k-z+y+b}}{e^{k-z+y+b} - \tau} \min(\sqrt{z}, \sqrt{k})\, dk.$$

Note that

$$\zeta_b(z) \leq \int_0^\infty e^{z-k} \min(\sqrt{z}, \sqrt{k})dk = \gamma(z)e^z + \sqrt{z} \equiv \zeta(z)$$

with $\gamma(x) = \int_0^x e^{-y}\sqrt{y}dy$, and $\xi_b(z,y) \leq \zeta(z)$. Assume first that $\varepsilon \to 0$. Then,

$$\frac{1}{\sqrt{\epsilon}} \int_0^\epsilon \zeta_b(\epsilon')\, e^{-\epsilon'}\, d\epsilon' \leq \frac{1}{\sqrt{\epsilon}} \int_0^\epsilon \zeta(\varepsilon')e^{-\epsilon'}\, d\epsilon' \to 0 \quad \text{as } \varepsilon \to 0.$$

On the other hand,

$$\frac{1}{\sqrt{\epsilon}} \int_\epsilon^\infty \xi_b(\epsilon, \varepsilon')\, e^{-\epsilon'}\, d\epsilon'$$

$$= \frac{1}{\sqrt{\epsilon}} \int_\varepsilon^\infty e^{-\varepsilon'} \int_0^\varepsilon e^{\varepsilon - \varepsilon'_*} \frac{e^{\varepsilon' + \varepsilon'_* - \varepsilon + b}}{e^{\varepsilon' + \varepsilon'_* - \varepsilon + b} - \tau} \sqrt{\varepsilon'_*}d\varepsilon'_*d\varepsilon' +$$

$$+ \int_\varepsilon^\infty e^{-\varepsilon'} \int_\varepsilon^\infty e^{\varepsilon - \varepsilon'_*} \frac{e^{\varepsilon' + \varepsilon'_* - \varepsilon + b}}{e^{\varepsilon' + \varepsilon'_* - \varepsilon + b} - \tau} \sqrt{\varepsilon'_*}d\varepsilon'_*d\varepsilon'$$

$$\to \int_0^\infty e^{-\varepsilon'} \int_0^\infty e^{-\varepsilon'_*} \frac{e^{\varepsilon' + \varepsilon'_* + b}}{e^{\varepsilon' + \varepsilon'_* + b} - \tau} \sqrt{\varepsilon'_*}d\varepsilon'_*d\varepsilon' = C_2(b, \tau) \quad \text{as } \varepsilon \to 0.$$

Suppose now that $\varepsilon \to \infty$. We first note that,

$$\frac{1}{\sqrt{\epsilon}} \int_0^\epsilon \xi_b(\epsilon, \varepsilon')\, e^{-\epsilon'}\, d\epsilon' \leq \frac{1}{\sqrt{\epsilon}} \int_0^\epsilon \zeta(\varepsilon')\, e^{-\epsilon'}\, d\epsilon' \to 0 \quad \text{as } \varepsilon \to \infty.$$

Finally,

$$\frac{1}{\sqrt{\varepsilon}}\int_0^\varepsilon e^{-\varepsilon'}\zeta_b(\varepsilon')d\varepsilon' = \frac{1}{\sqrt{\varepsilon}}\int_0^\varepsilon\int_0^{\varepsilon'} e^{-k}\frac{e^{k+b}}{e^{k+b}-\tau}\sqrt{k}dk\,d\varepsilon'$$
$$+\frac{1}{\sqrt{\varepsilon}}\int_0^\varepsilon\sqrt{\varepsilon'}\int_{\varepsilon'}^\infty e^{-k}\frac{e^{k+b}}{e^{k+b}-\tau}dkd\varepsilon' \equiv I_1+I_2$$

and,

$$I_2 \leq \frac{1}{\sqrt{\varepsilon}}\int_0^\varepsilon\sqrt{\varepsilon'}e^{-\varepsilon'}d\varepsilon' = O(\frac{1}{\sqrt{\varepsilon}}),$$

$$\lim_{\varepsilon\to\infty}\frac{1}{\varepsilon}\int_0^\varepsilon\int_0^{\varepsilon'}e^{-k}\sqrt{k}dkd\varepsilon' = \int_0^\infty e^{-k}\sqrt{k}dk \equiv \gamma$$

To prove (4.38) we just have to consider the two cases: If $\min(\epsilon,\epsilon')\to 0$ then

$$\chi \sim \frac{e^{-\max}-e^{-\min}}{\sqrt{\max}}\xrightarrow[\max\to 0,\infty]{} 0 \quad \text{uniformly in } \min.$$

If $\min(\epsilon,\epsilon')\to\infty$ then $\epsilon,\epsilon'\to\infty$ and

$$|\chi| \leq \frac{e^{-\min}-e^{-\max}}{\sqrt{\max}\sqrt{\min}}e^{\min} \leq 1.$$

$\square$

The particle number of the solutions to (4.34) is constant in time (at least formally) and this gives a first a priori bound in L^1. But we have still need to control higher moments of F. We show in the next Lemma how this may be formally done for polynomial momments. Exponential moments may be controlled in a similar way.

Lemma 4.5 *Let F be a solution to equation (4.34). Then, there exists a positive constant C such that, formally:*

$$\frac{d}{dt}Y_1(F)+\frac{\gamma}{4}\int_0^\infty F(1+\sqrt{\epsilon})\epsilon\,d\epsilon \leq C_{\text{in}}, \tag{4.39}$$

with

$$Y_\theta(F) := \int_0^\infty \epsilon^\theta\,d|F|(\epsilon). \tag{4.40}$$

Proof of Lemma 4.5 Let F be a solution to (4.34). A formal straightforward calculation gives

$$\frac{d}{dt}Y_1(F) = \int_0^\infty \int_0^\infty bFF'[e^{-\epsilon} - e^{-\epsilon'}]\epsilon \, d\epsilon' d\epsilon$$
$$+ \int_0^\infty \int_0^\infty b[\sqrt{\epsilon}F'\epsilon \, e^{-\epsilon} - \sqrt{\epsilon'}F \, e^{-\epsilon'}\epsilon] \, d\epsilon' d\epsilon$$
$$= \int_0^\infty \int_0^\infty \frac{b}{2}F \, F'[e^{-\epsilon} - e^{-\epsilon'}](\epsilon - \epsilon') \, d\epsilon' d\epsilon$$
$$+ \int_0^\infty \int_0^\infty b\sqrt{\epsilon'}F \, e^{-\epsilon'}[\epsilon' - \epsilon] \, d\epsilon' d\epsilon.$$

Since the first term is non positive, we just have to manage with the second term. In order to estimate it we compute

$$I(\epsilon) := \int_0^\infty b\sqrt{\epsilon'} \, e^{-\epsilon'} \, [\epsilon' - \epsilon] \, d\epsilon'$$
$$= \int_0^\epsilon \frac{\zeta(\epsilon')}{\sqrt{\epsilon}} \, e^{-\epsilon'} \, [\epsilon' - \epsilon] \, d\epsilon' + \frac{\zeta(\epsilon)}{\sqrt{\epsilon}} \int_\epsilon^\infty e^{-\epsilon'} \, [\epsilon' - \epsilon] \, d\epsilon' = I_1 + I_2.$$

On the one hand,

$$I_2 = \frac{\zeta(\epsilon)}{\sqrt{\epsilon}}\left(\int_\epsilon^\infty \epsilon' \, e^{-\epsilon'} \, d\epsilon' - \epsilon \int_\epsilon^\infty e^{-\epsilon'} \, d\epsilon'\right) = \frac{\zeta(\epsilon)}{\sqrt{\epsilon}} \, e^{-\epsilon} = e^{-\epsilon} + \frac{\gamma(\epsilon)}{\sqrt{\epsilon}} \leq C.$$

On the other hand,

$$I_1 \leq \frac{1}{\sqrt{\epsilon}}\int_0^\epsilon \gamma(\epsilon')(\epsilon' - \epsilon) \, d\epsilon' \leq \frac{\gamma(m)}{\sqrt{\epsilon}}\int_m^\epsilon (\epsilon' - \epsilon) \, d\epsilon' = \frac{\gamma(m)}{\sqrt{\epsilon}}\left(-\frac{\epsilon^2}{2} + m\epsilon - \frac{m^2}{2}\right).$$

Since $\gamma(m) \to \gamma$ when $m \to \infty$ and the term of greater order have a minus sign, we obtain

$$I \leq C - \frac{\gamma}{4}(1 + \sqrt{\epsilon})\epsilon,$$

and (4.39) follows. $\qquad\qquad\square$

We state now the global existence result for the equation (1.26).

Theorem 4.6 *For any initial datum $F_{\text{in}} \in L_1^1(\mathbb{R}_+)$, $F \geq 0$, there exists a solution $F \in C([0,\infty), L_{1/2}^1))$ to the equation*

$$\frac{\partial F}{\partial t} = Q_{BQ}(F), \quad \text{for} \quad \varepsilon > 0, \, t > 0$$

with Q_{BQ} defined in (4.32), and such that

$$\lim_{t \to 0}\|F(t) - F_{\text{in}}\|_{L^1(\mathbb{R}^+)} = 0.$$

Proof of Theorem 4.6 We first introduce the regularized problem:

$$\frac{\partial F_n}{\partial t} = \int_0^\infty b_n(\epsilon, \epsilon') \big[F_n'(\sqrt{\epsilon_n} + F_n)\, e^{-\epsilon} - F_n(\sqrt{\epsilon_n'} + F_n')\, e^{-\epsilon'} \big]\, d\epsilon', \quad (4.41)$$

with

$$b_n(\epsilon, \epsilon') = \frac{\zeta(\epsilon \wedge \epsilon' \wedge n)}{\sqrt{\epsilon \wedge \epsilon' \wedge n}\sqrt{\epsilon \vee \epsilon' \vee 1/n}}, \quad \epsilon_n = \epsilon \vee \frac{1}{n}.$$

The existence of a solution F_n to this equation follows from the existence result in [EM] since $b_n \in L^\infty$. We obtain a priori estimates on the sequence (F_n) in the two following Lemmas and then pass to the limit using Lemma 4.4

Lemma 4.7 *The sequence of solutions (F_n) satisfies*

$$\frac{d}{dt} Y_1(F_n) + \frac{\gamma}{3} \int_0^\infty F_n \sqrt{\epsilon}(\epsilon \wedge n)\, d\epsilon \le C_{\text{in}} + Y_1(F_n). \quad (4.42)$$

for n large enough. This implies that $Y_1(F_n)$ is bounded in $L^\infty(0,T)$ and $\int_0^\infty F_n \sqrt{\epsilon}(\epsilon \wedge n)\, d\epsilon$ is bounded in $L^1(0,T)$ for any $T > 0$.

Proof of Lemma 4.7 By the same computation as in Lemma 4.5, which makes now perfectly sense by the properties of F_n, we deduce

$$\frac{d}{dt} Y_1(F_n) = \int_0^\infty \int_0^\infty \frac{b_n}{2} F_n\, F_n'[e^{-\epsilon} - e^{-\epsilon'}](\epsilon - \epsilon')\, d\epsilon' d\epsilon + \int_0^\infty F_n\, I_n\, d\epsilon.$$

The first term in the right hand side is non positive and the second satisfies:

$$I_n(\epsilon) := \int_0^\infty b_n \sqrt{\epsilon_n'}\, e^{-\epsilon'}\, [\epsilon' - \epsilon]\, d\epsilon'$$

$$\le \int_0^\epsilon \frac{\gamma(\epsilon' \wedge n)\sqrt{\epsilon_n'}}{\sqrt{\epsilon_n}\sqrt{\epsilon' \wedge n}}\, e^{\epsilon' \wedge n - \epsilon'}[\epsilon' - \epsilon]d\epsilon' + \frac{\zeta(\epsilon \wedge n)}{\sqrt{\epsilon \wedge n}}\, e^{-\epsilon}.$$

Note that the last term is bounded. Let fix $\ell \ge 1$ such that $\gamma(\ell) > 2\gamma/3$. For n and ϵ such that $\epsilon \wedge n \ge \ell$ we write

$$\int_0^\epsilon \frac{\gamma(\epsilon' \wedge n)\sqrt{\epsilon_n'}}{\sqrt{\epsilon_n}\sqrt{\epsilon' \wedge n}}\, e^{\epsilon' \wedge n - \epsilon'}[\epsilon' - \epsilon]d\epsilon'$$

$$\le \frac{2\gamma}{3\sqrt{\epsilon}} \int_0^\epsilon \frac{\sqrt{\epsilon'}}{\sqrt{\epsilon' \wedge n}}\, e^{\epsilon' \wedge n - \epsilon'}(\epsilon' - \epsilon)\, d\epsilon' + \gamma \int_0^\ell \frac{\epsilon}{\sqrt{\epsilon' \wedge n}}\, d\epsilon'$$

$$\le \frac{2\gamma}{3\sqrt{\epsilon}} \int_0^{\epsilon \wedge n} (\epsilon' - \epsilon)\, d\epsilon' + \gamma\, 2\sqrt{\ell}\epsilon \le 2\sqrt{\ell}\gamma\epsilon - \frac{\gamma}{3}\sqrt{\epsilon}(\epsilon \wedge n).$$

We then conclude as in the proof of Lemma 4.5. $\qquad \square$

Lemma 4.8 *The set (F_n) is a Cauchy sequence in $C([0,T]; L^1_{1/2}(\mathbb{R}_+))$ for any $T > 0$.*

Proof of Lemma 4.8   We just compute

$$\frac{\partial}{\partial t}(F_m - F_n) = Q_n(F_m) - Q_n(F_n) + Q_m(F_m) - Q_n(F_m)$$

and

$$
\begin{aligned}
\frac{d}{dt}&\|F_m - F_n\|_{L^1_{1/2}} \\
&\leq \int_0^\infty |F_m - F_n| \int_0^\infty b_n \sqrt{\epsilon'_n}\, e^{-\epsilon'} [\sqrt{\epsilon} - \sqrt{\epsilon'}]\, d\epsilon'\, d\epsilon \\
&+ \int_0^\infty \int_0^\infty |F_m F'_m - F_n F'_n||e^{-\epsilon} - e^{-\epsilon'}|\, b_n(1 + \sqrt{\epsilon})\, d\epsilon'\, d\epsilon \\
&+ \int_0^\infty \int_0^\infty (b_m - b_n) F_m\, F'_m |e^{-\epsilon} - e^{-\epsilon'}|(1 + \sqrt{\epsilon})\, d\epsilon'\, d\epsilon \\
&+ \int_0^\infty \int_0^\infty |b_m \sqrt{\epsilon'_m} - b_n \sqrt{\epsilon'_n}| F_m\, e^{-\epsilon'} [2 + \sqrt{\epsilon'} + \sqrt{\epsilon}]\, d\epsilon'\, d\epsilon, \\
&\equiv J_1 + J_2 + J_3 + J_4
\end{aligned}
\tag{4.43}
$$

with $b_{m,n} = b_m - b_n$. We now estimate each of the terms J_i, $i = 1, 2, 3, 4$ separately.

1. Estimate of J_1. The first term is nothing but

$$J_1(n) = \int_0^\infty |F_m - F_n|\, I_n\, d\epsilon$$

with

$$
\begin{aligned}
I_n(\epsilon) &:= \int_0^\infty b_n \sqrt{\epsilon'_n}\, e^{-\epsilon'} [\sqrt{\epsilon'} - \sqrt{\epsilon}]\, d\epsilon' \\
&\leq \frac{\zeta(\epsilon \wedge n)}{\sqrt{\epsilon \wedge n}} \int_\epsilon^\infty e^{-\epsilon'} (\sqrt{\epsilon'} - \sqrt{\epsilon})\, d\epsilon' =: I'_n.
\end{aligned}
$$

For $\epsilon \geq 1$, integration by parts gives

$$I'_n = \frac{\zeta(\epsilon \wedge n)}{\sqrt{\epsilon \wedge n}} \int_\epsilon^\infty \frac{e^{-\epsilon'}}{\sqrt{\epsilon'}}\, d\epsilon' \leq \frac{\zeta(\epsilon \wedge n)}{\sqrt{\epsilon \wedge n}}\, e^{-\epsilon} \in L^\infty(\mathbb{R}_+).$$

Since I'_n is bounded for $\epsilon \leq 1$, there exists a constant $K_1 > 0$ such that

$$J_1 \leq K_1 \|F_m - F_n\|_{L^1}. \tag{4.44}$$

2. Estimate of J_2. We notice that

$$
\begin{aligned}
\chi_n &:= b_n(e^{-\epsilon} - e^{\epsilon'}) \\
&\leq \frac{\zeta(\epsilon \wedge \epsilon')}{\sqrt{\epsilon \wedge \epsilon'}} \frac{e^{-\epsilon} - \epsilon^{-\epsilon'}}{\sqrt{\epsilon \vee \epsilon'}} \mathbf{1}_{\epsilon \wedge \epsilon' \leq 1} + \frac{\zeta(\epsilon \wedge \epsilon' \wedge n)}{\sqrt{\epsilon \wedge \epsilon' \wedge n}} \max(e^{-\epsilon}, \epsilon^{-\epsilon'}) \mathbf{1}_{\epsilon \vee \epsilon' \geq 1}
\end{aligned}
$$

is uniformly bounded in $\mathbb{R}_+^2$ as we have already shown in the proof of Lemma 4.4. Then, we deduce

$$
\begin{aligned}
J_2 &\leq \|\chi_n\|_{L^\infty_{\epsilon,\epsilon'}} [\|(F_m - F_n)(1 + \sqrt{\epsilon})\|_{L^1}\|F_m\|_{L^1} + \|F_m - F_n\|_{L^1}\|F_m(1 + \sqrt{\epsilon})\|_{L^1} \\
&\leq K_2 \|F_m\|_{L^1_{1/2}} \|F_m - F_n\|_{L^1_{1/2}},
\end{aligned}
\tag{4.45}
$$

for some constant $K_2 > 0$.

3. *Estimate of J_3.* Since $x \mapsto \zeta(x)/\sqrt{x}$ is increasing (at least for the large values of x) we have $0 \leq b_n \leq b_m$. We then notice that

$$
\begin{aligned}
0 \leq b_m - b_n = {}&\frac{\zeta(\epsilon \wedge \epsilon')}{\sqrt{\epsilon \wedge \epsilon'}}\Big(\frac{1}{\sqrt{\epsilon \vee \epsilon' \vee 1/m}} - \sqrt{n}\Big)\mathbf{1}_{\epsilon\vee\epsilon'<1/n} \\
&+ \frac{1}{\sqrt{\epsilon \vee \epsilon'}}\Big(\frac{\zeta(\epsilon \wedge \epsilon' \wedge m)}{\sqrt{\epsilon \wedge \epsilon' \wedge m}} - \frac{\zeta(n)}{\sqrt{n}}\Big)\mathbf{1}_{\epsilon\wedge\epsilon' n}.
\end{aligned}
$$

On the one hand, we have

$$
\begin{aligned}
&(b_m - b_n)|e^{-\epsilon} - e^{-\epsilon'}|\mathbf{1}_{\epsilon\vee\epsilon'<1/n} \\
&\leq \frac{\zeta(\epsilon \wedge \epsilon')}{\sqrt{\epsilon \wedge \epsilon'}}\frac{1}{\sqrt{\epsilon \vee \epsilon' \vee 1/m}}|e^{-\epsilon} - e^{-\epsilon'}|\mathbf{1}_{\epsilon\vee\epsilon'<1/n} \\
&\leq \|\frac{\zeta(x)}{\sqrt{x}}\|_{L^\infty(0,1)}\sqrt{\epsilon \vee \epsilon'}\mathbf{1}_{\epsilon\vee\epsilon'<1/n} \leq \frac{K_3}{\sqrt{n}}\mathbf{1}_{\epsilon\vee\epsilon'<1/n}.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
(b_m - b_n)|e^{-\epsilon} - e^{-\epsilon'}|\mathbf{1}_{\epsilon\wedge\epsilon'\geq n} &\leq \frac{\zeta(\epsilon \wedge \epsilon' \wedge m)}{\sqrt{\epsilon \wedge \epsilon' \wedge m}}\frac{1}{\sqrt{\epsilon \vee \epsilon'}}|e^{-\epsilon} - e^{-\epsilon'}|\mathbf{1}_{\epsilon\wedge\epsilon'\geq n} \\
&\leq \frac{2}{\sqrt{\epsilon \vee \epsilon'}}\|\frac{\zeta(x)\,e^{-x}}{\sqrt{x}}\|_{L^\infty}\mathbf{1}_{\epsilon\wedge\epsilon'\geq n} \leq \frac{K_3}{\sqrt{n}}\mathbf{1}_{\epsilon\wedge\epsilon'>n}
\end{aligned}
$$

for some constant $K_3 > 0$. We deduce that

$$
J_3 \leq \frac{K_3}{\sqrt{n}}\,Y_0(F_m)\|F_m\|_{L^1_{1/2}}.
\tag{4.46}
$$

4. *Estimate of J_4.* Since $0 \leq \sqrt{\epsilon'_n}\,b_n \leq \sqrt{\epsilon'_m}\,b_m$ for n large enough, we may write

$$
\begin{aligned}
&\sqrt{\epsilon'_m}\,b_m - \sqrt{\epsilon'_n}\,b_n \\
&\leq \frac{\sqrt{\epsilon' \vee 1/m}}{\sqrt{\epsilon \vee \epsilon' \vee 1/m}}\frac{\zeta(\epsilon \wedge \epsilon')}{\sqrt{\epsilon \wedge \epsilon'}}\mathbf{1}_{\epsilon\vee\epsilon'<1/n} + \frac{\sqrt{\epsilon'}}{\sqrt{\epsilon \vee \epsilon'}}\frac{\zeta(\epsilon \wedge \epsilon' \wedge m)}{\sqrt{\epsilon \wedge \epsilon' \wedge m}}\mathbf{1}_{\epsilon\wedge\epsilon'>n}.
\end{aligned}
$$

On the one hand we compute

$$
\begin{aligned}
&\int_0^\infty (b_m\sqrt{\epsilon'_m} - b_n\sqrt{\epsilon'_n})\,e^{-\epsilon'}[2 + \sqrt{\epsilon} + \sqrt{\epsilon'}]\mathbf{1}_{\epsilon\vee\epsilon'<1/n}\,d\epsilon' \leq \\
&\leq 4\int_0^{1/n}(b_m\sqrt{\epsilon'_m} - b_n\sqrt{\epsilon'_n})\,d\epsilon' \leq 4\|\frac{\zeta(x)}{\sqrt{x}}\|_{L^\infty(0,1)}\frac{1}{n}.
\end{aligned}
\tag{4.47}
$$

On the other hand,

$$\int_0^\infty (b_m \sqrt{\epsilon'_m} - b_n \sqrt{\epsilon'_n})\, e^{-\epsilon'} [2 + \sqrt{\epsilon} + \sqrt{\epsilon'}] \mathbf{1}_{\epsilon \wedge \epsilon' \geq n}\, d\epsilon'$$

$$\leq \int_0^\epsilon \frac{\sqrt{\epsilon'}}{\sqrt{\epsilon}} \frac{\zeta(\epsilon' \wedge m)}{\sqrt{\epsilon' \wedge m}}\, e^{-\epsilon'} [2 + \sqrt{\epsilon} + \sqrt{\epsilon'}]\, d\epsilon' \mathbf{1}_{\epsilon \geq n}$$

$$+ \frac{\zeta(\epsilon \wedge m)}{\sqrt{\epsilon \wedge m}} \int_\epsilon^\infty e^{-\epsilon'} [2 + \sqrt{\epsilon} + \sqrt{\epsilon'}]\, d\epsilon' \mathbf{1}_{\epsilon \geq n}$$

$$\leq \int_0^\epsilon \frac{\zeta(\epsilon' \wedge m)}{\sqrt{\epsilon' \wedge m}}\, e^{-\epsilon'} \sqrt{\epsilon'} [3 + \frac{\sqrt{\epsilon'}}{\sqrt{\epsilon}}]\, d\epsilon' \mathbf{1}_{\epsilon \geq n} + \|\frac{\zeta(x)}{\sqrt{x}}\, e^{-x}\|_{L^\infty} (3 + 2\sqrt{\epsilon}) \mathbf{1}_{\epsilon \geq n}.$$

To estimate the first term in the last right hand side, we successively consider the cases $\epsilon \leq m$ and $\epsilon \geq m$ and make use of the following elementary estimate

$$\int_m^\epsilon (\epsilon')^\gamma\, e^{-\epsilon'}\, d\epsilon' \leq (1 + m^\gamma)\, e^{-m},$$

with $\gamma = 1/2$ and $\gamma = 1$. When $\epsilon \leq m$ we have

$$\int_0^\epsilon \frac{\zeta(\epsilon' \wedge m)}{\sqrt{\epsilon' \wedge m}}\, e^{-\epsilon'} \sqrt{\epsilon'} [3 + \frac{\sqrt{\epsilon'}}{\sqrt{\epsilon}}]\, d\epsilon' = \int_0^\epsilon \zeta(\epsilon')\, e^{-\epsilon'} [3 + \frac{\sqrt{\epsilon'}}{\sqrt{\epsilon}}]\, d\epsilon' \leq 4 \|\zeta(x)\, e^{-x}\|_{L^\infty} \epsilon.$$

$$(4.48)$$

Now, when $\epsilon \geq m$, we get

$$\int_0^\epsilon \frac{\zeta(\epsilon' \wedge m)}{\sqrt{\epsilon' \wedge m}}\, e^{-\epsilon'} \sqrt{\epsilon'} [3 + \frac{\sqrt{\epsilon'}}{\sqrt{\epsilon}}]\, d\epsilon' =$$

$$= \int_0^m \zeta(\epsilon')\, e^{-\epsilon'} [3 + \frac{\sqrt{\epsilon'}}{\sqrt{\epsilon}}]\, d\epsilon' + \frac{\zeta(m)}{\sqrt{m}} \int_m^\epsilon e^{-\epsilon'} \sqrt{\epsilon'} [3 + \frac{\sqrt{\epsilon'}}{\sqrt{\epsilon}}]\, d\epsilon'$$

$$\leq 4 \|\zeta(x)\, e^{-x}\|_{L^\infty} m + \|\frac{\zeta(x)}{\sqrt{x}}\, e^{-x}\|_{L^\infty} (1 + \sqrt{m}),$$

using (4.48). As a conclusion, we have proved that

$$\int_0^\infty (b_m \sqrt{\epsilon'_m} - b_n \sqrt{\epsilon'_n})\, e^{-\epsilon'} [2 + \sqrt{\epsilon} + \sqrt{\epsilon'}] \mathbf{1}_{\epsilon \wedge \epsilon' \geq n}\, d\epsilon' \leq K'_4 ((\epsilon \wedge m) + \sqrt{\epsilon} + 1) \mathbf{1}_{\epsilon \geq n}.$$

$$(4.49)$$

Therefore, combining (4.47) and (4.49), we get

$$J_4 \leq \frac{K_4}{\sqrt{n}} \int_0^\infty F_m (1 + \sqrt{\epsilon}(\epsilon \wedge m))\, d\epsilon. \qquad (4.50)$$

From (4.42)-(4.46), (4.50) and Lemma 4.7 we obtain

$$\frac{d}{dt} \|F_m - F_n\|_{L^1_{1/2}} \leq A(t) \|F_m - F_n\|_{L^1_{1/2}} + \frac{B(t)}{\sqrt{n}}$$

with $A \in L^\infty(0, T)$ and $B \in L^1(0, T)$ for any $T > 0$. We end the proof using the Gronwall Lemma. □

The equation (4.15)-(4.32) is of the form given by equation (1.1) in [22, 23]:

$$\varepsilon^2 \frac{\partial F}{\partial t}(\varepsilon) = \int_0^\infty b(\varepsilon, \varepsilon')(F'(1+F)e^{-\varepsilon} - F(1+F')e^{-\varepsilon'})d\varepsilon'$$

with $\varepsilon^2 \varepsilon'^2 b(\varepsilon, \varepsilon') = S(\varepsilon, \varepsilon')$. We then refer to [23, Theorem 2] for the proof of the following proposition.

Proposition 4.9 *Let* $0 \leq F \in M^1_{\mathrm{rad}}([0, \infty)$ *such that*

$$M(F) := \int_0^\infty v^2 dF(v) = N \geq 0 \tag{4.51}$$

Then the following two assertions are equivalent:
(i) $F = \mathcal{B}_N$ *with*

$$\varepsilon^2 \mathcal{B}_N(\varepsilon) = \begin{cases} \frac{\varepsilon^2}{e^{\varepsilon + \mu} - 1}, & \text{if} \quad N \leq N_0 \equiv \int_0^\infty \frac{\varepsilon^2 d\varepsilon}{e^\varepsilon - 1}, \\ \frac{\varepsilon^2}{e^\varepsilon - 1} + (N - N_0)\delta_0 & \text{otherwise.} \end{cases}$$

(ii) F *is the solution of the maximization problem:*

$$\mathcal{H}(F) = \max\{\mathcal{H}(F'), \ F' \ \text{satisfying (4.51)}\},$$

where the entropy H given by (4.21) reads:

$$H(F) = \int_0^\infty [(1+F)\ln(1+F) - F\ln F - \varepsilon F]\varepsilon^2 d\varepsilon. \tag{4.52}$$

Theorem 4.10 (Asymptotic behaviour) *For every* $F_{\mathrm{in}} \in L^1_1(\mathbb{R}^+)$, $F_{\mathrm{in}} \geq 0$, *let* $N = M(F_{\mathrm{in}})$, *and* $F \in C([0, \infty), L^1_{1/2}(\mathbb{R}^+))$ *the corresponding solution to (4.33) with initial data* F_{in}. *Then*

$$F(t, .) \underset{t \to \infty}{\rightharpoonup} \mathcal{B}_N \ \text{weakly} \star \ \text{in} \ \big(C_c(\mathbb{R}_+)\big)'$$
$$\lim_{t \to \infty} \|g(t, .) - \mathcal{B}_N\|_{L^1((k_0, \infty))} = 0 \quad \forall k_0 > 0. \tag{4.53}$$

The proof of this theorem is the same as that of [23, theorem 6]; thus we skip it.

5 The collision integral for relativistic quantum particles

In this section, we follow the books by de Groot, van Leeuwen and van Weert [29] and Glassey [26]

5.1 Parametrizations

In this Section we introduce the Boltzmann equation for relativistic particles. A particle is now determined by the pair (X, P) representing the position and momentum in the time-position and momentum space $\mathbb{R}^4 \times \mathbb{R}^4$ where we write $X = (X^\mu)$, with $X^0 = t$, $(X^1, X^2, X^3) = x$, $P = (P^\mu)$ with $(P^1, P^2, P^3) = p$. Moreover, since the particle has to be on its mass shell, we have

$$P^0 \equiv p^0 := \sqrt{|p|^2 + m^2 c^2}. \tag{5.1}$$

where c is the speed of the light. The gas is described by its density $F = F(X, P)$. In this context, the Boltzmann equation as it may be found in [29, 26] is

$$\langle P, \nabla_X F \rangle = \mathcal{Q}(F), \tag{5.2}$$

where the collision kernel reads

$$\mathcal{Q}(F)(P) = \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \mathcal{W} q(F) \delta_{P+P_*-P'-P'_*} \chi(P_*) \chi(P') \chi(P'_*) \, dP' dP'_* dP_*. \tag{5.3}$$

Here, $\nabla_X = (c^{-1} \partial_t, \nabla_x)$ and $\langle \cdot, \cdot \rangle$ represents the Lorentz inner product in $\mathbb{R}^4$ and we use the abbreviation $|P|^2 = P \cdot P$ for any $P \in \mathbb{R}^4$. We refer to the Appendix 7 for more details on the Lorentz space. Notice nevertheless that

$$\langle P, \nabla_X F \rangle = \sum_{\mu, \nu = 1}^{4} \eta_{\mu\nu} P^\mu (\nabla_X F)^\nu, \quad \text{and} \quad (\nabla_X F)^\nu = (c^{-1} \partial_t, -\nabla_x)$$

Moreover, $\mathcal{W} = \mathcal{W}(P, P_*, P', P'_*)$ is a given non negative function related to the differential cross-section σ, see (5.10) and (5.11) below, and as before,

$$q(F) = F' F'_* (1 + \tau F)(1 + \tau F_*) - F F_* (1 + \tau F')(1 + \tau F'_*) \tag{5.4}$$

with $\tau \in \{-1, 0, 1\}$, $F = F(X, P)$, $F_* = F(X, P_*)$, $F' = F(X, P')$, $F'_* = F(X, P'_*)$.

Finally, we have defined

$$\forall R \in \mathbb{R}^4 \quad \chi(R) = \delta_{R^2 - m^2 c^2} H(R^0) \tag{5.5}$$

where H stands for the Heaviside function. With these notations, if we denote $f(t, x, p) := F(X, P)$, we have

$$\langle P, \nabla_X \rangle = \frac{p^0}{c} \frac{\partial}{\partial t} + p \cdot \nabla_x. \tag{5.6}$$

Therefore, equation (5.2) reads

$$\frac{\partial}{\partial t} f + \frac{cp}{p^0} \cdot \nabla_x f = Q(F(t, x, .))(p^0, p) := \frac{c}{p^0} \mathcal{Q}(F(t, x, .))(p^0, p). \tag{5.7}$$

By Lemma 6.1

$$\chi(P) = \frac{1}{2p^0} \delta_{P^0 = p^0} \tag{5.8}$$

we obtain, performing the integration in variables P_*^0, P'^0, P'^0_*,

$$Q(F)(p^0, p) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{c\mathcal{W}}{8p^0 p_*^0 p'^0 p'^0_*} q(f)\delta_{\mathcal{C}}\, dp'\, dp'_*\, dp_*, \qquad (5.9)$$

where $\mathcal{C}$ is defined by (1.1) with of course $\mathcal{E}(p) = p^0/c$. Gathering (5.7) and (5.9) we then recover (1.5), (1.6) with

$$w = \frac{c\,\mathcal{W}}{8p^0 p_*^0 p'^0 p'^0_*}. \qquad (5.10)$$

The function w is determined by the differential cross section as follows

$$c\frac{s\sigma(s,\theta)}{2p^0 p_*^0 p'^0 p'_*} = w \qquad (5.11)$$

where

$$s = (P + P_*)^2, \quad \cos\theta = \frac{(P_* - P)\cdot(P'_* - P')}{(P_* - P)^2}. \qquad (5.12)$$

The parameter s times c^2 is the square of the energy in the center of momentum system and θ is the scattering angle (see Remark 5.1). The differential cross section $\sigma = \sigma(s,\theta)$ is a function of the energy and the scattering angle. Since we consider identical particles it satisfies the symmetry relation:

$$\sigma(s,\theta) = \sigma(s, \pi - \theta).$$

Thus, the differential cross section determines the structure of the collision integral. See Appendix 8 for more details about this function and its physical meaning. The 12-fold integral in (5.3) can be reduced to a 5-fold integral by carrying out the delta function integrations. This can be done in two different ways. They correspond to the two different parametrizations of the collision manifold which are well known for the classical Boltzmann equations.

5.1.1   The center of mass parametrization

We deduce this parametrization starting from the formulation (5.3) of the collision kernel. To this end we first perform a change of variables in the P' and P'_* integrals. So, given P and P_* fixed, consider the Lorentz transformation Λ from $\mathbb{R}^4$ into itself defined by:

$$\Lambda\begin{pmatrix}\sqrt{s}\\0\end{pmatrix} = (P + P_*). \qquad (5.13)$$

It is given by the 4×4 matrix

$$\Lambda = \begin{pmatrix}\gamma(v) & \gamma(v)v^\top \\ \gamma(v)v & I + \frac{\gamma(v)-1}{|v|^2}vv^\top\end{pmatrix} \qquad (5.14)$$

with

$$v = \frac{p + p_*}{\sqrt{|p + p_*|^2 + s}} \in \mathbb{R}^3, \quad \gamma(v) = \frac{1}{\sqrt{1 - |v|^2}} \in \mathbb{R} \tag{5.15}$$

and

$$(vv^\top)_{ik} = v_i v_k, \quad vv^\top \in \mathcal{M}_{3\times3}. \tag{5.16}$$

Define now,

$$P' = \Lambda Q', \quad P'_* = \Lambda Q'_*. \tag{5.17}$$

For the sake of brevity we shall denote:

$$P' = P'(Q'), \quad P'_* = P'_*(Q'_*), \quad p' \equiv p'(Q'), \quad p'_* = p'_*(Q'_*). \tag{5.18}$$

By the definition of Lorentz transform we have that s and θ are invariant (see Appendix 7) by the change of variable. Then:

$$Q(f)(p)$$
$$= \frac{4c}{p^0} \int_{\mathbb{R}^4}\int_{\mathbb{R}^4}\int_{\mathbb{R}^4} s\sigma(s,\theta)q(f)\delta_{\Lambda^{-1}(P+P_*)-Q'-Q'_*}\chi(P_*^0)\chi(Q_*'^0)\chi(Q'^0)\,dQ'dQ'_*dP_*, \tag{5.19}$$

with now

$$q(f) = f(p'_*(Q'_*))f(p'(Q'))(1 + \tau f(p))(1 + \tau f(p_*))\cdot$$
$$- f(p)f(p_*)(1 + \tau f(p'(Q')))(1 + \tau f(p'_*(Q'_*))), \tag{5.20}$$

where we have used that if $(u_0, u_*, u_2, u_3) = \Lambda(v_0, v_*, v_2, v_3)$, then $\text{sign } u_0 = \text{sign } v_0$, and that

$$\delta(P + P_* - \Lambda Q' - \Lambda Q'_*) = \delta(\Lambda^{-1}(P + P_*) - Q' - Q'_*).$$

We integrate with respect to Q'^0, $Q_*'^0$, and $Q_*'^0$. Thanks to (5.8) we have

$$Q(f)(p) = \frac{c}{2p^0} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} s\sigma(s,\theta)q(f)\delta_{\{(\sqrt{s},0)-Q'-Q'_*\}}\frac{dq'}{q'^0}\frac{dq'_*}{q_*'^0}\frac{dp_*}{p_*^0}, \tag{5.21}$$

with $P = (p^0, p)$, $P_* = (p_*^0, p_*)$, $Q' = (q'^0, q')$, $Q'_* = (q_*'^0, q'_*)$. Observing that

$$(q'^0, q') + (q_*'^0, q'_*) = (\sqrt{s}, 0) \quad \text{if and only if} \quad q' = -q'_*, \quad q'^0 + q_*'^0 = \sqrt{s},$$

we get

$$q'^0 = q_*'^0 = \frac{\sqrt{s}}{2}. \tag{5.22}$$

Therefore, integrating in q'_* we get

$$Q(f)(p) = \frac{c}{p^0} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \sigma q(f)\delta_{\frac{\sqrt{s}}{2}-\sqrt{|q'|^2+m^2c^2}}\,dq'\frac{dp_*}{p_*^0}. \tag{5.23}$$

We finally, change to spherical coordinates in the q' integral:

$$dq' = |q'|^2\,d|q'|\,d\Omega \tag{5.24}$$

and use, again by Lemma 6.1,

$$\delta(\frac{\sqrt{s}}{2} - \sqrt{|q'|^2 + m^2 c^2}) = \sqrt{\frac{s}{s - 4m^2 c^2}} \delta(|q'| - \frac{1}{2}\sqrt{s - 4m^2 c^2}) \qquad (5.25)$$

to obtain

$$Q(f)(p) = \frac{c}{p^0} \int_{\mathbb{R}^3} \int_0^\infty \int_{S^2} \sqrt{\frac{s}{s - 4m^2 c^2}} \sigma q(f) \delta_{|q'| - \frac{1}{2}\sqrt{s - 4m^2 c^2}} |q'|^2 \, d|q'| \, d\Omega \frac{dp_*}{p_*^0}$$

and finally

$$Q(f)(p) = \frac{c}{4p^0} \int_{\mathbb{R}^3} \int_{S^2} \sqrt{s(s - 4m^2 c^2)} \sigma q(f) \, d\Omega \frac{dp_*}{p_*^0}. \qquad (5.26)$$

Let us go back to the expression of $q(f)$ in order to give an explicit formula in terms of p, p_* and Ω. Remember that

$$q(f) = f(p'_*(Q'_*)) f(p'(Q'))(1 + \tau f(p))(1 + \tau f(p_*))$$
$$- f(p)f(p_*)(1 + \tau f(p'(Q')))(1 + \tau f(p'_*(Q'_*)))$$

where $P' = \Lambda Q'$, $P'_* = \Lambda Q'_*$, and $\Lambda(\sqrt{s}, 0) = (P + P_*) = (p^0 + p_*^0, p + p_*)$, so that we just have to express $p' = p'(Q')$ and $p'_* = p'_*(Q'_*)$. As we have said just before

$$Q' = (\frac{\sqrt{s}}{2}, |q'|\Omega), \quad Q'_* = (\frac{\sqrt{s}}{2}, -|q'|\Omega).$$

Therefore,

$$p' = \Lambda(\frac{\sqrt{s}}{2}, q')$$
$$= \left(\gamma(v)\frac{\sqrt{s}}{2} + \gamma(v)|q'|v^\top \Omega, \gamma(v)v\frac{\sqrt{s}}{2} + |q'|\Omega + \frac{\gamma(v) - 1}{v^2}|q'|vv^\top \Omega\right)$$
$$p'_* = \Lambda(\frac{\sqrt{s}}{2}, -q')$$
$$= \left(\gamma(v)\frac{\sqrt{s}}{2} - \gamma(v)|q'|v^\top \Omega, \gamma(v)v\frac{\sqrt{s}}{2} - |q'|\Omega - \frac{\gamma(v) - 1}{v^2}|q'|vv^\top \Omega\right)$$

with

$$v = \frac{p + p_*}{\sqrt{|p + p_*|^2 + s}} = \frac{p + p_*}{p^0 + p_*^0}, \quad \gamma(v) = \frac{1}{\sqrt{1 - |v|^2}} = \frac{p^0 + p_*^0}{\sqrt{(p^0 + p_*^0)^2 - |p + p_*|^2}}, \qquad (5.27)$$

and

$$|q'| = \frac{1}{2}\sqrt{s - 4m^2 c^2} = \frac{1}{2}\sqrt{(p^0 + p_*^0)^2 - |p + p_*|^2 - 4m^2 c^2}. \qquad (5.28)$$

Finally we obtain

$$p' = \frac{p + p_*}{2} + \frac{1}{2}\sqrt{(p^0 + p_*^0)^2 - |p + p_*|^2 - 4m^2 c^2}(\Omega + \frac{\gamma(v) - 1}{|v|^2} vv^\top \Omega),$$
$$p'_* = \frac{p + p_*}{2} - \frac{1}{2}\sqrt{(p^0 + p_*^0)^2 - |p + p_*|^2 - 4m^2 c^2}(\Omega + \frac{\gamma(v) - 1}{|v|^2} vv^\top \Omega).$$
$$\qquad (5.29)$$

Since the variable Ω describes the whole sphere S^2, it can then be parametrized in spherical coordinates of polar axis q:

$$\Omega = \frac{q}{|q|}\cos\theta + (\cos\phi\,\vec{\imath} + \sin\phi\,\vec{\jmath})\sin\theta \tag{5.30}$$

in such a way that

$$d\Omega = \sin\theta d\theta d\phi. \tag{5.31}$$

To see this, we observe that

$$\cos\theta = \frac{(P_* - P)\cdot(P'_* - P')}{(P_* - P)^2} = \frac{(Q_* - Q)\cdot(Q'_* - Q')}{(Q_* - Q)^2} \tag{1}$$

$$= \frac{(q_*^0 - q^0)(q'^{\,0}_* - q'^{\,0}) - (q_* - q)\cdot(q'_* - q')}{(q_*^0 - q^0)^2 - |q_* - q|^2} = \frac{q\cdot q'}{|q|^2}, \tag{2}$$

where we use fact that $q_*^0 = q^0 = q'^{\,0}_* = q'^{\,0} = \sqrt{s}/2$ and $q = -q_*$, $q' = -q'_*$. Finally since

$$|q| = \sqrt{q^{0^2} - m^2c^2} = \sqrt{q'^{0^2} - m^2c^2} = |q'|,$$

we get $\cos\theta = \frac{qq'}{|q||q'|}$.

Remark 5.1 *The reference frame of the variables (Q, Q_*, Q', Q'_*) in the previous calculation is called the center of momentum system. The geometry of the collision is particularly simple in this frame since*

$$Q + Q_* = Q' + Q'_* = \begin{pmatrix} \sqrt{s} \\ 0 \end{pmatrix} \tag{5.32}$$

and the scattering angle θ is

$$\cos\theta = \frac{qq'}{|q||q'|}. \tag{5.33}$$

It is easily seen that this is the angle formed by the trajectories before and after the collision for each of the particles.

Finally the Møller velocity is

$$v = \frac{c}{2p^0 p_*^0}\sqrt{s(s - 4m^2c^2)} \tag{5.34}$$

and the collision integral in (5.26) may be written as

$$Q(f)(p) = \frac{1}{2}\int_{\mathbb{R}^3}\int_{S^2} v\sigma(s,\theta)q(f)d\Omega dp_*. \tag{5.35}$$

5.1.2 Another expression for the collision integral

(We closely follow in this part the Appendix II of Glassey and Strauss [GS], see also [26]). There is another way to reduce the 12-fold original integral to a 5-fold by using a slightly different parametrization. Let us start again from the original equation, perform directly the integration in the p'^0, p'^0_* and p^0_* variables to obtain:

$$Q(f)(p) = \frac{c}{2\,p^0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \delta\big((P + P_*) - (\sqrt{|p'|^2 + m^2c^2}, p')$$
$$- (\sqrt{|p'_*|^2 + m^2c^2}, p'_*))s\sigma q(f) \frac{dp'}{p'^0} \frac{dp'_*}{p'^0_*} \frac{dp_*}{p_*^0}. \tag{5.36}$$

We integrate now in p'_* to obtain

$$Q(f)(p) = \frac{c}{16p^0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \delta(p^0 + p^0_* - p'^0 - p'^0_*)s\sigma q(f) \frac{1}{p'^0_*} \frac{dp'}{p'^0} \frac{dp_*}{p_*^0}. \tag{5.37}$$

We perform now the change of variables

$$p' = p + r \cdot \omega, \quad p'_* = p_* - r \cdot \omega \tag{5.38}$$

with $\omega \in S^2$ and $r \in \mathbb{R}$ in such a way that $p + p_* = p' + p'_*$ for every p and p_* in $\mathbb{R}^3$. We use now Lemma 6.1 (ii') to obtain

$$\delta(p^0 + p^0_* - p'^0 - p'^0_*)$$
$$= 2(p^0 + p^0_*)\delta((p'^0 + p'^0_*)^2 - (p^0 + p^0_*)^2) \tag{5.39}$$
$$= 4(p^0 + p^0_*)p'^0 p'^0_* \delta(4p'^0{}^2 p'^0_*{}^2 - [(p^0 + p^0_*)^2 - p'^0{}^2 - p'^0_*{}^2]^2).$$

Observe that

$$\mathcal{P}(r) \equiv 4p'^0{}^2 p'^0_*{}^2 - [(p^0 + p^0_*)^2 - p'^0{}^2 - p'^0_*{}^2]^2$$
$$= 4p'^0{}^2 p'^0_*{}^2 - [(p^0 + p^0_*)^2 - (p'^0{}^2 + p'^0_*{}^2)]^2 \tag{5.40}$$
$$= -(p^0 + p^0_*)^4 + 2(p^0 + p^0_*)^2(p'^0{}^2 + p'^0_*{}^2) - (p'^0{}^2 - p'^0_*{}^2)^2,$$

where by the change of variables (5.38),

$$p'^0{}^2 - p'^0_*{}^2 = |p|^2 - |p_*|^2 + 2r(p + p_*) \cdot \omega$$

It is now a simple matter to check that $\mathcal{P}$ is a polynomial of degree two whose roots are $r = 0$ and

$$a(p, p_*, \omega) = \frac{2(p^0 + p^0_*)(\omega \cdot (\hat{p} - \hat{p}_*))p^0 p_*^0}{(p^0 + p_*^0)^2 - (\omega \cdot (p + p_*))^2} \equiv 2\frac{N}{D},$$
$$\hat{p} = \frac{p}{p^0}, \; \hat{p}_* = \frac{p_*}{p_*^0}. \tag{5.41}$$

Therefore, $\mathcal{P}(r) = Dr(r - a(p, p_*, \omega))$. We may then write

$$\delta(p^0 + p_*{}^0 - p'^0 - p'_*{}^0)\frac{1}{p'_*{}^0}\frac{1}{p'^0}dp'$$
$$= 4(p^0 + p_*^0)\delta(4p'^{02}p'^{02}_* - [(p^0 + p_*^0)^2 - p'^{02} - p'^{02}_*]^2)dp'$$
$$= 4(p^0 + p_*^0)\delta(Dr(r - a(p, p_*, \omega)))\, r^2\, dr\, d\omega. \tag{5.42}$$

Using Lemma **??**, we get

$$\delta(p^0 + p_*^0 - p'^0 - p'_*{}^0)\frac{1}{p'_*{}^0}\frac{1}{p'^0}dp'$$
$$= 4(p^0 + p_*^0)|Da|^{-1}[\delta(r) + \delta(r - a)]\, r^2\, dr\, d\omega. \tag{5.43}$$

The first delta function drops because of the factor r^2 and we are led them to:

$$\delta(p^0 + p_*{}^0 - p'^0 - p'_*{}^0)\frac{1}{p'_*{}^0} \qquad\qquad \frac{1}{p'^0}dp'$$
$$= 4(p^0 + p_*{}^0)2|N|D^{-2}\delta(r - a)\, dr\, d\omega \tag{5.44}$$
$$= 8\frac{(p^0 + p_*{}^0)^2|\omega \cdot (\hat{p} - \hat{p}_*)|p^0 p_*{}^0}{[(p^0 + p_*{}^0)^2 - (\omega \cdot (p + p_*))^2]^2}\delta(r - a)\, dr\, d\omega.$$

Finally,

$$Q(f)(p) = \int_{\mathbb{R}^3}\int_{\mathcal{S}^2}\Gamma(p, p_*, \omega)q(f)\, d\omega dp_*,$$
$$\Gamma(p, p_*, \omega) = 4c\, s\sigma\frac{(p^0 + p_*^0)^2|\omega \cdot (\hat{p} - \hat{p}_*)|}{[(p^0 + p_*^0)^2 - (\omega \cdot (p + p_*))^2]^2}. \tag{5.45}$$

Observe that $\Gamma(p, p_*, \omega) = \Gamma(p_*, p, \omega)$ and since $D \geq 2$,

$$0 \leq \Gamma(p, p_*, \omega) \leq c\, s\sigma(p^0 + p_*^0)^2|\omega \cdot (\hat{p} - \hat{p}_*)|.$$

5.2 Particles with different masses

Similar calculations can be performed if we consider collisions of two quantum relativistic particles with different masses m_*, m_2 and with momenta $P = (p^0, p)$ and $P_* = (p_*^0, p_*)$ such that

$$p^0 = \sqrt{|p|^2 + m_*^2 c^2}, \quad p_*^0 = \sqrt{|p_*|^2 + m_2^2 c^2}. \tag{5.46}$$

Let us denote by f de density distribution of the particles P of mass m_* and g that of the particles P_* of mass m_2. These functions satisfy the coupled system (4.2). Let us consider here only the integral $Q_{1,2}(f, g) \equiv Q(f, g)$ since $Q_{2,1}(g, f)$ is completely similar:

$$Q(f, g)(p) = \frac{8c}{p^0}\int_{\mathbb{R}^4}\int_{\mathbb{R}^4}\int_{\mathbb{R}^4} s\sigma q(f, g)\delta_{P + P_* - P' - P'_* = 0}$$
$$\times \chi_2(P_*^0)\chi_1(P'^0)\chi_2(P'^0_*)\, dP'dP'_*dP_* \tag{5.47}$$
$$q(f, g) = g(p'_*)f(p')(1 + \tau f(p))(1 + \tau' g(p_*))$$
$$- f(p)g(p_*)(1 + \tau f(p'))(1 + \tau' g(p'_*)),$$

with $\chi_i(P) = \delta_{P^2-m_i^2c^2} H(P^0)$ for $i = 1, 2$, and $\tau, \tau' \in \{-1, 0, 1\}$.

As before, the integral collision may be written as

$$Q(f, g) = \int_{\mathbb{R}^3} \int_{S^2} \nu\sigma q(f, g) \, d\Omega dp_*, \tag{5.48}$$

where the Møller velocity is in this case: (where Ω and θ are defined as above)

$$\nu = \frac{c}{2p^0 p_*^0} \sqrt{(s - (m_* - m_2)^2 c^2)(s - (m_* + m_2)^2 c^2)}. \tag{5.49}$$

and the dependence of p' and p'_* in function of p, p_* and Ω in the expression (5.47) of $q(f, g)$ is

$$p' = \frac{p + p_*}{2} + \frac{\sqrt{(s - (m_* - m_2)^2 c^2)(s - (m_* + m_2)^2 c^2)}}{2\sqrt{s}} \Big(\Omega + \frac{\gamma(v) - 1}{|v|^2} vv^\top \Omega\Big),$$

$$p'_* = \frac{p + p_*}{2} - \frac{\sqrt{(s - (m_* - m_2)^2 c^2)(s - (m_* + m_2)^2 c^2)}}{2\sqrt{s}} \Big(\Omega + \frac{\gamma(v) - 1}{|v|^2}, vv^\top \Omega\Big).$$

An expression similar to (5.45) may also be obtained for the collision integral $Q(f, g)$. From the expression in (5.47), we perform an integration in the p'^0, $p'^{\,0}_*$ and p_*^0 variables to obtain

$$Q(f, g)(p) = \frac{c}{p^0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \delta\big((P + P_*) - (\sqrt{|p'|^2 + m_*^2 c^2}, p')$$

$$- (\sqrt{|p'_*|^2 + m_2^2 c^2}, p'_*)\big) s\sigma q(f, f) \frac{dp'}{p'^0} \frac{dp'_*}{p'^{\,0}_*} \frac{dp_*}{p_*^{\,0}}.$$

We integrate now in p'_* to obtain

$$Q(f, g)(p) = \frac{c}{p^0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \delta(p^0 + p_*^0 - p'^0 - p_*'^{\,0}) s\sigma q(f, g) \frac{1}{p_*'^{\,0}} \frac{dp'}{p'^0} \frac{dp_*}{p_*^{\,0}}. \tag{5.50}$$

We may then apply the calculations from (5.38) to (5.44) and obtain finally

$$Q(f, g)(p) = \int_{\mathbb{R}^3} \int_{S^2} \Gamma(p, p_*, \omega) q(f, g) \, d\omega dp_*,$$

$$\Gamma(p, p_*, \omega) = 8c \, s\sigma \frac{(p^0 + p_*^0)^2 |\omega \cdot (\hat{p} - \hat{p}_*)|}{[(p^0 + p_*^0)^2 - (\omega \cdot (p + p_*))^2]^2}. \tag{5.51}$$

Remark 5.2 The expressions (5.34)-(5.35) and (5.48)-(5.49) correspond to the center of mass angular parametrizations of the collisions in the classical and non quantum Boltzmann equation.

5.3 Boltzmann-Compton equation for photon-electron scattering

In this section, we consider the system of Boltzmann equations for a dilute gas of low energy electrons and weakly dense photons. We assume that particles P

are photons and so are massless, $m_* = 0$ and particles P_* are electrons of mass $m_2 > 0$ that we take to be $m_2 = m$. From the preceding sub Section 5.2, if f describes the density of photons and g the density of electrons the collision integral writes

$$Q(f,g)(p) = \frac{c}{2|p|} \int_{\mathbb{R}^3} \int_{S^2} (s - m^2 c^2) \sigma(s,\theta) q(f,g) d\Omega \frac{dp_*}{p_*^0} \qquad (5.52)$$

where,

$$\begin{aligned} s = (P + P_*)^2 &= (|p| + \sqrt{|p_*|^2 + m^2 c^2})^2 - |p + p_*|^2 \\ &= c^2 + 2|p|\sqrt{|p_*|^2 + m^2 c^2} - 2(p, p_*) \end{aligned} \qquad (5.53)$$

and

$$p' = \frac{p + p_*}{2} + \frac{s - m^2 c^2}{2\sqrt{s}}\left(\Omega + \frac{\gamma(v) - 1}{v^2} v v^\top \Omega\right), \qquad (5.54)$$

$$p'_* = \frac{p + p_*}{2} - \frac{s - m^2 c^2}{2\sqrt{s}}\left(\Omega + \frac{\gamma(v) - 1}{v^2} v v^\top \Omega\right). \qquad (5.55)$$

In this context, the differential cross section σ is given by the Klein Nishina formula given in Appendix 8.

We briefly consider now the so called classical limit, which amounts to consider the higher order term in (5.52) when $c \to \infty$. Notice first that

$$s - m^2 c^2 \sim 2|p|mc, \quad \text{and} \quad v \sim 1 \quad \text{as} \quad c \to \infty. \qquad (5.56)$$

Since the photons have low energy we have $p^0 \sim p'^0$ from where we deduce

$$\sigma(s,\theta) \sim \frac{1}{2} r_0^2 \{1 + \cos^2 \theta\}, \quad \text{as} \quad c \to \infty, \qquad (5.57)$$

(c.f. [46, p. 163]) and finally

$$\lim_{c \to \infty} \frac{s - m^2 c^2}{2\sqrt{s}} = |p|. \qquad (5.58)$$

Then, in the classical limit we obtain

$$Q(f,g)(p) = \frac{c\, r_0^2}{2} \int_{\mathbb{R}^3} \int_{S^2} \{1 + \cos^2 \theta\} q(f,g) d\Omega\, dp_* \qquad (5.59)$$

with

$$p' = \frac{p + p_*}{2} + |p|\,\Omega, \qquad p'_* = \frac{p + p_*}{2} - |p|\,\Omega. \qquad (5.60)$$

5.3.1 Dilute and low energy electron gas at equilibrium

We assume moreover that the dilute and low energy electron gas is at equilibrium. Then, the density of electrons is given by a Maxwell Boltzmann distribution. It is therefore possible to write the corresponding collision integral in a more explicit way as it was already done in Section 4. To this end it is

convenient to express this collision integral in a Carleman's type parametrization. It should be possible to obtain it from (5.52) but nevertheless we proceed in a simpler way. We consider then the classical limit, $c \to \infty$ directly in the integral collision (5.47), where σ is again the Klein Nishina differential cross section. Arguing as above we obtain that the collision integral reads

$$Q(f)(p) = \frac{c\, r_0^2}{2|p||p'|} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (1 + \cos^2 \theta) q(f) \delta_\Sigma dp_* dp'_* dp'$$

where Σ is now the manifold of 4-uplets (p, p_*, p', p'_*) such that,

$$p + p_* = p' + p'_*, \quad \text{and} \quad |p| + \frac{|p_*|^2}{2mc} = |p'| + \frac{|p'_*{}^2|}{2mc}. \tag{5.61}$$

By the hypothesis on the electron gas, we may take $g(p'_*) = e^{-\beta^0 \frac{|p'_*|^2}{2m}} e^\mu$ for some $\beta^0 > 0$ and $\mu \in \mathbb{R}$ constants . By (5.61) we deduce

$$g(p'_*) = e^{-\beta^0(|p|-|p'|)} e^{-\beta^0 \frac{|p_*|^2}{2m}} e^\mu, \tag{5.62}$$

and the integral collision reads,

$$Q(f)(p) = \frac{r_0^2}{2} e^\mu \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\{1 + \cos^2 \theta\}}{|p||p'|} e^{-\beta^0 \frac{|p_*|^2}{2m} + \beta^0 |p'|} q(f) \delta_\Sigma dp' dp_* dp'_*, \tag{5.63}$$

with

$$q(f) = e^{-\beta^0 |p|} f(p')(1 + f(p)) - e^{-\beta^0 |p'|} f(p)(1 + f(p')).$$

Note that the integration in the p'_* variable is straightforward. Let us state the following auxiliary Lemma.

Lemma 5.3 *With* $A = |p| - |p'| + \frac{|p-p'|^2}{2mc}$ *and* $w = p' - p$, *we have*

$$S(p, p') = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \delta_\Sigma e^{-\beta^0 \frac{|p_*|^2}{2mc}} dp_* dp'_* = \frac{2\pi\, m^2\, c}{\beta^0 |w|} e^{-\beta^0 \frac{A^2 mc^2}{2|w|^2}}.$$

Proof. Since

$$|p| + \frac{|p_*|^2}{2mc} - |p'| - \frac{|p + p_* - p'|^2}{2mc} = A - \frac{1}{mc}(p_*, w),$$

we have

$$S(p, p') = \int_0^\infty \int_{\mathbb{S}^2} \delta_{A - \frac{|p_*|}{mc}(\Omega_*, w)} d\Omega_* e^{-\beta^0 \frac{|p_*|^2}{2m}} |p_*|^2 d|p_*|.$$

From

$$\int_{\mathbb{S}^2} \delta_{A - \frac{|p_*|}{mc}(\Omega_*, w)} d\Omega_* = \frac{2\pi mc}{|p_*||w|} H(1 - \frac{A^2 m^2 c^2}{|p_*|^2 |w|^2})$$

(where H stands for the Heaviside's function) we obtain

$$S(p,p') = \frac{2\pi mc}{|w|} \int_{\frac{Amc}{|w|}}^{\infty} |p_*| e^{-\beta^0 \frac{|p_*|^2}{2m}} d|p_*| \qquad (5.64)$$

which completes the proof. $\qquad \square$

The collision kernel reads now

$$Q(f)(p) = \frac{cr_0^2}{2} e^{\mu} \int_{\mathbb{R}^3} \frac{S(p,p')}{|p||p'|} \{1 + \cos^2 \theta\} e^{\beta^0 |p'|} q(f) dp'.$$

Now, we assume that the photon distribution is radial. This is a further simplification which allows to write the equation in polar coordinates. We denote $\varepsilon = |p|$, $\varepsilon' = |p'|$, and, with a slight abuse of notation, $S(p,p') = S(\varepsilon, \varepsilon', \theta)$,

$$Q(f)(\varepsilon) = \frac{cr_0^2}{2\varepsilon^2} e^{\mu} \int_0^{\infty} B(\varepsilon, \varepsilon') q(f) e^{\beta_0 \varepsilon'} d\varepsilon' \qquad (5.65)$$

with

$$B(\varepsilon, \varepsilon') = 2\pi \, \varepsilon \varepsilon' \int_0^{\pi} (1 + \cos^2 \theta) S(\varepsilon, \varepsilon', \theta) \sin \theta \, d\theta. \qquad (5.66)$$

Finally, the Boltzmann-Compton equation reads:

$$\partial_t f = \int_0^{\infty} b(\varepsilon, \varepsilon') \left(f'(1+f) e^{-\beta^0 \varepsilon} - f(1+f') e^{-\beta^0 \varepsilon'} \right) \varepsilon'^2 d\varepsilon \, d\varepsilon', \qquad (5.67)$$

where

$$\begin{aligned}
b(\varepsilon, \varepsilon') &= \frac{c\, r_0^2}{2} e^{\mu} \frac{B(\varepsilon, \varepsilon')}{\varepsilon^2 \varepsilon'^2} e^{\beta^0 \varepsilon'} \\
&= 2 \frac{m^2 c^2 \, r_0^2 \pi^2}{\beta^0 \varepsilon \varepsilon'} e^{\mu} e^{\beta^0 \varepsilon'} \int_0^{\pi} (1 + \cos^2 \theta) \frac{1}{|w|} e^{-\beta^0 \frac{A^2 mc}{2|w|^2}} \sin \theta \, d\theta
\end{aligned}$$

with $|w|^2 = \varepsilon^2 + \varepsilon'^2 - 2\varepsilon \, \varepsilon' \cos \theta$.

Remark 5.4 Note that equation (5.67) is formally the same as (4.15)-(4.32) in Section 4 above and (1.1) of [23]. In particular, these equations have the same entropy function, given by (4.52). Nevertheless, in each of these cases, the functions $b(\varepsilon, \varepsilon')$ which appear in the collision have different, although similar, behaviour in the domain $\varepsilon > 0, \varepsilon' > 0$. It is in particular easy to check that the function b in (5.67) does not satisfy any of the conditions (2.1), (2.2) (2.3) imposed in the existence theorems obtained in [23]. The global existence of solutions in L^1 is then still an open question for this equation.

5.4 The Kompaneets equation

In this subsection we present the deduction of the Kompaneets equation from the Boltzmann-Compton equation, and we mainly follow [33] and [35, Vol. 3]. The Kompaneets limit is valid for

$$\varepsilon, \varepsilon' \ll mc^2.$$

We by start performing the change of variables $\beta^0 \varepsilon = \bar{\varepsilon}$, $f(\varepsilon) = g(\bar{\varepsilon})$ in (5.67) and, by abuse of notation, still denote the variable as ε. This gives

$$\varepsilon^2 \frac{\partial g}{\partial t} = \beta^{0-3} \int_0^\infty \left(g'(1+g) e^{-\varepsilon} - g(1+g')e^{-\varepsilon'} \right) \varepsilon^2 \varepsilon'^2 b(\frac{\varepsilon}{\beta^0}, \frac{\varepsilon'}{\beta^0}) \, d\varepsilon'.$$

In a similar way as in the usual Fokker Planck approximation (cf. [34, vol. 10, Ch. 2]), one notices that the main contribution to the integral in (5.67) should come from the region where $|\varepsilon' - \varepsilon|/|\varepsilon| << 1$. This is due to the fact that the kernel $b(\varepsilon, \varepsilon')$ given in (5.67) is rather peaked around $\varepsilon' = \varepsilon$. More precisely, as we shall see below (cf. (5.75)), for any $\varepsilon > 0$ fixed and $\theta \in [0, \pi]$:

$$\lim_{m \to \infty} b(\varepsilon, \varepsilon') \frac{\varepsilon'}{\varepsilon} (\varepsilon' - \varepsilon)^2 = C(\theta) \delta_{\varepsilon' = \varepsilon}$$

where $C(\theta)$ is a constant depending on θ and also on m, c, β^0. I.e., in the non relativistic limit of Compton scattering, the dominant contribution to the collision integral comes from the region where $|p| - |p'|(= \varepsilon - \varepsilon')$ is small, and not from the region where $p - p'$ is small. This is consistent with the fact that the cross section for the non relativistic limit of Compton scattering, as described by the Klein-Nishina formula (cf. Appendix 8, Relativistic case), has no singular behaviour when the momentum transfer $p' - p$ goes to zero.

We may then first expand $g(\varepsilon')$ and $e^{-\varepsilon'}$ around ε to obtain

$$g(\varepsilon')(1 + g(\varepsilon)) e^{-\varepsilon} - g(\varepsilon)(1 + g(\varepsilon')) e^{-\varepsilon'}$$
$$\sim \left\{ (\varepsilon' - \varepsilon)g'(\varepsilon) + [(\varepsilon' - \varepsilon) - \frac{1}{2}(\varepsilon' - \varepsilon)^2]g(\varepsilon) \right.$$
$$\left. - [(\varepsilon' - \varepsilon) - \frac{1}{2}(\varepsilon' - \varepsilon)^2]g^2 + \frac{1}{2}(\varepsilon' - \varepsilon)^{2g''} + (\varepsilon' - \varepsilon)^2 gg' \right\} e^{-\varepsilon}.$$

From where,

$$\beta^{0^3} \varepsilon^2 \frac{\partial g}{\partial t} \sim a(\varepsilon)g(\varepsilon) + (a(\varepsilon) + b(\varepsilon))g'(\varepsilon) + b(\varepsilon)g''(\varepsilon) + 2b(\varepsilon)gg' + a(\varepsilon)g^2$$

with,

$$a(\varepsilon) = \varepsilon^2 \int_0^\infty ((\varepsilon' - \varepsilon) - \frac{1}{2}(\varepsilon' - \varepsilon)^2)b(\frac{\varepsilon}{\beta^0}, \frac{\varepsilon'}{\beta^0})e^{-\varepsilon}\varepsilon'^2 d\varepsilon'$$

$$b(\varepsilon) = \varepsilon^2 \frac{1}{2} \int_0^\infty (\varepsilon' - \varepsilon)^2 b(\frac{\varepsilon}{\beta^0}, \frac{\varepsilon'}{\beta^0})e^{-\varepsilon}\varepsilon'^2 d\varepsilon'.$$

Therefore, if we set

$$\alpha(\varepsilon) = \exp \int_0^\varepsilon \frac{a(\sigma)}{b(\sigma)} d\sigma \tag{5.68}$$

and

$$d(\varepsilon) = b(\varepsilon) \exp(-\int_0^\varepsilon \frac{a(\sigma)}{b(\sigma)} d\sigma) \tag{5.69}$$

we have

$$\beta^{03}\varepsilon^2\frac{\partial g}{\partial t} \sim d(\varepsilon)\frac{\partial}{\partial\varepsilon}[\alpha(\varepsilon)(\frac{\partial g}{\partial\varepsilon} + g + g^2)]. \tag{5.70}$$

The second step is to identify the two functions d and α. To this end one argues as follows. We first assume that the equation (5.70) preserves the total number of particles, namely that

$$\frac{d}{dt}\int_0^\infty \varepsilon^2\, g(\varepsilon, t)\, d\varepsilon = 0. \tag{5.71}$$

This is reasonable as far as (5.70) is a good approximation of the Boltzmann-Compton equation. It requires that the following boundary conditions are satisfied:

$$\lim_{\varepsilon\to 0}\alpha(\varepsilon)(\frac{\partial g}{\partial\varepsilon} + g + g^2) = \lim_{\varepsilon\to\infty}\alpha(\varepsilon)(\frac{\partial g}{\partial\varepsilon} + g + g^2) = 0 \tag{5.72}$$

i.e. that the particle flux is zero at the boundary of the domain. Then, by a formal integration by parts on $\mathbb{R}^+$, $d(\varepsilon)$ has to be constant. Let us denote it by d.

We have now to identify α. By (5.68) and (5.69), $b(\varepsilon) = d\alpha(\varepsilon)$ where, we recall

$$b(\varepsilon) = \varepsilon^2\frac{1}{2}\int_0^\infty (\varepsilon' - \varepsilon)^2 b(\frac{\varepsilon}{\beta^0}, \frac{\varepsilon'}{\beta^0})e^{-\varepsilon}\varepsilon'^2 d\varepsilon'$$

and

$$b(\frac{\varepsilon}{\beta^0}, \frac{\varepsilon'}{\beta^0}) = 2\beta^0\frac{m^2c^2\, r_0^2\pi^2}{\varepsilon\varepsilon'}e^\mu e^{\varepsilon'}\int_0^\pi (1 + \cos^2\theta)\frac{1}{|w|}e^{-\beta^0\frac{A^2 mc}{2|w|^2}}\sin\theta d\theta\,,$$

with

$$|\omega| = \frac{1}{\beta^0}(\varepsilon^2 + \varepsilon'^2 - 2\varepsilon\varepsilon'\cos\theta)^{1/2}, \quad A = \frac{\varepsilon}{\beta^0} - \frac{\varepsilon'}{\beta^0} + \frac{|\omega|^2}{2mc}.$$

By the change of variables, $s = \cos\theta$ we slightly simplify this expression and obtain

$$b(\varepsilon) = m^2c^2r_0^2\pi^2 e^\mu\, e^{-\varepsilon}\varepsilon\int_{-1}^1 (1 + s^2)\int_0^\infty (\varepsilon' - \varepsilon)^2\frac{e^{\varepsilon'}\, e^{-\beta^0\frac{A^2 mc}{2|\omega|^2}}}{|\omega|}\varepsilon'd\varepsilon'ds. \tag{5.73}$$

To handle with the expression under the integral in (5.73) first notice that by (5.61),

$$\frac{\varepsilon'^2}{2mc} + \varepsilon'[1 - \frac{\varepsilon s}{mc} - (\frac{p_*}{mc}\cdot\frac{p'}{|p'|})] + \frac{\varepsilon^2}{2mc} - \varepsilon + \varepsilon(\frac{p_*}{mc}\cdot\frac{p}{|p|}) = 0. \tag{5.74}$$

A straightforward calculation gives

$$\Delta = 1 + (\frac{\varepsilon s}{mc} + (\frac{p_*}{mc}\cdot\frac{p'}{|p'|}))^2 - \frac{\varepsilon^2}{m^2c^2} - \frac{2\varepsilon}{mc}(\frac{p_*}{mc}\cdot\frac{p}{|p|}) - 2(\frac{\varepsilon s}{mc} + (\frac{p_*}{mc}\cdot\frac{p'}{|p'|})) + \frac{2\varepsilon}{mc}.$$

Since all the terms in the right hand side, except 1, are small when $m \gg 1$, $\varepsilon = \mathcal{O}(1)$, $|p^*| = \mathcal{O}(1)$, we may use $\sqrt{1+z} \sim 1 + z/2 - z^4/8 + \cdots$ to obtain

$$\sqrt{\Delta} \sim 1 - \frac{\varepsilon s}{mc} - \left(\frac{p_*}{mc} \cdot \frac{p'}{|p'|} \right) + \frac{\varepsilon}{mc} - \frac{\varepsilon}{mc} \frac{p_*}{mc} \cdot \left(\frac{p}{|p|} - \frac{p'}{|p'|} \right) - \frac{\varepsilon^2}{m^2 c^2}(1-s).$$

Then, the two solutions of (5.74) are

$$\varepsilon' = mc \left(\frac{\varepsilon\, s}{mc} + \left(\frac{p_*}{mc} \cdot \frac{p'}{|p'|} \right) - 1 \right) \pm mc\sqrt{\Delta}$$

$$= \varepsilon s + p_* \cdot \frac{p'}{|p'|} - mc \pm mc\sqrt{\Delta}.$$

The root with minus sign is negative, which is inconsistent with the fact that $\varepsilon' \geq 0$. The root with the plus sign gives,

$$\varepsilon' \sim \varepsilon - \frac{\varepsilon^2}{mc}(1-s) - \frac{\varepsilon}{mc} p_* \cdot \left(\frac{p}{|p|} - \frac{p'}{|p'|} \right).$$

Therefore,

$$|w|^2 = \frac{1}{\beta^{0\,2}}(\varepsilon^2 + \varepsilon'^2 - 2\varepsilon\varepsilon' s) \sim \frac{1}{\beta^{0\,2}} \left(\varepsilon^2 + \left(\varepsilon + \frac{\varepsilon^2}{2mc} \right)^2 - 2\varepsilon \left(\varepsilon + \frac{\varepsilon^2}{2mc} \right) s \right)$$

$$= \frac{1}{\beta^{0\,2}} \left(\varepsilon^2 + \varepsilon^2 \left(1 + \frac{\varepsilon}{2mc} \right)^2 - 2\varepsilon^2 \left(1 + \frac{\varepsilon}{2mc} \right) s \right) \sim 2\frac{\varepsilon^2}{\beta^{0\,2}}(1-s)$$

Let us define

$$\varepsilon_0 = \varepsilon + \frac{\varepsilon^2}{\beta^0\, mc}(1-s).$$

The quantity $\varepsilon - \varepsilon_0$ is the energy transfer of the photon as measured in the rest frame of the initial electron (lab frame). We have then

$$|w|^2 \sim 2mc\frac{(\varepsilon_0 - \varepsilon)}{\beta^0}, \quad \text{and} \quad A = \frac{\varepsilon}{\beta^0} - \frac{\varepsilon'}{\beta^0} + \frac{|w|^2}{2mc} \sim \frac{\varepsilon_0 - \varepsilon'}{\beta^0}.$$

and therefore,

$$\int_0^\infty (\varepsilon' - \varepsilon)^2 \frac{e^{-\beta^0 \frac{A^2 mc}{2|w|^2}}\, e^{\varepsilon'}}{|w|} \varepsilon' d\varepsilon' \sim \frac{\beta^0}{\sqrt{2(1-s)}\varepsilon} \int_0^\infty (\varepsilon' - \varepsilon)^2 e^{-\frac{|\varepsilon' - \varepsilon_0|^2}{4|\varepsilon - \varepsilon_0|}}\, e^{\varepsilon'} \varepsilon' d\varepsilon'$$

$$\sim \beta^0 \frac{|\varepsilon - \varepsilon_0|}{\sqrt{2(1-s)}\varepsilon} \int_0^\infty \frac{(\varepsilon' - \varepsilon_0)^2}{|\varepsilon - \varepsilon_0|} e^{-\frac{|\varepsilon' - \varepsilon_0|^2}{4|\varepsilon - \varepsilon_0|}}\, e^{\varepsilon'} \varepsilon' d\varepsilon'$$

$$\sim \frac{C\beta_0\, e^\varepsilon}{\sqrt{2(1-s)}\varepsilon} |\varepsilon - \varepsilon_0|^{3/2} \varepsilon$$

$$\sim C\frac{\varepsilon^3\, e^\varepsilon(1-s)}{m^{3/2}c^{3/2}\sqrt{2\beta_0}},$$

where we use the fact that $\varepsilon \sim \varepsilon_0$ and that

$$\lim_{m \to 0} \frac{1}{|\varepsilon - \varepsilon_0|^{1/2}} \frac{(\varepsilon' - \varepsilon_0)^2}{|\varepsilon - \varepsilon_0|} e^{-\frac{|\varepsilon' - \varepsilon_0|^2}{4|\varepsilon - \varepsilon_0|}} = C \delta_{\varepsilon'=\varepsilon}, \quad C = \int_{-\infty}^{\infty} r^2 \, e^{-r^2} \, dr. \quad (5.75)$$

Using this in (5.73) we obtain

$$b(\varepsilon) = m^2 c^2 r_0^2 \pi^2 e^\mu \, e^{-\varepsilon} \varepsilon \int_{-1}^{1} (1 + s^2) \int_0^\infty (\varepsilon' - \varepsilon)^2 \frac{e^{\varepsilon'} \, e^{-\beta^0 \frac{A^2 mc}{2|\omega|^2}}}{|\omega|} \varepsilon' d\varepsilon' ds$$

$$\sim C \sqrt{\frac{mc}{2\beta^0}} \, r_0^2 \pi^2 e^\mu \, \varepsilon^4 \int_{-1}^{1} (1 + s^2)(1 - s) \, ds \equiv \Lambda \varepsilon^4,$$

where Λ is independent of ε and s. The limiting equation reads,

$$\varepsilon^2 \frac{\partial g}{\partial t} = \Lambda \beta^{0-3} d^{-1} \frac{\partial}{\partial \varepsilon} [\varepsilon^4 (\frac{\partial g}{\partial \varepsilon} + g + g^2)].$$

Finally, we perform the change of time variable $t \to \Lambda t / d\beta^{0^3}$ and still denote by t the new variable. The function $h(t, \varepsilon)$ of the new variables satisfy then,

$$\varepsilon^2 \frac{\partial h}{\partial t} = \frac{\partial}{\partial \varepsilon} [\varepsilon^4 (\frac{\partial h}{\partial \varepsilon} + h + h^2)].$$

This is the so called Kompaneets equation such as it appears for example in [33, 35, 44]. See also [7, 21].

6 Appendix: A distributional lemma

For a function $\Psi : \mathbb{R}^m \to \mathbb{R}$ we define $\delta_\Psi = \delta_{\Psi=0}$ by

$$\langle \delta_{\Psi=0}, \varphi \rangle = \lim_{\epsilon \to 0} \int_{\mathbb{R}^m} \rho_\epsilon(\Psi(y)) \varphi(y) \, dy \quad \forall \varphi \in C_c(\mathbb{R}^m) \tag{3}$$

where (ρ_ϵ) is any approximation of the 1-dimensional Dirac measure at the origin. Then, we have:

Lemma 6.1 *For any* $a, b \in \mathbb{R}$, $a \neq 0$

$$\delta_{a\,x-b} = \frac{1}{a} \delta_{x=b/a}. \tag{4}$$

For $a \neq b$,

$$\delta_{(x-a)(x-b)} = \frac{1}{|b-a|} (\delta_{x-a} + \delta_{x-a}). \tag{5}$$

Remark 6.2 As two particular cases we obtain

$$\forall a > 0 \quad \delta_{x^2 - a^2} \mathbf{1}_{x>0} = \frac{1}{2\,a} \delta_{x=a}, \tag{6}$$

$$\forall a > 0 \quad \delta_{x(x-a)} \mathbf{1}_{x>0} = \frac{1}{a} \delta_{x=a}. \tag{7}$$

Proof of Lemma 6.1 Since (4) is evident and (6), (7) are immediate conse-
quences of (5), we just prove (5). Let (ρ_ϵ) be a sequence of $L^1(\mathbb{R})$ such that
$\rho_\epsilon \rightharpoonup \delta$ in $\mathcal{D}'(\mathbb{R})$. Then

$$\langle \delta_{(x-a)(x-b)}, \phi \rangle := \lim_{\epsilon \to 0} \langle \rho_\epsilon((x-a)(x-b)), \phi \rangle$$

$$= \lim_{\epsilon \to 0} \int_{\mathbb{R}} \rho_\epsilon((x-a)(x-b))\phi \, dx = \lim_{\epsilon \to 0} I_\epsilon + J_\epsilon,$$

with

$$I_\epsilon = \int_{-\infty}^{\frac{a+b}{2}} \rho_\epsilon((x-a)(x-b))\phi \, dx, \quad J_\epsilon = \int_{\frac{a+b}{2}}^{\infty} \rho_\epsilon((x-a)(x-b))\phi \, dx.$$

Without any loss of generality we may assume that $a < b$. Set $y = (x - a)(x - b) = x^2 - (a + b)x + ab$. The function $x \mapsto y(x)$ is monotone for
any $x \leq (a + b)/2$ so that it is an allowed change of variable. We compute
$2x = a + b \pm \sqrt{4y + (a-b)^2}$ and $dy = [2x - (a+b)]dx = \pm\sqrt{4y + (a-b)^2}\,dx$,
and hence

$$I_\epsilon = \int_{\frac{-(b-a)^2}{4}}^{\infty} \rho_\epsilon(y)\phi\Big(\frac{a+b}{2} \pm \sqrt{y + \big(\frac{a-b}{2}\big)^2}\Big)\frac{dy}{\sqrt{4y + (a-b)^2}}$$

$$\to \frac{1}{|b-a|}\phi\Big(\frac{a+b-|a-b|}{2}\Big) = \frac{\phi(a)}{|b-a|}.$$

Similarly, we prove $J_\epsilon \to \phi(a)/|b-a|$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

7 Appendix: Minkowsky space and Lorentz transform

We call $P = (P^0, p) \in \mathbb{R}^4$ with $P^0 \in \mathbb{R}$, $p \in \mathbb{R}^3$, or indifferently $P = (P^\mu)$, the
Lorentz metric which is defined by

$$\langle P, Q \rangle = P^0 Q^0 - p \cdot q \quad \forall P, Q \in \mathbb{R}^4. \tag{8}$$

We also write

$$\langle P, Q \rangle = P^\mu Q_\mu = P^\top \eta Q = \sum_{\mu, \nu = 0}^{4} \eta_{\mu\nu} P^\mu Q^\nu, \quad \text{with} \quad Q_\mu = \eta_{\mu\nu} Q^\mu,$$

where

$$\eta = (\eta_{\mu\nu}) = \begin{pmatrix} 1 & 0^\top \\ 0 & -\mathbf{I}_3 \end{pmatrix} \tag{9}$$

is the Minkowsky matrix. The inner product $\langle , \rangle$ on $\mathbb{R}^4$ is symmetric, non
degenerated but not positive.

Definition A Lorentz transform is a linear operator $\Lambda : \mathbb{R}^4 \to \mathbb{R}^4$ such that

$$\langle \Lambda P, \Lambda Q \rangle = \langle P, Q \rangle \quad \forall P, Q \in \mathbb{R}^4. \tag{10}$$

7.1 Examples of Lorentz transforms

Rotations For a rotation $\mathbf{R}$ of $\mathbb{R}^3$ ($\mathbf{R} \in SO(3)$),

$$\Lambda = \begin{pmatrix} 1 & 0^\top \\ 0 & \mathbf{R} \end{pmatrix} \tag{11}$$

is a Lorentz transform.

Boosts For $v \in \mathbb{R}^3$ such that $v := |v| < 1$,

$$\Lambda = \begin{pmatrix} \gamma & \gamma v^\top \\ \gamma v & \mathbf{I} + \frac{\gamma - 1}{v^2} v v^\top \end{pmatrix}, \quad \gamma = \frac{1}{\sqrt{1 - v^2}}. \tag{12}$$

is a Lorentz transform.

Remark 7.1 Any Lorentz transform is the composition of a boost and a rotation, see [42].

For $(\beta^0, \beta) \in \mathbb{R}^4$ with $\beta^0 > |\beta|$ we define $\mathbf{b} > 0$ and $u \in \mathbb{R}^3$ by

$$\mathbf{b}^2 := (\beta^0)^2 - |\beta|^2, \quad \frac{u}{c} := \frac{\beta}{\beta^0}. \tag{13}$$

Then, setting

$$\gamma := \frac{1}{\sqrt{1 - (u/c)^2}}, \quad \text{we have} \quad (\beta^\mu) = (\beta^0, \beta) = \left(\gamma \bar{\beta}, \gamma \bar{\beta} \frac{u}{c} \right). \tag{14}$$

For such a 4-vector (β^0, β) we define the boost transform $\Lambda = \Lambda_{(\beta^\mu)}$ associated to $v := u/c$. It satisfies

$$\Lambda \begin{pmatrix} \mathbf{b} \\ 0 \end{pmatrix} = \begin{pmatrix} \beta^0 \\ \beta \end{pmatrix}. \tag{15}$$

Lemma 7.2 *Any Lorentz transform Λ satisfies* $\det \Lambda = \pm 1$.

Proof For

$$\Lambda = \begin{pmatrix} a & b^\top \\ c & d \end{pmatrix} \quad \text{define} \quad \Lambda^\star = \begin{pmatrix} a & -c^\top \\ -b & d^\top \end{pmatrix}$$

with $a \in \mathbb{R}$, $b, c \in \mathbb{R}^3$ and $d \in M(\mathbb{R}^3)$. We easily verify that $\langle \Lambda P, Q \rangle = \langle P, \Lambda^\star Q \rangle$ for all $P, Q \in \mathbb{R}^4$. In particular, by definition of a Lorentz transform, one has

$$\langle \Lambda^\star \Lambda P, Q \rangle = \langle \Lambda P, \Lambda Q \rangle = \langle P, Q \rangle \quad \text{for aall } P, Q \in \mathbb{R}^4,$$

so that $\Lambda^\star \Lambda = Id_4$. Since $\det \Lambda^\star = \det \Lambda$, we get $(\det \Lambda)^2 = 1$. $\qquad \square$

Definition 7.3 For $s \in \mathbb{R}$ we define the set

$$M_+^s := \{P \in \mathbb{R}^4, : \langle P, P \rangle = s, : P^0 > 0\} = \{(\sqrt{s + |p|^2}, p), \quad p \in \mathbb{R}^3\}. \quad (16)$$

We write $\Lambda \in \mathcal{L}_+^\uparrow$, if Λ is a Lorentz transform such that $\det \Lambda = +1$ and $(\Lambda P)^0 > 0$ for any $P \in M_+^s$, with $s > 0$.

The boost Λ associated to (β^0, β) belongs to $\mathcal{L}_+^\uparrow$ since $\det \Lambda = 1$ and

$$(\Lambda P)^0 = \gamma p^0 + \gamma u \cdot p = \frac{1}{\mathbf{b}}[\beta^0 p^0 + \beta \cdot p] \geq \frac{1}{\mathbf{b}}[\beta^0 |p| - |\beta||p|] \geq 0$$

for any $P \in M_+^s$ with $s > 0$.

Lemma 7.4 *Let $f : \mathbb{R} \to \mathbb{R}$ and $(\beta^\mu) = (\beta^0, \beta) \in \mathbb{R}^4$ such that $\mathbf{b}^2 := \beta^{02} - |\beta|^2 > 0$. We define $F : \mathbb{R}^3 \to \mathbb{R}$ by $F(p) = f(\beta^\mu p^\mu) = f(\beta^0 p^0 - \beta \cdot p)$ with $p^0 := \sqrt{s + |p|^2}$, $s > 0$. Then, there are some constants $A_i = A_i(f, \mathbf{b})$ such that*

$$\int_{\mathbb{R}^3} p^\mu F \frac{dp}{p^0} = A_1 \beta^\mu \quad (17)$$

$$\int_{\mathbb{R}^3} p^\mu p^\nu F \frac{dp}{p^0} = A_2 \eta^{\mu\nu} + A_3 \beta^\mu \beta^\nu. \quad (18)$$

In particular, one has

$$N(F) := \int_{\mathbb{R}^3} F \, dp = A_1 \beta^0 \quad (19)$$

$$P(F) := \int_{\mathbb{R}^3} F p \, dp = A_3 \beta^0 \beta \quad (20)$$

$$E(F) := \int_{\mathbb{R}^3} F p^0 \, dp = A_2 + A_3 (\beta^0)^2 \quad (21)$$

$$G(F) := s \int_{\mathbb{R}^3} F \frac{dp}{p^0} = 4 A_2 + A_3 \mathbf{b}^2. \quad (22)$$

Proof Using Lemma 7.2 we have

$$\delta_{P^2 = s} H(P^0) = \left(\delta_{(P^0 - \sqrt{s + |p|^2})(P^0 + \sqrt{s + |p|^2})} \right) H(P^0)$$

$$= \frac{1}{2\sqrt{s + |p|^2}} \left(\delta_{P^0 - \sqrt{s + |p|^2}} + \delta_{P^0 + \sqrt{s + |p|^2}} \right) H(P^0)$$

$$= \frac{1}{2\sqrt{s + |p|^2}} \delta_{P^0 - \sqrt{s + |p|^2}}.$$

where H is the Heaviside function. Therefore, we get the fundamental identity

$$\int_{\mathbb{R}^4} F(P) \delta_{P^2 = s} H(p^0) \, dP = \int_{\mathbb{R}^3} \left\{ \int_{\mathbb{R}} F(P) \frac{1}{2\sqrt{s + |p|^2}} \delta_{P^0 - \sqrt{s + |p|^2}} P^0 \right\} dp$$

$$= \int_{\mathbb{R}^3} F(p^0, p) \frac{dp}{p^0}.$$

For $\Lambda \in \mathcal{L}_+^\uparrow$ we get

$$\int_{\mathbb{R}^3} F(\Lambda(p^0,p))\frac{dp}{p^0} = \int_{\mathbb{R}^4} F(\Lambda\,P)\delta_{P^2=s}\,H(p^0)\,dP$$

$$= \int_{\mathbb{R}^4} F(Q)\delta_{Q^2=s}\,H(q^0)\,dQ = \int_{\mathbb{R}^3} F(p^0,p)\frac{dp}{p^0}.$$

Now we choose as Λ the boost associated to (β^μ) and using (15) we have, setting $P = \Lambda\,Q$, we get

$$\int_{\mathbb{R}^3} p^\mu f(\beta^\nu p^\nu)\frac{dp}{p^0} = \int_{\mathbb{R}^3} p^\mu f(\mathbf{b}(\Lambda^{-1}P)^0)\frac{dp}{p^0}$$

$$= \Lambda^\mu \int_{\mathbb{R}^3} q f(\mathbf{b}q^0)\frac{dq}{q^0} = \Lambda^\nu \begin{pmatrix} \mathbf{b}\,A_1 \\ 0 \end{pmatrix},$$

with

$$A_1 := \int_{\mathbb{R}^3} q^0 f(\mathbf{b}q^0)\frac{dq}{q^0},$$

since

$$\int_{\mathbb{R}^3} q^i f(\mathbf{b}q^0)\frac{dq}{q^0} = 0$$

for $i = 1, 2, 3$ by rotation symmetry. This proves (17). Similarly,

$$\int_{\mathbb{R}^3} p^\mu p^\nu f(\beta^\sigma p^\sigma)\frac{dp}{p^0} = \Lambda^\mu_{\mu'}\Lambda^\nu_{\nu'} \int_{\mathbb{R}^3} q^{\mu'} q^{\nu'} f(\bar\beta q^0)\frac{dq}{q^0} = \Lambda^\mu_{\mu'}\Lambda^\nu_{\nu'}\alpha_{\mu'}\delta_{\mu'\nu'}$$

with

$$\alpha_0 = \int_{\mathbb{R}^3} (q^0)^2 f(\bar\beta q^0)\frac{dq}{q^0} \quad \alpha_1 = \alpha_2 = \alpha_3 = \int_{\mathbb{R}^3} (q^1)^2 f(\bar\beta q^0)\frac{dq}{q^0}.$$

By simple computations,

$$\Lambda^2 = \begin{pmatrix} 2\gamma^2 - 1 & 2\gamma^2 v^\top \\ 2\gamma^2 v & \mathbf{I} + 2\gamma^2 vv^\top \end{pmatrix} = -(\eta)^{\mu\nu} + 2\begin{pmatrix} \gamma^2 & \gamma(\gamma u/c)^\top \\ \gamma(\gamma u/c) & (\gamma u/c)(\gamma u/c)^\top \end{pmatrix}$$

$$= -(\eta)^{\mu\nu} + \frac{2}{\mathbf{b}^2}\beta^\mu\beta^\nu,$$

and $\Lambda_0^\mu = \begin{pmatrix} \gamma \\ \gamma u \end{pmatrix} = \frac{\beta^\mu}{\mathbf{b}}$. Therefore

$$\int_{\mathbb{R}^3} p^\mu p^\nu f(\beta^\sigma p^\sigma)\frac{dp}{p^0} = \alpha_1\Lambda^\mu_{\mu'}\Lambda^\nu_{\mu'} + (\alpha_0 - \alpha_1)\Lambda_0^\mu\Lambda_0^\nu$$

$$= \alpha_1(\Lambda^2)^{\mu\nu} + (\alpha_0 - \alpha_1)\Lambda_0^\mu\Lambda_0^\nu$$

$$= -\alpha_1\eta^{\mu\nu} + \frac{\alpha_0 + \alpha_1}{\mathbf{b}^2}\beta^\mu\beta^\nu,$$

and (18) is proved. The statements (19), (20) and (21) follow. Finally, we compute, thanks to (18),

$$\int_{\mathbb{R}^3} \eta_{\mu\nu} p^\mu p^\nu f \frac{dp}{p^0} = A_1 \eta^{\mu\nu} \eta_{\mu\nu} + A_2 \beta^\mu \beta^\nu \eta_{\mu\nu},$$

so that

$$\int_{\mathbb{R}^3} ((p^0)^2 - |p|^2) f \frac{dp}{p^0} = 4\,A_1 + A_2[(\beta^0)^2 - |\beta|^2] = 4\,A_1 + A_2 \bar{\beta}^2,$$

and (22) follows, remarking that $(p^0)^2 - |p|^2 = s$. $\qquad\qquad\square$

8 Appendix: Differential cross section

We present in this Appendix some short notes about differential cross sections, mainly taken from the volumes 3 and 10 of the Landau and Lifschitz.

Following Landau and Lifshitz vol.10, §2 let us consider collisions between two molecules of a monatomic gas, one of which has momentum p in a given range dp and the other in a range dp_* and which acquire in the collision values in the ranges dp' and dp'_* respectively. For brevity we refer simply to a collision of molecules with p and p_*, resulting in p' and p'_*. The total number of such collisions per unit time and unit volume of the gas may be written as a product of the number of molecules per unit volume, $f(t, p)dp$, and the probability that any of them has a collision of the type concerned. This probability is always proportional to the number of molecules p_* per unit volume, $f(t, p_*)dp_*$ and to the number of molecules p' and p'_* per unit volume, $(1 + \tau f(t, p'))dp'$ and $(1 + \tau f(t, p'_*))dp'_*$. Thus the number of collisions $p, p_* \to p', p'_*$ per unit time and volume may be written as

$$W(p', p'_*; p, p_*) f f_*(1 + \tau f')(1 + \tau f'_*) dp dp_* dp' dp'_*,$$

where as usual, $\tau = 1$ for Bose particles, $\tau = -1$ for Fermi particles and $\tau = 0$ for non quantum particles. The coefficient W is a function of all its arguments. The ratio of $W dp' dp'_*$ to the absolute value of the relative velocity $v - v_*$ of the colliding molecules has the dimensions of area, and is the effective collision cross section:

$$d\sigma = \frac{W(p', p'_*; p, p_*)}{|p - p_*|} dp' dp'_*$$

The function W can in principle be determined only by solving the mechanical problem of collision of particles interacting according to some given law. However, certain properties of this function can be elucidated from general arguments.

The first property is called detailed balance and reads:

$$W(p', p'_*; p, p_*) = W(p, p_*; p', p'_*);$$

i.e. each microscopic collision process is balanced by the reverse process. It is a direct consequence of the following two symmetries:

1. The symmetry of the laws of the mechanic (classical or quantum) under time reversal: according to this, the number of collisions $p, p_* \to p', p'_*$ is equal, in equilibrium, to the number $-p', -p'_* \to -p, -p_*$ from where one obtains:

$$W(p', p'_*; p, p_*) = W(-p, -p_*; -p', -p'_*).$$

2. The symmetry of the molecules under spatial inversion i.e. change of the signs of all coordinates. This symmetry implies

$$W(-p, -p_*; -p', -p'_*) = W(p, p_*; p', p'_*)$$

and the detailed balance follows.

Moreover, we may also use the fact that, we do not integrate in the whole momentum space but only along the manifold determined by the conservation of energy and conservation of the momentum. Let us assume for the sake of brevity that the two particles have the same mass equal to one. Then, the two conservation properties read:

$$p' + p'_* = p + p_*$$

$$\frac{|p|^2}{2} + \frac{|p_*|^2}{2} = \frac{|p'|^2}{2} + \frac{|p'_*|^2}{2}.$$

The expression of W may therefore be written

$$W(p', p'_*; p, p_*) dp' dp'_*$$

$$= H(p', p'_*; p, p_*) \delta(p' + p'_* - p - p_*) \delta\left(\frac{|p'|^2 + |p'_*|^2 - |p|^2 - |p_*|^2}{2}\right) dp' \, dp'_*$$

$$= H(p + p_* - p'_*, p'_*; p, p_*) \delta\left(\frac{|p'|^2 + |p'_*|^2 - |p|^2 - |p_*|^2}{2}\right) dp'_*$$

On this manifold we can write

$$p' = q(p, p_*, \omega) = p - (p - p_*, \omega)\omega$$
$$p'_* = q_*(p, p_*, \omega) = p_* + (p - p_*, \omega)\omega, \qquad \omega \in \mathbf{S}^2$$

from where,

$$W(p', p'_*; p, p_*) dp' dp'_* = B(p, p_*, \omega) d\omega.$$

Finally, since the two interacting particles constitute a closed physical system, W has to be Galilean invariant (or Lorentz invariant in the relativistic case). Consider for the sake of simplicity the classical case. This implies

$$W(Tp', Tp'_*; Tp, Tp_*) = W(p', p'_*; p, p_*)$$

for any rotation $T = R \in SO(3)$ and any translation $T(p) = a + p$ with $a \in \mathbb{R}^3$. Also we have

$$(q(p + a, p_* + a, \omega), q_*(p + a, p_* + a, \omega)) = (q(p, p_*, \omega) + a, q_*(p, p_*, \omega) + a)$$

in such a way that

$$B(p+a, p_*+a, \omega) = W(p+a, p_*+a, p'+a, p'_*+a) = W(p, p_*, p', p'_*) = B(p, p_*, \omega).$$

Therefore, $B(p, p_*, \omega) = B(0, p_* - p, \omega)$ that we denote

$$B(0, p_* - p, \omega) \equiv B(p_* - p, \omega).$$

On the other hand, we also have

$$(q(Rp, Rp_*, R\omega), q_*(Rp, Rp_*, R\omega)) = (Rq(p, p_*, \omega), Rq_*(p, p_*, \omega))$$

in such a way that

$$\begin{aligned} B(R(p_* - p), R\omega) &= B(Rp, Rp_*, R\omega) \\ &= W(Rp, Rp_*, Rp', Rp'_*) = W(p, p_*, p', p'_*) = B(p_* - p, \omega) \end{aligned}$$

for every rotation $R \in SO(3)$. Therefore

$$B(p_* - p, \omega) = B(|p_* - p|, \frac{p_* - p}{|p_* - p|}, \omega).$$

Note that $B(p_* - p, \omega)f(p_*)d\omega dp_*$ is the probability per unit time and unit volume that any of the colliding particles p has a collision of the type considered. The effective collision cross section is then defined by

$$d\sigma = \frac{B(p_* - p, \omega)}{|p - p_*|} d\omega.$$

8.1   Scattering theory

The differential cross section depends on a crucial way on the kind of interaction between the two colliding particles that one considers. If the interaction between particles only depends on the distance between the two particles, we may assume without any loss of generality that

$$B(|p_* - p|, \frac{p_* - p}{|p_* - p|}, \omega) = B(|p_* - p|, \frac{p_* - p}{|p_* - p|} \cdot \omega),$$

i.e. the function B only depends on the modulus of the difference of momentum of the two incident particles and of the angle α formed by the two directions $p_* - p$ and $p'_* - p'$.

On the other hand, like any problem of two bodies, the problem of elastic collision amounts to a problem, of the scattering of a single particle with the reduced mass in the field U of a fixed centered force. This simplification is effected by changing to a system of coordinates in which the center of mass of the two particles is at rest. We set

$$\Omega = \frac{p - p_*}{|p - p_*|} - 2(\frac{p - p_*}{|p - p_*|}, \omega)\omega$$

in such a way that with this new variable,

$$p' = \frac{p + p_*}{2} + \frac{|p - p_*|}{2}\Omega,$$

$$p'_* = \frac{p + p_*}{2} - \frac{|p - p_*|}{2}\Omega.$$

Then we have $d\omega = \frac{1}{4\cos\alpha}d\Omega$, for $\alpha =$ angle between $p - p_*$ and ω.

Then, we use a spherical coordinate system with axis $p - p_*$:

$$\Omega = \frac{p - p_*}{|p - p_*|}\cos\theta + (\cos\phi h + \sin\phi i)\sin\theta.$$

With these new variables,

$$|(p - p_*, \omega)| = |p - p_*|\sin\frac{\theta}{2} \quad \text{and} \quad \cos\alpha = \sin\frac{\theta}{2}.$$

Therefore,

$$d\omega = \frac{d\Omega}{4\cos\alpha} = \frac{1\sin\theta d\theta d\phi}{4\cos\alpha} = \frac{\sin\theta}{4\sin(\theta/2)}d\theta d\phi = \frac{1}{2}\cos(\theta/2)d\theta d\phi$$

from where,

$$B(z, \omega)d\omega = \frac{B(z, \omega)}{4\cos\alpha}d\Omega = \frac{\cos(\theta/2)}{2}B(z, \omega)d\theta d\phi.$$

The differential cross section, $\sigma(z, \theta)$ is then such that

$$B(z, \omega)d\omega = \sigma(z, \theta)\, d\Omega \equiv \sigma(z, \theta)\sin\theta d\theta d\phi.$$

Then write

$$d\sigma(p, p_*, \theta) = \sigma\frac{(|p - p_*|, \theta)}{|p - p_*|}d\Omega.$$

The angle θ is the angle formed by the incident and scattered trajectory of the particle interacting with the central potential. In classical mechanics, collisions of two particles are entirely determined by their velocities and impact parameter. For a detailed study of the different differential cross section depending on the potential U considered in the classical case the reader may consult the detailed work by Cercignani [8]. It is shown in particular that:

1.- If U is a power law potential: $U(\rho) = |\rho|^{1-n}$, $n \neq 2, 3$, then

$$B(|p_* - p|, \frac{p_* - p}{|p_* - p|} \cdot \omega) = |p_* - p|^{\gamma}\beta(\frac{p_* - p}{|p_* - p|} \cdot \omega),$$

where p and p_* are the velocities of the two particles.

2.- *Coulomb potential.* If $U(\rho) = \alpha|\rho|^{-1}$, then

$$\sigma(|p - p_*|, \theta) = \frac{\alpha^2}{16|p_* - p|^4 \sin^4\frac{\theta}{2}}.$$

This is known as the Rutherford's formula (see Landau & Lifschitz vol. 1 §19).
3. *Hard sphere potential.* For the so called Hard sphere potential U defined as

$$U(\rho) = \lim_{n \to \infty} U_n(\rho), \quad U_n(\mathbf{r}) = \begin{cases} n & \text{if } |\rho| < a \\ 0 & \text{if } |\rho| > a \end{cases}$$

we have $\sigma = a^2$.

Remark 8.1 In the general case, where the two interacting particles p and p_* have masses m_1 and m_2 respectively , similar arguments and calculations can be performed. In particular, the center of mass parametrization may be written:

$$p' = \frac{m_1 m_2}{m_1 + m_2}(p + p_*) + \frac{m_1}{m_1 + m_2}|p - p_*|\Omega,$$
$$p'_* = \frac{m_1 m_2}{m_1 + m_2}(p + p_*) - \frac{m_2}{m_1 + m_2}|p - p_*|\Omega.$$

One defines again the angle θ by

$$\Omega = \frac{p - p_*}{|p - p_*|}\cos\theta + (\cos\phi h + \sin\phi i)\sin\theta.$$

As before, $W(p', p'_*; p, p_*)dp'dp'_* = \sigma(z, \theta)d\theta d\phi$ and

$$d\sigma(p, p_*, \theta) = \sigma \frac{(|p - p_*|, \theta)}{|p - p_*|}d\theta d\phi.$$

Remark 8.2 ([34, Vol.10, §2.]) Although the free motion of particles is assumed to be classical, this does not at all mean that their collision cross section need not be determined quantum mechanically; in fact, it usually must be so determined.

In quantum mechanics the very wording of the scattering problem must be changed, since in motion with definite velocities the concept of path is meaningless, and so is the impact parameter. The purpose of the theory is then only to calculate the probability that, as a result of the collision, the particles will scattered through any given angle. It is not our purpose to present in detail the derivation and the properties of the differential cross section from the scattering theory in detail. We only want to present the general idea, the relevant results for our study and some precise references for the interested readers.

We only give here a brief description of what M. Reed and B. Simon present as "naïve" scattering theory, or a stationary picture of it ([47], Vol.3, Notes on §X1.6). It is nevertheless the usual in the textbooks of quantum mechanics as for instance vol.3, §123 of Landau Lifschitz.

For a fixed $\mathbb{R}^3$-vector p and a positive real number E let us consider the function of $\rho = (x, y, z)$ and t called plane wave:

$$e^{-\frac{i}{\hbar}Et}e^{\frac{i}{\hbar}p\cdot\rho}.$$

This plane wave describes a state in which the particle has a definite energy E and momentum p. The angular frequency of this wave is $E/\hbar$ and its wave vector $k = p/\hbar$; the corresponding wavelength $2\pi\hbar/|p|$ is the de Broglie wavelength of the particle. The mass is $m = |p|^2/2E$ and the velocity is $v = p/m$.

Consider now a free particle, with mass m, total energy E, moving in the direction of the z-axis and by abuse of notation ley us denote its plane wave as

$$\psi_1(\rho) = e^{ikz}, \quad \forall \rho = (x, y, z) \in \mathbb{R}^3.$$

Assume that it is scattered by a radially symmetric potential $U(r)$ ($\rho = (r, \theta, \varphi)$ in polar coordinates).

The basic ansatz of naïve scattering theory is that the scattering state is the solution of the linear Schrödinger equation

$$\Delta\psi(\rho) + k^2\psi(\rho) - \frac{2m}{\hbar^2}U(r)\psi(\rho) = 0$$

such that

$$\psi(\rho) \sim e^{ikz} + f(k, \theta)\frac{e^{ikr}}{r} \quad \text{as} \quad r \to \infty.$$

This is, at large r we see the incident plane wave moving in the positive sense of the z- axis and a spherical divergent wave, modulated by the function $f \equiv f(k, \theta)$, called the scattering amplitude. This extra term describes the "scattered particle".

The scattered particle is described far from the center, as a spherical divergent wave, i.e. a wave moving in the "increasing" sense of the radial direction

$$f(k, \theta)e^{ikr}/r.$$

Remember that the square of the modulus of the wave ψ is the density of probability to find the particle at the point ρ. The presence of the factor $1/r$ is just to preserve that property since we are in $\mathbb{R}^3$. The density of probability is not necessarily the same in all the points but has to be independent of the φ and r variables due to the spherical symmetry of the potential. This is taken into account by the coefficient $f(k, \theta)$ which is the amplitude diffusion and depends only on the angle θ between the direction of the incoming particle, which is $e_3 = (0, 0, 1)$, and the direction where we are looking for the scattered particle, i.e. ρ/r.

As it is pointed out in [47], at first sight, this ansatz looks absurd, for, if $\psi \sim e^{ikz} + f(\theta)r^{-1}e^{ikr}$ for $r \to \infty$, then, for all the time ψ has both a plane wave coming in and an outgoing spherical wave. The point of the argument is to consider an initial state which is more localized i.e. given by the function:

$$\psi(\rho) = \int g(k)\left(e^{ikz} + f(\theta)r^{-1}e^{ikr}\right)dk$$

and g peaked around $k = k_0$. Then following the same idea, for r and t large, the wave would be given by

$$\psi(\rho) \sim \int g(k)e^{ik(z-kt)}dk + r^{-1}f(\theta)\int g(k)e^{ik(r-kt)}\,dk.$$

Then, essentially by the Riemann-Lebesgue lemma, the first integral for z and t large has appreciable size only for $z \sim k_0 t$ and the second integral if $r \sim k_0 t$. Therefore, if $t \to -\infty$, the second term is negligible for all $r \geq 0$ and we recover, asymptotically only the incident wave.

Therefore, the probability per unit time that the scattered particle will pass through the surface element $dS = r^2 d\Omega$ is $(v/r^2)|f|^2 dS = v|f|^2 d\Omega$, where v is the velocity of the particle. We have then:

$$\sigma(k, \theta)d\Omega = |f(k, \theta)|^2 d\Omega$$

and we recover the well known formula $d\sigma = |f(\theta)|^2 d\Omega$.

We are then lead to see how do we get information about the function $f(k, \theta)$. It is well known that the solutions of the linear Schrödinger equation may be written as

$$\psi(r) = \sum_{l=0}^{\infty} A_l P_l(\cos\theta) R_{k,l}(r)$$

where A_l are constants and the $R_{k,l}(r)$ are radial functions satisfying the equation

$$\frac{1}{r^2}\frac{d}{dr}(r^2\frac{dR}{dr}) + [k^2 - \frac{l(l+1)}{r^2} - \frac{2m}{\hbar}U(r)]R = 0$$

and the P_l are the Legendre polynomials. The coefficients A_l are constants which have to be chosen so that the condition at $r \to \infty$ is fulfilled. This implies that:

$$A_l = \frac{1}{2k}(2l+1)i^l \exp(i\delta_l)$$

where for every l, δ_l is a constant called phase shift.

To see this observe that, the asymptotic form of each of the functions $R_{k,l}$ is

$$R_{k,l}(r) \sim \frac{2}{r}\sin(kr - \frac{l\pi}{2} + \delta_l) = \frac{1}{ir}\{(-i)^l \exp[i(kr + \delta_l)] - i^l \exp[-i(kr + \delta_l)]\}$$

Therefore, it is formally deduced that

$$\psi(r) \sim \sum_{l=0}^{\infty} A_l P_l(\cos\theta)\frac{i}{r}\left\{\exp[-i(kr - \frac{l\pi}{2} + \delta_l)] - \exp[i(kr - \frac{l\pi}{2} + \delta_l)]\right\}.$$

On the other hand, the plane wave expansion in spherical harmonics gives

$$e^{ikz} = \sum_{l=0}^{\infty}(-i)^l(2l+1)P_l(\cos\theta)(\frac{r}{k})^l(\frac{1}{r}\frac{d}{dr})^l\frac{\sin kr}{kr}.$$

As $r \to \infty$ we have

$$e^{ikz} \sim \frac{1}{kr}\sum_{l=0}^{\infty}i^l(2l+1)P_l(\cos\theta)\sin(kr - \frac{l\pi}{2})$$

$$= \sum_{l=0}^{\infty}i^l(2l+1)P_l(\cos\theta)\frac{i}{2kr}\left\{\exp[-i(kr - \frac{l\pi}{2})] - \exp[i(kr - \frac{l\pi}{2})]\right\}$$

For $A_l = \frac{1}{2k}(2l+1)i^l \exp(i\delta_l)$, as indicated above,

$$\psi - e^{ikz} \sim \frac{i}{2kr} \sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)[(-1)^l e^{-ikr} - S_l e^{ikr}]$$

with $S_l = e^{2i\delta_l}$. Then

$$f(k,\theta) = \frac{1}{2\,i\,k} \sum_{l=0}^{\infty}(2l+1)[S_l - 1]P_l(\cos\theta).$$

Integrating $d\sigma$ over all the values of the angles we obtain the total cross section

$$\sigma = 2\pi \int_0^\pi |f(k,\theta)|^2 \sin\theta d\theta = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l$$

and since the Legendre polynomials are orthonormal,

$$\int_0^\pi P_l^2(\cos\theta)\sin\theta d\theta = \frac{2}{2l+1}.$$

The coefficients $f_l(k,\theta) = \frac{1}{2i\,k}(S_l - 1)$ are called partial amplitude diffusion.

8.2 Study of the general formula of $f(k,\theta)$

This formula is valid for all radial potential $U(r)$ vanishing at infinity. Its study reduces to that of the phases δ_l. (We quote [34, Vol. 10, §124]).

Under the assumption that $U(r) \sim r^{-n}$ as $r \to \infty$, we have the following two statements:

1.- If $n > 2$ then the total cross section is finite and the differential cross section is integrable.

2.- If $n \leq 2$ the total cross section is infinite and the differential cross section is not integrable. From a physical point of view this is due to the fact that, since the field is slowly decreasing with the distance, the probability of diffusion of very small angles becomes very large. Remember that in classical mechanics, for every positive potential $U(r)$ vanishing only at infinity, any particle with large but finite impact parameter is deviated by a small but non zero angle and so the total cross section is infinite, whatever is the decay of $U(r)$. (In that sense, it may be considered that the quantum scattering is more regular, or less singular, than the classical one).

Concerning the differential cross section itself we have the following two statements:

1.- If $n \leq 3$ the differential cross section becomes infinite as $\theta \to 0$

2.- If $n > 3$ the differential cross section is finite as $\theta \to 0$

Finally, if $n \leq 1$, then the total cross section is infinite, i.e. the differential cross section is not integrable; the differential cross section is singular as $\theta \to 0$ but it is well defined for $\theta \neq 0$ and is given by

$$f(k,\theta) = \frac{1}{2\,i\,k} \sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)(e^{2i\delta_l} - 1).$$

8.3   Non radial interaction

Let us consider again the incident plane wave e^{ikz} in the radial potential considered above, moving in the $(0, 0, 1)$ direction, which is reflected and which at a large distance point $\rho = (r, \theta, \phi)$ is seen as $e^{ikz} + f(\theta)e^{ikr}/r$. Observe that, given the vector $\rho = (x, y, z) = (r, \theta, \phi)$, we have

$$z = r\, e_3 \cdot \frac{\rho}{r}, \quad e_3 = (0, 0, 1),$$

i.e. z may be seen as r times the scalar product of the two unitary vectors giving the directions of the incident and scattered particle's velocities. Consider now a general potential $U(\rho)$, and n a unitary vector of $\mathbb{R}^3$. Consider then an incident particle in the direction n, scattered by $U(\rho)$. The wave describing this particle would then be the solution of the Schrödinger equation

$$-\Delta\psi(\rho) - k^2\psi(\rho) + U(\rho)\psi(\rho) = 0$$

such that at the point $\rho = (r, \theta, \psi)$ with $|\rho| \to \infty$,

$$\psi(\rho) \sim e^{ikrnn'} + \frac{1}{r}f(n, n')e^{ikr}$$

where $n' = r/\rho$. The amplitude diffusion depends on the two directions of the incident and scattered particles and not only on the angle that they form.

Born's formula   ([34, Vol. 3, §126]) This s formula gives an explicit relation between the differential cross section and the potential $U(\rho)$, non necessarily radial. As we have seen, in that case the amplitude diffusion depends on the incident and scattered directions and not only on the angle they form. The explicit expression is:

$$f(q, q') = -\frac{m}{2\pi\hbar^2} \int_{\mathbb{R}^3} U(\mathbf{r})e^{-i(q'-q)\cdot\rho}dV(\rho)$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{4\pi^2\hbar^4} |\int_{\mathbb{R}^3} U(\rho)e^{-i(q'-q)\cdot\rho}dV(\rho)|^2, \quad |q'-q| = 2k\sin\frac{\theta}{2},$$

where θ is the angle between the two vectors q and q'.

One may approximate the differential cross section by the Born's formula whenever the perturbation field $U(\rho)$, not necessarily spherically symmetric, may be considered as a perturbation. [This corresponds to the case where all the phases δ_l are small]. This is possible when one of the following conditions are fulfilled:

$$|U| \ll \frac{\hbar^2}{ma^2} \quad \text{or} \quad |U| \ll \frac{\hbar v}{a} = \frac{\hbar^2}{ma^2}ka$$

where a is the rayon of action of $U(\rho)$ and U its order of magnitude in the main region of its existence. In the first case, the Born's approximation may be applied for all the velocities. In the second case, it may be applied for particles with sufficiently large velocities.

Moreover, if the potential is spherically symmetric, $U = U(r)$, then we obtain

$$f(k, \theta) = -\frac{m}{\hbar^2} \int_0^\infty U(r) \frac{\sin[2rk \sin \frac{\theta}{2}]}{k \sin \frac{\theta}{2}} r dr.$$

When $\theta = 0$, this integral diverges provided $U(r)$ decreases as r^{-3} or slower when $r \to \infty$.

8.4   Scattering of slow particles:

([34, Vol. 3, §132]). We consider the limiting case where:
1.- The potential $U(r)$ is radial and decreases at large distances more rapidly than $1/r^3$.
2.- The velocities of the particles undergoing scattering are so small that their wavelength is large compared with the radius of action a of the field $U(r)$, i.e. $ka << 1$, and their energy is small compared with the field within that radius,

Under these conditions, the total amplitude may be approximated as $f(\theta) \sim f_0$ which is the first partial amplitude. Therefore, $d\sigma(k, \theta) = |f_0|^2 d\Omega$. At low velocities, the scattering is isotropic, and the differential cross section is independent of the particle energy.

If the potential decreases as $U(r) \sim r^{-n}$ with $n < 3$ the above approximation is not valid.

8.5   Some examples of differential cross sections

(i) *Coulomb interaction*: (Rutherford formula ([34, Vol. 3 §135].) The scattering by central coulomb potential is particularly important with respect to the applications in physics. Moreover, the quantumechanic problem of the collisions may be solved explicitly until the end. So, if we assume the potential to be $U(\rho) = |\rho|^{-1}$, the differential cross section is

$$\sigma(k, \theta) = \frac{1}{4k^4 \sin^4 \frac{\theta}{2}}.$$

Observe that it is the same as the differential cross section obtained for the classical Coulomb interaction.
(ii) *Hard sphere potential.* (Slow particles; [34, Vol. 3 §132, Problem 2].) For the Hard sphere potential $U(r) = \lim_{n \to \infty} U_n(r)$, with

$$U_n(r) = \begin{cases} n & \text{if } r < a \\ 0 & \text{if } r > a, \end{cases}$$

the differential cross section, under the conditions $ka \ll 1$ and for small values of k is $\sigma = a^2$. Observe that this implies that the total cross section is $4\pi a^2$, which is four times the result following classical mechanics.

(iii) *Yukawa potential.* (Born approximation; [34, Vol. 3 §126, Problem 3]) Assume that $U(r) = \alpha \frac{e^{-r/a}}{r}$. Then the Born approximation is

$$\sigma = (\frac{\alpha m a}{\hbar^2})^2 \frac{4a^2}{(q^2 a^2 + 1)^2},$$

with $q = 2k \sin(\theta/2)$ This approximation is valid whenever $\alpha m a/\hbar^2 \ll 1$ or $\alpha/\hbar v \ll 1$. In the first case it is valid for all the velocities. In the second, only for velocities sufficiently large.

One may find in [34], Vol. 3, §132, problem 1 and problem 2 the Born approximations of the differential cross sections corresponding to the spherical well and uniform potential barrier and in §126 problem 2 to the Gaussian potential.

Collisions of identical particles   ([34, Vol. 3 §137].) If the two particles are identical, then they are indiscernible and

$$W(p', p'_*; p, p_*) = W(p'_*, p'; p, p_*).$$

The wave function of a system of two particles has to be symmetric or anti-symmetric with respect to the particles depending whether their total spin is odd or even. Therefore, the wave function, describing the scattering, obtained by solving the Schrödinger equation has to be symmetrized or antisymmetrized. Their asymptotic expansion has then to be written as

$$\psi \sim e^{ikz} \pm e^{-ikz} + \frac{1}{r}e^{ikr}[f(\theta) \pm f(\pi - \theta)].$$

In that way, if the total spin of the particles in the collision is even, the differential section is

$$d\sigma_s = |f(\theta) + f(\pi - \theta)|^2 d\Omega.$$

If the total spin is odd, then the differential section is

$$d\sigma_a = |f(\theta) - f(\pi - \theta)|^2 d\Omega.$$

We have assumed in these two formulas that the total spin of the particles in the collision has a fixed value. But in general, we deal with collisions in which the particles do not have their spins in a determined state. In order to find the differential cross section one has then to take the mean over all the possible states of the spin where we consider all of them equiprobable. For an half integer s, the probability for a system of two particles of spin s to have a spin S even is $s/(2s + 1)$. The probability to have a spin S odd is $(s + 1)/(2s + 1)$. Then the differential cross section for interacting identical particles of half integer spin s is

$$d\sigma = \frac{s}{2s + 1}d\sigma_s + \frac{s + 1}{2s + 1}d\sigma_a.$$

Similarly, for interacting identical particles of integer spin s:

$$d\sigma = \frac{s + 1}{2s + 1}d\sigma_s + \frac{s}{2s + 1}d\sigma_a.$$

8.6 Relativistic case

The differential cross sections in the relativistic case are calculated in a completely different way. Let us first mention here that for short range interaction, one still has $s\sigma(s,\theta) \equiv$ constant (cf. [46]). We conclude with the following example.

Photon-electron scattering Let $P = (p^0, p)$ and $P_* = (p_*^0, p_*)$ be the 4-momenta of the photon and electron before collision, and $P' = (p'^0, p')$ and $P'_* = (p'^0_*, p'_*)$ their 4-momenta after the collision. Define the center of mass coordinates:

$$s = (P + P_*)^2, \quad t = (P - P')^2.$$

The differential Compton cross section is given by the Klein-Nishina formula:

$$\sigma(s,\theta) = \frac{1}{2}r_0^2(1-\xi)\{1 + \frac{1}{4}\frac{\xi^2(1-x)^2}{1-\frac{1}{2}\xi(1-x)} + [\frac{1-(1-\frac{1}{2}\xi)(1-x)}{1-\frac{1}{2}\xi(1-x)}]^2\}$$

where m is the mass of the electron,

$$x = 1 + \frac{2st}{(s-m^2)^2}, \quad \xi = \frac{((P+P_*)^2 - m^2c^2)}{(P+P_*)^2}, \quad r_0 = \frac{e^2}{4\pi mc^2}.$$

See e.g. [29]. If the energy of the photons is low, the non relativistic limit of the Klein Nishina differential cross section is

$$\xi \sim \frac{2|p|mc}{m^2c^2} = \frac{2|p|}{mc}$$

and so it gives

$$\sigma(s,\theta) \sim \frac{1}{2}r_0^2\{1 + \cos^2\theta\}, \quad \text{as} \quad c \to \infty.$$

This is the Thomson formula for the efficient cross section of the diffusion of an incident electromagnetic wave diffused by a single free charge at rest, (in that case, θ is the angle formed by the direction of the diffusion and the direction of the electric field of the incident wave [34, Vol. 2, §78.7].

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