# CONVERGENCE TO EQUILIBRIUM FOR THE CONTINUOUS COAGULATION-FRAGMENTATION EQUATION

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**Abstract**

We establish an H-Theorem for solutions to the continuous coagulation-fragmentation equation under the detailed balance condition. We deduce the convergence of the solution to an equilibrium state via a LaSalle invariance principle.

**Keywords:** H-Theorem, convergence to equilibrium, LaSalle invariance principle

# 1 Introduction

The purpose of this work is to investigate the long time convergence to an equilibrium state for solutions to the continuous coagulation-fragmentation model with reaction rates satisfying the so-called detailed balance condition. Recall that coagulation and fragmentation processes occur in the dynamics of cluster growth and describe the mechanisms by which clusters can coalesce to form larger ones or fragment into smaller pieces. Coagulation-fragmentation models then aim at describing the time evolution of the cluster-size distribution function. We refer to the survey papers [9, 2] and the references therein for a detailed account on the physical derivation and properties of coagulation-fragmentation models.

Denoting by $f(t,y) \geq 0$ the density of clusters of size $y \in \mathbb{R}_+ := (0,\infty)$ at time $t \geq 0$, the continuous coagulation-fragmentation equation (hereafter refered to as the CCF equation) reads

$$\partial_t f = Q(f), \quad (t,y) \in (0,\infty) \times \mathbb{R}_+ \,, \tag{1.1}$$

$$f(0,y) = f^{in}(y), \quad y \in \mathbb{R}_+ \,. \tag{1.2}$$

Here the coagulation-fragmentation reaction term $Q(f)$ is given by

$$Q(f) = Q_1(f) - Q_2(f) - Q_3(f) + Q_4(f)\,, \tag{1.3}$$

with

$$\begin{aligned}
Q_1(f)(y) &= \frac{1}{2} \int_0^y a(y',y-y')\ f(y')\ f(y-y')\ dy'\,, \\
Q_2(f)(y) &= \frac{1}{2} \int_0^y b(y',y-y')\ dy'\ f(y)\,, \\
Q_3(f)(y) &= \int_0^\infty a(y,y')\ f(y)\ f(y')\ dy'\,, \\
Q_4(f)(y) &= \int_0^\infty b(y,y')\ f(y+y')\ dy'\,,
\end{aligned}$$

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where $a$ and $b$ are the coagulation and fragmentation rates, respectively. Throughout the paper we make the following assumptions on the coagulation and fragmentation rates.

**(H1)** *Symmetry and growth assumption:* the rates $a, b \ : \mathbb{R}_+^2 \to \mathbb{R}_+$ are measurable functions satisfying the symmetry condition

$$0 < a(y,y') = a(y',y), \quad 0 < b(y,y') = b(y',y)\,, \quad \forall\,(y,y') \in \mathbb{R}_+^2\,, \tag{1.4}$$

and the growth conditions

$$0 \le a(y,y') \le A_0\ \left((1+y)^\alpha\ (1+y')^\beta + (1+y)^\beta\ (1+y')^\alpha\right)\,, \tag{1.5}$$

$$0 \le b(y,y') \le B_0\ (1+y+y')^\gamma\,, \tag{1.6}$$

for $(y,y') \in \mathbb{R}_+^2$, where $\alpha \in [0,1]$, $\beta \in [0,\alpha]$, $\gamma \in \mathbb{R}$ and $A_0$ and $B_0$ are positive real numbers. We further assume that, either

$$\alpha + \beta \le 1 \qquad \text{(weak coagulation)} \tag{1.7}$$

or there is $b_0 > 0$ such that

$$\begin{aligned} &\gamma > \alpha + \beta - 2, \quad \gamma > -1 \qquad \text{(strong fragmentation)} \\ &b(y,y') \ge b_0\ (1+y+y')^\gamma\,, \quad (y,y') \in (1,\infty)^2\,. \end{aligned} \tag{1.8}$$

The symmetry condition (1.4) is physically natural for the coagulation and fragmentation rates. The growth conditions (1.5)–(1.8) are more restrictive but still include a wide class of coagulation and fragmentation rates. Moreover, these conditions imply that there exists a solution $f$ to (1.1), (1.2) which is mass-conserving, that is,

$$Y_1(f(t)) := \int_0^\infty f(t,y)\,y\,dy = Y_1(f^{in})\,, \qquad \forall t \ge 0\,. \tag{1.9}$$

A more precise statement will be given in Theorem 1.1. Let us emphasize here that, in general, solutions to (1.1), (1.2) need not fulfil (1.9) and that we may have $Y_1(f(t)) < Y_1(f^{in})$ for $t$ large, a phenomenon known as gelation. However, the assumptions (1.7) and (1.8) prevent the occurrence of gelation, so that (1.9) holds true in our case. We finally refer to [14, 12, 11] and the references therein for more details about the gelation phenomenon.

We next make a structural assumption on the coagulation and fragmentation rates which guarantees the existence of stationary solutions to (1.1).

**(H2)** *Detailed balance condition:* there exists a positive function $M \in L^1_\mu(\mathbb{R}_+)$ such that

$$a(y,y')\,M(y)\,M(y') = b(y,y')\,M(y+y')\,, \quad \forall\,(y,y') \in \mathbb{R}_+^2\,, \tag{1.10}$$

$$M \in L^\infty(\mathbb{R}_+)\,, \quad \operatorname*{ess\,inf}_{y \in \mathbb{R}_+} \frac{\ln M(y)}{y} \ge -M_0 \tag{1.11}$$

for some positive constant $M_0$, where

$$\mu := \max\{1+\alpha+\beta, 2+\gamma\} \tag{1.12}$$

and $L^p_s(\mathbb{R}_+) := L^p(\mathbb{R}_+, (1+y)^s dy)$ for $s \in \mathbb{R}$ and $p \in [1,\infty]$.

Clearly, the function $M_z$ defined by $M_z(y) := M(y)\ z^y$, $y \in \mathbb{R}_+$, also satisfies (1.10) for each $z \in \mathbb{R}_+$ and $M_z$ is a stationary solution to (1.1) usually called an *equilibrium*. Observe that $M_z$ does not necessarily belong to $L^1_1(\mathbb{R}_+)$ for $z$ large. Furthermore, the detailed balance condition (H2) ensures that the "entropy"

$$H(f) := \int_0^\infty \left[f\ \left(\ln\frac{f}{M} - 1\right) + M\right]\,dy \ge 0 \tag{1.13}$$

is (at least formally) a strict Liapunov functional. That is, if $f$ is a solution to (1.1), either $t \mapsto H(f(t))$ is a strictly decreasing function of time or there exists a time $T \geq 0$ such that $H(f(t))$ is constant and $f$ is identically equal to an equilibrium for $t \geq T$.

A typical example of coagulation and fragmentation rates enjoying the detailed balance condition is the following:

$$\tag{1.14} \left\{ \begin{array}{lcl} a(y,y') & = & A_0\ (1+y)^\alpha\ (1+y')^\alpha, \\ & & \\ b(y,y') & = & B_0\ a(y,y')\ \dfrac{\exp\left(\lambda\ (y+y')^p\right)}{\exp\left(\lambda\ \ (y^p+y'^p)\right)}\ \dfrac{(1+y+y')^\tau}{(1+y)^\tau\ (1+y')^\tau}, \end{array} \right.$$

where $\alpha \in [0,1]$, $\lambda > 0$, $p \in [0,1)$, $\tau \in [0,+\infty)$, and $A_0$, $B_0$ are positive real numbers. In that case, $M(y) = (1+y)^{-\tau}\ \exp\left(-\lambda\ y^p - y\right)$, $y \in \mathbb{R}_+$. For instance, (H1) and (H2) are fulfilled with $\beta = \alpha \in [0,1/2]$ and $\gamma = \alpha + |\tau - \alpha|$. Notice also that the case of constant coefficients $a$ and $b$ is included in the above example (with $\alpha = \tau = p = 0$).

Before stating our results, let us recall an existence result for (1.1), see [10, 17, 11].

**Theorem 1.1** *Assume that a, b satisfy (H1)–(H2) and that the initial condition satisfies*

$$\tag{1.15} 0 \leq f^{in} \in L^1_\mu(\mathbb{R}_+) \quad \text{and} \quad H(f^{in}) < \infty\ .$$

*There is a solution f to the CCF equation (1.1), (1.2) satisfying*

$$\tag{1.16} f \in \mathcal{C}([0,\infty); L^1_\mu(\mathbb{R}_+)) \quad \text{and} \quad H(f) \in L^\infty_{loc}([0,\infty))\,,$$

$$\tag{1.17} D(f) := (a\, f\, f' - b\, f'')\ (\ln(a\, f\, f') - \ln(b\, f'')) \in L^1_{loc}([0,\infty) \times \mathbb{R}^2_+)\,,$$

*and the conservation of mass (1.9). Here and below, we use the notations $f = f(y)$, $f' = f(y')$ and $f'' = f(y+y')$.*

The existence of a mass-conserving solution to (1.1), (1.2) satisfying also the first assertion of (1.16) follows from [10, 17, 11]. The second assertion of (1.16) and (1.17) rely on the lower semicontinuity of $H$ and $D$ and may be proved by similar arguments as [16, Theorem 5.1]. The class of coefficients $a$ and $b$ considered in [16] is actually more restrictive than (H1) but the proof still works in the present framework.

We may now state an H-Theorem for (1.1).

**Theorem 1.2** *Under the assumptions of Theorem 1.1, the following identity (H-Theorem) is true:*

$$\tag{1.18} H(f(t_2)) - H(f(t_1)) = -\int_{t_1}^{t_2} \mathcal{D}(f(t))\, dt\,, \quad t_2 \geq t_1 \geq 0\,,$$

*where the dissipation of entropy $\mathcal{D}(f)$ is defined by*

$$\tag{1.19} \mathcal{D}(f) := \frac{1}{2} \int_0^\infty \int_0^\infty (a\, f\, f' - b\, f'')\, (\ln(a\, f\, f') - \ln(b\, f''))\ dy dy'\,.$$

As far as we know, the identity (1.18) has not been established previously for the CCF equation (1.1) but it is already known for the discrete coagulation-fragmentation equations [3, 5, 6]. The proofs however rely partly on different arguments, owing to the different functional settings ($\ell^1(\mathbb{N} \setminus \{0\})$ for the latter and $L^1(\mathbb{R}_+)$ for the former).

Theorem 1.2 is obviously the first step towards the study of the trend to equilibrium of solutions to (1.1) and we now investigate this issue. In order to get our next result, we need two additional assumptions.

**(H3)** *Monotonicity condition on the coagulation rate and control of the fragmentation rate by the coagulation rate:* there exists $A > 0$ such that

$$a(y', y - y') \le a(y', y) \quad \text{for} \;\; y \ge y' \ge 0\,, \tag{1.20}$$

$$b(y', y - y') \le A\; a(y, y') \quad \text{for} \;\; y \ge y' \ge 0\,. \tag{1.21}$$

Observe that the coagulation and fragmentation rates given in (1.14) satisfy (1.20), (1.21) if $\tau \le \alpha$. This assumption was introduced in [13, 4] and further developed in [16] and ensures that

$$\sup_{[0,\infty)} \int_0^\infty \Phi(f(t,y))\, dy < \infty$$

for a large family of convex functions $\Phi$ (see (3.1) below).

**(H4)** There exists $z_s \in (0, \infty)$ such that

$$\lim_{y\to\infty} M(y)^{1/y} = \frac{1}{z_s}\,. \tag{1.22}$$

This assumption allows us to introduce the modified entropy

$$H_{z_s}(f) := H(f) - Y_1(f)\, \ln z_s,$$

which has better continuity properties than the entropy $H$ [3]. The condition (H4) also implies that the function

$$\varrho(z) := \int_0^\infty z^y\, y\, M(y)\, dy \tag{1.23}$$

is increasing from $[0, z_s)$ onto $[0, \varrho(z_s))$ and satisfies

$$\begin{cases} \varrho(z) < \infty & \text{if} \quad 0 < z < z_s, \\ \varrho(z) = \infty & \text{if} \quad z > z_s\,. \end{cases} \tag{1.24}$$

We further define $\varrho_s := \varrho(z_s) \in (0, \infty]$. Our next result then reads as follows.

**Theorem 1.3** *Assume that $a$, $b$ satisfy (H1)–(H4) with (1.7), and that the initial condition satisfies (1.15). There is $z \in [0, z_s]$ such that the solution $f$ to (1.1), (1.2) given by Theorem 1.1 satisfies*

$$f(t) \underset{t\to\infty}{\longrightarrow} M_z \quad \textit{in} \quad L^1(\mathbb{R}_+) \tag{1.25}$$

*and*

$$H(f(t)) \underset{t\to\infty}{\longrightarrow} H(M_z)\,. \tag{1.26}$$

The proof of Theorem 1.3 is in the spirit of the LaSalle invariance principle. In the context of coagulation-fragmentation models, such an approach seems to have been introduced in [3] for the Becker-Döring equations. It was further developed in [5, 6] for the discrete coagulation-fragmentation equations and in [15] for the Becker-Döring equations with diffusion.

Theorem 1.3 completes the analysis performed in [16] where a result of stabilization has been proved under the sole assumptions (H1) and (H2). More precisely, it is shown in [16, Theorem 2.3] that the set $\{f(t),\ t \ge 0\}$ is strongly compact in $L^1(\mathbb{R}_+)$ and the *$\omega$-limit set* (that is, the set of cluster points of $f(t)$ as $t \to \infty$) only contains equilibria. The proof of this result relies on the lower semicontinuity of the dissipation of entropy $\mathcal{D}$ (see [19] and the references therein). Theorem 1.3 is thus a step further since it guarantees that the $\omega$-limit set reduces to a single point. In fact, the proof of Theorem 1.3 combines the results from [16, Theorem 2.3] with the additional information obtained from the assumptions made here. Unfortunately, it does not allow to conclude that

$$\begin{aligned} &\varrho(z) = Y_1(f^{in}) \quad \text{if} \quad Y_1(f^{in}) < \varrho_s\,, \\ &\varrho(z) = \varrho_s \qquad\quad\;\; \text{if} \quad Y_1(f^{in}) \ge \varrho_s\,, \end{aligned} \tag{1.27}$$

which is the natural conjecture following from (1.9) since there is no equilibrium with mass $Y_1 > \varrho_s$ when $\varrho_s < \infty$. This conjecture is known to be true for the Becker-Döring equations (under suitable assumptions on the coefficients, see [3, 8, 21] and the references therein) but is still a widely open question for the general coagulation-fragmentation equations (1.1) and its discrete counterpart. The main difficulty encountered is the possibility of a loss of mass as $t \to \infty$, that is, the lack of compactness of $\{f(t),\ t \ge 0\}$ in $L^1_1(\mathbb{R}_+)$. There are however some situations for which the above conjecture can be proved: either when $z_s = \infty$ or under the assumption of strong fragmentation (1.8) which was introduced in [5]. In both cases, $\varrho_s = \infty$ and it is possible to control the behaviour of $f(t,y)$ for large values of $y$ in $L^1_1(\mathbb{R}_+)$, the control being uniform with respect to $t > 0$ [22, 16, 11]. Consequently, any element of the $\omega$-limit set has the same mass $Y_1(f^{in})$ as the initial datum, from which the validity of (1.27) follows. For completeness, we state below the analogue of Theorem 1.3 obtained under the strong fragmentation assumption.

**Proposition 1.4** *Assume that $a$, $b$ satisfy (H1)–(H4) with (1.8). Then the solution $f$ to (1.1), (1.2) given by Theorem 1.1 satisfies (1.25) with $z \in [0, z_s]$ given by $\varrho(z) = Y_1(f^{in})$.*

Notice that the case where $a$ and $b$ are both constants is included in Proposition 1.4. A more precise result is actually available in that case: assuming for simplicity that $f^{in}$ is such that $Y_1(f^{in}) = 1$, it is proved in [1] that $f(t)$ converges strongly towards $M(y) = e^{-y}$ in $L^1(\mathbb{R}_+)$ with an exponential rate. The approach in [1] is completely different and is based on an estimate from below of the dissipation of entropy $\mathcal{D}(f)$ in terms of the entropy $H(f)$.

To conclude, let us mention that an analogue of Theorem 1.3 is not known for the coagulation-fragmentation equations with diffusion (except for the Becker-Döring equations with diffusion [15], see also [7] for stabilization results). An important feature of our approach is that, as soon as one obtains Theorem 1.2 for the coagulation-fragmentation equations with diffusion, the proof of Theorem 1.3 extends straightforwardly to this situation. In other words, under assumptions (H1)–(H4), proving the convergence to an equilibrium reduces to the questions of mass conservation and H-theorem.

## 2 H-Theorem

The proof follows the lines of the proof of Lu for the Boltzmann equation [20]. Let $T \in \mathbb{R}_+$. For $n \ge 1$ and $y \in \mathbb{R}_+$, we define

$$h_n(f) = (f + \phi_n)\left(\ln\left(\frac{f \wedge n + \phi_n}{M}\right) - 1\right) + M \quad \text{with} \quad \phi_n(y) = \frac{M(y)}{n}\,, \tag{2.1}$$

and compute the time derivative of

$$H_n(f) \ := \int_0^\infty h_n(f)\, dy$$

for $t \in [0,T]$. We first observe that, since $f \in L^\infty(0,T; L^1_\mu(\mathbb{R}_+))$, it follows from (1.5), (1.6) and the Fubini theorem that $Q(f) \in L^\infty(0,T; L^1_1(\mathbb{R}_+))$ while (1.11) ensures that

$$\ln X_n \in L^\infty(0,T; L^\infty_{-1}(\mathbb{R}_+)) \quad \text{where} \quad X_n := \frac{f \wedge n + \phi_n}{M}\,.$$

We therefore deduce from (1.1) that $\partial_t f \in L^\infty(0,T;L^1_1(\mathbb{R}_+))$ and it follows from (1.1) that

$$\frac{d}{dt}H_n(f) = \int_0^\infty \frac{\partial f}{\partial t}\,\frac{\partial h_n(f)}{\partial f}\,dy = \int_0^\infty Q(f)\,\left[\ln X_n - \mathbf{1}_{\{f\geq n\}}\right]\,dy$$

$$= -\frac{1}{2}\int_0^\infty\int_0^\infty (a\,f\,f' - b\,f'')\,\left(\ln\left(X_n\,X_n'\right) - \ln\left(X_n''\right)\right)\,dydy'$$

$$-\int_0^\infty Q(f)\mathbf{1}_{\{f\geq n\}}\,dy\,,$$

the last equality being a consequence of the Fubini Theorem. We next put

$$D_n = (a\,f\,f' - b\,f'')\,\left(\ln\left(X_n\,X_n'\right) - \ln\left(X_n''\right)\right)$$

and write $D_n = D_n^+ - D_n^-$ with $D_n^+ = \max\{D_n,0\}$, $D_n^- = \max\{-D_n,0\}$. Integrating the previous identity over $(0,t)$ with $t\in[0,T]$, we obtain

$$H_n(f(t)) - H_n(f(0)) \;=\; -\int_0^t \left\{\frac{1}{2}\int_0^\infty\int_0^\infty \left(D_n^+ - D_n^-\right)dydy' + \int_0^\infty Q(f)\,\mathbf{1}_{\{f\geq n\}}\,dy\right\}\,dt\,. \tag{2.2}$$

To pass to the limit as $n\to\infty$ in (2.2), we need some properties of $D_n^+$ and $D_n^-$ which we gather in the next lemma.

**Lemma 2.1** *For $n\geq \|M\|_{L^\infty}$, there holds*

$$0\leq D_n^- \leq F\,, \qquad 0\leq D_n^+ \leq D(f) + F, \tag{2.3}$$

*with*

$$\begin{aligned} F \;&:=\; 2\,(a+b)\,\left\{(f''+M'')\,(1+f+M+f'+M')\right.\\ &\qquad \left. +\,(f+M)\,(f'+M')\right\} \in L^1((0,T)\times\mathbb{R}_+^2)\,. \end{aligned}$$

*Proof.* We consider four cases:

• $a\,f\,f' \geq b\,f''$ and $X_n\,X_n' \geq X_n''$. Then $D_n^- = 0$ and $D_n^+ = D_n = D(f) + A$ with

$$\begin{aligned} A \;&:=\; (a\,f\,f' - b\,f'')\,\ln\left(\frac{b\,f''\,X_n\,X_n'}{a\,f\,f' X_n''}\right)\\ &\leq\; (a\,f\,f' - b\,f'')\,\frac{b\,f''\,X_n\,X_n'}{a\,f\,f' X_n''} \leq b\,f''\,\frac{X_n\,X_n'}{X_n''}\\ &\leq\; a\,f''\,\frac{(f\wedge n + \phi_n)\,(f'\wedge n + \phi_n')}{f''\wedge n + \phi_n''}\\ &\leq\; a\,(f+M)\,(f'+M')\,\mathbf{1}_{\{f''\leq n\}} + 2\,a\,f''\,(f'+M')\,\mathbf{1}_{\{f''>n\}} \leq F\,. \end{aligned}$$

• $a\,f\,f' \leq b\,f''$ and $X_n\,X_n' \geq X_n''$. Then $D_n^+ = 0$ and

$$D_n^- = -D_n = (b\,f'' - a\,f\,f')\,\ln\left(\frac{X_n\,X_n'}{X_n''}\right) \leq b\,f''\,\frac{X_n\,X_n'}{X_n''} \leq F$$

by the same argument as above.

• $a\,f\,f' \geq b\,f''$ and $X_n\,X_n' \leq X_n''$. Then $D_n^+ = 0$ and

$$\begin{aligned} D_n^- \;&=\; -D_n = (a\,f\,f' - b\,f'')\,\ln\left(\frac{X_n''}{X_n\,X_n'}\right)\\ &\leq\; a\,f\,f'\,\frac{X_n''}{X_n\,X_n'} \leq b\,f\,f'\,\frac{f''\wedge n + \phi_n''}{(f\wedge n + \phi_n)\,(f'\wedge n + \phi_n')}\\ &\leq\; b\,\left[(f''+M'')\,\mathbf{1}_{\{f,f'\leq n\}} + f\,f'\,\mathbf{1}_{\{f,f'\geq n\}}\right.\\ &\qquad \left. +(f+f')\,(f''+M'')\,\mathbf{1}_{\{f\leq n\leq f'\}\cup\{f\geq n\geq f'\}}\right] \leq F\,. \end{aligned}$$

• $a\, f\, f' \le b\, f''$ and $X_n\, X_n' \le X_n''$. Then $D_n^- = 0$ and $D_n^+ = D_n = D(f) + B$ with

$$B := (b\, f'' - a\, f\, f')\, \ln\left(\frac{a\, f\, f'\, X_n''}{b\, f''\, X_n\, X_n'}\right) \le a\, f\, f'\, \frac{X_n''}{X_n\, X_n'} \le F$$

by the same argument as above.

Finally, by (H1), we have

$$F \le C\,(1+y+y')^{\mu-1}\, [(f''+M'')\,(1+f+M+f'+M') + (f+M)\,(f'+M')]$$

which belongs to $L^\infty(0,T;L^1(\mathbb{R}_+^2))$ thanks to (H2) and (1.16). ◻

Since $D(f) \in L^1((0,t)\times\mathbb{R}_+^2)$ by (1.17), $(D_n)$ converges almost everywhere to $D(f) \ge 0$ and it follows from Lemma 2.1, the regularity of $Q(f)$ and $f$, and the Lebesgue dominated convergence theorem that

$$\begin{aligned} &D_n^+ \;\to\; D(f) \quad\text{and}\quad D_n^- \;\to\; 0 \quad\text{in}\quad L^1((0,t)\times\mathbb{R}_+^2)\,,\\ &Q(f)\,\mathbf{1}_{\{f\ge n\}} \;\to\; 0 \quad\text{in}\quad L^1((0,t)\times\mathbb{R}_+)\,. \end{aligned}$$

Consequently, the right-hand side of (2.2) converges to the right-hand side of (1.18) (with $t_1 = 0$ and $t_2 = t$) as $n \to \infty$.

Finally, the convexity of $r \mapsto r\;\ln r$ entails that

$$\begin{aligned} 0 \le h_n(f) &\le (f+M)\;\left(\ln\left(\frac{f+M}{M}\right)-1\right) + M\\ &\le \frac{1}{2}\,(2\;f\;\ln(2f) + 2\;M\;\ln(2M))) - (f+M)\;\ln M - f\\ &\le (f+M)\;\ln 2 + f\;\left(\ln\left(\frac{f}{M}\right)-1\right) + M \in L^1(\mathbb{R}_+) \end{aligned}$$

by (1.16) and (H2). We then pass to the limit in the left-hand side of (2.2) with the Lebesgue dominated convergence theorem to conclude the proof of Theorem 1.2.

# 3 Convergence to equilibrium

We now turn to the proof of Theorem 1.3 which is divided into five steps.

*Step 1.* We claim that the solution $f$ given by Theorem 1.1 satisfies: there exists $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\frac{\Phi(r)}{r\ln r} \underset{r\to\infty}{\longrightarrow} \infty \quad\text{and}\quad \sup_{t\ge 0}\int_0^\infty \Phi(f(t,y))\,dy < \infty\,. \tag{3.1}$$

Such a property has been used in [16] for a different purpose and we recall its main steps for completeness. It first follows from (1.15) and a refined version of the de la Vallée Poussin theorem [18, Proposition I.1.1] that we can build a non-negative and convex function $\Phi \in W^{1,\infty}_{\mathrm{loc}}([0,+\infty))$ such that $\Phi(r)/(r\ln r) \to \infty$ when $r \to \infty$, $\Phi(r) \le r\,\Phi'(r)$ for any $r \ge 0$, $\Phi(r) = 0$ for $r \in [0,4A]$ and $\Phi(f^{in}) \in L^1$ (see [16, Corollary 3.2 & Appendix B] for details). Then, arguing as in [16, Lemma 3.5 & Lemma 3.7] there holds

$$\int_0^\infty (Q_1(f) - Q_3(f))\,\Phi'(f)\,dy \le -\frac{1}{2}\int_0^\infty\int_0^\infty a(y,y')\,f'\,f\,\Phi'(f)\,dy'dy\,,$$

$$\int_0^\infty Q_4(f)\;\Phi'(f)\;dy \le \frac{1}{4}\int_0^\infty\int_0^\infty a(y,y')\;f\;f'\;\Phi'(f)\;dy'dy\,,$$

so that

$$\frac{d}{dt}\int_0^\infty \Phi(f(t,y))\,dy = \int_0^\infty Q(f)\,\Phi'(f)\,dy \le 0\,,$$

and (3.1) is proved. Notice that the arguments given above are mainly formal but can be justified by approximating $\Phi$ by convex functions which increase linearly at infinity.

*Step 2.* Let $(t_n)$ be a sequence such that $t_n \to +\infty$. We may argue as in the proof of [16, Theorem 2.3] to prove that there are a subsequence $(t_{n'})$ and an equilibrium $\bar{M}$ such that

$$f(t_{n'}+1) \to \bar{M} \quad \text{in} \quad L^1(\mathbb{R}_+)\,. \tag{3.2}$$

In addition, (1.9) and (3.2) imply that

$$\int_0^\infty \bar{M}(y)\, y\, dy \le Y_1(f^{in})\,. \tag{3.3}$$

*Step 3.* Thanks to the properties of $f$, we are in a position to apply the H-Theorem (Theorem 1.2) and conclude that $t \mapsto H(f(t))$ is non-increasing. We then infer from the non-negativity of $H$ and the mass conservation (1.9) that there is $\bar{H} \in \mathbb{R}$ such that

$$H_{z_s}(f(t_{n'}+1)) \to \bar{H} := \inf_{t\ge 0} H_{z_s}(f(t))\,, \tag{3.4}$$

where

$$H_{z_s}(f) := H(f) - Y_1(f)\, \ln z_s\,.$$

*Step 4.* We then claim that

$$H_{z_s}(f(t_{n'}+1)) \to H_{z_s}(\bar{M}). \tag{3.5}$$

In order to see this, we write

$$H_{z_s}(f) = \int_0^\infty f\, \ln f\, dy - \int_0^\infty f\, \left[ y\, \ln(z_s\, M(y)^{1/y}) + 1 \right]\, dy\,.$$

On the one hand, observing that

$$\begin{aligned} |f\, \ln f| &\le f\,(-\ln f)\, \mathbf{1}_{\{0\le f\le e^{-\sqrt{y}}\}} + f\,(-\ln f)\, \mathbf{1}_{\{e^{-\sqrt{y}}\le f\le 1\}} + f\, \ln f\, \mathbf{1}_{\{f\ge 1\}} \\ &\le e^{-\sqrt{y}/2} \sup_{r\in[0,1]} (r^{1/2}\, |\ln(r)|) + f\, \sqrt{y} + f\, (\ln f)^+\,, \end{aligned}$$

we deduce from (1.9) and (3.1) that, for any $R>1$,

$$\begin{aligned} & \int_R^\infty |f(t_{n'}+1,y)\, \ln f(t_{n'}+1,y)|\, dy \\ \le\ & \frac{2}{e} \int_R^\infty e^{-\sqrt{y}/2}\, dy + \frac{1}{R^{1/2}}\, Y_1(f^{in}) + R^{1/2} \int_R^\infty f(t_{n'}+1,y)\, dy \\ & + \int_R^\infty f(t_{n'}+1,y)\, \ln f(t_{n'}+1,y)\, \mathbf{1}_{\{f(t_{n'}+1,y)\ge e^{\sqrt{R}}\}}\, dy \\ \le\ & \frac{2}{e} \int_R^\infty e^{-\sqrt{y}/2}\, dy + \frac{2}{R^{1/2}}\, Y_1(f^{in}) \\ & + \left( \sup_{r\ge e^{\sqrt{R}}} \frac{r}{\Phi(r)} \right) \sup_{t\ge 0} \int_0^\infty \Phi(f(t,y))\, dy\,, \end{aligned}$$

and the right-hand side of the above inequality converges to zero as $R \to \infty$ uniformly with respect to $n'$ by (3.1). Using once more (3.1), we thus conclude that $\left(f(t_{n'}+1)\, \ln f(t_{n'}+1)\right)_{n'}$ is weakly compact in $L^1(\mathbb{R}_+)$ which, together with (3.2) and the Vitali theorem, ensures that

$$\int_0^\infty f(t_{n'}+1,y)\, \ln f(t_{n'}+1,y)\, dy \to \int_0^\infty \bar{M}\, \ln \bar{M}\, dy\,.$$

On the other hand, owing to (1.22), $\ln(z_s\, M(y)^{1/y}) \to 0$ as $y \to \infty$ and it follows from (1.9) and (3.2) that

$$\int_0^\infty f(t_{n'}+1,y)\, \Big[y\, \ln(z_s\, M(y)^{1/y})+1\Big]\, dy \to \int_0^\infty \bar{M}\, \Big[y\, \ln(z_s\, M(y)^{1/y})+1\Big]\, dy\,.$$

Combining the previous two assertions yields the claim (3.5).

*Step 5.* Clearly, the conditions

$$\bar{M} \quad \text{is an equilibrium and} \quad H_{z_s}(\bar{M}) = \bar{H} \tag{3.6}$$

determine a unique $\bar{z} \in [0, z_s]$ (depending only on $\bar{H}$) such that $\bar{M} = M_{\bar{z}}$. Indeed, arguing as in [3, Proposition 4.3] shows that $z \mapsto H_{z_s}(M_z)$ is decreasing on $[0, z_s)$. We have thus proved that $\{f(t)\}$ has only one cluster point in $L^1(\mathbb{R}_+)$ as $t \to \infty$. Consequently, $f(t)$ converges to $M_{\bar{z}}$ in $L^1(\mathbb{R}_+)$ as $t \to \infty$ and we have proved (1.25).

**Proof of Proposition 1.4**. We proceed as in [5]. Arguing as in step 2 of the proof of Theorem 1.3, we have (3.2). On the other hand, it follows from [11] that

$$\sup_{t \geq 1} \int_0^\infty f(t,y)\, y^2\, dy < \infty.$$

This implies that $Y_1(\bar{M}) = Y_1(f^{in})$ which uniquely determines $\bar{M}$. □

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