ON THE CONVERGENCE OF NUMERICAL SCHEMES FOR THE BOLTZMANN EQUATION

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Abstract

We consider a time and spatial explicit discretisation scheme for the Boltzmann equation. We prove some Maxwellian bounds on the resulting approximated solution and deduce its convergence using a new time-discrete averaging lemma.

1 Introduction

This article is devoted to the proof of the convergence of a time and spatial explicit discretisation scheme for the Boltzmann equation. The Boltzmann equation provides a time evolution of a gas described by the distribution of particles $f(t,x,v) \geq 0$ which at time $t \geq 0$ and at position $x \in \mathbb{R}^d$ move with velocity $v \in \mathbb{R}^d$. The Boltzmann equation reads

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f,f) \quad \text{in} \quad (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \tag{1.1}$$

$$f(0,x,v) = f_{in}(x,v) \quad \text{on} \quad \mathbb{R}^d \times \mathbb{R}^d, \tag{1.2}$$

where $Q(f,f)$ is the quadratic Boltzmann collision operator describing the collision interactions between particles (binary elastic shock). We refer to [3] for a detailed presentation of the equation and to [27] and the references therein for recent results concerning its analysis. Let us just summarize now the fundamental properties of the collision kernel that we shall use in the sequel. First, the collision kernel splits into two parts

$$Q(f,f) = Q^+(f,f) - Q^-(f,f)$$

where the gain term Q^+ and the loss term Q^- are positive operators. Next, it vanishes on Maxwellian functions, namely

$$Q^+(M,M) = Q^-(M,M) \quad \text{if} \quad M(v) = \exp(a\,|v|^2 + b \cdot v + c) \tag{1.3}$$

with $a, c \in \mathbb{R}$, $a < 0$, $b \in \mathbb{R}^d$. Last, the loss term writes

$$Q^-(f,f) = f\,L(f), \qquad L(f) = A *_v f, \tag{1.4}$$

and we assume here that the so-called *total cross-section* A satisfies

$$0 \leq A(z) \leq K_0\,|z|^\gamma, \qquad K_0 > 0,\ \gamma \in (-d,1]. \tag{1.5}$$

A particular case for which the above condition holds is the cross-section associated to an inverse potential (except the Coulomb potential) with the angular cut-off condition of Grad and the cross-section associated to hard sphere collisions. Before describing the scheme investigated here, we recall what is known about several partial discretisations of (1.1). A first step of the discretisation

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(usually used in numerical simulation) is to split the transport part

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = 0 \tag{1.6}$$

and the collision part

$$\frac{\partial f}{\partial t} = Q(f,f), \tag{1.7}$$

and to solve each equation one after another in small intervals $(k\,\Delta_t,(k+1)\,\Delta_t)$ for any $k \in \mathbb{N}$. This splitting algorithm has been proved to converge in [4]: constructing a approximate solution f_{Δ_t}, one may prove that (f_{Δ_t}) converges (up to the extraction of a subsequence) to a solution of the Boltzmann equation (1.1) when $\Delta_t \to 0$. A crucial step is the velocity discretisation, which means to approximate (1.1) by a family of equations

$$\frac{\partial f_j}{\partial t} + v_j \cdot \nabla_x f_j = Q_j((f_j)_j,(f_j)_j) \quad \text{in} \quad (0,\infty)\times\mathbb{R}^d \tag{1.8}$$

where $f_j = f_j(t,x) \geq 0$ represents the density of particles with velocity v_j (or with velocity in a neighborhood of v_j), (v_j) is a family of given velocities and the operator $(Q_j)_j$ is an approximation of $Q(f,f)$ (with the help of quadrature formula). Such a scheme (construction of a good approximation operator Q_j) have been proposed by [10], [15], [21] and their convergence have been proved in [17], [19], [21], [20], [16]. Another step is to perform a time Euler explicit discretisation of (1.7):

$$\frac{f^{k+1}-f^k}{\Delta_t} = Q_{R_v}(f^k,f^k), \qquad f^0 = f_{in}. \tag{1.9}$$

Here Q_{R_v} denote a velocity truncation of Q which guarantees the positivity of f^k and can be relaxed in the limit $\Delta_t \to 0$. Convergence of the Euler scheme has been proved in [18]. See also [9] for other time discretisations. The scheme we consider here consists in an explicit time and space discretisation of the splitting algorithm of (1.1). Full discretisations including velocity discretisation is postponed to future works. We successively perform (and iterate):

1. solve explicitly the transport equation (1.6),
2. project on space mesh,
3. perform the time explicit Euler scheme (1.9).

In order to be more precise, let us introduce a partition of $\mathbb{R}^d$ in cells:

$$\mathbb{R}^d = \cup_{a\in\mathbb{Z}^d}\Lambda_a, \quad \Lambda_a = \prod_{i=1}^d [a_i\,\Delta_{x,n},(a_i+1)\,\Delta_{x,n}[, \tag{1.10}$$

for some $\Delta_{x,n} > 0$; and let us define the projection operator on the meshes $(\Lambda_a)_{a\in\mathbb{Z}^d}$:

$$P_n\,\phi = \sum_a P^a\,\phi \quad \text{with} \quad P^a\,\phi(x) := \frac{1}{(\Delta_{x,n})^d}\int_{\Lambda_a}\phi(y)\,dy\,\mathbf{1}_{\Lambda_a}(x). \tag{1.11}$$

Let also define $Q_{R_{v,n}}$ a velocity truncated Boltzmann operator such that its total cross-section $A_{R_{v,n}}$ satisfies

$$A_{R_{v,n}}(z) \leq A(z)\,\mathbf{1}_{|z|\leq R_{v,n}}.$$

Starting from the initial datum

$$f_n^0 = (P_n f_{in})\,\mathbf{1}_{B_{R_{v,n}}}(v)\,\mathbf{1}_{B_{R_{x,n}/4}}(x) \tag{1.12}$$

we define

$$\begin{aligned}
&(f_n^{k+1/3})^\sharp(x,v) = f_n^k(x,v),\\
&f_n^{k+2/3} = P_n\,f_n^{k+1/3},\\
&\frac{f_n^{k+1}-f_n^{k+2/3}}{\Delta_{t,n}} = Q_{R_{v,n}}(f_n^{k+2/3},f_n^{k+2/3})
\end{aligned}$$

where we use the notation $\phi^{k\sharp}(x,v) := \phi(x + k\,\Delta_{t,n}\,v, v)$ for $k \in \mathbb{Z}$ (for a given $\Delta_{t,n} > 0$). In other words, we define

$$f_n^{k+1} = P_n(f_n^{k-\sharp}) + \Delta_{t,n}\,Q_{R_{v,n}}(P_n(f_n^{k-\sharp}), P_n(f_n^{k-\sharp})). \tag{1.13}$$

We finally define the approximate solution f_n by

$$f_n(t,x,v) = \sum_k f_n^k(x - v\,(t - k\,\Delta_{t,n}), v)\,\mathbf{1}_{t\in[k\,\Delta_{t,n},(k+1)\,\Delta_{t,n})}\,\mathbf{1}_{t\in[0,T_n]} \tag{1.14}$$

for a given choice of $\Delta_{t,n}$, $\Delta_{x,n}$, $R_{x,n}$, $R_{v,n}$, $T_n > 0$.

This paper is devoted to the proof of the following result.

Theorem 1.1 *Let consider an initial datum f_{in} such that*

$$0 \le f_{in} \le M^0 = C^0\,\xi, \quad \xi(x,v) = \exp(-\alpha\,|x|^2 - \beta\,|v|^2), \tag{1.15}$$

with

(i) *local case:* $\gamma \in (-d, 0]$

or

(i) *global case:* $\gamma \in (-d+1, 0]$ *and* C^0 *small enough (depending on* α,β*)*
or $\gamma \in (0,1]$ *and* α *large enough (depending on* C^0,β*).*

There exists $T^* = T^*(M^0) > 0$, and we may choose $T^* = +\infty$ in case (ii), and there exists a sequence of the discretisation parameters $(\Delta_{t,n})$, $(\Delta_{x,n})$, $(R_{x,n})$, $(R_{v,n})$, (T_n) satisfying

$$\Delta_{t,n},\,\Delta_{x,n} \to 0, \quad R_{x,n},\,R_{v,n} \to +\infty, \quad T_n \nearrow T^*, \tag{1.16}$$

such that the sequence (f_n) defined by (1.14) satisfies

$$\sup_n \sup_{[0,T^*]} \left\| f_n\,\xi^{-1} \right\|_{L^\infty} < \infty, \tag{1.17}$$

and, up to the extraction of a subsequence, (f_n) converges weakly to a solution f of the Boltzmann equation (1.1).

Remark 1.2 *The same result holds for different versions of time and space discretisations such that replacing (1.13) by*

$$(f^{k+1})^\sharp = P_n(f^k) + \Delta_{t,n}\,Q_{R_{v,n}}(P_n(f^k), P_n(f^k)).$$

Let us briefly explain the strategy of the proof. First, remark that though the convergence proof for the splitting algorithm and for the velocity discretisation scheme can be performed in the general framework of DiPerna-Lions renormalized solutions (and thus for general initial data) such a framework seems difficult to use in the present situation at least for two reasons. On one hand, for an explicit scheme we loose the entropy-dissipation entropy bound which is a fundamental information in the weak stability result for renormalized solutions. On the other hand, even for a modified implicit scheme (for which entropy-dissipation entropy bound is available) time (and position) discretisation seems to be inadapted to the renormalization technic. We then choose the (less general) framework of distributional solution bounded above by a Maxwellian function introduced by Illner and Shinbrot.

The first step is thus to build for any $n \in \mathbb{N}$ a sequence of Maxwellians $(M_n^k)_k$ which are subsolution of the discrete scheme (1.13) in the following sense

$$M_n^{k+1} \ge P_n\,M_n^{k-\sharp} + \Delta_{t,n}\,Q^+_{R_{v,n}}(P_n\,M_n^{k-\sharp}, P_n\,M_n^{k-\sharp}) \tag{1.18}$$

in the case of soft potential ($\gamma \le 0$) and

$$M_n^k \ge \pi_n^k \, M^0 + \sum_{j=0}^{k-1} \Delta_{t,n} \, \pi_n^{k-1-j} \, Q^+_{R_v,n}(\pi_n M_n^j, \pi_n M_n^j), \tag{1.19}$$

in the case of hard potential ($\gamma > 0$), where we use the notation $\pi_n \, \phi = P_n(\phi^{-\sharp})$. These subsolutions (M_n^k) can be constructed locally or globally in time (depending on the size of the initial datum and of γ). We then easily verify that they are indeed subsolutions: if $0 \le f_n^0 \le M_n^0$ then $0 \le f_n^k \le M_n^k \; \forall k, n$ and that provides the strong bounds (1.17).

A second step is to write the kinetic equation satisfied by f_n, namely

$$\frac{\partial f_n}{\partial t} + v \cdot \nabla_x f_n = \sum_k \delta_{k\,\Delta_{t,n}}(t) \int_{k\,\Delta_{t,n}}^{(k+1)\,\Delta_{t,n}} \left(\frac{P_n \, g_n - g_n}{\Delta_{t,n}} + Q_{R_v,n}(P_n \, g_n, P_n \, g_n) \right) d\tau \tag{1.20}$$

with

$$g_n(t, \cdot, \cdot) = f_n((t_{\Delta_{t,n}})^-, \cdot, \cdot), \quad t_{\Delta_{t,n}} = E(t/\Delta_{t,n}) \, \Delta_{t,n}, \tag{1.21}$$

where E denotes the truncation function, and to pass to the limit in (1.20) when $n \to \infty$. In order to do it, the main difficulty is to prove that the velocity averages of g_n converge strongly. Of course, the so-called "compactness lemma on velocity averaging" of solutions of continuous transport equation has been introduced by [12], [11], [1] at the middle of the 80's and has been extensively developped by [5], [6], [8], [22], [2]. Discrete versions in velocity have been proved in [17] and time discrete version for the splitting algorithm have been introduced in [4]. See also [2] for an alternative and simpler proof. We need here such a discrete version of averaging lemmas (which means for velocity averaging of g_n instead of velocity averaging of f_n) extended to this time and position discrete context. Gathering the "ultimate" version of averaging lemma due to [22], the previous "time" discrete version of averaging lemma by [4] and [2] and the scale techniques developed by Vasseur in [25, 26], we prove the following result.

Theorem 1.3 *Consider a sequence $\Delta_{t,n} \to 0$ and a sequence f_n uniformly bounded in $L^\infty(\mathbb{R}^+ \times \mathbb{R}^{2d})$ which satisfies*

$$\frac{\partial}{\partial t} f_n + v \cdot \nabla_x f_n = \sum_{i \in \mathbb{Z}} \delta_{i\,\Delta_{t,n}}(t) \left(\int_{i\,\Delta_{t,n}}^{(i+1)\,\Delta_{t,n}} H_n(\tau, x, v) \, d\tau \right) = J_n. \tag{1.22}$$

We assume that

$$f_n \rightharpoonup f \quad \textit{weakly in } L^\infty(\mathbb{R}^+ \times \mathbb{R}^{2d}) * \tag{1.23}$$

$$J_n \textit{ is relatively compact in } W^{-1,p}(\mathbb{R}^+ \times \mathbb{R}^{2d}) \textit{ for some } p > 1, \tag{1.24}$$

$$\textit{there exists a sequence } \; \epsilon_n \to 0 \textit{ with } \epsilon_n/\Delta_{t,n} \to +\infty \; \textit{ such that:} \tag{1.25}$$

$$\|{\epsilon_n}^2 \, H_n\|_{L^2} \overset{n \to +\infty}{\longrightarrow} 0.$$

Then, for any $\psi \in \mathcal{D}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} g_n(t, x, v) \, \psi(v) \, dv \;\to\; \int_{\mathbb{R}^d} f(t, x, v) \, \psi(v) \, dv \tag{1.26}$$

strongly in $L^p_{\rm loc}((0, T) \times \mathbb{R}^{2d}) \; \forall p \in [1, \infty)$.

It remains to verify that Theorem 1.3 may be used for the sequence (f_n) built in the statement of Theorem 1.1, and then it is classical to pass to the limit $n \to \infty$ in the formulation (1.20) and obtain Theorem 1.1. For the sake of completeness we present in the appendix a different version of Theorem 1.3 where Hypothesis (1.24) is slightly generalized.

The outline of the paper is the following. In section 2 we prove Theorem 1.3. In section 3 we built the subsolution (M_n) for the discrete scheme (1.13). In section 4 we then prove Theorem 1.1.

2 Proof of Theorem 1.3

Let us begin giving the idea of the proof. We first use the classical compactness averaging lemma to prove compactness for the continuous functions with respect to time. Indeed we are able to show that $\{\int \psi(v) f_n(\cdot,\cdot,v)\,dv\}$ is relatively compact in $L^2_{\text{loc}}(\mathbb{R}^{d+1})$. Let us recall this result due to Perthame-Souganidis [22] in our framework:

Theorem 2.1 *Let f_n be a sequence of functions bounded in $L^q(\mathbb{R}^{1+2d})$ for some $1 < q < +\infty$ and $\{J_n\}$ be relatively compact in $W^{-1,p}(\mathbb{R}^{1+2d})$ verifying:*

$$\partial_t f_n + v.\nabla_x f_n = J_n.$$

Then, for every function $\psi \in \mathcal{D}(\mathbb{R}^d)$, the average:

$$\rho^n_\psi(t,x) := \int_{R^d} \psi(v) f_n(t,x,v)\,dv$$

is relatively compact in $L^q(\mathbb{R}^{1+d})$.

On the other hand, Property (1.25) allows us to show a result of the kind (1.26) "at a local scale" thanks to the following Theorem due to Desvillettes and Mischler [4].

Theorem 2.2 *Consider a sequence $\overline{\Delta}_n \to 0$ and a sequence of functions $\overline{f}_n$ bounded in $L^2_{\text{loc}}([0,T] \times \mathbb{R}^{2d})$ which verify:*

$$\partial_s \overline{f}_n + v \cdot \nabla_y \overline{f}_n = \sum_{i\in\mathbb{Z}} \delta_{i\overline{\Delta}_n}(s) \left(\int_{i\overline{\Delta}_n}^{(i+1)\overline{\Delta}_n} \overline{H}_n(\tau,y,v)\,d\tau \right) \tag{2.1}$$

with $\overline{H}_n$ bounded in $L^2([0,T] \times \mathbb{R}^{2d})$. Then, for every $\psi \in \mathcal{D}(\mathbb{R}^d)$, the average:

$$\overline{\eta}^n_\psi(s,y) = \int_{\mathbb{R}^d} \psi(v) \overline{g}_n(s,y,v)\,dv$$

is relatively compact in $L^2_{\text{loc}}([0,T] \times \mathbb{R}^d)$, where

$$\overline{g}_n(s,y,v) = \overline{f}_n(s_{\overline{\Delta}_n}, y, v),$$

with $s_{\overline{\Delta}_n} = E(s/\overline{\Delta}_n)\overline{\Delta}_n$.

More precisely, if we denote $\overline{\Delta}_n = \Delta_{t,n}/\epsilon_n$ with $\Delta_{t,n} \ll \epsilon_n \ll 1$, and:

$$\begin{aligned}
&\overline{f}_n(t,x,s,y,v) = f_n(t_{\Delta_{t,n}} + s\epsilon_n, x + y\epsilon_n, v)\\
&\overline{g}_n(t,x,s,y,v) = g_n(t_{\Delta_{t,n}} + s\epsilon_n, x + y\epsilon_n, v) = f_n(t_{\Delta_{t,n}} + \epsilon_n\, s_{\overline{\Delta}_n}, x + y\epsilon_n, v),
\end{aligned}$$

then for every fixed point (t,x) the function $\overline{f}_n(t,x,\cdot,\cdot,\cdot)$ verifies (2.1). So we conclude that $\{\int \psi(v)\overline{g}_n(t,x,\cdot,\cdot,v)\,dv\}$ is relatively compact in $L^2([0,T] \times \mathbb{R}^d)$ when (t,x) is fixed. The following lemma allows us to compare the results at the global scale (in variables (t,x)) and at the local scale (in variables (s,y)) in order to carry the desired result from the result at the local scale using the compactness result on the continuous function in time at the global scale:

Lemma 2.3 *(From local scale to global scale) Let ρ^n, $\rho \in L^p_{\text{loc}}([0,T] \times \mathbb{R}^d)$ with $1 \le p < +\infty$, $\Delta_{t,n} \to 0$ and $\Delta_{t,n}/\epsilon_n \to 0$. Then ρ^n converges strongly to ρ in $L^p_{\text{loc}}([0,T] \times \mathbb{R}^d)$ if and only if for every $R > 0$:*

$$\int_0^T \int_{B_d(0,R)} \int_0^1 \int_{B_d(0,1)} \left|\rho^n(t_{\Delta_{t,n}} + \epsilon_n s, x + \epsilon_n y) - \rho(t,x)\right|^p dy\,ds\,dx\,dt \overset{n\to+\infty}{\longrightarrow} 0, \tag{2.2}$$

where $B_d(0,R)$ is the d-dimensional ball of center 0 and radius R.

This lemma is a slight generalization of a result of [25] (in the case $\epsilon_n \neq \Delta_{t,n}$). For the sake of completeness we give its proof in the appendix.

Proof of Theorem 2. We denote:

$$\eta_\psi^n(t,x) = \int_{\mathbb{R}^d} \psi(v) g_n(t,x,v)\, dv.$$

We split the proof into several parts.

(i) Compactness at the global scale. For every $j \in \mathbb{N}$ we consider a regular function $\Phi_j \in C^\infty(\mathbb{R}\times\mathbb{R}^d)$ defined such that $\mathrm{Supp}\Phi_j \subset [0,2j] \times B_d(0,2j)$ and $\Phi_j(t,x) = 1$ if $(t,x) \in [1/j, j] \times B_d(0,j)$. From Equation (1.22) we get:

$$\frac{\partial \Phi_j f_n}{\partial t} + v \cdot \nabla_x \Phi_j f_n = \Phi_j J_n + f_n \left[\partial_t \Phi_j + v \cdot \nabla_x \Phi_j\right].$$

Since $f_n \in L^\infty$ and Φ_j is regular the right-hand-side term is compact in $W^{-1,p}(\mathbb{R}\times\mathbb{R}^{2d})$. Moreover $\Phi_j f_n \in L^p(\mathbb{R}\times\mathbb{R}^{2d})$ (since $f_n \in L^\infty$ and Φ_j is compactly supported). Therefore $\{\Phi_j(t,x)\int \psi(v) f_n(t,x,v)\, dv\}$ is relatively compact in $L^p(\mathbb{R}^{1+2d})$ thanks to Theorem 2.1. By diagonal extraction, up to a subsequence, there exists $\rho_\psi \in L^\infty([0,T]\times\mathbb{R}^d)$ such that $\rho_\psi^n(t,x) = \int \psi(v) f_n(t,x,v)\, dv$ converges to $\rho_\psi(t,x)$ in L^p_{loc}. Because of Hypothesis (1.23), $\rho_\psi(t,x) = \int \psi(v) f\, dv$. By uniqueness of the limit the entire sequence is converging. Finally the convergence holds true in $L^q_{\mathrm{loc}}([0,T]\times\mathbb{R}^{2d})$ for $1 \leq q < +\infty$ as well since

$$\sup_{n\geq 0} \|\rho_\psi^n\|_{L^\infty} \leq \sup_{n\geq 0} \|f_n\|_{L^\infty} \|\psi\|_{L^1} < \infty.$$

In short we have proved

$$\rho_\psi^l \longrightarrow \int f\psi dv \tag{2.3}$$

in $L^p_{\mathrm{loc}}([0,T]\times\mathbb{R}^d, \quad \forall p \in [1,+\infty[$.

(ii) From global scale to local scale.

We consider the local functions depending on the local variables (s,y). We introduce the two new ones:

$$\overline{\rho}_\psi^n(t,x,s,y) = \int \psi(v) \overline{f}_n(t,x,s,y,v)\, dv$$
$$\overline{\eta}_\psi^n(t,x,s,y) = \int \psi(v) \overline{g}_n(t,x,s,y,v)\, dv.$$

From Lemma 2.3 and (2.3), we deduce that for every $R,T>0$, $1 \leq p < +\infty$:

$$\int_0^T \int_{B_d(0,R)} \left[\int_0^1 \int_{B_d(0,1)} \left|\overline{\rho}_\psi^n(t,x,s,y) - \rho_\psi(t,x)\right|^p \, ds\, dy\right] dx\, dt \overset{n\to+\infty}{\longrightarrow} 0.$$

From hypothesis (1.25), $\int_{\mathbb{R}^d} |{\epsilon_n}^2 H_n|^2(\cdot,\cdot,v)\, dv$ converges to 0 in $L^1(\mathbb{R}^+\times\mathbb{R}^d)$. Then Lemma 2.3 with $p=1$ implies that:

$$\int_0^T \int_{B_d(0,R)} \left[\int_{\mathbb{R}^d}\int_0^1 \int_{B_d(0,1)} \left|\overline{H}_n(t,x,s,y,v)\right|^2 \, ds\, dy\, dv\right] dx\, dt \overset{n\to+\infty}{\longrightarrow} 0,$$

where we denote $\overline{H}_n = {\epsilon_n}^2 H_n(t_{\Delta_{t,n}} + \epsilon_n s, x + \epsilon_n y, v)$. This leads to the following proposition:

Proposition 2.4 *Up to a subsequence (still denoting $\Delta_{t,n}$), there exists $\Omega \subset [0,T] \times \mathbb{R}^d$ with $\mathcal{L}([0,T] \times \mathbb{R}^d \setminus \Omega) = 0$ such that for every $(t,x) \in \Omega$:*

$$\int_0^1 \int_{B_d(0,1)} \left|\overline{\rho}_\psi^n(t,x,s,y) - \rho_\psi(t,x)\right|^p \, ds\, dy \overset{n\to+\infty}{\longrightarrow} 0$$
$$\int_{\mathbb{R}^d} \int_0^1 \int_{B_d(0,1)} \left|\overline{H}_n(t,x,s,y,v)\right|^2 \, ds\, dy\, dv \overset{n\to+\infty}{\longrightarrow} 0,$$

for every $1 \le p < +\infty$, where $\mathcal{L}$ denotes the Lebesgue measure.

From now on we fix $(t,x) \in \Omega$.

(iii) Strong convergence at the local scale.

Since the point $(t,x) \in \Omega$ is fixed, let us skip it in the notation (so we denote $\overline{f}_n(s,y,v)$ for $\overline{f}_n(t,x,s,y,v)$). We have:

Lemma 2.5 *Local functions $\overline{f}_n$ are bounded in $L^\infty([0,T] \times \mathbb{R}^{2d})$ and verify:*

$$\frac{\partial \overline{f}_n}{\partial s} + v.\nabla_y \overline{f}_n = \sum_{i\in\mathbb{Z}} \delta_{i\overline{\Delta}_n}(s) \left(\int_{i\overline{\Delta}_n}^{(i+1)\overline{\Delta}_n} \overline{H}_{\epsilon_n}(\tau,y,v)\, d\tau \right). \tag{2.4}$$

Proof of the Lemma. We just compute:

$$\begin{aligned}
\frac{\partial \overline{f}_n}{\partial s} + v.\nabla_y \overline{f}_n &= \epsilon_n \sum_{i\in\mathbb{Z}} \delta_{i\Delta_{t,n}}(t_{\Delta_{t,n}} + \epsilon_n s) \left(\int_{i\Delta_{t,n}}^{(i+1)\Delta_{t,n}} H_n(\tau, x+\epsilon_n y, v)\, d\tau \right) \\
&= \sum_{i\in\mathbb{Z}} \delta_{i\Delta_{t,n}}(\epsilon_n s) \left(\int_{i\Delta_{t,n}+t_{\Delta_{t,n}}}^{(i+1)\Delta_{t,n}+t_{\Delta_{t,n}}} \epsilon_n H_n(\tau, x+\epsilon_n y, v)\, d\tau \right) \\
&= \sum_{i\in\mathbb{Z}} \delta_{i\Delta_{t,n}}(\epsilon_n s) \left(\int_{i\overline{\Delta}_n}^{(i+1)\overline{\Delta}_n} {\epsilon_n}^2 H_n(t_{\Delta_{t,n}} + \epsilon_n\tau, x+\epsilon_n y, v)\, d\tau \right) \\
&= \sum_{i\in\mathbb{Z}} \delta_{i\overline{\Delta}_n}(s) \left(\int_{i\overline{\Delta}_n}^{(i+1)\overline{\Delta}_n} \overline{H}_{\epsilon_n}(\tau,y,v)\, d\tau \right).
\end{aligned}$$

In the second equality we do the change of indice $i \to i + t_{\Delta_{t,n}}/\Delta_{t,n}$, in the third equality we do the change of variables $\tau \to t_{\Delta_{t,n}} + \epsilon_n \tau$ and in the last equality we use the definition of $\overline{H}_{\epsilon_n}$ and the remark that $\epsilon_n s = i\Delta_{t,n}$ if and only if $s = i\overline{\Delta}_n$. □

This lemma gives the hypothesis needed to apply Theorem 2.2 (with Proposition 2.4). Therefore we conclude that:

Proposition 2.6 *The sequence $\{\overline{\eta}_\psi^n\}$ is relatively compact in $L^2_{\mathrm{loc}}([0,T] \times \mathbb{R}^d)$.*

(iv) Uniqueness of the limit at the local scale.

Proposition 2.7 *The entire sequence $\overline{\eta}_\psi^n(s,y)$ converges to ρ_ψ (which is constant with respect to (s,y)) in $L^2_{\mathrm{loc}}([0,T] \times \mathbb{R}^d)$ when n goes to $+\infty$.*

Proof of the proposition. We have

$$\begin{aligned}
f_n(t,x,v) &= f_n(t_{\Delta_{t,n}}, x + (t - t_{\Delta_{t,n}})v, v) \\
&= g_n(t, x + (t - t_{\Delta_{t,n}})v, v).
\end{aligned}$$

Notice that

$$(t_{\Delta_{t,n}} + \epsilon_n s)_{\Delta_{t,n}} = t_{\Delta_{t,n}} + \epsilon_n s_{\overline{\Delta}_n},$$

So $\overline{f}_n(s,y,v) = \overline{g}_n(s, y - \epsilon_n s_{\overline{\Delta}_n} v, v)$ and

$$\overline{\rho}_\psi^n(s,y) - \overline{\eta}_\psi^n(s,y) = \int_{\mathbb{R}^d} \psi(v) \left[\overline{g}_n(s, y - \epsilon_n s_{\overline{\Delta}_n} v, v) - \overline{g}_n(s,y,v) \right] \, dv.$$

If we consider a test function $\phi \in C_c^\infty([0,T] \times \mathbb{R}^{2d})$ we find:

$$\begin{aligned} & \int \phi(s,y) \left[\overline{\rho}_\psi^n(s,y) - \overline{\eta}_\psi^n(s,y) \right] \, ds \, dy \\ = & \int \psi(v) \overline{g}_n(s,y,v) [\phi(s,y) - \phi(s, y + \epsilon_n s_{\overline{\Delta}_n} v)] \, ds \, dy \, dv \\ \leq & \; \Delta_{t,n} \|\nabla \phi\|_{L^\infty} \, (\mathrm{Supp}\, \phi) \, \|\psi\|_{L^1} \|\overline{f}_n\|_{L^\infty} \overset{n \to \infty}{\longrightarrow} 0. \end{aligned}$$

Therefore:

$$\overline{\eta}_\psi^n - \overline{\rho}_\psi^n \overset{\mathcal{D}'}{\longrightarrow} 0.$$

Since $\overline{\rho}_\psi^n$ converges to ρ_ψ thanks to Proposition 2.4, the entire sequence $\overline{\eta}_\psi^n$ converges to ρ_ψ in the sense of distribution and we conclude gathering this information with Proposition 2.6. □

(v) Back to the global scale.

We have shown that for every $(t,x) \in \Omega$:

$$\int_0^1 \int_{B_d(0,1)} \left| \overline{\eta}_\psi^n(t,x,s,y) - \rho_\psi(t,x) \right|^2 \, ds \, dy \overset{n \to +\infty}{\longrightarrow} 0.$$

Therefore, since $\mathcal{L}([0,T] \times \mathbb{R}^d \setminus \Omega) = 0$, for every $1 \leq p < 2$:

$$\int_0^T \int_{B_d(0,R)} \int_0^1 \int_{B_d(0,1)} \left| \overline{\eta}_\psi^n(t,x,s,y) - \rho_\psi(t,x) \right|^p \, ds \, dy \, dx \, dt \overset{n \to +\infty}{\longrightarrow} 0.$$

Using Lemma 2.3 we conclude that η_ψ^n converges to ρ_ψ in $L^p_{\mathrm{loc}}([0,T] \times \mathbb{R}^d)$ which ends the proof of Therorem 1.3. □

3 Subsolution.

In this section we fix some positive real Δ_t, Δ_x, R_x, R_v, T (without dependence in n) and we prove several estimates on the sequence (f^k) defined by the discrete scheme (1.13). We treat separatly the case of the soft potential and the case of the hard potential. In all what follows we define

$$\xi(x,v) = \exp(-\alpha\,|x|^2 - \beta\,|v|^2), \quad \xi^{k\sharp}(x,v) = \exp(-\alpha\,|x + v\,k\Delta_t|^2 - \beta\,|v|^2). \tag{3.1}$$

Case 1 - Soft potential $\gamma \in (-d, 0]$. We begin with some technical lemmas that we will use in the construction of subsolutions.

Lemma 3.1 *There exists $K_1 = K_1(\alpha, \beta, \gamma)$ such that*

$$0 \leq L(\xi^{-k\sharp}) \leq \frac{\ell(k\Delta_t)}{4} \quad \forall\, x, v \in \mathbb{R}^d,\, k \leq k^* := E(\frac{T}{\Delta t}), \tag{3.2}$$

where

$$\ell(t) := \frac{K_1}{(1+t)^{d+\gamma}}.$$

Proof of Lemma 3.1. We write

$$\xi^{-k\sharp} = \exp\bigl(-|A(k\Delta_t)\,x - B(k\Delta_t)\,v|^2 - C(k\Delta_t)\,|x|^2\bigr)$$

with

$$C(t) = \frac{\alpha\,\beta}{\alpha\,t^2+\beta}, \quad B(t) = \sqrt{\alpha\,t^2+\beta} \quad \text{and} \quad A(t) = \frac{t\alpha}{\sqrt{\alpha t^2+\beta}}.$$

By a change of variables, we have

$$\begin{aligned}
(\xi^{-k\sharp} *_v |.|^\gamma)(v) &= e^{-C\,|x|^2} \int_{\mathbb{R}^d} \exp\bigl(-|A\,x - B\,(v-z)|^2\bigr)\,|z|^\gamma\,dz \\
&= \frac{e^{-C\,|x|^2}}{B^{\gamma+d}} \int_{\mathbb{R}^d} \exp\bigl(-|A\,x - B\,v - w|^2\bigr)\,|w|^\gamma\,dw \\
&= \frac{e^{-C\,|x|^2}}{B^{\gamma+d}} \bigl(e^{-|.|^2} * |.|^\gamma\bigr)\,(A\,x - B\,v).
\end{aligned}$$

We conclude that

$$\|\,L(\xi^{-k\sharp})\,\|_{L^\infty(\mathbb{R}^d_v)} \;\le\; K_0\,\frac{C_\gamma}{B^{\gamma+d}}$$

since $|.|^\gamma \in L^1(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$ and $e^{-|.|^2} \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. □

Lemma 3.2 *For any $C^0 > 0$ there exists T, $C^*, \tau > 0$ (depending on C^0, $\gamma+d$, K_1) such that the sequence $(C^k)_{k\ge1}$ recursively defined by*

$$C^{k+1} = (1+\Delta_t\,\tau)\,C^k + \Delta_t\,(C^k)^2\,\ell((k+1)\,\Delta_t) \tag{3.3}$$

satisfies

$$0 \le C^k \le C^* \quad \textit{for any } k = 0, 1, ..., k^*. \tag{3.4}$$

Moreover, there exists $\bar{C} = \bar{C}(d+\gamma, K_1) \in (0,1)$ such that if $\gamma \in (-d+1, 0]$ and $C^0 \in (0, \bar{C})$, then, for any $T > 0$, setting

$$\tau = \frac{K_2}{(1+T)^{\gamma+d}}, \quad K_2 = K_2(K_1, d, \gamma) > 0, \tag{3.5}$$

the sequence (C^k) satisfies (3.4) with $C^* = 1$.

Remark 3.3 *We define the interval $(0, T^*)$ on which a uniform bound on (C^k) is obtained thanks to Lemma 3.2, setting $T^* = T$ in the first case (arbitrary C^0) and setting $T^* = +\infty$ in the second case (C^0 small enough and $\gamma \in (-d+1, 0]$). In both cases, for any $T \in (0, T^*)$ we may define the parameter τ by (3.5).*

Proof of Lemma 3.2. Noticing that if, for some $C^* > 0$, we have

$$C^j \le C^* \qquad \text{for any} \quad j = 0, ..., k, \tag{3.6}$$

then $C^{j+1} \le (1 + (\tau + C^*\,\ell((j+1)\,\Delta_t))\,\Delta_t)\,C^j$ for any $j = 0, ..., k$. We deduce that

$$\begin{aligned}
C^{k+1} &\le \prod_{j=1}^{k} (1 + (\tau + C^*\,\ell((j+1)\,\Delta_t))\,\Delta_t)\,C^0 \\
&\le \exp\left(\int_0^{k\,\Delta_t} (\tau + C^*\,\ell(s))\,ds\right)\,C^0 \\
&\le \exp\left(\tau\,T + C^* \int_0^T \ell(s))\,ds\right)\,C^0, \quad \text{if } \; k\,\Delta_t \le T.
\end{aligned} \tag{3.7}$$

Consider first the general case and take $C^0 > 0$ arbitrary. Let choose $C^* > 4\,C^0$ and $T > 0$ (small enough) so that

$$\exp\left(C^* \int_0^T \ell(s)\,ds\right) \le \frac{C^*}{2\,C_0}.$$

Then for $\tau = \ln 2/T$ and $k \le k^*$ we deduce from (3.7) that $C^{k+1} \le C^*$. Thus, by induction, (3.4) holds.

Now, consider the case $\gamma \in (-d+1, 0]$ (so that $\ell \in L^1(\mathbb{R}_+)$). We remark that if

$$C^* = 1, \quad C^0 \le \bar{C} := \exp\left(-\int_0^\infty \ell(s)\,ds\right),$$

and since, for an appropriate choice of K_2 (small enough),

$$\tau \le \frac{1}{T}\int_T^\infty \ell(s)\,ds,$$

then we also have $C^{k+1} \le C^*$ and we conclude again by induction. □

Lemma 3.4 *There is $K_3 = K_3(\alpha)$ such that for any $\theta \in]0,1[$ the condition*

$$\Delta_x\,(R_x + T\,R_v) \le K_3\,\theta \tag{3.8}$$

implies

$$|P\,\xi^{-k\sharp} - \xi^{-k\sharp}| \le \theta\,\xi^{-k\sharp} \quad \forall\ x \in B_{R_x},\ v \in B_{R_v}\ k \le k^*, \tag{3.9}$$

and

$$P((L_{R_v}\xi^{j\sharp})^{k\sharp}) \le (1+\theta)\,(L_{R_v}\xi^{j\sharp})^{k\sharp} \quad \forall\ x \in B_{R_x},\, v \in B_{R_v},\, k, j+k \le k^*. \tag{3.10}$$

Proof of Lemma 3.4. Let $x \in \Lambda_a$ and write

$$(P\xi^{-k\sharp} - \xi^{-k\sharp})(x,v) = \frac{\xi^{-k\sharp}}{|\Lambda_a|}\int_{\Lambda_a - x}[\exp(\alpha\,|x - k\Delta_t\,v|^2 - \alpha\,|z + (x - k\Delta_t\,v)|^2) - 1]\,dz.$$

Since $\Lambda_a - x \subset [-\Delta_x, \Delta_x]^d$ by definition of Λ_a and x, one has on $B_{R_x} \times B_{R_v}$, taking $\Delta_x \le 1$,

$$\begin{aligned}
&|\exp(\alpha\,|x - k\Delta_t\,v|^2 - \alpha\,|z + (x - k\Delta_t\,v)|^2) - 1| \\
&\le |\exp(4\,\alpha\,(R_x + T\,R_v)\,\Delta_x) - 1| \\
&\le 4\,\alpha\,(R_x + T\,R_v)\,\Delta_x \exp(4\,\alpha\,(R_x + T\,R_v)\,\Delta_x).
\end{aligned}$$

The inequality (3.9) follows taking for instance $K_3^{-1} := 4\,\alpha\,e^{4\,\alpha}$.

In order to abreviate the notation we put $\phi := (L_{R_v}\xi^{j\sharp})^{k\sharp}$. Let $x \in \Lambda_a$ and write

$$(P\phi - \phi)(x,v) = \frac{1}{|\Lambda_a|}\int_{\mathbb{R}^d}\int_{\Lambda_a - x}\xi(x + \Delta_t(jw + kv), w) K(x,z,v,w)\,A_{R_v}(v-w)\,dwdz,$$

where, similarly,

$$\begin{aligned}
&K(x,z,v,w) = \\
&= |\exp(\alpha\,|x + \Delta_t\,(j\,v + k\,w)|^2 - \alpha\,|z + (x + \Delta_t\,(j\,v + k\,w))|^2) - 1| \\
&\le 4\,\alpha\,(R_x + 2\,T\,R_v)\,\Delta_x \exp(4\,\alpha\,(R_x + 2\,T\,R_v)\,\Delta_x).
\end{aligned}$$

Hence

$$\begin{aligned}
(P\phi - \phi)(x,v) &\le \theta\int_{\mathbb{R}^d}\xi(x + \Delta_t(jw + kv), w)\,A_{R_v}(v-w)\,dw \\
&= \theta\phi(x,v).
\end{aligned}$$

□

Proposition 3.5 *For any $C^0 > 0$ there exists $T^* > 0$ (defined in Remark 3.3), $K_4 > 0$ such that, for any choice of discretisation parameters $\Delta_t, \Delta_x, R_x, R_v, T$ satisfying*

$$T < T^*, \quad T\,R_v \le R_x/4, \quad \Delta_t \le K_4 \tag{3.11}$$

$$\Delta_x\,(R_x + T\,R_v)\,T^{\gamma+d} \le K_4\,\Delta_t, \tag{3.12}$$

the sequence $(M^k)_{k\in\mathbb{N}}$ defined by

$$M^k(x,v) = C^k\,\xi^{-k\sharp}, \qquad k = 0,...,k^* = E(T/\Delta_t), \tag{3.13}$$

with $(C^k)_{k\in\mathbb{N}}$ given by Lemma 3.2, satifies

$$M^{k+1} \ge P\,M^{k-\sharp} + \Delta_t\,Q^+(P\,M^{k-\sharp}, P\,M^{k-\sharp}) \quad on \quad \{1,..,k_*\} \times B_{3\,R_x/4} \times B_{R_v}. \tag{3.14}$$

Moreover, if

$$0 \le f_{in} \le M^0 \tag{3.15}$$

then

$$0 \le f^k \le M^k \quad on\ \{1,..,k_*\} \times B_{R_x/2} \times B_{R_v}. \tag{3.16}$$

Proof of Proposition 3.5. Fix k, x, v so that $k\,\Delta_t \le T$, $x \in B_{3\,R_x/4}$ and $v \in B_{R_v}$. Since $|x + \Delta_t\,v| \le R_x$, one has, according to lemma 3.4, (1.3), (1.4) and the fact that Q^+ is a positive operator

$$\begin{aligned} Q^+(P\,M^{k-\sharp}, P\,M^{k-\sharp}) &= (C^k)^2\,Q^+(P\,(\xi^{-k\sharp}), P(\xi^{-k\sharp})) \\ &\le (1+\theta)^2\,(C^k)^2\,Q^+(\xi^{-(k+1)\sharp}, \xi^{-(k+1)\sharp}) \\ &\le (1+\theta)^2\,(C^k)^2\,L(\xi^{-(k+1)\sharp})\,\xi^{-(k+1)\sharp}. \end{aligned}$$

Since if K_4 is chosen small enough then $\theta = \tau\,\Delta_t \le 1$, we infer from (3.2) and (3.3) that

$$\begin{aligned} PM^{k-\sharp} + \Delta_t Q^+(PM^{k-\sharp}, PM^{k-\sharp}) &\le ((1+\tau\Delta_t)C^k + \Delta_t(C^k)^2\ell((k+1)\,\Delta_t))\,\xi^{-(k+1)\sharp} \\ &\le C^{k+1}\,\xi^{-(k+1)\sharp} = M^{k+1}, \end{aligned}$$

and (3.14) holds. Let us now assert that

$$0 \le f^k \le M^k \text{ on } B_{3\,R_x/4 - k\,\Delta_t\,R_v} \times B_{R_v}. \tag{3.17}$$

According to (3.15) it is obviously true for $k = 0$. Assume it is true for some k. Then $f^{k\sharp} \le M^{k\sharp}$ on $B_{3\,R_x/4-(k+1)\,\Delta_t\,R_v} \times B_{R_v}$ and therefore by definitions of f^{k+1} and M^{k+1} we also have

$$\begin{aligned} f^{k+1} &\le P(f^{k^{-\sharp}}) + \Delta_t\,Q^+(P(f^{k-\sharp}), P(f^{k-\sharp})) \\ &\le P(M^{k^{-\sharp}}) + \Delta_t\,Q^+(P(M^{k-\sharp}), P(M^{k-\sharp})) \le M^{k+1} \end{aligned}$$

on $B_{3\,R_x/4-(k+1)\,\Delta_t\,R_v} \times B_{R_v}$. Moreover, we have from (3.15)

$$\begin{aligned} f^{k+1} &\ge P(f^{k^{-\sharp}}) - \Delta_t\,P(f^{k-\sharp})\,L(P(f^{k-\sharp})) \\ &\ge P(f^{k^{-\sharp}}) - \Delta_t\,P(f^{k-\sharp})\,L(P(M^{k-\sharp})) \\ &\ge P(f^{k^{-\sharp}})[1 - \Delta_t\,C^k\,\ell((k+1)\,\Delta_t)], \end{aligned}$$

and the last term is non negative with a conveniant choice of K_4 (for instance $K_4 \le (C^*\,\ell(0))^{-1}$). Then (3.17) follows by induction and (3.16) is proved. □

Case 2 - Hard potential $\gamma \in (0, 1]$.

Lemma 3.6 *There exists $K_5 = K_5(\gamma, d, \beta, K_0)$ such that under the condition*

$$\Delta_t \, R_v \le \alpha^{-1/2} \tag{3.18}$$

there holds

$$\sum_{k\in\mathbb{Z}} \Delta_t \, (L_{R_v} \, \xi^{-k\sharp})^{k\sharp} \le \frac{K_5}{\sqrt{\alpha}} \qquad \forall x, v \in \mathbb{R}^d. \tag{3.19}$$

Proof of Lemma 3.6. One has

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \Delta_t \, (L_{R_v} \xi^{-k\sharp})^{k\sharp} &= \sum_{k\in\mathbb{Z}} \Delta_t \int_{\mathbb{R}^3} \xi^{-k\sharp}(x + v \, \Delta_t \, k, w) \, A(v-w) \, \mathbf{1}_{|v-w|\le R_v} \, dw \\ &= \Delta_t \int_{\mathbb{R}^3} \sum_{k\in\mathbb{Z}} \xi(x + (v-w) \, \Delta_t \, k, w) \, A(v-w) \, \mathbf{1}_{|v-w|\le R_v} \, dw.\end{aligned}$$

Since

$$\Delta_t \sum_{k\in\mathbb{Z}} e^{-\alpha \, |x+(v-w) \, \Delta_t \, k|^2} \le \Delta_t + \int_{\mathbb{R}} e^{-\alpha \, |x+(v-w) \, t|^2} \, dt \le \Delta_t + \frac{\sqrt{\pi}}{\sqrt{\alpha} \, |v-w|},$$

we get

$$\begin{aligned}\sum_{k\in\mathbb{Z}} \Delta_t \, (L_{R_v}(\xi^k))^{k\sharp} &\le \int_{\mathbb{R}^3} A_R(v-w) \, \Delta_t \, e^{-\beta|w|^2} \, dw + \frac{\sqrt{\pi}}{\sqrt{\alpha}} \int_{\mathbb{R}^3} |v-w|^{\gamma-1} e^{-\beta|w|^2} \, dw \\ &\le \Delta_t \, K_0 \, R_v^\gamma \, \|e^{-\beta \, |.|^2}\|_{L^1} + \frac{\sqrt{\pi}}{\sqrt{\alpha}} \, K_0 \, \| \, |.|^{\gamma-1} * e^{-\beta \, |.|^2} \|_{L^\infty}.\end{aligned}$$

We deduce (3.19) thanks to (3.18) and that $|.|^{\gamma-1} \in L^1 + L^\infty$, $e^{-\beta \, |.|^2} \in L^1 \cap L^\infty$. □

Proposition 3.7 *There exists $\alpha > 0$ and $K_6 > 0$ (depending of β, K_0, γ, d) such that for any choice of the discretisation parameters $\Delta_t, \Delta_x, R_x, R_v, T$ satisfying*

$$T \, R_v \le R_x/4, \quad R_v^{d+\gamma} \, \Delta_t \le K_6, \tag{3.20}$$

$$\Delta_x \, (R_x + T \, R_v) \, T \le K_6 \, \Delta_t, \tag{3.21}$$

the sequence $(M_k)_{k\in\mathbb{N}}$ defined by

$$M^0 = \frac{\xi}{4}, \qquad M^k = \xi^{-k\sharp} \quad k \ge 1,$$

satisfies

$$M^k \ge \pi^k \, M^0 + \sum_{j=0}^{k-1} \Delta_t \, \pi^{k-1-j} \, Q^+_{R_v}(\pi M^j, \pi M^j) \ \text{on} \ \{0, .., k_*\} \times B_{R_x/3} \times B_{R_v}, \tag{3.22}$$

where we have introduced the notation $\pi \, \phi = P(\phi^{-\sharp})$ and again $k^* = E(T/\Delta_t)$. Moreover, if f_{in} satisfies (3.15) then estimate (3.16) holds.

Proof of Proposition 3.7. We fix $x \in B_{2 \, R_x/3}$, $v \in B_{R_v}$, $0 \le j \le k \le k^*$ and we define $\theta = \Delta_t \, (\ln\sqrt{2})/T$. By repeating use of Lemma 3.4 we get

$$\begin{aligned}\pi^{k-1-j} \, Q^+_{R_v}(\pi M^j, \pi M^j) &\le (1+\theta)^2 \, (\pi^{k-1-j} \, M^j) \, (\pi^{k-1-j} \, L \, M^j) \\ &\le (1+\theta)^{2\,(k-j)} \, (M^j)^{-(k-1-j)\sharp} \, (L \, M^j)^{-(k-1-j)\sharp}.\end{aligned}$$

Therefore, we deduce from Lemma 3.6 and condition (3.21) that

$$
\begin{aligned}
\left(\pi^k M^0 + \sum_{j=0}^{k-1} \Delta_t \, \pi^{k-1-j} \, Q^+_{R_v}(\pi M^j, \pi M^j)\right)^{k\sharp} &\leq \\
&\leq (1+\theta)^{2k} \left(M^{0\,-k\sharp} + \sum_{j=0}^{k-1} \Delta_t \, Q^+(M^{j-\sharp}, M^{j-\sharp})^{-(k-1-j)\sharp}\right)^{k\sharp} \\
&\leq e^{2k\theta} \left(M^0 + \sum_{j=0}^{k-1} \Delta_t \, L(M^{j-\sharp})^{(j+1)\sharp} \, (M^j)^{j\sharp}\right) \\
&\leq 2\left(\frac{1}{4} + \frac{K_5}{\sqrt{\alpha}}\right) \xi \leq (M^k)^{k\sharp},
\end{aligned}
$$

with $\alpha = (8\,K_5)^2$, and (3.22) holds.

We now assert that for any $k \leq k^*$

$$
0 \leq f^k \leq M^k \text{ on } B_{3R_x/4-(T-k\,\Delta_t)\,R_v} \times B_{R_v}. \tag{3.23}
$$

It is of course true at the rank $k = 0$ by assumption. Assume it is true at the rank $k-1$ and remarking that f^k may be writen

$$
f^k = \pi^k \, f^0 + \sum_{j=0}^{k-1} \Delta_t \, \pi^{k-1-j} \, Q_{R_v}(\pi f^j, \pi f^j),
$$

we deduce thanks to (3.22) that (3.23) holds at rank k, and we conclude by induction. Finally, we have

$$
\begin{aligned}
f^{k+1} &\geq \pi\, f^k - \Delta_t\,(\pi\, f^k)\,(L_{R_v}\pi\, f^k) \\
&\geq \pi\, f^k\,(1 - \Delta_t\,(L_{R_v}1)],
\end{aligned}
$$

and the last term is nonnegative thanks to the condition (3.20). $\square$

4 Proof of Theorem 1.1

Let us consider sequences of real numbers $\Delta_{t,n}, \Delta_{x,n}, T_n, R_{x,n}, R_{v,n}$ such that (1.16) holds and

$$
T_n\,R_{v,n} \leq R_{x,n}/4, \quad \frac{(\Delta_{x,n})^s}{\Delta_{t,n}} \leq 1, \quad \Delta_{t,n}^{1/s-1}\,T_n^d\,R_{x,n}^{d+2} \to 0, \tag{4.1}
$$

with $s \in (1/2, 1)$. For example, in dimension $d = 3$, we may take $\Delta_{t,n} = n^{-1}$, $\Delta_{x,n} = n^{-2}$, $s = 3/4$ and $R_{x,n} = n^{1/21}$, $R_{v,n} = T_n = n^{1/42}/2$.

Lemma 4.1 *For every $T \in (0, T^*)$ there exist A_T, α_T, β_T such that:*

$$
0 \leq f_n,\, g_n,\, Q^{\pm}_{R_v}(P_n g_n, P_n g_n) \leq \xi_T = A_T \exp(-\alpha_T |x|^2 - \beta_T |v|^2). \tag{4.2}
$$

Proof of the lemma. We easily check that (1.16), (4.1) imply that the conditions (3.11), (3.12) and (3.20), (3.21) hold. Therefore, according to Propositions 3.5 and 3.7, there exists $T^* \in (0, +\infty]$ as stated in Theorem 1.1 such that for every $T < T*$ there exists $C_T > 0$ such that:

$$
0 \leq f_n \leq M_n \leq C_T\,\xi \qquad \text{on} \quad [0,T] \times B_{R_{x,n}/2} \times B_{R_{v,n}}, \tag{4.3}
$$

where M_n is defined from M_n^k thanks to formula (1.14). Since, by construction (see (1.12)), $f_n(t,x,v) = 0$ on $[0,T] \times (B_{R_{x,n}/2})^c \times \mathbb{R}^d$, the same holds on $[0,T] \times \mathbb{R}^d \times \mathbb{R}^d$ and that proves that (f_n), (g_n) and (M_n) satisfy (4.2).

Moreover, we have

$$A *_v \xi(v) \leq K_0\, (|.|^\gamma * \xi)(v) \leq K_0'\, (1 + |v|),$$

and therefore $\xi\, L(\xi)$ satisfies (4.2). Finally, since $P_n\, g_n \leq P_n\, M_n \leq 2\, M_n$, we have

$$Q^{\pm}_{R_v}(P_n g_n, P_n g_n) \leq 4\, M_n\, L(M_n) \tag{4.4}$$

and we conclude gathering (4.4) with the bound (4.2) on (M_n). □

So, there exists a function $f \in L^\infty([0,T] \times \mathbb{R}^{2d})$, $0 \leq f \leq \xi_T$, such that, up to a subsequence (still denoted f_n): f_n converges weakly to f in $L^\infty *$.

Let then show that (f_n) satisfies (1.20). Indeed, with the notation $t_k = k\, \Delta_{t,n}$, we have

$$\begin{aligned}
\frac{\partial f_n}{\partial t} + v.\nabla f_n &= \sum_{k \geq 0} \delta_{t_k}\, \left(f_n(t_k^+) - f_n(t_k^-)\right) \\
&= \sum_{k \geq 0} \delta_{t_k}\, \left(f_n^k - (f_n^{k-1})^{-\sharp}\right) \\
&= \sum_{k \geq 0} \delta_{t_k}\, \left(P_n\left((f_n^{k-1})^{-\sharp}\right) - (f_n^{k-1})^{-\sharp} + \Delta_{t,n}\, P_n\, Q((f_n^{k-1})^{-\sharp}, (f_n^{k-1})^{-\sharp})\right),
\end{aligned}$$

and (1.20) follows remarking that $g_n(t) = (f_n^{k-1})^{-\sharp}$ for $t \in [t_k, t_{k+1})$.

We split the right-hand side term of (1.20) in the following way:

$$\frac{\partial f_n}{\partial t} + v.\nabla f_n = J_1^n + J_2^n,$$

with

$$\begin{aligned}
J_1^n &= \sum_k \delta_{k\Delta_{t,n}}(t) \int_{k\Delta_{t,n}}^{(k+1)\Delta_{t,n}} \frac{P_n g_n - g_n}{\Delta_{t,n}}\, d\tau \\
J_2^n &= \sum_k \delta_{k\Delta_{t,n}}(t) \int_{k\Delta_{t,n}}^{(k+1)\Delta_{t,n}} Q_{R_{v,n}}(P_n g_n, P_n g_n)\, d\tau.
\end{aligned}$$

Let us show the following lemma:

Lemma 4.2 *We have for every $p < \frac{2}{1+s}$:*

$$\|J_1^n\|_{W^{-1,p}([0,T] \times \mathbb{R}^{2d})} \overset{n \to +\infty}{\longrightarrow} 0.$$

Proof of Lemma 4.2: For every $\Phi_1 \in \mathcal{D}([0,T])$, $\Phi_2 \in \mathcal{D}(\mathbb{R}^d)$, $\Phi_3 \in \mathcal{D}(\mathbb{R}^d)$, we have:

$$\begin{aligned}
&\left| \int J_1^n(t,x,v) \Phi_1(t) \Phi_2(x) \Phi_3(v)\, dv\, dx\, dt \right| \\
&= \left| \sum_k \Phi_1(k\Delta_{t,n}) \int \int_{k\Delta_{t,n}}^{(k+1)\Delta_{t,n}} \left[\int \Phi_2(x) \frac{P_n g_n - g_n}{\Delta_{t,n}}\, dx \right] d\tau\, \Phi_3(v)\, dv \right| \\
&= \left| \sum_k \Phi_1(k\Delta_{t,n}) \int \int_{k\Delta_{t,n}}^{(k+1)\Delta_{t,n}} \left[\int \frac{P_n \Phi_2(x) - \Phi_2(x)}{\Delta_{t,n}} g_n\, dx \right] d\tau\, \Phi_3(v)\, dv \right| \\
&\leq \|\Phi_1\|_{L^\infty} \|g_n\|_{L^1([0,T], L^2(\mathbb{R}^{2d}))} \left\| \frac{P_n \Phi_2 - \Phi_2}{\Delta_{t,n}} \right\|_{L^2(\mathbb{R}^d)} \|\Phi_3\|_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

Thanks to (4.1):
$$\left\|\frac{P_n\Phi_2-\Phi_2}{\Delta_{t,n}}\right\|_{L^2(\mathbb{R}^d)}\leq\left\|\frac{P_n\Phi_2-\Phi_2}{\Delta_{x,n}^{\frac{1+s}{2}}}\right\|_{L^2(\mathbb{R}^d)}(\Delta_{x,n})^{\frac{1-s}{2}},$$
and:
$$\left\|\frac{P_n\Phi_2-\Phi_2}{\Delta_{x,n}^{\frac{1+s}{2}}}\right\|^2_{L^2(\mathbb{R}^d)}=\sum_{a\in\mathbb{Z}^d}\frac{1}{\Delta_{x,n}^{1+s}}\int_{\Lambda_a}(P_n\Phi_2-\Phi_2)^2\,dx.$$

$$\begin{aligned}\sum_{a\in\mathbb{Z}^d}\frac{1}{\Delta_{x,n}^{1+s}}\int_{\Lambda_a}(P_n\,\Phi_2-\Phi_2)^2\,dx &\leq \sum_{\alpha}\frac{1}{\Delta_{x,n}^{d+1+s}}\iint_{\Lambda_a\times\Lambda_a}(\Phi_2(x)-\Phi_2(y))^2\,dxdy\\ &\leq \sum_{\alpha}\int_{\Lambda_a}\int_{\mathbb{R}^d}\frac{(\Phi_2(x)-\Phi_2(y))^2}{|y-x|^{1+s+d}}\,dxdy\\ &\leq \int_{\mathbb{R}^{2d}}\frac{(\Phi_2(x)-\Phi_2(y))^2}{|y-x|^{1+s+d}}\,dxdy=\|\Phi_2\|^2_{H^{\frac{1+s}{2}}(\mathbb{R}^d)}.\end{aligned}$$

Finally:
$$\begin{aligned}&\left|\int J_1^n(t,x,v)\Phi_1(t)\Phi_2(x)\Phi_3(v)\,dv\,dx\,dt\right|\\ \leq\;& \|\Phi_1\|_{L^\infty([0,T])}\|g_n\|_{L^1([0,T],L^2(\mathbb{R}^{2d}))}\|\Phi_3\|_{L^2(\mathbb{R}^d)}\|\Phi_2\|_{H^{\frac{1+s}{2}}(\mathbb{R}^d)}\Delta_{x,n}^{\frac{1-s}{2}}\\ \leq\;& C\|\Phi_1\|_{W^{\frac{1-s}{2},p}}\|\Phi_3\|_{L^2(\mathbb{R}^d)}\|\Phi_2\|_{H^{\frac{1+s}{2}}(\mathbb{R}^d)}\Delta_{x,n}^{\frac{1-s}{2}}\\ \leq\;& C\Delta_{x,n}^{\frac{1-s}{2}}\|\Phi_1\Phi_2\Phi_3\|_{W^{1,p}([0,T]\times\mathbb{R}^{2d})}\overset{n\to+\infty}{\longrightarrow}0,\end{aligned}$$
for $p<\frac{2}{1+s}$, since $\frac{1-s}{2}>0$. Therefore:
$$\|J_1^n\|_{W^{-1,p}([0,T]\times\mathbb{R}^{2d})}\overset{n\to+\infty}{\longrightarrow}0.$$

□

We denote $\mathcal{M}$ the set of bounded measures on $[0,T]\times\mathbb{R}^{2d}$. We have:
$$\begin{aligned}\|J_2^n\|_{\mathcal{M}} &= \sum_k\left|\int_{k\Delta_{t,n}}^{(k+1)\Delta_{t,n}}Q_{R_{v,n}}(P_ng_n,P_ng_n)\,d\tau\right|\,dx\,dv\\ &\leq \|Q_{R_{v,n}}(P_ng_n,P_ng_n)\|_{L^1([0,T]\times\mathbb{R}^{2d})}\\ &\leq C,\end{aligned}$$
thanks to Lemma 4.1. The set of bounded measures is compactly embedded in $W^{-1,p}([0,T]\times\mathbb{R}^{2d})$ for $p<\frac{2d+1}{2d}$. Therefore, with Lemma 4.2 we conclude that $J^n=J_1^n+J_2^n$ belongs to a compact subset of $W^{-1,p}([0,T]\times\mathbb{R}^{2d})$ for $p<\inf(\frac{2d+1}{2d},\frac{2}{1+s})$. Equation (1.20) can be written in the form (1.22) with:
$$H_n(\tau,x,v)=\frac{P_ng_n-g_n}{\Delta_{t,n}}+Q_{R_{v,n}}(P_ng_n,P_ng_n).$$
For $\varepsilon_n=(\Delta_{t,n})^\alpha$, $\alpha\in(1/2,1)$ one has
$$\|\varepsilon_n^2\,H_n\|_{L^2}\leq 2\,\frac{\varepsilon_n^2}{\Delta_{t,n}}\,\|f_n\|_{L^2}+\varepsilon_n^2\,\|Q_{R_{v,n}}(P_ng_n,P_ng_n)\|_{L^2}\to 0,$$
thanks to Lemma 4.1. Gathering the above estimates we conclude that the conditions of Theorem 1.3 hold.

One may apply the compactness Theorem 1.3 and we get that for any $\psi \in \mathcal{D}(\mathbb{R}^d)$ $\int_{\mathbb{R}^d} g_n\psi(v)\,dv$ converges in $L^2_{\text{loc}}([0,T]\times\mathbb{R}^d)$ to $\int_{\mathbb{R}^d} f\psi(v)\,dv$. Since $\int \xi|\psi(v)|\,dv \in L^2([0,T]\times\mathbb{R}^d)$, from (4.3) and Lebesgue's theorem we find:

$$\int_{\mathbb{R}^d} g_n\,\psi(v)\,dv \;\longrightarrow\; \int_{\mathbb{R}^d} f\psi(v)\,dv \text{ in } L^2((0,T)\times\mathbb{R}^d) \tag{4.5}$$

for every $\psi \in \mathcal{D}(\mathbb{R}^d)$. The linear operator P_n on $L^2([0,T]\times\mathbb{R}^d)$ verifies:

$$\begin{aligned}
&\|P_n\|_{\mathcal{L}(L^2,L^2)} \le 1\\
&\|P_n\Phi - \Phi\|_{L^2} \overset{n\to+\infty}{\longrightarrow} 0 \text{ for every } \Phi \in L^2([0,T]\times\mathbb{R}^d).
\end{aligned}$$

Notice that:

$$P_n\left(\int_{\mathbb{R}^d} g_n\psi(v)\,dv\right) = \int_{\mathbb{R}^d} (P_n g_n)\psi(v)\,dv.$$

Therefore:

$$\begin{aligned}
&\left\|\int_{R^d} P_n g_n\psi(v)\,dv - \int f\psi(v)\,dv\right\|_{L^2([0,T]\times\mathbb{R}^d)}\\
\le\ &\left\|P_n\left(\int_{R^d} g_n\psi(v)\,dv - \int f\psi(v)\,dv\right)\right\|_{L^2} + \left\|(P_n - I)\int f\psi(v)\,dv\right\|_{L^2}\\
\le\ &\left\|\int_{\mathbb{R}^d} g_n\psi(v)\,dv - \int f\psi(v)\,dv\right\|_{L^2} + \left\|(P_n - I)\int f\psi(v)\,dv\right\|_{L^2},
\end{aligned}$$

which converges to 0 when n goes to $+\infty$. And so:

$$\int_{\mathbb{R}^d} P_n\,g_n\,\psi\,dv \;\to\; \int_{\mathbb{R}^d} f\,\psi\,dv, \tag{4.6}$$

strongly in L^2. In particular:

$$P_n g_n \rightharpoonup f \text{ in } L^2 \text{ w.} \tag{4.7}$$

By standard argument, see for instance [5], we deduce from (4.7) and (4.6) that

$$Q_{R_{v,n}}(P_n g_n, P_n g_n) \rightharpoonup Q(f,f) \quad \text{in} \quad L^1. \tag{4.8}$$

For every test function $\Phi_1 \in \mathcal{D}([0,T])$, $\Phi_2 \in \mathcal{D}(\mathbb{R}^{2d})$:

$$\begin{aligned}
&\int J_2^n(t,x,v)\Phi_1(t)\Phi_2(x,v)\,dx\,dt\,dv\\
=\ &\sum_k \Phi_1(k\Delta_{t,n})\int_{\mathbb{R}^d}\int_{k\Delta_{t,n}}^{(k+1)\Delta_{t,n}} Q_{R_{v,n}}(P_n g_n, P_n g_n)\Phi_2(x,v)\,d\tau\,dx\,dv\\
=\ &\int \Phi_1(t)Q_{R_{v,n}}(P_n g_n, P_n g_n)\Phi_2(x,v)\,dt\,dx\,dv\\
&+\int [\Phi_1(t) - \Phi_1(t_{\Delta_{t,n}})]Q_{R_{v,n}}(P_n g_n, P_n g_n)\Phi_2(x,v)\,dt\,dx\,dv.
\end{aligned}$$

The first term converges to:

$$\int Q(f,f)\Phi_1(t)\Phi_2(x,v)\,dt\,dx\,dv,$$

and the second term converges to 0 since it is smaller than:

$$\|Q_{R_{v,n}}(P_n g_n, P_n g_n)\|_{L^1}\|\Phi_2\|_{L^\infty}\text{Sup}|\Phi_1(t) - \Phi_1(t_{\Delta_{t,n}})| \overset{n\to+\infty}{\longrightarrow} 0.$$

Hence, thanks to Lemma 4.2, $\int J^n(t,x,v)\Phi_1(t)\Phi_2(x)\Phi_3(v)\,dt\,dx\,dv$ converges to:

$$\int Q(f,f)\Phi_1(t)\Phi_2(x)\Phi_3(v)\,dt\,dx\,dv,$$

and we conclude that f is a solution of (1.1) in the distributional sense. This concludes the proof of Theorem 1.1. $\square$

A Proof of Lemma 2.3

We introduce the linear operator L_n defined from $L^p_{\text{loc}}([0,T]\times\mathbb{R}^d)$ to $L^p_{\text{loc}}([0,T]\times\mathbb{R}^d\times[0,1]\times\mathbb{R}^d)$ by:

$$L_n\rho(t,x,s,y) = \rho(t_{\Delta_{t,n}} + \epsilon_n s, x+\epsilon_n y).$$

For every fixed $R>0$ we denote (to simplify the notation):

$$\begin{aligned}
\|\rho\|^p_{L^p_{\text{loc}}} &= \int_0^T \int_{B_d(0,R)} |\rho(t,x)|^p\,dx\,dt \\
\|L_n\rho\|^p_{L^p_{\text{loc}}} &= \int_0^T \int_{B_d(0,R)} \int_0^1 \int_{B_d(0,1)} |\rho(t,x,s,y)|^p\,dy\,ds\,dx\,dt.
\end{aligned}$$

Using Fubini's theorem we show that for every $\rho \in L^p_{\text{loc}}([0,T]\times\mathbb{R}^d)$:

$$\begin{aligned}
\|L_n\rho\|^p_{L^p_{\text{loc}}} &= \int_0^T \int_0^1 \int_{B_d(0,1)} \left(\int_{B_d(0,R)} |\rho(t_{\Delta_{t,n}}+\epsilon_n s, x)|^p\,dx \right) dy\,ds\,dt \\
&= |B_d(0,1)| \int_0^T \int_0^1 \int_{B_d(0,R)} |\rho(t_{\Delta_{t,n}}+\epsilon_n s, x)|^p\,dx\,ds\,dt \\
&= |B_d(0,1)| \int_{B_d(0,R)} \left[\int_0^T \int_0^1 |\rho(t_{\Delta_{t,n}}+\epsilon_n s, x)|^p\,ds\,dt \right] dx.
\end{aligned}$$

But:

$$\begin{aligned}
\int_0^T \int_0^1 |\rho(t_{\Delta_{t,n}}+\epsilon_n s, x)|^p\,ds\,dt &= \sum_{0\le i\Delta_{t,n}\le T} \Delta_{t,n} \int_0^1 |\rho(i\Delta_{t,n}+\epsilon_n s, x)|^p\,ds \\
&= \sum_{0\le i\Delta_{t,n}\le T} \frac{\Delta_{t,n}}{\epsilon_n} \int_{i\Delta_{t,n}}^{i\Delta_{t,n}+\epsilon_n} |\rho(\tau,x)|^p\,d\tau.
\end{aligned}$$

If we denote $\alpha_n = E(\epsilon_n/\Delta_{t,n})$ we have:

$$\begin{aligned}
\sum_{0\le i\Delta_{t,n}\le T} \frac{\Delta_{t,n}}{\epsilon_n} \int_{i\Delta_{t,n}}^{i\Delta_{t,n}+\epsilon_n} |\rho(\tau,x)|^p\,d\tau &\le \sum_{0\le i\Delta_{t,n}\le T} \frac{\Delta_{t,n}}{\epsilon_n} \int_{i\Delta_{t,n}}^{(i+\alpha_n+1)\Delta_{t,n}} |\rho(\tau,x)|^p\,d\tau \\
\sum_{0\le i\Delta_{t,n}\le T} \frac{\Delta_{t,n}}{\epsilon_n} \int_{i\Delta_{t,n}}^{i\Delta_{t,n}+\epsilon_n} |\rho(\tau,x)|^p\,d\tau &\ge \sum_{0\le i\Delta_{t,n}\le T} \frac{\Delta_{t,n}}{\epsilon_n} \int_{i\Delta_{t,n}}^{(i+\alpha_n)\Delta_{t,n}} |\rho(\tau,x)|^p\,d\tau.
\end{aligned}$$

Therefore:

$$\begin{aligned}
\sum_{0\le i\Delta_{t,n}\le T} \frac{\Delta_{t,n}}{\epsilon_n} \int_{i\Delta_{t,n}}^{i\Delta_{t,n}+\epsilon_n} |\rho(\tau,x)|^p\,d\tau &\le (\alpha_n+1)\frac{\Delta_{t,n}}{\epsilon_n} \int_0^T |\rho(\tau,x)|^p\,d\tau, \\
&\le \left(1+\frac{\Delta_{t,n}}{\epsilon_n}\right) \int_0^T |\rho(\tau,x)|^p\,d\tau,
\end{aligned}$$

and:

$$\begin{aligned}\sum_{0\le i\Delta_{t,n}\le T}\frac{\Delta_{t,n}}{\epsilon_n}\int_{i\Delta_{t,n}}^{i\Delta_{t,n}+\epsilon_n}|\rho(\tau,x)|^p\,d\tau &\ge \alpha_n\frac{\Delta_{t,n}}{\epsilon_n}\int_0^T|\rho(\tau,x)|^p\,d\tau,\\ &\ge \left(1-\frac{\Delta_{t,n}}{\epsilon_n}\right)\int_0^T|\rho(\tau,x)|^p\,d\tau.\end{aligned}$$

Finally:

$$\Big|\|L_n\rho\|_{L^p_{\rm loc}}-|B_d(0,1)|\|\rho\|_{L^p_{\rm loc}}\Big|\le |B_d(0,1)|\frac{\Delta_{t,n}}{\epsilon_n}\|\rho\|_{L^p_{\rm loc}},$$

which implies, since $\Delta_{t,n}/\epsilon_n\to 0$, that there exists $C>0$ such that:

$$\frac{1}{C}\|\rho\|_{L^p_{\rm loc}}\le\|L_n\rho\|_{L^p_{\rm loc}}\le C\|\rho\|_{L^p_{\rm loc}}.$$

Therefore, for every functions $\rho_n,\rho\in L^p_{\rm loc}([0,T]\times\mathbb{R}^d)$ we have:

$$\begin{aligned}\|\rho_n-\rho\|_{L^p_{\rm loc}} &\le C\|L_n\rho_n-L_n\rho\|_{L^p_{\rm loc}}\\ &\le C\left(\|L_n\rho_n-\rho\|_{L^p_{\rm loc}}+\|L_n\rho-\rho\|_{L^p_{\rm loc}}\right),\end{aligned}$$

and:

$$\begin{aligned}\|L_n\rho_n-\rho\|_{L^p_{\rm loc}} &\le \|L_n\rho_n-L_n\rho\|_{L^p_{\rm loc}}+\|L_n\rho-\rho\|_{L^p_{\rm loc}}\\ &\le C\|\rho_n-\rho\|_{L^p_{\rm loc}}+\|L_n\rho-\rho\|_{L^p_{\rm loc}}.\end{aligned}$$

Therefore we just have to show that for every $\rho\in L^p_{\rm loc}$, $L_n\rho-\rho$ converges to 0 in $L^p_{\rm loc}$ when n tends to $+\infty$. Indeed for every $\varepsilon>0$ we can choose $\phi\in C_c^\infty([0,T]\times\mathbb{R}^d)$ such that $\|\phi-\rho\|_{L^p_{\rm loc}}\le\varepsilon$. Since ϕ is regular:

$$\|L_n\phi-\phi\|_{L^\infty}\le\|\nabla\phi\|_{L^\infty}\sqrt{{\epsilon_n}^2+(\Delta_{t,n}+\epsilon_n)^2},$$

so:

$$\|L_n\phi-\phi\|_{L^p_{\rm loc}}\le C\|\nabla\phi\|_{L^\infty}\epsilon_n\overset{n\to+\infty}{\longrightarrow}0,$$

so is less that ε for n big enough and:

$$\begin{aligned}\|L_n\rho-\rho\|_{L^p_{\rm loc}} &\le \|L_n\rho-L_n\phi\|_{L^p_{\rm loc}}+\|\rho-\phi\|_{L^p_{\rm loc}}+\|L_n\phi-\phi\|_{L^p_{\rm loc}}\\ &\le (C+1)\|\rho-\phi\|_{L^p_{\rm loc}}+\|L_n\phi-\phi\|_{L^p_{\rm loc}}\\ &\le (C+2)\varepsilon.\end{aligned}$$

So finally $\|L_n\rho-\rho\|_{L^p_{\rm loc}}$ converges to 0 which ends the proof. □

B Extended averaging compactness Theorem

We prove in this appendix the following version of Theorem 1.3:

Theorem B.3 *Consider a sequence $\Delta_{t,n}\to 0$ and a sequence of functions $f_n\in L^\infty(\mathbb{R}^+\times\mathbb{R}^{2d})$ which satisfies*

(B.9)
$$\frac{\partial}{\partial t}f_n+v\cdot\nabla_x f_n=\sum_{i\in\mathbb{Z}}\delta_{i\,\Delta_{t,n}}(t)\Big(\int_{i\,\Delta_{t,n}}^{(i+1)\,\Delta_{t,n}}H_n(\tau,x,v)\,d\tau\Big).$$

We assume that

- *(i) $f_n\rightharpoonup f$ weakly in $L^\infty(\mathbb{R}^+\times\mathbb{R}^{2d})*$,*

- *(ii) $H_n = h_0^n + \sum_{j=1}^d \partial_{x_j} h_j^n$, with $\{h_j^n\}$ relatively compact in L^p (for some $p > 1$) and (h_0^n) bounded in L^2,*
- *(iii) there exists a sequence $\epsilon_n \to 0$ with $\epsilon_n/\Delta_{t,n} \to +\infty$ such that:*

$$\|{\epsilon_n}^2 \, H_n\|_{L^2} \overset{n\to+\infty}{\longrightarrow} 0. \tag{B.10}$$

Then, for any $\psi \in \mathcal{D}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} g_n(t,x,v)\,\psi(v)\,dv \;\to\; \int_{\mathbb{R}^d} f(t,x,v)\,\psi(v)\,dv \tag{B.11}$$

strongly in $L^p_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^{2d})$ $\forall p \in [1,\infty)$.

Proof of the Theorem. We follow the structure of section 2:

(i) Compactness at the global scale.

Notice that we cannot apply directly Theorem 2.1 because of the singularity with respect to time on the right hand side term of (B.9). But since we are concerned only with values of f_n at time $t = i\Delta_{t,n}$, we consider a new function f_n^* (a mean value of f_n with respect to time) which verifies:

$$f_n^*(i\Delta_{t,n},\cdot,\cdot) = f_n(i\Delta_{t,n},\cdot,\cdot),$$

and such that f_n^* verifies an equation of the form (B.9) without singularity with respect to time on the right-hand side term. In order to do so, we consider the function $S(t) = \text{Sup}(0, 1-|t|)$. In particular, $S(0) = 1$ and $\text{Supp}S \subset [-1,1]$. We denote f_n^* the function defined by:

$$f_n^*(t,x,v) = \sum_{i\in\mathbb{Z}} S\left(\frac{t - i\Delta_{t,n}}{\Delta_{t,n}}\right) f_n(i\Delta_{t,n}-, x - (t - i\Delta_{t,n})v, v).$$

Notice that f_n^* is bounded in $L^\infty([0,T]\times\mathbb{R}^{2d})$ and:

$$f_n^*(t_{\Delta_{t,n}}-, x, v) = f_n(t_{\Delta_{t,n}}-, x, v) = g_n(t,x,v).$$

We have:

$$\begin{aligned}
\frac{\partial}{\partial t} f_n^* + v.\nabla_x f_n^* &= \frac{1}{\Delta_{t,n}} \sum_{i\in\mathbb{Z}} S'\left(\frac{t-i\Delta_{t,n}}{\Delta_{t,n}}\right) f_n(i\Delta_{t,n}-, x-(t-i\Delta_{t,n})v, v)\\
&= \frac{1}{\Delta_{t,n}}\big[f_n((t_{\Delta_{t,n}} + \Delta_{t,n})-, x - (t - t_{\Delta_{t,n}} - \Delta_{t,n})v, v)\\
&\qquad - f_n(t_{\Delta_{t,n}}-, x - (t - t_{\Delta_{t,n}})v, v)\big].
\end{aligned}$$

Equation (B.9) is equivalent to:

$$\frac{\partial}{\partial t}\{f_n(t, x+tv, v)\} = \sum_{i\in\mathbb{Z}} \delta_{i\Delta_{t,n}}(t)\left(\int_{i\Delta_{t,n}}^{(i+1)\Delta_{t,n}} H_n(\tau, x+tv, v)\,d\tau\right).$$

Integrating with respect to t on $[t_{\Delta_{t,n}}, t_{\Delta_{t,n}} + \Delta_{t,n}[$ this equation leads to:

$$\begin{aligned}
&f_n((t_{\Delta_{t,n}} + \Delta_{t,n})-, x + (t_{\Delta_{t,n}} + \Delta_{t,n})v, v) - f_n(t_{\Delta_{t,n}}-, x + t_{\Delta_{t,n}}v, v)\\
=\;& \int_{t_{\Delta_{t,n}}}^{t_{\Delta_{t,n}}+\Delta_{t,n}} H_n(\tau, x + t_{\Delta_{t,n}}v, v)\,d\tau.
\end{aligned}$$

Therefore:

$$\frac{\partial}{\partial t} f_n^* + v.\nabla_x f_n^* = \frac{1}{\Delta_{t,n}} \int_{t_{\Delta_{t,n}}}^{t_{\Delta_{t,n}}+\Delta_{t,n}} H_n(\tau, x + (t_{\Delta_{t,n}} - t)v, v)\,d\tau. \tag{B.12}$$

The right-hand side term can be written in the form:

$$\sum_{i=1}^{d} \frac{\partial k_j^n}{\partial x_j}$$

where:

$$k_j^n = \frac{1}{\Delta_{t,n}} \int_{t_{\Delta_{t,n}}}^{t_{\Delta_{t,n}}+\Delta_{t,n}} h_j^n(\tau, \cdot - (t - t_{\Delta_{t,n}})v), \cdot)\, d\tau,$$

is relatively compact in $L^2(\mathbb{R}^{1+2d})$. So thanks to the classical averaging lemma of Perthame-Souganidis (Theorem 2.1), up to a subsequence there exists $\rho_\psi \in L^2_{\text{loc}}([0,T] \times \mathbb{R}^d)$ such that $\rho_\psi^{*,n}(t,x) = \int \psi(v) f_n^*(t,x,v)\, dv$ converges to $\rho_\psi(t,x)$ in L^2_{loc}. Thanks to Hypothesis (i) and since $f_n^* - f_n$ converges to 0 in the sense of distribution, $\rho_\psi = \int \psi(v) f\, dv$. By uniqueness of the limit, the entire sequence converges.

(ii) From global scale to local scale.

Replacing $\overline{f}_n$ by:

$$\overline{f}_n^*(t,y,v) = f^*(t_{\Delta_{t,n}} + \epsilon_n s, x + \epsilon_n y, v),$$

and $\overline{\rho}_\psi^n$ by:

$$\overline{\rho}_\psi^{n,*}(s,y) = \int \psi(v) \overline{f}_n^*(s,y,v)\, dv,$$

in the same way that in the section 2 we find the following proposition related to Proposition 2.4:

Proposition B.4 *Up to a subsequence (still denoting $\Delta_{t,n}$),there exists $\Omega \subset [0,T] \times \mathbb{R}^d$ with $\mathcal{L}([0,T] \times \mathbb{R}^d \setminus \Omega) = 0$ such that for every $(t,x) \in \Omega$:*

$$\int_0^1 \int_{B_d(0,1)} \left| \overline{\rho}_\psi^{n,*}(t,x,s,y) - \rho_\psi^*(t,x) \right|^p ds\, dy \overset{n\to+\infty}{\longrightarrow} 0$$

$$\int_{\mathbb{R}^d} \int_0^1 \int_{B_d(0,1)} \left| \overline{H}_n(t,x,s,y,v) \right|^2 ds\, dy\, dv \overset{n\to+\infty}{\longrightarrow} 0,$$

for every $1 \le p < +\infty$, where $\mathcal{L}$ denotes the Lebesgue measure.

(iii) Strong convergence at the local scale.

No change with section 2. This gives Proposition 2.6.

(iv) Uniqueness of the limit at the local scale.

Notice that: $(t_{\Delta_{t,n}} + \epsilon_n s)_{\Delta_{t,n}} = t_{\Delta_{t,n}} + \epsilon_n s_{\overline{\Delta}_n}$. Therefore, from (B.12) we find that:

$$\begin{aligned}
\frac{\partial \overline{f}_n^*}{\partial s} + v.\nabla_y \overline{f}_n^* &= \frac{1}{\overline{\Delta}_n} \int_{t_{\Delta_{t,n}}+\epsilon_n s_{\overline{\Delta}_n}}^{t_{\Delta_{t,n}}+\epsilon_n s_{\overline{\Delta}_n}+\Delta_{t,n}} H_n(\tau, x + \epsilon_n y + (\epsilon_n s_{\overline{\Delta}_n} - \epsilon_n s)v, v)\, d\tau \\
&= \frac{\epsilon_n}{\overline{\Delta}_n} \int_{s_{\overline{\Delta}_n}}^{s_{\overline{\Delta}_n}+\overline{\Delta}_n} H_n(t_{\Delta_{t,n}} + \epsilon_n \tau, x + \epsilon_n(y + (s_{\overline{\Delta}_n} - s)v), v)\, d\tau \\
&= \frac{1}{\epsilon_n \overline{\Delta}_n} \int_{s_{\overline{\Delta}_n}}^{s_{\overline{\Delta}_n}+\overline{\Delta}_n} \overline{H}_{\epsilon_n}(\tau, y + (s_{\overline{\Delta}_n} - s)v, v)\, d\tau.
\end{aligned}$$

In the second equation we do the change of variables $\tau \to t_{\Delta_{t,n}} + \epsilon_n \tau$ and in the last equation we use the definition of $\overline{H}_n$. We deduce that:

$$\begin{aligned}
&\left| \overline{f}_n^*(s, y + (s - s_{\overline{\Delta}_n})v, v) - \overline{f}_n^*(s_{\overline{\Delta}_n}, y + (s - s_{\overline{\Delta}_n})v, v) \right| \\
&= \left| \frac{s - s_{\overline{\Delta}_n}}{\epsilon_n \overline{\Delta}_n} \int_{s_{\overline{\Delta}_n}}^{s_{\overline{\Delta}_n}+\overline{\Delta}_n} \overline{H}_n(\tau, y, v)\, d\tau \right| \le \frac{\Delta_{t,n}}{\overline{\Delta}_n} \left| \int_{s_{\overline{\Delta}_n}}^{s_{\overline{\Delta}_n}+\overline{\Delta}_n} \overline{H}_n(\tau, y, v)\, d\tau \right|.
\end{aligned}$$

Therefore this term converges to 0 in $L^2([0,1] \times B_d(0,1) \times \mathbb{R}^d)$ thanks to Proposition B.4.

We denote:

$$r_n(s,y,v) = \overline{f}_n^*(s,y,v) - \overline{g}_n(s,y,v) = \overline{f}_n^*(s,y,v) - \overline{f}_n^*(s_{\overline{\Delta}_n}, y, v).$$

We have:

$$\begin{aligned} r_n(s,y,v) &= r_n(s,y,v) - r_n(s, y + (s - s_{\overline{\Delta}_n})v, v) \\ &\quad + \overline{f}_n^*(s, y + (s - s_{\overline{\Delta}_n})v, v) - \overline{f}_n^*(s_{\overline{\Delta}_n}, y + (s - s_{\overline{\Delta}_n})v, v). \end{aligned}$$

We have just proved that the second term converges to zero in L^2_{loc}. If we consider a test function $\phi \in C_c^\infty([0,T] \times \mathbb{R}^{2d})$ we find:

$$\begin{aligned} &\left| \int \left[r_n(s,y,v) - r_n(s, y - (s - s_{\overline{\Delta}_n})v, v) \right] \phi(s,y,v) \, ds \, dy \, dv \right| \\ &= \left| \int \left[\phi(s,y,v) - \phi(s, y - (s - s_{\overline{\Delta}_n})v, v) \right] r_n(s,y,v) \, ds \, dy \, dv \right| \\ &\le \overline{\Delta}_n \sqrt{|\text{Supp}\phi|} \|\nabla \phi\|_{L^\infty} \|r_n\|_{L^2_{\text{loc}}} \overset{n \to +\infty}{\longrightarrow} 0. \end{aligned}$$

So finally, r_n converges to 0 in the sense of distribution and therefore:

$$\overline{\eta}_\psi^{n,*} - \overline{\rho}_\psi^n \xrightarrow{\mathcal{D}'} 0.$$

Since $\overline{\rho}_\psi^{n,*}$ converges to ρ_ψ, the entire sequence $\overline{\eta}_\psi^n$ converges to ρ_ψ in the sense of distribution. Finally thanks to Proposition 2.6, we have Proposition 2.7.

(v) Back to the global scale.

No change with section 2. This ends the proof.

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