Université de Versailles – Saint-Quentin
Département de MATHÉMATIQUES

UVSQ 73 – 01 octobre 2001

# Uniqueness and semigroup for the Vlasov equation with elastic-diffusive reflexion boundary conditions

Antoine MELLET† & Stéphane MISCHLER‡

Prépublication ou rapport de recherche numéro 73 (01)

Laboratoire de Mathématiques Appliquées
Université de Versailles – Saint-Quentin et CNRS - umr 7641
VERSAILLES (FRANCE)

# Uniqueness and semigroup for the Vlasov equation with elastic-diffusive reflexion boundary conditions

Antoine MELLET[†] & Stéphane MISCHLER[‡]

*Abstract* - In this paper, we investigate some uniqueness results for the Vlasov equation with elastic-diffusive boundary conditions. As an application, we build the associated semigroup in a $L^1$ setting.

*Keywords* - Vlasov equation, elastic-diffuse reflection, Darrozès-Guiraud inequality, trace Theorems.

[‡] Mathématiques pour l'Industrie et la Physique, UMR 5640,
`Site Internet: http://www.math.uvsq.fr/~mischler/`

# 1 Introduction

Let $\Omega$ be an open subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and define $\mathcal{O} = \Omega \times \mathbb{R}^N$, and $\Sigma = \partial\Omega \times \mathbb{R}^N$. We introduce the outgoing and incoming trace subset $\Sigma_\pm = \{(x,v) \in \Sigma\,;\, \pm n(x)\cdot v > 0\,\}$, where $n(x)$ denotes the unit outward normal vector on the boundary $\partial\Omega$, and we denote by $\gamma_\pm f$ the restriction of the trace of $f$ on $\Sigma_\pm$. The equation we are concerned with in this paper is the following Vlasov equation:

$$\begin{aligned} &\Lambda_E f = \frac{\partial f}{\partial t} + v\cdot\nabla_x f + E\cdot\nabla_v f = 0\,, \quad (x,v)\in\mathcal{O}\,,\ t\in[0,T]\,,\\ &f(x,v,0) = \varphi(x,v)\,, \quad (x,v)\in\mathcal{O}\,, \end{aligned} \tag{1}$$

together with boundary conditions of the form:

$$\gamma_- f(t) = K\gamma_+ f(t)\,, \qquad \forall (x,v)\in\Sigma_-\,, \forall t\in(0,T)\,. \tag{2}$$

This equation describes the evolution of the distribution function of a cloud of particles confined in the domain $\Omega$. $v\in\mathbb{R}^N$ is the velocity of the particles, $E(x)$ is the electric field which satisfies:

(H0) $\quad E(x)$ is given, time-independent and belongs to $W^{1,1}_{loc}(\bar\Omega)\cap L^\infty(\Omega)$.

The expression and properties of the operator $K$ depend on the model we choose for the reflexion, absorption and emission of particles. This note is concerned with the case of diffusive reflexion by the boundary, which gives rise to the following expression:

$$K(\varphi)(x,v) = \int_{v\in\mathbf{R}^N,\, n(x)\cdot v'>0} k(x,v',v)\varphi(x,v')n(x)\cdot v'\,dv'\,, \quad \forall (x,v)\in\Sigma_-\,.$$

From physical considerations, the kernel (or cross-section) $k(x,v',v)$ has to satisfy:

(H1) positivity : $k(x,v',v)\geq 0$,

(H2) mass conservation : $\displaystyle\int_{n(x)\cdot v<0} k(x,v',v)|n(x)\cdot v|dv = 1, \quad$ for $n(x)\cdot v' > 0$.

One usually also adds the following hypothesis, which ensures the existence of a thermodynamical equilibrium:

(H3) There exists a Maxwellian distribution $M(x,v)$ satisfying $KM = M$.

The Maxwellian distribution reads $M(x,v) = \frac{1}{2\pi\Theta^2}\exp(-\frac{|v|^2}{2\Theta})$, where $\Theta(x)$ is the temperature of the boundary. This last hypothesis is actually a consequence of (H2) when the following detailed balance principle (or reciprocity relation) holds:

$$k(x,-v,-v')M(x,v) = k(x,v',v)M(x,v')\,. \tag{3}$$

However, in this paper, we investigate the case of elastic reflexion. Therefore, from now on, we shall assume that the cross-section reads:

(H3') $\quad k(x,v',v) = k_0(x,v',v)\delta(|v|^2-|v'|^2)\,.$

In this context, (H3) gives rise to the following normalization condition:

$$\int_{n(x)\cdot v'>0} k(x,v',v)|n(x)\cdot v'|dv' = 1, \qquad \text{for } n(x)\cdot v < 0\,.$$

In particular, $K\Phi = \Phi$ holds for every function depending on the velocity through the energy only.

Under these hypothesese, we investigate in this paper the properties of equations (1)-(2): Firstly, we establish in Proposition 1 the existence and uniqueness of solutions for initial data in $L^1(\mathcal{O}) \cap L^2(\mathcal{O})$, that satisfies $\gamma f \in L^1_{\text{loc}}([0,T]\times\Sigma, |n(x)\cdot v|dv\,d\sigma_x\,dt)$. Then, in Proposition 2, we deduce the existence of a semigroup $S(t)$ on $L^1(\mathcal{O})$, such that $S(t)\varphi \in L^\infty(0,T;L^1(\mathcal{O}))$ is a weak solution of (1), and satisfies (2) in a sense that has to be precised (see Remark 2.1). However we do not know whether the so-constructed solution, the trace of which belongs to $L^1_{\text{loc}}([0,T]\times\Sigma, |n(x)\cdot v|^2 dv\,d\sigma_x\,dt)$ is unique in the class of weak solutions in $L^1(\mathcal{O})$.

Many ideas used below have been first developed by S. Mischler in [4] and [5] in the case of specular and maxwellian reflexion on the boundary. We refer to these papers for reference about the Vlasov equation and boundary conditions. We also stress the fact that the proof of Proposition 1 can be adapted in order to provide the uniqueness of the solution in [2].

The main tool that will be used throughout this note is the so-called Darrozès-Guirraud inequality [3], which reads, under the general framework of (H1)-(H3):
For all convex non negative functions $\beta \in \mathcal{C}^0(\mathbb{R})$, we have

$$\int_{n(x)\cdot v<0} \beta\left(\frac{K(\varphi)}{M}\right) M(v)|n(x)\cdot v|dv \le \int_{n(x)\cdot v>0} \beta\left(\frac{\varphi}{M}\right) M(v)|n(x)\cdot v|dv\,, \tag{4}$$

with equality for $\beta(y) = y$ (this is the expression of the flux conservation).

As a consequence, any function $f$ satisfying the boundary conditions (2), satisfies (at least formally):

$$\int_\Sigma \beta\left(\frac{\gamma f}{M}\right) M(v)(n(x)\cdot v)\,dv\,d\sigma_x \ge 0\,.$$

In the next section, we state our main results, the proofs of which are detailed in Section 3 and 4.

# 2 Main results

From now on, we assume that (H0), (H1), (H2), (H3) and (H3') hold. We also assume that $n(x)$ can be extended to $\mathbb{R}^n$ in a regular way (such that $n(x) \in W^{1,\infty}_{loc}(\mathbb{R}^N)$). For technical purpose, it is convenient to decompose the velocity set $\mathbb{R}^N$ as $\mathbb{R}^+ \times S^{N-1}$ by writing $v = |v|\omega$, where $\omega$ is the angular velocity. With these notations, the operator $K$ reads:

$$K(\varphi)(x,v) = \int_{n(x)\cdot\omega'>0} k_0(x,|v|,\omega',\omega)\,\varphi(x,|v|\omega')\,|v|\,n(x)\cdot\omega'\,d\omega'\,, \quad \forall (x,v)\in\Sigma_-\,, \tag{5}$$

and the normalization condition yields:

$$\int_{n(x)\cdot\omega'>0} k_0(x,u,\omega',\omega)|n(x)\cdot v'|d\omega' = 1\,.$$

Let us also rewrite the Darrozès-Guiraud inequality (4) in this case:
For all convex non negative functions $\beta \in \mathcal{C}^0(\mathbb{R})$, we have

$$\int_{n(x)\cdot\omega<0} \beta\left(K\varphi(|v|\omega)\right)|n(x)\cdot\omega|d\omega \leq \int_{n(x)\cdot\omega>0} \beta\left(\varphi(|v|\omega)\right)|n(x)\cdot\omega|d\omega\,. \tag{6}$$

In particular, with $\beta(y)=y^p$, we deduce that

$$\|K(\varphi)\|_{L^p(\Sigma_-)} \leq \|\varphi\|_{L^p(\Sigma_+)}\,, \qquad \forall p<+\infty\,.$$

**Remark 2.1** From Theorem 1 in [4], for any function $f \in L^\infty(0,T;L^p_{\mathrm{loc}}(\mathcal{O}))$ solution of (1), we can define its trace $\gamma f$, which belongs to $L^1_{\mathrm{loc}}([0,T]\times\Sigma,(n(x)\cdot v)^2 dv\,d\sigma_x\,dt)$. However, we need more integrability in order to give sense to (2).
In view of (6), $K(x,|v|)$ is a bounded operator on $L^1(S^{N-1}_+)$, for any $x,|v| \in \partial\Omega\times\mathbb{R}^+$. Therefore $K\gamma_+ f$ is well-defined (and (2) has a meaning) as soon as $\gamma f \in L^1_{\mathrm{loc}}([0,T]\times\partial\Omega\times\mathbb{R}^+;L^1(S^{N-1});|n(x)\cdot v|dv\,d\sigma_x\,dt)$. ∎

The first result we are aiming at is the following proposition:

**Proposition 1** *For all initial data $\varphi \in L^1(\mathcal{O})\cap L^2(\mathcal{O})$, there exists a unique solution $f(x,v,t)$ of (1)-(2) in $L^\infty(0,T;L^1(\mathcal{O})\cap L^2(\mathcal{O}))$ satisfying $\gamma f \in L^1(0,T;L^1_{loc}(\Sigma,|n(x)\cdot v|dvd\sigma_x))$. Moreover, we have*

$$\|f(t)\|_{L^1(\mathcal{O})} \leq \|\varphi\|_{L^1(\mathcal{O})} \tag{7}$$

*(with equality if $\varphi\geq 0$), and for all $U$ compact subset of $\mathcal{O}$,*

$$\int_0^T\int_{U\cap\Sigma} |\gamma f||n(x)\cdot v|dv\,d\sigma_x\,dt \;\leq C_U\left(1+\|E\|_{L^\infty(U_x)}+\|\nabla_x n\|_{L^\infty(U_x)}\right)\|\varphi\|_{L^2(\mathcal{O})}\,.$$

This first result obviously defines a semigroup $S(t)$ on $L^1(\mathcal{O})\cap L^2(\mathcal{O})$, which satisfies $\|S(t)\|_{\mathcal{L}(L^1(\mathcal{O}))}\leq 1$. $L^1(\mathcal{O})\cap L^2(\mathcal{O})$ being a dense subset of $L^1(\mathcal{O})$, there exists a unique extension $S(t): L^1(\mathcal{O})\to L^1(\mathcal{O})$. However, in Proposition 1, we controle the $L^1_{\mathrm{loc}}$-norm of the trace by the $L^2$-norm of the initial data, and we shall therefore need further estimate in order to take the limit in (2). To that purpose, we introduce the following hypothesis:

(H4) For all compact set $U\subset\partial\Omega\times\mathbb{R}^+$, there exists a constant $\beta_U>0$, such that:

$$\int_{n(x)\cdot\omega<0} k_0(x,u,\omega',\omega)(n(x)\cdot\omega)^2\,d\omega \geq \beta_U \qquad \forall (x,u)\in U\,,\quad \omega'\cdot n(x)>0\,.$$

This so-called 'spreading condition', which can also be written $K^*|n(x)\cdot\omega|\geq\beta_V$, is often used to get controle on the trace. It is satisfied in particular when $k_0$ is bounded by below by a Maxwellian distribution.

Under (H0)-(H4), we now have:

**Proposition 2** *For any initial data $\varphi\in L^1(\mathcal{O})$, there exists a function $f(t)=S(t)\varphi\in L^\infty(0,T;L^1(\mathcal{O}))$, solution of (1) which satisfies*

$$\|S(t)\varphi\|_{L^1(\mathcal{O})}\leq\|\varphi\|_{L^1(\mathcal{O})}\,.$$

*Moreover, its trace is such that $\gamma_+S(t)\varphi\in L^1([0,T]\times V\times S^{N-1};|n(x)\cdot v|\,dv\,d\sigma_x\,dt)$ for all compact subset $V$ of $\partial\Omega\times\mathbb{R}^+_*$, and satisfies (2).*

## 3 Proof of Propositions 1.

The proof of Proposition 1 will be divided as follows: First, we prove the uniqueness part of the result, by establishing the estimate (7) (under the general setting of solutions in $L^1(\mathcal{O})$ with trace in $L^1(0,T;L^1_{\rm loc}(|n(x)\cdot v|dvd\sigma_x)))$. Then we shall prove the existence part, through an iterative process.

*Uniqueness:* Let $f$ be a solution satisfying $f|_{t=0}\in L^1(\mathcal{O})$, and $\gamma f\in L^1(0,T;L^1_{\rm loc}(|n(x)\cdot v|dvd\sigma_x))$. First of all, it has been proved in [4] (Theorem 1) that such a solution actually belongs to $\mathcal{C}^0(0,T;L^1_{\rm loc}(\mathcal{O}))$. From [4], we also know that $f(t)$ is a renormalized solution of the Vlasov equation (1): For all $\beta\in W^{1,\infty}_{loc}(\mathbb{R})$, we have:

$$\Lambda_E\beta(f)=0\,,\quad\text{and } \gamma\beta(f)=\beta(\gamma f)\,.$$

We define a sequence of smooth convex and non negative functions $\beta_\varepsilon$ as follows: $\beta_\varepsilon(y)=|y|-\varepsilon$ for $|y|\geq 2\varepsilon$ and $\beta_\varepsilon(y)=y^2/(4\varepsilon)$ for $|y|\leq 2\varepsilon$. As in [4], we also introduce $\chi_R(x,|v|)=\chi(x/R,|v|/R)$, with $\chi$ a smooth function satisfying $0\leq\chi\leq 1$, $\chi=1$ on $B_1\times[0,1]$, and $supp\,\chi_R\subset B_2\times[0,2]$ (where $B_r$ denotes the ball of radius $r$, center at the origin in $\mathbb{R}^N$). Then the Green formula leads to:

$$\begin{aligned}\left[\int_{\mathcal{O}}\beta_\varepsilon(f)\chi_R\,dv\,dx\right]_0^t &= \int_0^t\int_{\mathcal{O}}\beta_\varepsilon(f)\Lambda_E\chi_R\,dv\,dx\,ds\\ &\quad+\int_0^t\int_{\Sigma_-}\beta_\varepsilon(\gamma_- f)\chi_R(|v|)|n(x)\cdot v|\,dv\,d\sigma_x\,ds\\ &\quad-\int_0^t\int_{\Sigma_+}\beta_\varepsilon(\gamma_+ f)\chi_R(|v|)|n(x)\cdot v|\,dv\,d\sigma_x\,ds\,.\end{aligned}$$

Since $\chi_R$ does not depend on the angular velocity $\omega$, Fubini's theorem and (6) yields:

$$\left[\int_{\mathcal{O}}\beta_\varepsilon(f)\chi_R\,dv\,dx\right]_0^t\leq\int_0^t\int_{\mathcal{O}}\beta_\varepsilon(f)\Lambda_E\chi_R\,dv\,dx\,ds\,,$$

with equality if $f\geq 0$. We deduce (7) by taking successively the limits $\varepsilon\to 0$ and $R\to\infty$ (see [4] for details). The uniqueness follows by standard argument. ∎

*Existence:* Let now $\varphi$ be in $L^1(\mathcal{O})\cap L^2(\mathcal{O})$. In order to prove the existence of a solution, we first assume that $\varphi$ is positive (for general function, we decompose into positive and negative part), and we define a sequence $(f_n)_{n\in\mathbb{N}}$ of solutions of the Vlasov equation (1) in $L^2(\mathcal{O})$, with initial data $\varphi$, and the following boundary conditions:

$$\begin{aligned}&\gamma_- f_0=0\,,\\ &\gamma_- f_n=K\gamma_+ f_{n-1}\,,\quad\forall\, n\geq 1\,.\end{aligned}$$

Such a sequence is well-defined since $\gamma_+ f_{n-1}$, and therefore $K\gamma_+ f_{n-1}$ lies in $L^2(\Sigma_-)$ for all $n\geq 1$ (see S. Ukai [7]).

Thanks to the monotonicity of the operator $K$ and the maximum principle for the transport equation, it is easy to check that the sequence $(f_n)_{n\in\mathbb{N}}$ is non-decreasing. We deduce that for all convex functions $\beta$ such that $\beta$ is non-decreasing on $\mathbb{R}^+$, we have:

$$\begin{aligned}\int_{\Sigma_-} \beta(\gamma_- f_n)|n(x)\cdot v|\, dv\, d\sigma_x &= \int_{\Sigma_-} \beta(K\gamma_+ f_{n-1})|n(x)\cdot v|\, dv\, d\sigma_x \\ &\leq \int_{\Sigma_+} \beta(\gamma_+ f_{n-1})|n(x)\cdot v|\, dv\, d\sigma_x \\ &\leq \int_{\Sigma_+} \beta(\gamma_+ f_n)|n(x)\cdot v|\, dv\, d\sigma_x\,.\end{aligned}$$

The first inequality is a consequence of the Darrozès-Guirraud inequality (6), and the second one a consequence of the monotonicity of the sequence. It follows:

$$\int_\Sigma \beta(\gamma f_n)(n(x)\cdot v)\, dv\, d\sigma_x \geq 0\,. \tag{8}$$

Now, multiplying (1) by $f_n$ and integrating, the Green formula yields:

$$\left[\int_{\mathcal{O}} |f_n|^2\, dv\, dx\right]_0^t = -\int_0^t \int_\Sigma (\gamma f_n)^2 (n(x)\cdot v)\, dv\, d\sigma_x\, ds\,,$$

and (8) with $\beta(y) = y^2$ implies:

$$\|f_n(t)\|_{L^2(\mathcal{O})} \leq \|\varphi\|_{L^2(\mathcal{O})}\,. \tag{9}$$

The similar $L^1$ estimate is obtained by proceeding as in the proof of the uniqueness, using inequality (8).

It remains to show that we can controle the trace, at least locally. Let $U$ be a compact subset of $\mathcal{O}$, and let $\psi(x,|v|)$ be a compactly supported function on $\mathcal{O}$ such that $\psi = 1$ on $U$. In the spirit of [4], we multiply (1) by $(n(x)\cdot v) f_n \psi(x,|v|)$; using (9), it yields:

$$\int_0^T \int_{\Sigma\cap U} (\gamma f_n)^2 |n(x)\cdot v|^2 \psi(x,|v|)\, dv\, d\sigma_x\, dt \leq C_U\Big(1 + \|E\|_{L^\infty(U_x)} + \|\nabla_x n\|_{L^\infty(U_x)}\Big)\|\varphi\|^2_{L^2(\mathcal{O})}\,,$$

and the Cauchy-Schwartz inequality implies:

$$\int_0^T \int_{U\cap\Sigma} |\gamma f_n||n(x)\cdot v|\, dv\, d\sigma_x\, dt \leq C_U\Big(1 + \|E\|_{L^\infty(U_x)} + \|\nabla_x n\|_{L^\infty(U_x)}\Big)\|\varphi\|^2_{L^2(\mathcal{O})}\,.$$

These estimates allow us to pass to the limit $n$ goes to infinity, and conclude the proof of Proposition 1. ∎

**Remark 3.1** The existence part of the proof holds for more general boundary operators. Actually, we only used that $K$ satisfies the Darrozès-Guirraud inequality, and is non-negative ($f \leq g$ implies $Kf \leq Kg$). ∎

## 4 Proof of Proposition 2

The main issue in the proof of Proposition 2 is concerned with the boundary condition (2): Let $\varphi$ be in $L^1(\mathcal{O})$, and $\varphi_n \in L^1(\mathcal{O}) \cap L^2(\mathcal{O})$ be a sequence of function such that $\varphi_n \to \varphi$ in $L^1(\mathcal{O})$. Assume moreover, as in the previous section, that $\varphi$ and $\varphi_n$ are non negative. Thanks to Proposition 1, the sequence $f_n = S(t)\varphi_n$ strongly converges in $L^1(\mathcal{O})$ toward $f(t) = S(t)\varphi$. Moreover, $\gamma f_n \in L^1(0,T; L^1_{\text{loc}}(|n(x)\cdot v|\, dv\, d\sigma_x))$ satisfies:

$$\gamma_- f_n = K\gamma_+ f_n\,. \tag{10}$$

It remains to prove that we can take the limit in (10).

First of all, multiplying (1) by $(n(x)\cdot v)\psi(x,|v|)$ for some compactly supported function $\psi$, we prove (as in the proof of Proposition 1) that, for all $U$ compact subset of $\mathcal{O}$, we have

$$\int_0^T \int_{U\cap\Sigma} |\gamma f_n|(n(x)\cdot v)^2\, dv\, d\sigma_x\, dt \le C_U \left(1 + \|E\|_{L^\infty(U_x)} + \|\nabla_x n\|_{L^\infty(U_x)}\right) \|\varphi_n\|_{L^1(\mathcal{O})}\,,$$

and therefore $\gamma f \in L^1_{\text{loc}}([0,T]\times\Sigma, |n(x)\cdot v|^2\, dv\, d\sigma_x\, dt)$. As discussed previously, this is not enough to take the limit in (10); however, thanks to (H4), we are going to derive another a-priori estimate. Let now $V$ be a compact subset of $\partial\Omega\times\mathbb{R}^+_*$, from (10) and the previous estimate, we get

$$\int_0^T \int_{(V\times\mathcal{S}^{N-1})\cap\Sigma_-} |K\gamma_+ f_n|(n(x)\cdot v)^2 dv\, d\sigma_x\, dt \le C_V \|\varphi_n\|_{L^1(\mathcal{O})}\,,$$

for some constant $C_V$. Hence, $\gamma f_n$ being non negative, we have:

$$\int_0^T \int_{V\cap\Sigma_-} (n(x)\cdot v)^2 \int_{n(x)\cdot\omega'>0} k_0(\omega',\omega)\gamma_+ f_n(\omega')\, |n(x)\cdot v'|\, d\omega'\, dv\, d\sigma_x\, dt \le C_V \|\varphi_n\|_{L^1(\mathcal{O})}\,,$$

and Fubini's Theorem yields:

$$\int_0^T \int_{V\cap\Sigma_+} \gamma_+ f_n(v')\, |n(x)\cdot v'|\, |v|^2 \int_{n(x)\cdot\omega<0} k_0(\omega',\omega)(n(x)\cdot\omega)^2\, d\omega\, dv'\, d\sigma_x\, dt \le C_V \|\varphi_n\|_{L^1(\mathcal{O})}\,.$$

Noticing that $|v|^2$ is bounded by below on $V$, it follows from (H4) that:

$$\int_0^T \int_{(V\times\mathcal{S}^{N-1})\cap\Sigma_+} \gamma_+ f_n(v')\, |n(x)\cdot v'|\, dv'\, d\sigma_x\, dt \le \frac{C_V}{\beta_V} \|\varphi_n\|_{L^1(\mathcal{O})}\,.$$

We deduce that $\gamma_+ f_n$ strongly converges to $\gamma_+ f$ in $L^1([0,T]\times V\times S^{N-1}, |n(x)\cdot v|\, dv\, d\sigma_x\, dt)$ for all compact subset $V$ of $\partial\Omega\times\mathbb{R}^+_*$ (and so does $\gamma_- f_n$ thanks to (10)), and, in view of Remark 2.1, this is enough to give sense to $K\gamma_+ f$, and pass to the limit in (10). ∎

## 5 Remark and extension

i) *Duality method:*
The uniqueness result state in Proposition 1 could also be proved by duality: For all $\Phi \in \mathcal{D}(\mathcal{O}\times(0,T))$, we solve the backward problem:

$$\begin{cases} \Lambda_E g = \Phi\,, & (x,v,t)\in\mathcal{O}\times(0,T)\,,\\ g(x,v,t=T) = 0\,, & (x,v)\in\mathcal{O}\,,\\ \gamma_+ g(t) = K^*\gamma_- g\,, & (x,v)\in\Sigma_+\,,\ t\in(0,T)\,. \end{cases} \tag{11}$$

The solvability of (11) is obtained as in Proposition 2, and since $|K^*\varphi| \leq |\varphi|_{L^\infty}$, we have $g \in L^\infty(\mathcal{O} \times (0,T))$. Assume now that $f \in L^\infty((0,T), L^1(\mathcal{O}))$ solves (1)-(2) with $\gamma f \in L^1(0,T; L^1_{\mathrm{loc}}(|n(x) \cdot v| dv d\sigma_x))$ and $f_{in} = 0$. Let $\chi_R$ be as in the proof of Proposition 1, then we get:

$$\begin{aligned} 0 = \int_0^T \int_{\mathcal{O}} \Lambda_E(f) g\, \chi_R \, dv\, dx\, dt &= -\int_0^T \int_{\mathcal{O}} f \Lambda_E(g\chi_R)\, dv\, dx\, dt \\ &\quad + \int_0^T \int_{\Sigma_-} (\gamma f)(\gamma g)\chi_R(|v|)|n(x) \cdot v|\, dv\, d\sigma_x\, ds \\ &\quad + \int_{\mathcal{O}} ((fg)_{t=T} - (fg)_{t=0})\, \chi_R(|v|)\, dv\, dx\,. \end{aligned}$$

And the boundary conditions yield:

$$\int_0^T \int_{\mathcal{O}} f(\Lambda_E g)\chi_R\, dv\, dx\, dt + \int_0^T \int_{\mathcal{O}} fg(\Lambda_E \chi_R)\, dv\, dx\, ds = 0\,.$$

When $R$ goes to infinity, we get

$$\int_0^T \int_{\mathcal{O}} f\Phi\, dv\, dx\, dt = 0\,,$$

which implies $f = 0$.

ii)*Specular reflexion:*
As in [4], we can also consider mixed boundary conditions of the form:

$$\gamma_- f = R(\gamma_+ f) = (1-\alpha)\mathcal{J}\gamma_+ f + \alpha K \gamma_+ f\,, \tag{12}$$

with $\mathcal{J}(\varphi)(x,v) = \varphi(x, \mathcal{R}_x v)$ where $\mathcal{R}_x$ is the symmetry defined by $\mathcal{R}_x \xi = \xi - 2(\xi \cdot n(x))n(x)$. $\alpha$ satisfies $0 \leq \alpha \leq 1$ and may depends on $x$ and $|v|$. It is easy to check that the Darrozès-Guirraud inequality (6) holds for the operator $R$, and that Proposition 1 is still valid. Proposition 2 needs the further assumption that $0 < \alpha_0 \leq \alpha(x,|v|)$ for some positive constant $\alpha_0$.

iii) *Non-elastic reflexion:*
Throughout this note, we considered elastic colisions, which gives rise to the very simplified expression of the Darrozès-Guirraud inequality. There is another particular case in which similar results can be obtained: We assume that the velocity set ($\mathbb{R}^n$ in this note) is now a bounded domain $V$ (e.g. the first Brillouin zone in semiconductor device), see [2]. Moreover, we assume that the electric field derives from a potential $E(x) = \nabla_x \Phi(x)$, and that the cross section satisfies (3). Therefore, (H2) is satisfied with the following modified maxwellian distribution:

$$M(v) = Ce^{-\frac{|v|^2}{2} + \Phi(x)}\,.$$

Furthermore, we notice that $M$ satisfies $\Lambda_E M = 0$. In this case, we obtain similar result to Proposition 1 and 2, replacing $L^2$ by:

$$E = \{ f(x,v)\ s.t. \int_{\mathcal{O}} \frac{|f(x,v)|^2}{M(x,v)}\, dx\, dv\ < \infty \}\,.$$

**Acknowledgments:**
The authors wish to thank A. Heintz, F. Poupaud and C. Cercignani for their remarks and advices.

## References

[1] C. Bardos, Problèmes aux limites pour les EDP du premier ordre à coefficients réels; Théorèmes d'approximation; application à l'équation de transport, *Ann. Scient. Éc. Norm. Sup.*, $4^{ème}$ *série,* **3** *(1970), pp. 185-233.*

[2] N. Ben Abdallah, P. Degond, A. Mellet, F. Poupaud, Electron transport in semiconductor superlattices, *to appear in Quarterly Appl. Math.*

[3] J.S. Darrozès, J.P. Guiraud, Généralisation formelle du théorème H en présence de parois. *C.R.A.S. Serie I,* **262** *(1966), pp. 1368-1371.*

[4] S. Mischler, On the trace problem for solutions of the Vlasov equation, *Comm. in P.D.E.,* **25** *(2000), pp. 1415-1443.*

[5] S. Mischler, Kinetics equations with Maxwell boundary conditions. Preprint 2000.

[6] F. Poupaud, The transport operator $a \cdot \nabla$, *unpublished.*

[7] S. Ukai, Solutions of the Boltzmann equations, Paterns and Waves Qaulitatives Analysis of Nonlinear Differential Equation, *Stud. Math. Appl.,* **18**, *North-Holland, (1986), pp. 37-96.*