

# GELATION AND MASS CONSERVATION IN COAGULATION-FRAGMENTATION MODELS

M. Escobedo<sup>1</sup>, Ph. Laurençot<sup>2</sup>,

S. Mischler<sup>3,4</sup> and B. Perthame<sup>4</sup>

**Abstract.** The occurrence of gelation and the existence of mass-conserving solutions to the continuous coagulation-fragmentation equation are investigated under various assumptions on the coagulation and fragmentation rates, thereby completing the already known results. A non-uniqueness result is also established and a connection to the modified coagulation model of Flory is also made.

**Key words.** Coagulation, strong fragmentation, gelation, non-uniqueness, Flory's coagulation equation.

## 1 Introduction

Coagulation-fragmentation models describe the dynamics of cluster growth, the sizes of the clusters evolving with time as the clusters undergo coagulation and fragmentation events. Hereafter, we restrict ourselves to binary reactions, that is, we only take into account the merging of two clusters to form a larger one and the break-up of a cluster into two smaller ones, without any loss of mass during these events. Denoting by  $f(t, y)$  the size distribution function at time  $t$ , the continuous coagulation-fragmentation equation reads [6]

$$(1.1) \quad \frac{\partial f}{\partial t} = Q(f), \quad (t, y) \in (0, +\infty) \times \mathbb{R}_+,$$

$$(1.2) \quad f(0, y) = f^{in}(y), \quad y \in \mathbb{R}_+,$$

where the coagulation-fragmentation reaction term  $Q(f) = Q_c(f) - Q_f(f)$  is given by

$$Q_c(f) = Q_1(f) - Q_2(f), \quad Q_f(f) = Q_3(f) - Q_4(f),$$

$$Q_1(f)(y) := \frac{1}{2} \int_0^y a(y', y - y') f(y') f(y - y') dy',$$

$$Q_2(f)(y) := f(y) \int_0^\infty a(y, y') f(y') dy',$$

$$Q_3(f)(y) := \frac{1}{2} \int_0^y b(y', y - y') dy' f(y),$$

$$Q_4(f)(y) := \int_0^\infty b(y, y') f(y + y') dy'.$$

---

<sup>1</sup>Departamento de Matemáticas, Universidad del País Vasco, Apartado 644, E-48080 Bilbao, Spain. E-mail: [mtpesmam@lg.ehu.es](mailto:mtpesmam@lg.ehu.es)

<sup>2</sup>Mathématiques pour l'Industrie et la Physique, CNRS UMR 5640, Université Paul Sabatier – Toulouse 3, 118 route de Narbonne, F-31062 Toulouse cedex 4, France. E-mail : [laurenco@mip.ups-tlse.fr](mailto:laurenco@mip.ups-tlse.fr)

<sup>3</sup>Laboratoire de Mathématiques Appliquées, Université de Versailles – Saint Quentin, 45 avenue des Etats-Unis, F-78035 Versailles, France. E-mail: [mischler@math.uvsq.fr](mailto:mischler@math.uvsq.fr)

<sup>4</sup>DMA, CNRS UMR 8553, Ecole Normale Supérieure, 45 rue d'Ulm, F-75230 Paris cedex 05, France. E-mail: [perthame@dma.ens.fr](mailto:perthame@dma.ens.fr)

Here  $y \in \mathbb{R}_+$  denotes the mass (or volume, or size) of the clusters and  $a$  and  $b$  denote the coagulation and fragmentation rates, respectively. The rates  $a$  and  $b$  are assumed to depend only on the sizes of the clusters involved in the reactions and satisfy

$$(1.3) \quad 0 \leq a(y, y') = a(y', y) \leq A (1 + y) (1 + y'), \quad (y, y') \in \mathbb{R}_+^2,$$

$$(1.4) \quad 0 \leq b(y, y') = b(y', y), \quad (y, y') \in \mathbb{R}_+^2.$$

Also, throughout the paper, the following assumption is made on the initial datum  $f^{in}$ :

$$(1.5) \quad f^{in} \in L^1_1(\mathbb{R}_+) := L^1(\mathbb{R}_+; (1 + y)dy) \text{ and is non-negative a.e.}$$

Introducing the (total) number  $M_0(f(t))$  and the (total) mass  $M_1(f(t))$  of clusters at time  $t$  defined by

$$(1.6) \quad M_\ell(t) = M_\ell(f(t)) := \int_0^\infty y^\ell f(t, y) dy, \quad \ell \geq 0,$$

it is clear that  $M_0(f)$  is increased by coagulation events and decreased by fragmentation events, while  $M_1(f)$  does not vary during these events. It is however a physically relevant and mathematically challenging question to figure out whether the mass  $M_1(f)$  of solutions to (1.1) is kept constant throughout time evolution. In fact, several works in the physical literature have considered this question for the pure coagulation equation ( $b \equiv 0$ ) and either formal arguments or explicit solutions have been provided to show that the conservation of mass holds true for  $a(y, y') = (yy')^\alpha$  when  $\alpha \in [0, 1/2]$  and breaks down in finite time when  $\alpha \in (1/2, 1]$  [23, 17, 16, 10, 8]. In the latter case, we say that a *gelation transition* occurs. Still, mathematical proofs of the occurrence of gelation including larger classes of coagulation rates and initial data, and also fragmentation, have only been supplied recently, either by probabilistic arguments [11] (for the discrete model) or by deterministic arguments [9]. In particular, it is shown in [9] that, if

$$(1.7) \quad a(y, y') = y^\alpha y'^\beta + y^\beta y'^\alpha, \quad b(y, y') = (1 + y + y')^\gamma,$$

with  $0 \leq \alpha \leq \beta \leq 1$  and  $\gamma \in \mathbb{R}$ , gelation occurs if  $\lambda := \alpha + \beta > 1$  and  $\gamma < (\lambda - 3)/2$  (notice that the parameter  $\gamma$  in (1.7) corresponds to the parameter  $-\gamma$  in [9]). On the other hand, existence of mass-conserving solutions is known when  $a(y, y') \leq A_0 (1 + y + y')$  under various assumptions on the fragmentation rates [19, 7, 14] which include the rates given by (1.7) whenever  $\lambda \leq 1$  and  $\gamma \in \mathbb{R}$ . Furthermore, it is known for the discrete model that a sufficiently strong fragmentation prevents the occurrence of gelation [4] and a natural guess is that a similar result holds true for the continuous model (1.1).

One of the main results of this paper is thus to establish the existence of a mass-conserving solution to the continuous model (1.1) when the fragmentation is sufficiently strong with respect to the coagulation (see Section 3). In particular, for the model case (1.7) with  $\lambda > 1$ , this is true if  $\gamma > \lambda - 2$ . Since gelation is known to occur for  $\gamma < (\lambda - 3)/2$  by [9] and  $(\lambda - 3)/2 < \lambda - 2$ , it remains to check what happens when  $\gamma \in [(\lambda - 3)/2, \lambda - 2)$ . It turns out that a further development of the proof of [9] allows us to show that gelation also occurs in that case for sufficiently large initial data (Section 2). More precisely, our main results are the following.

**Theorem 1.1** *Assume that the reaction rates are given by (1.7) with  $0 \leq \alpha \leq \beta \leq 1$  and  $\gamma \in \mathbb{R}$ .*

1. *If  $\lambda := \alpha + \beta \leq 1$  or if  $\gamma > \lambda - 2$  there exists a weak solution to (1.1), (1.2) which conserves the mass.*

2. *If  $\lambda > 1$  and  $\gamma < \lambda - 2$  there exists  $M_1^*$  such that, if  $M_1(f^{in}) > M_1^*$  then gelation occurs for any weak solution to (1.1), (1.2), that is,*

$$T_{gel} := \inf \{t \geq 0, M_1(f(t)) < M_1(f^{in})\} < \infty.$$

We will actually prove Theorem 1.1 for a larger class of coefficients  $a$  and  $b$  and refer to Theorems 2.1 and 3.1 for precise statements.

**Remark 1.2** 1. In the case  $\lambda \leq 1$ , the existence of a mass-conserving solution has been previously established in [21, 7, 14] under additional assumptions on the fragmentation rates or the initial data. Let us emphasize that there is no growth condition on  $\gamma$  in contrast to the above mentioned results. On the other hand, the case  $\gamma > \lambda - 2$  is new. A similar result has been proved for the discrete model [4].

2. This result has been proved in [9] under the condition  $\lambda \in (1, 2]$  and  $\gamma < (\lambda - 3)/2$ . Herein, we fill the gap  $\gamma \in [(\lambda - 3)/2, \lambda - 2)$ .

Let us also point out that the “critical” exponent  $\gamma_c := \lambda - 2$  has already been noticed in the physical literature. In [20], the authors consider the coagulation-fragmentation equation under similar homogeneity hypotheses on the coagulation and fragmentation rates. Assuming that no gelation occurs, formal arguments lead them to a differential equation for the time evolution of the cluster mean size  $s(t) := M_2(f(t))/M_1$ . They observed that, for  $\gamma > \lambda - 2$ , the cluster mean size  $s(t)$  converges to a stable equilibrium as  $t \rightarrow \infty$ . Whilst, if  $\gamma < \lambda - 2$ , this equilibrium is unstable. Another related analysis has been performed in [18] with coagulation and fragmentation rates  $a$  and  $kb$ ,  $a$  and  $b$  being homogeneous functions of degree  $\lambda \leq 1$  and  $\gamma \geq -1$ , respectively, and  $k$  being a positive real number. The asymptotic behaviour of the solutions in the limit  $y \rightarrow \infty$ ,  $t \rightarrow \infty$  and  $k \rightarrow 0$  is analysed by means of formal scaling arguments, the quantity  $T := t k^{1/(\gamma+2-\lambda)}$  being fixed.

**Remark 1.3** As a final comment on Theorem 1.1 and [9, Theorem 1.4], we remark that, when  $\lambda > 1$  and  $\gamma < \lambda - 2$ , gelation is only known to take place if  $M_1(f^{in})$  is large enough and one may wonder whether gelation also occurs when  $M_1(f^{in})$  is small. This is actually true in some particular cases (see [9, Theorem 1.4]) but it is not yet understood in the general case. For homogeneous coagulation and fragmentation rates of degree  $\lambda \in (1, 2]$  and  $\gamma < \lambda - 2$ , formal scaling arguments which we perform in the appendix lead us to the following conjectures: for  $\gamma < (\lambda - 3)/2$ , gelation should occur for all initial data  $f^{in} \not\equiv 0$  while there shall be mass-conserving solutions when  $\gamma \in ((\lambda - 3)/2, \lambda - 2)$  for initial data  $f^{in}$  with  $M_1(f^{in})$  sufficiently small.

The remainder of the paper is devoted to some consequences of strong fragmentation: we first prove a non-uniqueness result.

**Theorem 1.4** Assume that

$$(1.8) \quad \begin{cases} a(y, y') \leq C_a (1 + y + y')^\lambda, \\ C_b (1 + y + y')^\gamma \leq b(y, y') \leq C'_b (1 + y + y')^{\gamma'}, \end{cases}$$

with  $\lambda \in [0, 2]$ ,  $\gamma > \max\{\lambda - 2, -1\}$  and  $\gamma' \in [\gamma, 2 + \gamma)$ . There exists a solution  $f$  to (1.1), (1.2) such that

$$M_1(f(t)) > M_1(f^{in}) \quad \text{for every } t > 0.$$

Since Theorem 3.1 guarantees the existence of a mass-conserving solution to (1.1), (1.2) under the assumptions of Theorem 1.4, we realize that there is no uniqueness of the solution to (1.1), (1.2) in that case. This non-uniqueness phenomenon was already known for the pure fragmentation equation [2, 22, 3] and we thus extend it to the coagulation-fragmentation equation. Observe that the solution constructed in Theorem 1.4 is unphysical as its mass increases.

We finally derive a modified coagulation model, the so-called Flory model [26], from the coagulation-fragmentation equation with strong fragmentation. An alternative derivation has been proposed in [1].

**Theorem 1.5** Let  $\alpha \in [0, 1]$ . For  $\varepsilon \in (0, 1)$ , we consider

$$(1.9) \quad a(y, y') = y^\alpha y' + y (y')^\alpha \quad \text{and} \quad b_\varepsilon(y, y') = \varepsilon (1 + y + y')^{1/2}.$$

Then there exists a family of mass-conserving solutions  $(f_\varepsilon)$  to (1.1), (1.2) and a sequence  $(\varepsilon_k)$ ,  $\varepsilon_k \rightarrow 0$ , such that

$$f_{\varepsilon_k} \longrightarrow f \text{ in } \mathcal{C}([0, T]; w - L^1(\mathbb{R}_+))$$

for every  $T > 0$ , where  $f$  is a solution to the modified coagulation equation

$$(1.10) \quad \frac{\partial f}{\partial t} = Q_{mc}(f), \quad (t, y) \in (0, +\infty) \times \mathbb{R}_+,$$

$$(1.11) \quad f(0) = f^{in}, \quad y \in \mathbb{R}_+,$$

where

$$\begin{aligned} Q_{mc}(f)(y) &:= \frac{1}{2} \int_0^y a(y', y - y') f(y') f(y - y') dy' \\ &\quad - y f(y) \int_0^\infty y'^\alpha f(y') dy' - y^\alpha f(y) M_1(f^{in}) \end{aligned}$$

if  $\alpha \in [0, 1)$ , and

$$Q_{mc}(f)(y) := \frac{1}{2} \int_0^y a(y', y - y') f(y') f(y - y') dy' - 2 y f(y) M_1(f^{in})$$

if  $\alpha = 1$ .

Here and below, if  $X$  is a Banach space and  $T > 0$ ,  $\mathcal{C}([0, T]; w - X)$  denotes the space of weakly continuous functions from  $[0, T]$  in  $X$ .

**Remark 1.6** The choice  $b_\varepsilon(y, y') = \varepsilon(1 + y + y')^{1/2}$  is in some sense arbitrary and could be replaced by  $b_\varepsilon(y, y') = \varepsilon(1 + y + y')^\gamma$  for  $\gamma \in (0, 1)$ .

Before proving the above mentioned results, we point out that the key point in the analysis of (1.1) are the following identities: for every measurable functions  $f$  and  $\psi$ , we have

$$(1.12) \quad \int_0^\infty Q_c(f) \psi dy = \frac{1}{2} \int_0^\infty \int_0^\infty a(y, y') f(y) f(y') \tilde{\psi}(y, y') dy dy',$$

$$(1.13) \quad \tilde{\psi}(y, y') = \psi(y + y') - \psi(y) - \psi(y'),$$

and

$$(1.14) \quad \begin{aligned} \int_0^\infty Q_f(f) \psi dy &= \frac{1}{2} \int_0^\infty \int_0^\infty b(y, y') f(y + y') \tilde{\psi}(y, y') dy dy' \\ &= -\frac{1}{2} \int_0^\infty f(z) k_\psi(z) dz, \end{aligned}$$

with

$$(1.15) \quad k_\psi(z) = \int_0^z b(y, z - y) (\psi(y) + \psi(z - y) - \psi(z)) dy.$$

As an immediate consequence, when we choose  $\psi(y) = y$  in (1.12)-(1.14), so that  $\tilde{\psi}$  vanishes, we get formally the conservation of mass

$$(1.16) \quad \frac{d}{dt} \int_0^\infty y f(t, y) dy = 0.$$

**Acknowledgements.** We gratefully acknowledge the partial support of the European Research Training Network *HYKE* HPRN-CT-2002-00282 during this work. The first and third authors were partially supported by CNRS and UPV/EHU through a PICS between the Universidad del País Vasco and the Ecole Normale Supérieure.

## 2 On the occurrence of gelation

In this section, we prove the second assertion of Theorem 1.1. We actually consider a larger class of rates  $a$  and  $b$  and prove the following result.

**Theorem 2.1** *Assume that*

$$(2.1) \quad a(y, y') \geq y^\alpha (y')^\beta + y^\beta (y')^\alpha \quad \text{and} \quad b(y, y') \leq B (1 + y + y')^\gamma,$$

where

$$(2.2) \quad 0 \leq \alpha \leq \beta \leq 1, \quad \lambda := \alpha + \beta > 1 \quad \text{and} \quad (\lambda - 3)/2 \leq \gamma < \lambda - 2.$$

Consider  $f^{in} \in L^1_1(\mathbb{R}_+)$  and denote by  $f$  a weak solution to (1.1), (1.2). There exists  $M_1^*$  such that, if  $M_1(f^{in}) > M_1^*$ , then gelation occurs.

Notice that (2.2) implies that  $-1 < \gamma \leq 0$ . The cornerstone of the proof of Theorem 2.1 is the following proposition.

**Proposition 2.2** *Let  $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be a non-decreasing function satisfying  $\Phi(0) = 0$  and  $f$  be a weak solution to (1.1), (1.2). Then, for every  $t_1 \geq t_0 \geq 0$ ,*

$$(2.3) \quad \int_{t_0}^{t_1} \left( \int_0^\infty f(t, y) y^{\lambda/2} \Phi(y) dy \right)^2 dt \leq C_{\Phi, 1+\gamma-\lambda/2} (4 M_1(t_0) C_{\Phi, (\lambda/2)-\gamma-2} + B^2 (t_1 - t_0) C_{\Phi, 1+\gamma-\lambda/2})$$

where

$$(2.4) \quad C_{\Phi, k} := \int_0^\infty \Phi'(A) A^k dA.$$

**Remark 2.3** *Notice that (2.3) only gives relevant information on  $f$  when both  $C_{1+\gamma-\lambda/2}$  and  $C_{(\lambda/2)-\gamma-2}$  are finite.*

*Proof of Proposition 2.2.* We choose  $\psi(y) := \psi_A(y) = y \wedge A$  in (1.12) and (1.14) and notice that  $\tilde{\psi}_A$  and  $k_{\psi_A}$  defined by (1.13) and (1.15) satisfy

$$\begin{aligned} -\tilde{\psi}_A(y, y') &\geq A \mathbf{1}_{\{y \geq A, y' \geq A\}}, \\ k_{\psi_A}(z) &\leq A B (1 + z)^\gamma (z - A)^+, \end{aligned}$$

by (2.1). Using that  $a(y, y') \geq (yy')^{\lambda/2}$  by (2.1), we thus obtain

$$\frac{1}{2} \int_{t_0}^{t_1} \left( \int_A^\infty f(t, y) y^{\lambda/2} dy \right)^2 dt \leq \frac{M_1(t_0)}{A} + \frac{B}{2} \int_{t_0}^{t_1} \int_A^\infty f(t, z) z (1 + z)^\gamma dz dt.$$

Since  $1 + \gamma - (\lambda/2) < 0$  by (2.2), let  $\nu$  be such that

$$0 < \nu \quad \text{and} \quad 1 + \nu + \gamma - \lambda/2 \leq 0.$$

Thanks to the Young inequality, we get

$$\begin{aligned} \int_A^\infty f(t, z) z^{\lambda/2} \left\{ \frac{z^{1+\nu-\lambda/2} (1+z)^\gamma}{z^\nu} \right\} dz &\leq \frac{1}{A^\nu} \int_A^\infty f(t, z) z^{\lambda/2} dz \\ &\leq \frac{1}{2B} \left( \int_A^\infty f(t, z) z^{\lambda/2} dz \right)^2 + \frac{B}{2A^{2\nu}}. \end{aligned}$$

Gathering the preceding two estimates, we deduce that, for  $A > 0$ ,

$$\int_{t_0}^{t_1} \left( \int_A^\infty f(t, y) y^{\lambda/2} dy \right)^2 dt \leq \left\{ \frac{4}{A} M_1(t_0) + \frac{B^2}{A^{2\nu}} (t_1 - t_0) \right\}.$$

Using the Fubini theorem and the Cauchy-Schwarz inequality, we obtain, for each  $\mu \geq 0$ ,

$$\begin{aligned} & \int_{t_0}^{t_1} \left( \int_0^\infty f(t, y) y^{\lambda/2} \Phi(y) dy \right)^2 ds \leq \\ & \leq \int_{t_0}^{t_1} \left( \int_0^\infty \Phi'(A) \int_A^\infty f(t, y) y^{\lambda/2} dy dA \right)^2 ds \\ & \leq \int_{t_0}^{t_1} \left( \int_0^\infty \frac{\Phi'(A)}{A^\mu} dA \right) \left( \int_0^\infty \Phi'(A) A^\mu \left( \int_A^\infty f(t, y) y^{\lambda/2} dy \right)^2 dA \right) ds \\ & \leq C_{-\mu} \int_0^\infty \Phi'(A) A^\mu \int_{t_0}^{t_1} \left( \int_A^\infty f(t, y) y^{\lambda/2} dy \right)^2 ds dA \\ & \leq C_{-\mu} \int_0^\infty [4 M_1(t_0) A^{\mu-1} + B^2 (t_1 - t_0) A^{\mu-2\nu}] \Phi'(A) dA. \end{aligned}$$

The inequality (2.3) then follows with the choice  $\mu = \nu := (\lambda/2) - \gamma - 1 > 0$ .  $\square$

**Corollary 2.4** *Assume further that  $\lambda < 2$ . Then there exists a constant  $C_1 = C_1(\lambda, \gamma)$  such that*

$$(2.5) \quad \int_{t_0}^{t_1} \left( \int_R^\infty f(t, y) y dy \right)^2 dt \leq C_1 \left( M_1(t_0) R^{1-\lambda} + B^2 (t_1 - t_0) R^{2(2+\gamma-\lambda)} \right).$$

*Proof.* We define  $\Phi = \Phi_r$  by

$$\Phi_r(y) := (y^{1-\lambda/2} - r^{1-\lambda/2})^+ \geq (2^{1-\lambda/2} - 1) \mathbf{1}_{[2r, +\infty)}(y).$$

We have

$$C_{\Phi_r, k} = \frac{1 - \lambda/2}{(\lambda/2) - k - 1} r^{1-\lambda/2+k},$$

for  $k$  such that  $k - \lambda/2 < -1$ . The conditions  $C_{\Phi_r, 1+\gamma-\lambda/2} < \infty$  and  $C_{\Phi_r, (\lambda/2)-\gamma-2} < \infty$  are fulfilled since  $-1 < \gamma < \lambda - 2$  by (2.2). We then obtain (2.5) by taking  $r = R/2$ .  $\square$

**Corollary 2.5** *There exists a constant  $C_2 = C_2(\lambda, \gamma)$  such that*

$$(2.6) \quad \int_0^T M_1^2(t) dt \leq C_2 (M_1(0) + M_0(0) + B^2 T).$$

*Proof.* Since  $a(y, y') \geq (y y')^{\lambda/2}$  by (2.1), it follows from (1.1), (1.12) and (1.14) with  $\psi = 1$  that

$$\frac{1}{2} \int_0^T M_{\lambda/2}^2(t) dt \leq M_0(0) + \frac{B}{2} \int_0^T \int_0^\infty f(t, z) z (1+z)^\gamma dz dt.$$

Since  $\gamma \leq 0$ , we have  $1 + \gamma \leq 1 + \gamma/2 \leq \lambda/2$ . Therefore,

$$\frac{1}{2} \int_0^T M_{\lambda/2}^2(t) dt \leq M_0(0) + \frac{B}{2} \int_0^T M_{\lambda/2}(t) dt,$$

and the Young inequality yields

$$\int_0^T M_{\lambda/2}^2(t) dt \leq 4(M_0(0) + B^2 T).$$

If  $\lambda = 2$ , this is exactly the claim (2.6). If  $\lambda < 2$ , we combine the above inequality and (2.5) with  $R = 1$  to obtain (2.6).  $\square$

*Proof of Theorem 2.1.* We argue by contradiction and assume that  $M_1(t) = M_1(0)$  for  $t \geq 0$ . By (2.6) we deduce that

$$M_1(0)^2 T \leq C_2 (M_1(0) + M_0(0) + B^2 T),$$

whence a contradiction for  $T$  large enough if  $M_1(0) > C_2^{1/2} B$ .  $\square$

Proposition 2.2 actually enables us to obtain more precise information on the behaviour of  $f(t, y)$  for large values of  $y$ .

**Corollary 2.6** (i) For any  $\delta > 1$ , there exists a constant  $C = C(\lambda, \gamma, \delta)$  such that

$$(2.7) \quad \int_{t_0}^{t_1} \left( \int_e^\infty f(t, y) \frac{y^{\lambda-\gamma-1}}{(\ln(y))^\delta} dy \right)^2 dt \leq C (M_1(t_0) + B^2 (t_1 - t_0)).$$

Notice that  $1 < \lambda - \gamma - 1 < 2$ .

(ii) For any  $\tau > 2\gamma - (\lambda - 3) > 0$ , there exists a constant  $C = C(\lambda, \gamma, \tau)$  such that, for each  $R > 0$ ,

$$(2.8) \quad \int_{t_0}^{t_1} \left( \frac{1}{R^\tau} \int_0^R f(t, y) y^{\lambda-\gamma-1+\tau} dy \right)^2 dt \leq C \left( \frac{M_1(t_0)}{R^{3-\lambda+2\gamma}} + B^2 (t_1 - t_0) \right).$$

*Proof.* We first prove (i). We define  $\Phi(y) = (y^{(\lambda/2)-\gamma-1}/(\ln(y))^\delta - r^{(\lambda/2)-\gamma-1}/(\ln(r))^\delta)^+$  with  $\delta > 1$  and  $r = e/2$ . We easily verify that  $\Phi$  is increasing and that the associated constants  $C_{\Phi, (\lambda/2)-\gamma-2}$  and  $C_{\Phi, 1+\gamma-(\lambda/2)}$  are finite. The estimate (2.7) follows from (2.3) for this choice of  $\Phi$  and Corollary 2.4 with  $R = e$ .

We next proceed as in [9, Theorem 2.7] with  $\Phi(y) := (y \wedge R)^\ell$  to prove (ii), with  $\ell = \lambda/2 - \gamma - 1 + \tau$ .  $\square$

**Remark 2.7** Observe that, in (2.7) and (2.8), we have the weight  $y^{\lambda-\gamma-1}$  instead of the weight  $y^{(\lambda+1)/2}$  obtained in [9, Theorem 2.7] for the case  $\gamma < (\lambda - 3)/2$ . On the other hand, we do not know whether a bound from below similar to [9, Corollary 2.9] is available here.

### 3 Existence of mass-conserving solutions

In this section, we prove the first assertion of Theorem 1.1, that is, the existence of a mass-conserving solution to (1.1), (1.2) when  $a$  and  $b$  are given by (1.7) with either  $\alpha + \beta \leq 1$  or  $\gamma > \lambda - 2$ . Such a result is actually valid for a larger class of coefficients when the coagulation term  $Q_c(f)$  is either “weak” or suitably dominated by the fragmentation term  $Q_f(f)$ . More precisely, we assume that:

$$(3.1) \quad \left\{ \begin{array}{l} \text{The coagulation coefficient } a : \mathbb{R}_+^2 \rightarrow \mathbb{R} \text{ is a measurable function and there are} \\ \text{some real numbers } A_0 > 0 \text{ and } 0 \leq \alpha \leq \beta \leq 1 \text{ such that} \\ 0 \leq a(y, y') = a(y', y) \leq A_0 \{ (1+y)^\alpha (1+y')^\beta + (1+y)^\beta (1+y')^\alpha \} \\ \text{for } (y, y') \in \mathbb{R}_+^2. \end{array} \right.$$

$$(3.2) \quad \left\{ \begin{array}{l} \text{The fragmentation coefficient } b : \mathbb{R}_+^2 \rightarrow \mathbb{R} \text{ is a measurable function such that} \\ \text{(i) for each } R \in \mathbb{R}_+ \text{ there is } b_R > 0 \text{ such that} \\ \qquad \qquad \qquad 0 \leq b(y, y') = b(y', y) \leq b_R, \quad (y, y') \in (0, R)^2, \\ \text{(ii) there are } R_0 > 0, S_0 \geq R_0 \text{ and } C_0 > 0 \text{ such that} \\ \qquad \qquad \qquad \int_0^{R_0} b(y', y - y') dy' \leq C_0 \int_0^{R_0} y' b(y', y - y') dy' \text{ for } y \geq S_0. \end{array} \right.$$

Let us point out here that, in contrast to the previous works [7, 14, 21], no *a priori* growth condition on  $b$  is imposed by the assumption (3.2) (ii). Indeed, examples of fragmentation coefficients fulfilling (3.2) are

$$(y + y')^\kappa, \quad (yy')^\kappa, \quad y^\kappa + (y')^\kappa,$$

where  $\kappa$  is an arbitrary non-negative real number.

Introducing  $\lambda := \alpha + \beta \in [0, 2]$ , we notice that

$$(3.3) \quad a(y, y') \leq A_0 \{ (1 + y)^\lambda + (1 + y')^\lambda \}, \quad (y, y') \in \mathbb{R}_+^2,$$

by the Young inequality. As expected from the analysis of the discrete coagulation-fragmentation equations [2] and from the previous studies of (1.1) [21, 7], the coagulation term is sufficiently weak if  $\lambda \leq 1$ . Thus, as we shall see below, there is at least a mass-conserving solution to (1.1), (1.2) without any additional assumption on  $b$ . On the other hand, if  $\lambda \in (1, 2]$ , gelation occurs in the absence of fragmentation [9]. For the discrete coagulation-fragmentation equations, it has been noticed in [4] that a sufficiently strong fragmentation term prevents the occurrence of the gelation phenomenon. When  $\lambda \in (1, 2]$ , we will thus impose an additional condition on  $b$ , namely,

$$(3.4) \quad \left\{ \begin{array}{l} \text{there are } B_0 > 0 \text{ and } \gamma > \lambda - 2 \text{ such that} \\ b(y - y', y') \geq B(y) := B_0 (1 + y)^\gamma \text{ for } y \geq 1 \text{ and } y' \in (0, y). \end{array} \right.$$

The main result of this section then reads:

**Theorem 3.1** *Assume that the kinetic coefficients  $a$  and  $b$  satisfy (3.1), (3.2) and that*

- (i) *either  $\lambda \in [0, 1]$ ,*
- (ii) *or  $\lambda \in (1, 2]$  and  $b$  fulfils (3.4).*

*For any initial datum  $f^{in}$  satisfying (1.5), there is at least one density-conserving weak solution  $f$  to (1.1), (1.2), that is,*

$$(3.5) \quad \begin{aligned} 0 \leq f &\in \mathcal{C}([0, +\infty); L^1(\mathbb{R}_+)) \text{ with } f(0) = f^{in}, \\ Q_c(f), Q_f(f) &\in L^1_{loc}([0, +\infty) \times \mathbb{R}_+), \\ \int_0^\infty y f(t, y) dy &= \int_0^\infty y f^{in}(y) dy, \quad t \geq 0, \end{aligned}$$

*and (1.1) holds in the mild sense : for  $0 \leq t_0 < t_1$  there holds*

$$(3.6) \quad f(t_1, \cdot) - f(t_0, \cdot) = \int_{t_0}^{t_1} Q(f(t, \cdot)) dt \quad \text{a.e. in } \mathbb{R}_+.$$

Observe that the coefficients  $a$  and  $b$  given by (1.7) enjoy the properties (3.1) and (3.2). The first assertion of Theorem 1.1 then readily follows from Theorem 3.1.



The proof of Theorem 3.1 relies on a compactness method and requires to derive several estimates on the solutions  $f$  to approximations of (1.1), (1.2). The starting point is to obtain an  $L^1(0, T)$ -bound on  $M_{\lambda_1}$ , the moment of order  $\lambda_1$  of  $f$ , where  $\lambda_1 = \max\{1, \lambda\}$ . While such a bound is obvious for  $\lambda \in [0, 1]$ , it relies on the assumption (3.4) for  $\lambda \in (1, 2]$ . Thanks to the bound on  $M_{\lambda_1}$ , we may control the behaviour of  $yf(t, y)$  and the fragmentation term for large values of  $y$ . This in turn allows us to obtain a uniform bound on the  $L^1$ -norm of  $f$  on bounded time intervals. These estimates then enable us to proceed as in [14] to conclude that  $f$  lies in a relatively sequentially weakly compact subset of  $L^1((0, T) \times \mathbb{R}_+, (1+y)dydt)$  which does not depend on the approximation. Theorem 3.1 then follows by a compactness argument.

### 3.1 Estimates for compactly supported solutions

In this section, we assume that  $a$  and  $b$  are coagulation and fragmentation coefficients satisfying the assumptions of Theorem 3.1 together with the additional requirement that

$$a(y, y') = b(y, y') = 0 \quad \text{if } y + y' > \varrho$$

for some  $\varrho > S_0$ . We next assume that  $f^{in}$  satisfies (1.5) and has compact support with  $\text{Supp } f^{in} \subset [0, \varrho]$ . Thanks to these assumptions, we may argue as in [21, Section 3] (see also [19, 7]) and prove that there is a unique solution  $f \in \mathcal{C}([0, +\infty); L^1(\mathbb{R}_+))$  to (1.1), (1.2) such that

$$(3.7) \quad \text{Supp } f(t, \cdot) \subset [0, \varrho] \quad \text{and} \quad \int_0^\infty y f(t, y) dy = \int_0^\infty y f^{in}(y) dy := M_1$$

for each  $t \geq 0$ . We finally assume an additional integrability property on  $f^{in}$ , namely that there is a non-negative, convex and non-decreasing function  $\Phi \in \mathcal{C}^1([0, +\infty)) \cap W_{loc}^{2, \infty}([0, +\infty))$  such that

$$(3.8) \quad \begin{cases} \Phi(0) = 0, \quad \Phi'(0) \geq 0 \quad \text{and} \quad \Phi' \text{ is concave,} \\ \lim_{r \rightarrow +\infty} \Phi'(r) = \lim_{r \rightarrow +\infty} \frac{\Phi(r)}{r} = +\infty, \end{cases}$$

and

$$(3.9) \quad L_\Phi := \int_0^\infty \Phi(y) f^{in}(y) dy < \infty.$$

We now derive several properties enjoyed by  $f$  which do not depend on  $\varrho$ . In the following, we denote by  $C, (C_i)_{i \geq 1}$ , positive constants which depend only on  $A_0, \alpha, \beta, R_0, S_0, C_0, \|f^{in}\|_{L^1}, M_1, \Phi, L_\Phi$  and also on  $B_0$  and  $\gamma$  when (3.4) holds true. The dependence of  $C$  upon additional parameters will be indicated explicitly.

Following [4], we prove that, when  $\lambda \in (1, 2]$ , (3.4) entails a control on the second moment  $M_2$  of  $f$  for positive times.

**Lemma 3.2** *When (3.4) holds true with  $\gamma > -1$ , we have*

$$(3.10) \quad M_2(t) \leq C \left(1 + t^{-1/(1+\gamma)}\right), \quad t \in \mathbb{R}_+.$$

*Proof.* We take  $\psi(y) = y^2$  in (1.12) and use (3.3) and (3.7) to obtain

$$(3.11) \quad \begin{aligned} \int_0^\infty y^2 Q_c(f)(t, y) dy &\leq 2 A_0 \int_0^\infty \int_0^\infty (1+y)^\lambda yy' f(t, y) f(t, y') dy dy' \\ &\leq C (1 + M_{1+\lambda}(t)). \end{aligned}$$

On the other hand, it follows from (3.4), (3.7) and (1.14) with  $\psi(y) = y^2$  that

$$\begin{aligned}
\int_0^\infty y^2 Q_f(f)(t, y) dy &\geq C \int_1^\infty y^3 (1+y)^\gamma f(t, y) dy \\
&\geq C \int_0^\infty y^{3+\gamma} f(t, y) dy - C \int_0^1 y^{3+\gamma} f(t, y) dy \\
(3.12) \qquad \qquad \qquad &\geq C_1 (M_{3+\gamma}(t) - 1).
\end{aligned}$$

We then infer from (1.1), (3.11) and (3.12) that

$$\frac{dM_2}{dt} + C_1 M_{3+\gamma} \leq C (1 + M_{1+\lambda}).$$

Now, since  $\gamma > \lambda - 2$ , the Hölder inequality and (3.7) yield

$$M_{1+\lambda}^{2+\gamma} \leq M_1^{2+\gamma-\lambda} M_{3+\gamma}^\lambda \leq C M_{3+\gamma}^\lambda.$$

The previous differential inequality then becomes

$$\frac{dM_2}{dt} + C_1 M_{3+\gamma} \leq C \left(1 + M_{3+\gamma}^{\lambda/(2+\gamma)}\right).$$

We use once more the fact that  $\lambda < 2 + \gamma$  together with the Young inequality to conclude that

$$\frac{dM_2}{dt} + C_2 M_{3+\gamma} \leq C.$$

Finally, since  $\gamma > -1$ , we have  $2 \in (1, 3 + \gamma)$  and it follows from the Hölder inequality and (3.7) that

$$M_2^{2+\gamma} \leq M_1^{1+\gamma} M_{3+\gamma} \leq C M_{3+\gamma}.$$

Inserting this estimate in the previous differential inequality, we end up with

$$\frac{dM_2}{dt} + C_3 M_2^{2+\gamma} \leq C,$$

from which (3.10) readily follows (since  $2 + \gamma > 1$ ).  $\square$

The estimates (3.4) and (3.10) clearly allow us to control the behaviour of  $f$  and  $Q_c(f)$  for large values of  $y$  and positive times. Some further computations are needed to control the fragmentation term, and the short time behaviour as well. First, as a consequence of Lemma 3.2, we obtain the following integrability property of  $M_\lambda$ .

**Lemma 3.3** *Let  $T \in \mathbb{R}_+$ . When  $\lambda \in (1, 2]$  and (3.4) holds true, there is  $C(T)$  such that*

$$(3.13) \qquad \qquad \qquad \int_0^T M_\lambda(t) dt \leq C(T).$$

*Proof.* Since  $\lambda \in (1, 2]$ , the assumption (3.4) implies that  $\gamma > -1$  and we infer from (3.7), (3.10) and the Hölder inequality that

$$M_\lambda(t) \leq M_1(t)^{2-\lambda} M_2(t)^{\lambda-1} \leq C \left(1 + t^{-(\lambda-1)/(1+\gamma)}\right),$$

from which (3.13) readily follows as  $\lambda - 1 < 1 + \gamma$ .  $\square$

We next develop further a device already used in [14] to estimate the behaviour of  $f$  and  $Q_f(f)$  for large values of  $y$ . In [14], an additional growth condition was required on  $b$ , namely,  $b(y, y') \leq C (1+y) (1+y')$  and we show here that this condition can be replaced by (3.2) (ii) which is much weaker. We actually prove that the integrability property (3.8) enjoyed by  $f^{in}$  propagates through time evolution.

**Proposition 3.4** For  $T \in \mathbb{R}_+$ , we have

$$(3.14) \quad \int_0^\infty \Phi(y) f(t, y) dy \leq C(T), \quad t \in [0, T],$$

$$(3.15) \quad \int_0^T \int_0^\infty \left\{ \int_0^y b(y', y - y') \tilde{\Phi}(y', y - y') dy' \right\} f(s, y) dy ds \leq C(T),$$

where  $\tilde{\Phi}$  is given by (1.13).

*Proof.* Owing to the properties of  $\Phi$ , we have

$$(3.16) \quad 0 \leq \tilde{\Phi}(y, y') \leq 2 \frac{y' \Phi(y) + y \Phi(y')}{y + y'}, \quad (y, y') \in \mathbb{R}_+^2,$$

where the second inequality follows from [13, Lemma A.2]. A first consequence of (1.14) and (3.16) is that

$$(3.17) \quad \int_0^\infty \Phi(y) Q_f(f)(y) dy \geq 0.$$

We next derive an upper bound for the coagulation term: introducing

$$\Psi(y, y') := \tilde{\Phi}(y, y') (1 + y)^\alpha (1 + y')^\beta, \quad (y, y') \in \mathbb{R}_+^2,$$

we have

$$\Psi(y, y') \leq 2^\lambda \|\Phi''\|_{L^\infty(0,2)} yy' \leq C (yy'),$$

for  $(y, y') \in (0, 1)^2$ , while (3.16) entails that

$$\Psi(y, y') \leq 2^\lambda \tilde{\Phi}(y, y') (y')^\beta \leq C \{y' \Phi(y) + y \Phi(y')\}$$

for  $0 \leq y \leq 1 \leq y'$  and  $0 \leq y' \leq 1 \leq y$  since  $0 \leq \alpha \leq \beta \leq 1$ . Finally, if  $(y, y') \in (1, +\infty)^2$ , we infer from (3.16) that

$$\Psi(y, y') \leq 2^\lambda \tilde{\Phi}(y, y') y^\alpha (y')^\beta \leq \begin{cases} C \{y' \Phi(y) + y \Phi(y')\} & \text{if } \lambda \in [0, 1], \\ C \{(y')^\lambda \Phi(y) + y^\lambda \Phi(y')\} & \text{if } \lambda \in (1, 2]. \end{cases}$$

Inserting the estimates for  $\Psi$  in (1.12) with  $\psi = \Phi$  and using (3.7) leads to

$$(3.18) \quad \int_0^\infty \Phi(y) Q_c(f)(t, y) dy \leq C (1 + M_{\lambda_1}(t)) \left( 1 + \int_0^\infty \Phi(y) f(t, y) dy \right)$$

with  $\lambda_1 := \max\{1, \lambda\}$ . Consequently, we deduce from (1.1) and (3.18) that

$$\begin{aligned} & \frac{d}{dt} \int_0^\infty \Phi(y) f(t, y) dy + \int_0^\infty \Phi(y) Q_f(f)(y) dy \\ & \leq C (1 + M_{\lambda_1}(t)) \left( 1 + \int_0^\infty \Phi(y) f(t, y) dy \right). \end{aligned}$$

Owing to (3.17) and since  $M_{\lambda_1} \in L^1(0, T)$  by either (3.7) (if  $\lambda \in [0, 1]$ ) or Lemma 3.3 (if  $\lambda \in (1, 2]$ ), the assertions (3.14) and (3.15) follow from the previous differential inequality by the Gronwall lemma.  $\square$

Since no growth condition is required on  $b$ , the estimate (3.15) provides a control on  $Q_4(f)$ . More precisely, we have the following result:

**Corollary 3.5** For each  $R \geq R_0$  and  $S \geq S_0 + 2R$ , we have

$$(3.19) \quad \int_0^T \int_S^\infty \int_0^R b(y', y - y') dy' f(s, y) dy ds \leq \omega_{T,R}(S) \xrightarrow{S \rightarrow +\infty} 0.$$

*Proof.* For  $0 \leq R \leq S$ , we put

$$(3.20) \quad I(R, S, s) := \int_S^\infty \int_0^R b(y', y - y') dy' f(s, y) dy.$$

Consider now  $R > R_0$  and  $S \geq S_0 + 2R$ . By (3.2) we have

$$\int_0^R b(y', y - y') dy' \leq C_0 \int_0^{R_0} y' b(y', y - y') dy' + \frac{1}{R_0} \int_{R_0}^R y' b(y', y - y') dy',$$

whence

$$I(R, S, s) \leq C \int_S^\infty \int_0^R y' b(y', y - y') dy' f(s, y) dy.$$

Since  $\Phi$  is convex with  $\Phi(0) = 0$ , we have

$$\tilde{\Phi}(y - y', y') \geq y' (\Phi'(y - y') - \Phi'(y')) \geq y' (\Phi'(S - R) - \Phi'(R))$$

for  $(y, y') \in (S, +\infty) \times (0, R)$ . Therefore,  $I(R, S, s)$  is bounded from above by

$$\frac{C}{\Phi'(S - R) - \Phi'(R)} \int_S^\infty \int_0^R b(y', y - y') \tilde{\Phi}(y', y - y') dy' f(s, y) dy,$$

whence (3.19) by (3.8) and (3.15).  $\square$

We next derive an estimate on  $f$  in  $L^1(\mathbb{R}_+)$ . Observe that such a bound is not obvious as no growth condition is assumed on  $b$ . Indeed, from a physical point of view, the  $L^1$ -norm of  $f$  represents the number of clusters which is increased by fragmentation reactions and could thus grow without bound. However, the condition (3.2) (ii) allows us to exclude such a behaviour.

**Lemma 3.6** *For  $T \in \mathbb{R}_+$ , we have*

$$(3.21) \quad \|f(t)\|_{L^1} \leq C(T), \quad t \in [0, T].$$

*Proof.* We take  $\psi = \mathbf{1}_{[0, R_0]}$  in (1.12) and (1.14). Noticing that

$$-(\psi(y) + \psi(y')) \leq \tilde{\psi}(y, y') \leq 0,$$

we have

$$(3.22) \quad \int_0^{R_0} Q_c(f)(t, y) dy \leq 0,$$

and we infer from (3.2) and (3.7) that

$$\begin{aligned} - \int_0^{R_0} Q_f(f)(t, y) dy &\leq \int_0^{3S_0} f(t, y) \int_0^y b(y', y - y') dy' dy \\ &\quad + \int_{3S_0}^\infty f(t, y) \int_0^{R_0} b(y', y - y') dy' dy \\ &\leq b_{3S_0} M_1 + I(R_0, 3S_0, t). \end{aligned}$$

It then follows from (3.19) that

$$(3.23) \quad - \int_0^T \int_0^{R_0} Q_f(f)(t, y) dy dt \leq C(T) + \omega_{T, R_0}(3S_0) \leq C(T).$$

Combining (1.1), (3.22) and (3.23) yields

$$\int_0^{R_0} f(t, y) dy \leq \|f^{in}\|_{L^1} + C(T), \quad t \in [0, T].$$

The assertion (3.21) is now a straightforward consequence of (3.7) and the above estimate.  $\square$

Owing to Proposition 3.4 and Lemma 3.6, it remains to control the behaviour of  $f(t)$  on subsets of  $\mathbb{R}_+$  with small measure in order to establish that  $f(t)$  lies in a relatively weakly sequentially compact subset of  $L^1_1(\mathbb{R}_+)$  for each  $t \in [0, T]$ .

**Lemma 3.7** *For  $T \in \mathbb{R}_+$ ,  $R > R_0$ ,  $\delta \in (0, 1)$  and  $t \in [0, T]$ , we put*

$$\mathcal{E}^{\delta, R}(t) := \sup \left\{ \int_E f(t, y) dy, \quad E \subset (0, R), \quad |E| \leq \delta \right\}.$$

*Given  $\varepsilon \in (0, 1)$ , there is  $C_4(T, R, \varepsilon)$  such that, if  $\delta \leq C_4(T, R, \varepsilon)$ ,*

$$\sup_{t \in [0, T]} \mathcal{E}^{\delta, R}(t) \leq C_5(T, R) (\mathcal{E}^{\delta, R}(0) + \varepsilon).$$

*Proof.* Let  $E$  be a measurable subset of  $(0, R)$  with  $|E| \leq \delta$ ,  $t \in (0, T]$  and  $s \in (0, T)$ . We first infer from (3.3), (1.12) with  $\psi = \mathbf{1}_E$  and (3.21) that

$$\begin{aligned} \int_E Q_c(f)(s, y) dy &\leq C (1+R)^\lambda \int_0^R \int_0^{R-y} \mathbf{1}_E(y+y') f(s, y) f(s, y') dy' dy \\ &\leq C(R) \int_0^R f(s, y) \int_0^R \mathbf{1}_{E-y}(y') f(s, y') dy' dy \\ (3.24) \qquad \qquad &\leq C(R, T) \mathcal{E}^{\delta, R}(s), \end{aligned}$$

since the Lebesgue measure on  $\mathbb{R}_+$  is translation-invariant. It next follows from (3.2), (1.14) with  $\psi = \mathbf{1}_E$  and (3.21) that, for every  $S \geq S_0 + 2R$ ,

$$\begin{aligned} - \int_E Q_f(f)(s, y) dy &\leq b_S |E| \int_0^S f(s, y) dy \\ &\quad + \int_S^\infty f(s, y) \int_0^R b(y', y-y') dy' dy \\ (3.25) \qquad \qquad &\leq C(T, S) \delta + I(R, S, s), \end{aligned}$$

where  $I(R, S, s)$  is defined by (3.20). Combining (1.1), (3.19), (3.24) and (3.25), we obtain

$$\begin{aligned} \int_E f(t, y) dy &\leq \int_E f^{in}(y) dy + C(R, T) \int_0^T \mathcal{E}^{\delta, R}(s) ds \\ &\quad + C(T, S) \delta + \omega_{T, R}(S) \end{aligned}$$

for any measurable subset of  $(0, R)$  with  $|E| \leq \delta$ . Consequently,

$$\begin{aligned} \mathcal{E}^{\delta, R}(t) &\leq \mathcal{E}^{\delta, R}(0) + C(R, T) \int_0^T \mathcal{E}^{\delta, R}(s) ds \\ &\quad + C(T, S) \delta + \omega_{T, R}(S), \end{aligned}$$

whence, by the Gronwall lemma,

$$\mathcal{E}^{\delta, R}(t) \leq C(T, R) (\mathcal{E}^{\delta, R}(0) + C(T, S) \delta + \omega_{T, R}(S)), \quad t \in [0, T].$$

Lemma 3.7 then readily follows from (3.19) and the above inequality, choosing first  $S > S_0 + 2R$  large enough and then  $\delta$  small enough.  $\square$

To summarize the outcome of this section, we realize that the reaction terms  $Q_c(f)$  and  $Q_f(f)$  are bounded in  $L^1((0, T) \times (0, R))$  for any  $R \geq 1$ , thanks to the bounds (3.7), (3.19) and (3.21). Next, the estimates (3.14), (3.19) and Lemma 3.7 ensure the sequential weak compactness of  $f$  in  $L^1((0, T) \times \mathbb{R}_+, (1+y)dtdy)$  and of  $Q_c(f)$  and  $Q_f(f)$  in  $L^1((0, T) \times (0, R))$  for any  $R \geq 1$ .

### 3.2 Proof of Theorem 3.1

We are now ready to complete the proof of the existence of a mass-conserving solution to (1.1), (1.2). Let  $a$  and  $b$  be coagulation and fragmentation coefficients satisfying the assumptions of Theorem 3.1.

We introduce the following sequence of approximating equations: given an integer  $n \geq S_0$ , we put

$$a_n(y, y') := a(y, y') \mathbf{1}_{[0, n]}(y + y'), \quad b_n(y, y') := b(y, y') \mathbf{1}_{[0, n]}(y + y'),$$

for  $(y, y') \in \mathbb{R}_+^2$  and  $f_n^{in}(y) := f^{in}(y) \mathbf{1}_{[0, n]}(y)$ ,  $y \in \mathbb{R}_+$ . Then,  $a_n$ ,  $b_n$  and  $f_n^{in}$  clearly fulfil the requirements of Subsection 3.1 (with  $\varrho = n$ ) and we denote by  $f_n \in \mathcal{C}([0, +\infty); L^1(\mathbb{R}_+))$  the unique solution to (1.1), (1.2) with kinetic coefficients  $a_n$ ,  $b_n$  and initial data  $f_n^{in}$  such that

$$(3.26) \quad \text{Supp } f_n(t, \cdot) \subset [0, n] \quad \text{and} \quad \int_0^\infty y f_n(t, y) dy = \int_0^\infty y f_n^{in}(y) dy$$

for each  $t \geq 0$ . We next recall that (1.5) and a refined version of the de la Vallée-Poussin theorem [5, 15] ensure that there is a non-negative, convex and non-decreasing function  $\Phi \in \mathcal{C}^1([0, +\infty)) \cap W_{\text{loc}}^{2, \infty}([0, +\infty))$  such that (3.8) and (3.9) hold true for  $f^{in}$ . Since  $\Phi$  is non-decreasing and  $f_n^{in} \leq f^{in}$ , this last property is also enjoyed by  $f_n^{in}$ , that is,

$$(3.27) \quad \int_0^\infty \Phi(y) f_n^{in}(y) dy \leq L_\Phi$$

for  $n \geq S_0$ . We finally put

$$\mathcal{E}_n^{\delta, R}(t) := \sup \left\{ \int_E f_n(t, y) dy, \quad E \subset (0, R), \quad |E| \leq \delta \right\}$$

for  $T \in \mathbb{R}_+$ ,  $n \geq S_0$ ,  $R > R_0$ ,  $\delta \in (0, 1)$  and  $t \in [0, T]$ . Since  $f_n^{in} \leq f^{in} \in L^1(\mathbb{R}_+)$ , we clearly have

$$(3.28) \quad \lim_{\delta \rightarrow 0} \sup_{n \geq S_0} \mathcal{E}_n^{\delta, R}(0) = 0$$

for each  $R > R_0$ .

Since the data  $a_n$ ,  $b_n$  and  $f_n^{in}$  fulfil the requirements of Subsection 3.1 uniformly with respect to  $n \geq S_0$ , the analysis performed in Subsection 3.1 allows us to establish the weak compactness of the sequence  $(f_n(t))$  in  $L_1^1(\mathbb{R}_+)$  for each  $t \in [0, T]$ .

**Proposition 3.8** *Under the assumptions of Theorem 3.1, for each  $T \in \mathbb{R}_+$ , there is a weakly compact subset  $\mathcal{K}_T$  of  $L_1^1(\mathbb{R}_+)$  such that  $f_n(t) \in \mathcal{K}_T$  for every  $t \in [0, T]$  and  $n \geq S_0$ .*

*Proof.* Let  $t \in [0, T]$ . On the one hand, we infer from Lemma 3.6, Lemma 3.7, (3.26) and (3.28) that  $(f_n(t))_{n \geq S_0}$  is bounded in  $L_1^1(\mathbb{R}_+)$  and satisfies

$$\lim_{\delta \rightarrow 0} \sup_{n \geq S_0} \mathcal{E}_n^{\delta, R}(t) = 0$$

for each  $R > R_0$ . On the other hand, it follows from (3.8), (3.14) and (3.27) that

$$\lim_{S \rightarrow +\infty} \sup_{n \geq S_0} \int_S^\infty y f_n(t, y) dy = 0.$$

Proposition 3.8 is then a straightforward consequence of the Dunford-Pettis theorem, observing that the above bounds and limits are uniform with respect to  $t \in [0, T]$ .  $\square$

We next argue as in the proof of Proposition 3.8 along the lines of [12, Lemma 2.7] to check the time equicontinuity of the family  $\{f_n(t), t \in [0, T]\}$  in  $L^1(\mathbb{R}_+)$ .

**Lemma 3.9** For  $t \in [0, T)$ , there holds

$$(3.29) \quad \lim_{h \rightarrow 0} \sup_{n \geq S_0} \int_0^\infty |f_n(t+h, y) - f_n(t, y)| dy = 0.$$

We are now ready to complete the proof of the existence of a mass-conserving solution to (1.1), (1.2).

*Proof of Theorem 3.1.* Owing to Proposition 3.8 and Lemma 3.9, we may apply a variant of the Arzelà-Ascoli theorem (see, e.g., [24, Theorem 1.3.2]) to conclude that there are a subsequence of  $(f_n)$  (not relabeled) and a function

$$f \in \mathcal{C}([0, +\infty); w - L^1_1(\mathbb{R}_+))$$

such that

$$(3.30) \quad f_n \longrightarrow f \text{ in } \mathcal{C}([0, T]; w - L^1_1(\mathbb{R}_+))$$

for every  $T \in \mathbb{R}_+$ . Clearly, (3.26) and (3.30) ensure that  $f$  is non-negative with  $f(0) = f^{in}$  and

$$\int_0^\infty y f(t, y) dy = \int_0^\infty y f^{in}(y) dy, \quad t \geq 0.$$

Thanks to (3.1) and (3.30), we have  $Q_c(f) \in L^1_{loc}((0, +\infty) \times \mathbb{R}_+)$  and it is by now a standard matter to show that (3.30) implies that

$$Q_c(f_n) \rightharpoonup Q_c(f) \text{ weakly in } L^1((0, T) \times (0, R))$$

for every  $T \in \mathbb{R}_+$  and  $R \in \mathbb{R}_+$  (see, e.g., [21, 14]). Similarly,

$$Q_3(f_n) \rightharpoonup Q_3(f) \text{ weakly in } L^1((0, T) \times (0, R)).$$

It remains to pass to the limit in the fragmentation term  $Q_4$ . For that purpose, we first notice that (3.2), (3.19) and (3.30) ensure that

$$(3.31) \quad (t, y, y') \mapsto \mathbf{1}_{[0, y]}(y') b(y', y - y') f(t, y) \in L^1((0, T) \times \mathbb{R}_+ \times (0, R))$$

for any  $T \in \mathbb{R}_+$  and  $R \in \mathbb{R}_+$ . Indeed, by (3.8) and (3.19), there is  $S_1 > \max\{S_0, R\}$  such that

$$\int_0^T \int_{S_1}^\infty \int_0^R b_n(y', y - y') f_n(t, y) dy' dy ds \leq 1.$$

It then follows from the above inequality, (3.2) and Lemma 3.6 that, for  $S > S_1$  and  $n > S$ ,

$$\begin{aligned} & \int_0^T \int_0^S \int_0^R b(y', y - y') f_n(t, y) dy' dy dt \\ &= \int_0^T \int_0^S \int_0^R b_n(y', y - y') f_n(t, y) dy' dy dt \\ &\leq R b_{S_1} \int_0^T \int_0^{S_1} f_n(t, y) dy dt + 1 \leq C(T, R, S_1). \end{aligned}$$

Since the right-hand side of the above estimate does not depend on  $n$  neither on  $S$ , we may first let  $n \rightarrow +\infty$  by (3.30) and then  $S \rightarrow +\infty$  to conclude that (3.31) holds true. Consequently, by the Fubini theorem,  $Q_4(f) \in L^1((0, T) \times (0, R))$  for any  $R \geq 1$ . It is then not difficult to check that (3.2), (3.19), (3.30) and (3.31) imply that

$$Q_4(f_n) \rightharpoonup Q_4(f) \text{ weakly in } L^1((0, T) \times (0, R))$$

for every  $T \in \mathbb{R}_+$  and  $R \in \mathbb{R}_+$ . Consequently,  $f$  satisfies (3.6) and we may argue as in [14, Appendix] to conclude that  $f \in \mathcal{C}([0, +\infty); L^1(\mathbb{R}_+))$ , which completes the proof of Theorem 3.1.  $\square$

**Remark 3.10** Observe that the solution  $f$  to (1.1), (1.2) constructed in Theorem 3.1 satisfies (3.10) and  $M_k \in L^1(0, T)$  for  $k \in [0, 2 + \gamma)$ . Indeed, this follows at once from Lemma 3.2, Lemma 3.3 and (3.30).

## 4 Non-uniqueness

This section is devoted to the proof of Theorem 1.4 which is adapted from [25] where a similar result is proved for the Boltzmann equation. Let  $(\varphi_n)_{n \geq 1}$  be a sequence of mollifiers with  $\text{Supp } \varphi_n \subset [-1/n, 1/n]$  and put

$$f_n^{in}(y) := f^{in}(y) + \frac{\varphi_n(y-n)}{y}$$

for  $y \in \mathbb{R}_+$  and  $n \geq 1$ . Obviously,

$$(4.1) \quad f_n^{in} \rightharpoonup f^{in} \quad \text{in } L^1(\mathbb{R}_+),$$

$$(4.2) \quad M_1(f_n^{in}) = 1 + M_1(f^{in}),$$

so that  $(f_n^{in})$  does not converge weakly to  $f^{in}$  in  $L^1(\mathbb{R}_+, ydy)$ . The assumption (1.8) on  $a$  and  $b$  allows us to apply Theorem 3.1 to deduce that, for each  $n \geq 1$ , there is a weak solution  $f_n$  to (1.1) with initial datum  $f_n^{in}$  which satisfies  $M_{1,n}(t) = 1 + M_1(f^{in})$ ,

$$(4.3) \quad M_{2,n}(t) \leq C \left( 1 + t^{-1/(1+\gamma)} \right),$$

$$(4.4) \quad \int_0^T M_{\ell,n}(s) ds \leq C(\ell, T)$$

for each  $T > 0$  and  $\ell \in [0, 2 + \gamma)$ , where  $M_{\ell,n}(t) := M_\ell(f_n(t))$ .

We now prove that

$$(4.5) \quad \left\{ \begin{array}{l} (f_n) \text{ is relatively compact in } \mathcal{C}([0, T]; w - L^1(\mathbb{R}_+)) \text{ and } \mathcal{C}((0, T]; w - \\ L^1(\mathbb{R}_+, ydy)). \end{array} \right.$$

Since  $(f_n^{in})$  is weakly compact in  $L^1(\mathbb{R}_+)$  by (4.1), a refined version of the de la Vallée-Poussin theorem [15] ensures that there is a non-negative, convex and non-decreasing function  $\Phi \in \mathcal{C}^1([0, +\infty)) \cap W_{\text{loc}}^{2,\infty}([0, +\infty))$  satisfying (3.8) and

$$(4.6) \quad \sup_{n \geq 1} \int_0^\infty \Phi(f_n^{in}(y)) dy \leq C.$$

**Lemma 4.1** *For each  $T > 0$  and  $R \geq 1$ , there is  $C(T, R)$  such that, for  $t \in [0, T]$ ,*

$$(4.7) \quad \sup_{n \geq 1} \int_0^R (f_n(t, y) + \Phi(f_n(t, y))) \leq C(T, R).$$

Taking Lemma 4.1 for granted, we next argue as in Lemma 3.9 with the help of (4.4) to prove the time equicontinuity (3.29) of the sequence  $\{f_n(t), t \in [0, T]\}$ . We then proceed as in the proof of Theorem 3.1 to deduce the claim (4.5) from the above bounds and the Dunford-Pettis theorem. Observe that the weak compactness in  $L^1(\mathbb{R}_+, ydy)$  only holds for positive times because of the blow-up at  $t = 0$  of the estimate (4.3) on  $M_{2,n}$ . Arguing as in [14] with the help of (4.4) then ensure the existence of a non-negative function  $f$  such that

$$f_n \longrightarrow f \quad \text{in } \mathcal{C}([0, T]; w - L^1(\mathbb{R}_+)) \text{ and } \mathcal{C}((0, T]; w - L^1(\mathbb{R}_+, ydy)),$$

and  $f$  is a weak solution to (1.1) with  $f(0) = f^{in}$ . Moreover, since  $M_{1,n}(t) = 1 + M_1(f^{in})$  for  $t > 0$ , the second convergence result warrants that

$$M_1(f(t)) = 1 + M_1(f^{in}) > M_1(f^{in})$$

for each  $t > 0$ . □



*Proof of Lemma 4.1.* We proceed along the lines of [14] with the help of (4.4). We take  $\psi = \mathbf{1}_{[0,1]}$  in (1.12) and (1.14). Since  $\tilde{\mathbf{1}}_{[0,1]} \leq 0$ , we use (1.8) and the mass conservation to obtain

$$\begin{aligned} \frac{d}{dt} \int_0^1 f_n dy &\leq C \int_0^1 \int_y^\infty (1+y')^{\gamma'} f_n(y') dy' dy \\ &\leq C (M_{0,n} + M_{\gamma',n}) \\ &\leq C \left( 1 + M_{\gamma',n} + \int_0^1 f_n dy \right). \end{aligned}$$

Since  $\gamma' < 2 + \gamma$ , the Gronwall lemma and (4.4) imply that

$$\sup_{n \geq 1} \int_0^1 f_n(t, y) dy \leq C(T) \text{ for } t \in [0, T].$$

Consequently, by the mass conservation, we end up with

$$(4.8) \quad \sup_{n \geq 1} M_{0,n}(t) \leq C(T), \quad t \in [0, T].$$

Next, the convexity and non-negativity of  $\Phi$  imply that

$$(4.9) \quad u \Phi'(v) \leq \Phi(u) + v \Phi'(v), \quad u, v \geq 0.$$

For  $R \geq 1$ , it follows from (1.1), (1.8), (4.8) and (4.9) that

$$\begin{aligned} \frac{d}{dt} \int_0^R \Phi(f_n) dy &\leq C \int_0^R \int_0^y (1+y')^\lambda f_n(y') f_n(y-y') \Phi'(f_n(y)) dy' dy \\ &\quad + C \int_0^R \int_y^\infty (1+y')^{\gamma'} f_n(y') \Phi'(f_n(y)) dy' dy \\ &\leq C \int_0^R \int_0^y (1+y')^\lambda f_n(y') \Phi(f_n(y-y')) dy' dy \\ &\quad + C \int_0^R \int_0^y (1+y')^\lambda f_n(y') f_n(y) \Phi'(f_n(y)) dy' dy \\ &\quad + C \int_0^\infty \int_0^R (1+y')^{\gamma'} f_n(y') (\Phi(1) + f_n(y) \Phi'(f_n(y))) dy' dy \\ &\leq C (1 + M_{\lambda,n}) \int_0^R (\Phi(f_n(y)) + f_n(y) \Phi'(f_n(y))) dy \\ &\quad + C (1 + M_{\gamma',n}) \int_0^R (\Phi(1) + f_n(y) \Phi'(f_n(y))) dy. \end{aligned}$$

Now, owing to the convexity of  $\Phi$  and the concavity of  $\Phi'$ , we have  $u \Phi'(u) \leq 2 \Phi(u)$  for  $u \geq 0$ , whence

$$\frac{d}{dt} \int_0^R \Phi(f_n) dy \leq C (1 + M_{\lambda,n} + M_{\gamma',n}) \int_0^R \Phi(f_n) dy,$$

and we conclude as before by (4.4) and the Gronwall lemma.  $\square$

## 5 A modified coagulation model

In this section, we prove Theorem 1.5 and give some qualitative properties of solutions to the modified coagulation equation (1.10).

*Proof of Theorem 1.5.* For  $\varepsilon > 0$ , the coagulation and fragmentation rates given by (1.9) satisfy  $\lambda = 1 + \alpha$  and  $\gamma = 1/2 > \alpha - 1 = \lambda - 2$ . We are thus in a position to apply Theorem 1.1 to deduce that there is a mass-conserving solution  $f_\varepsilon$  to (1.1), (1.2). In particular,  $f_\varepsilon$  solves

$$\begin{aligned} \frac{\partial f_\varepsilon}{\partial t} &= \frac{1}{2} \int_0^y a(y', y - y') f_\varepsilon(y') f_\varepsilon(y - y') dy' \\ &\quad - y f_\varepsilon(y) \int_0^\infty (y')^\alpha f_\varepsilon(y') dy' - y^\alpha f_\varepsilon(y) M_1(f^{in}), \\ &\quad + \varepsilon \left( \int_y^\infty b_1(y, y' - y) f_\varepsilon(y') dy' - \frac{1}{2} \int_0^y b_1(y', y - y') dy' f_\varepsilon(y) \right). \end{aligned}$$

In addition, we may proceed as in [14] to show that the family  $(f_\varepsilon)$  is relatively compact in  $\mathcal{C}([0, T]; w - L^1(\mathbb{R}_+))$  for every  $T > 0$ . Consequently, there are a subsequence  $(f_{\varepsilon_k})$  of  $(f_\varepsilon)$  and a non-negative  $f$  in  $\mathcal{C}([0, +\infty); w - L^1(\mathbb{R}_+))$  such that

$$(5.1) \quad f_{\varepsilon_k} \longrightarrow f \text{ in } \mathcal{C}([0, T]; w - L^1(\mathbb{R}_+)),$$

for each  $T > 0$  and

$$(5.2) \quad M_1(f(t)) \leq M_1(f^{in}), \quad t \geq 0.$$

It is then straightforward to pass to the limit in the equation satisfied by  $f_{\varepsilon_k}$  and conclude that  $f$  is a solution to (1.10), (1.11).  $\square$

We now briefly discuss the occurrence of gelation for (1.10).

**Proposition 5.1** *Consider an initial datum  $f^{in}$  satisfying (1.5) and let  $f$  be a weak solution to (1.10), (1.11) such that*

$$(5.3) \quad M_1(f(t)) \leq M_1(f^{in}), \quad t \geq 0.$$

*Then gelation occurs, that is, there is  $T_{gel} \in [0, +\infty)$  such that  $M_1(f(t)) < M_1(f^{in})$  for  $t > T_{gel}$  and*

$$(5.4) \quad \int_{t_0}^{t_1} \left( \int_e^\infty \frac{y^{1+(\alpha/2)}}{\ln(y)^\delta} f(t, y) dy \right)^2 dt < \infty,$$

$$(5.5) \quad \int_{t_0}^{t_1} (M_1(f^{in}) - M_1(f(t))) \int_e^\infty \frac{y^{\alpha+1}}{\ln(y)^\delta} f(t, y) dy dt < \infty$$

for each  $\delta > 1$  and  $t_1 > t_0 \geq 0$ .

Note that the solution to (1.10), (1.11) constructed in Theorem 1.5 satisfies (5.3) by (5.2), so that Proposition 5.1 applies at least to this particular solution.

*Proof.* We first observe that, for  $\psi \in L^\infty(\mathbb{R}_+)$ , we have

$$(5.6) \quad \begin{aligned} \frac{d}{dt} \int_0^\infty f(y) \psi(y) dy &= \frac{1}{2} \int_0^\infty \int_0^\infty a(y, y') f(y) f(y') \tilde{\psi}(y, y') dy dy' \\ &\quad - (M_1(f^{in}) - M_1(f(t))) \int_0^\infty y^\alpha \psi(y) f(y) dy, \end{aligned}$$

where  $a$  and  $\tilde{\psi}$  are given by (1.9) and (1.13), respectively. The only difference between (5.6) and the corresponding expression for (1.1) is the last term of the above identity. Observe in particular that, thanks to (5.3), this term is non-positive whenever  $\psi \geq 0$ . We then proceed as in [9, Theorem 1.1 & Corollary 2.3] to prove that gelation occurs and that (5.4) holds true.

We next prove (5.5). Let  $A > 0$  and take  $\psi(y) = y \wedge A$  in (5.6) to obtain

$$(5.7) \quad \int_{t_0}^{t_1} (M_1(f^{in}) - M_1(f(t))) \int_A^\infty y^\alpha f(t, y) dy dt \leq \frac{M_1(f(t_0))}{A}$$

for  $t_1 > t_0 \geq 0$ . We now proceed as in [9] and consider a non-decreasing function  $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  with  $\Phi(0) = 0$  such that

$$(5.8) \quad C_\Phi := \int_0^\infty \frac{\Phi'(A)}{A} dA < \infty.$$

We multiply (5.7) by  $\Phi'(A)$  and integrate over  $(t_0, t_1) \times \mathbb{R}_+$  with respect to  $(t, A)$  and obtain

$$(5.9) \quad \int_{t_0}^{t_1} (M_1(f^{in}) - M_1(f(t))) \int_0^\infty f(t, y) y^\alpha \Phi(y) dy dt \leq C_\Phi M_1(t_0).$$

We choose  $\Phi(y) = \left( y^{\alpha+1}/\ln(y)^\delta - r^{\alpha+1}/\ln(r)^\delta \right)^+$  with  $\delta > 1$  and  $r = e^\delta$  and argue as in [9, Corollary 2.3] to conclude that (5.5) holds true.  $\square$

**Remark 5.2** Observe that, if  $t_0 \geq T_{gel}$ ,  $M_1(f^{in}) > M_1(f(t))$  for  $t \in (t_0, t_1]$  and (5.5) implies that

$$\int_e^\infty \frac{y^{\alpha+1}}{\ln(y)^\delta} f(t, y) dy < \infty$$

is finite for almost every  $t \in (t_0, t_1)$ . On the other hand, if  $t_0 < T_{gel}$ , this argument fails. For  $\alpha = 1$ , it is actually conjectured in [8, Section 4] that the solution  $f(t)$  to (1.10), (1.11) decays exponentially for large  $y$  for each  $t \neq T_{gel}$  and algebraically as  $y^{-5/2}$  for  $t = T_{gel}$ . Even the weaker result

$$\int_{t_0}^{t_1} \left( \int_0^\infty y^{1+(\alpha/2)} f(t, y) dy \right)^2 dt = \infty$$

for  $t_1 > T_{gel} > t_0$ , which is true for (1.1) (see [9]), is not available for (1.10).

We finally prove that  $M_1(f(t))$  decays more rapidly for large times than for (1.1), at least for initial data vanishing near  $y = 0$ . It is likely that this is also true for general initial data, but we restrict ourselves to this particular situation for simplicity.

**Proposition 5.3** For any weak solution to (1.10), (1.11) satisfying (5.3) and such that  $f^{in} \equiv 0$  on  $[0, \delta]$  for some  $\delta > 0$ , there is  $C = C(\delta, f^{in}) > 0$  such that

$$M_1(f(t)) \leq C e^{-Ct}, \quad t \geq 0.$$

*Proof.* On the one hand, we take  $\Phi(y) = (y^{1-\alpha} - (\delta/2)^{1-\alpha})^+$  in (5.9) to obtain

$$\int_{t_0}^{t_1} (M_1(f^{in}) - M_1(f(t))) \int_\delta^\infty y f(t, y) dy dt \leq C(\delta) M_1(t_0).$$

On the other hand, it is straightforward to check that  $f(t, y) \equiv 0$  for  $(t, y) \in \mathbb{R}_+ \times (0, \delta)$ . Consequently, we have

$$\int_{t_0}^\infty (M_1(f^{in}) - M_1(f(t))) M_1(f(t)) dt \leq C(\delta) M_1(t_0).$$

For  $t \geq 2 T_{gel}$ , we have  $(M_1(f^{in}) - M_1(f(t))) \geq \tau$  for some  $\tau > 0$  and Proposition 5.3 follows.  $\square$

Notice that the decay rate of  $M_1(f(t))$  obtained in Proposition 5.3 is in agreement with [8]. The corresponding result for (1.1) is that  $M_1(f(t))$  decays at the slower rate  $C(1+t)^{-1}$ .

## A Occurrence of gelation by scaling arguments

We assume that the coagulation and fragmentation rates  $a$  and  $b$  are homogeneous, for instance,

$$(A.1) \quad a(y, y') = y^\alpha (y')^\beta + y^\beta (y')^\alpha, \quad b(y, y') = (y + y')^\gamma,$$

with  $0 \leq \alpha \leq \beta \leq 1$  and  $\gamma \in \mathbb{R}$ . Putting  $\lambda := \alpha + \beta$ , it follows from Theorem 3.1 that gelation can only occur for  $\lambda > 1$ , and this will be assumed to be the case in all the following. Therefore, if only coagulation was present, we would have gelation for any non-zero solution to (1.1).

## A.1 Dominant coagulation

We start by considering the situation where the effects of the fragmentation are so small that we may expect that all the solutions are still gelling. In order to obtain some insight on this possibility, let  $f$  be a solution of the coagulation-fragmentation equation with gelation time  $T \geq 0$  and define the scaled function:

$$(A.2) \quad f_\mu(t, y) = \mu^a f(T + (t - T)\mu^{-1}, \mu^b y)$$

for all  $\mu > 1$ , with

$$a = \frac{3 + \lambda}{\lambda - 1} > 0, \quad b = \frac{2}{\lambda - 1} > 0.$$

Then,

$$\begin{aligned} Q_c(f)(\mu^b y) &= \frac{1}{2} \int_0^{\mu^b y} a(y', \mu^b y - y') f(y') f(\mu^b y - y') dy' \\ &\quad - f(\mu^b y) \int_0^\infty a(\mu^b y, y') f(y') dy' \\ &= \frac{1}{2} \mu^b \int_0^y a(\mu^b y', \mu^b (y - y')) f(\mu^b y') f(\mu^b (y - y')) dy' \\ &\quad - \mu^b f(\mu^b y) \int_0^\infty a(\mu^b y, \mu^b y') f(\mu^b y') dy' \\ &= \frac{1}{2} \mu^{(\lambda+1)b} \int_0^y a(y', y - y') f(\mu^b y') f(\mu^b (y - y')) dy' \\ &\quad - \mu^{(\lambda+1)b} f(\mu^b y) \int_0^\infty a(y, y') f(\mu^b y') dy' \\ &= \mu^{(\lambda+1)b-2a} Q_c(f_\mu)(y). \end{aligned}$$

A similar calculation gives

$$Q_f(f)(\mu^b y) = \mu^{(\gamma+1)b-a} Q_f(f_\mu)(y).$$

We deduce that the function  $f_\mu$  satisfies:

$$(A.3) \quad \frac{\partial f_\mu}{\partial t} = Q_c(f_\mu) - \mu^{(\gamma+1)b-1} Q_f(f_\mu), \quad (t, y) \in (0, +\infty) \times \mathbb{R}_+.$$

If

$$(\gamma + 1)b - 1 > 0 \left( \iff \gamma > \frac{\lambda - 3}{2} \right),$$

the coefficient in front of the fragmentation term can be made as large as we wish. This would make the fragmentation more and more important for the gelling solutions to (1.1). It then does not seem possible in that case that gelation occurs for all solutions to (1.1).

Suppose on the contrary that

$$(\gamma + 1)b - 1 < 0 \left( \iff \gamma < \frac{\lambda - 3}{2} \right).$$

Then, the coefficient in front of the fragmentation term can be made as small as we want. It is then reasonable to expect that the equation (A.3) behaves more and more like the pure coagulation equation for which we know that gelation occurs for all solutions. It is then reasonable to conjecture that this is also the case for the complete equation. Moreover, since the equation (A.3) tends formally to the coagulation equation as  $\mu \rightarrow \infty$ , we may even expect the profile of the gelling solutions at the gelation time to be the same as that of the pure coagulation equation.

## A.2 Balance between coagulation and fragmentation

We consider now the possibility for (1.1) to have a mass-conserving solution  $f$  and, according to the analysis of Section A.1, this requires  $\gamma > (\lambda - 3)/2$ . We then define the following scaled function

$$(A.4) \quad f_\mu(t, y) = \mu^a f(\mu t, \mu^b y)$$

for all  $\mu > 1$ , with

$$a = 2b, \quad b = -\frac{1}{\gamma + 1}.$$

These conditions ensure that  $(\gamma + 1)b + 1 = 0$  and

$$\int_0^\infty y f_\mu(t, y) dy = \int_0^\infty y f(t, y) dy, \quad (t, \mu) \in [0, +\infty) \times (1, +\infty).$$

As above,

$$Q_c(f)(\mu^b y) = \mu^{(\lambda+1)b-2a} Q_c(f_\mu)(y), \quad Q_f(f)(\mu^b y) = \mu^{(\gamma+1)b-a} Q_f(f_\mu)(y),$$

and therefore,

$$(A.5) \quad \frac{\partial f_\mu}{\partial t} = \mu^{(\lambda+1)b+1-a} Q_c(f_\mu) - Q_f(f_\mu), \quad (t, y) \in (0, +\infty) \times \mathbb{R}_+.$$

Notice that

$$(\lambda + 1)b + 1 - a = (\lambda - 1)b + 1 = \frac{\gamma - (\lambda - 2)}{\gamma + 1},$$

and we have  $1 + \gamma > 0$  since  $\lambda > 1$  and  $\gamma > (\lambda - 3)/2$ . Consequently, if

$$(A.6) \quad \gamma - \lambda + 2 < 0,$$

the coefficient in front of the coagulation term can be made as small as we wish. We may then conjecture that the effects of coagulation should be very small compared to those of fragmentation. This would allow for the existence of mass-conserving solutions. Since we have already shown that, if (A.6) holds true and  $M_1(f^{in})$  is large enough, gelation occurs, we may conjecture that only small initial data could give rise to mass-conserving solutions.

## A.3 Gelation

Let us finally recover formally the possibility for a solution to (1.1) to exhibit gelation at time  $T > 0$ , as it is proved in the second assertion of Theorem 1.1. To this end we consider the change of variables:

$$(A.7) \quad f_\mu(t, y) = \mu^a f(T + (t - T)\mu^{-1}, \mu^b y)$$

for all  $\mu > 1$  (so that  $T + (t - T)\mu^{-1} \geq 0$  for  $t \in [0, T]$ ), with

$$a = \frac{\lambda - \gamma}{\gamma + 1}, \quad b = \frac{1}{\gamma + 1}.$$

We then have

$$(A.8) \quad \frac{\partial f_\mu}{\partial t} = Q_c(f_\mu) - Q_f(f_\mu), \quad (t, y) \in (0, +\infty) \times \mathbb{R}_+,$$

and

$$\int_0^\infty y f_\mu(0, y) dy = \mu^{a-2b} \int_0^\infty f^{in}(y) dy.$$

Therefore, if  $a - 2b > 0$ , or equivalently if (A.6) holds, and  $\mu \gg 1$ , the first moment of  $f_\mu(0, y)$  is as large as we need. This is in complete agreement with the second assertion of Theorem 1.1 (Notice that, if  $a - 2b < 0$ , the first moment of  $f_\mu(0)$  cannot be made large since  $\mu$  cannot be too small in order for  $f_\mu$  to be well defined).

We summarize the above analysis in Figure A.1, the coagulation and fragmentation rates  $a$  and  $b$  being still given by (A.1). The parameter  $\lambda$  ranging in  $(1, 2]$ , there is a mass-conserving solution to (1.1), (1.2) for any initial datum  $f^{in}$  satisfying (1.5) when  $\gamma > \lambda - 2$  (region **I** in Figure A.1, see Theorem 3.1). When  $\gamma \in ((\lambda - 3)/2, \lambda - 2)$  (region **II** in Figure A.1), Theorem 1.1 and Section A.2 indicate that gelation occurs when  $M_1(f^{in})$  is large while there should be mass-conserving solutions when  $M_1(f^{in})$  is small. Finally, gelation should occur for every non-zero solution to (1.1), (1.2) when  $\gamma < (\lambda - 3)/2$  (region **III** in Figure A.1) as expected from the analysis in Section A.1.

Figure A.1: **(I)** mass conservation. **(II)** gelation for large initial data and mass conservation for small initial data. **(III)** gelation.

## References

- [1] T.A. Bak and O. Heilmann, *Post-gelation solutions to Smoluchowski's coagulation equation*, J. Phys. A **27** (1994), 4203–4209.
- [2] J.M. Ball and J. Carr, *The discrete coagulation-fragmentation equations : existence, uniqueness, and density conservation*, J. Statist. Phys. **61** (1990), 203–234.
- [3] J. Banasiak, *On a non-uniqueness in fragmentation models*, Math. Methods Appl. Sci. **25** (2002), 541–556.
- [4] F.P. da Costa, *Existence and uniqueness of density conserving solutions to the coagulation-fragmentation equations with strong fragmentation*, J. Math. Anal. Appl. **192** (1995), 892–914.
- [5] C. Dellacherie and P.A. Meyer, *Probabilités et Potentiel, Chapitres I à IV*, Hermann, Paris, 1975.
- [6] R.L. Drake, *A general mathematical survey of the coagulation equation*, in “Topics in Current Aerosol Research (part 2),” International Reviews in Aerosol Physics and Chemistry, Pergamon Press, Oxford, 1972, pp. 203–376.
- [7] P.B. Dubovskii and I.W. Stewart, *Existence, uniqueness and mass conservation for the coagulation-fragmentation equation*, Math. Methods Appl. Sci. **19** (1996), 571–591.
- [8] M.H. Ernst, R.M. Ziff and E.M. Hendriks, *Coagulation processes with a phase transition*, J. Colloid Interface Sci. **97** (1984), 266–277.
- [9] M. Escobedo, S. Mischler and B. Perthame, *Gelation in coagulation and fragmentation models*, Comm. Math. Phys., to appear.
- [10] E.M. Hendriks, M.H. Ernst and R.M. Ziff, *Coagulation equations with gelation*, J. Statist. Phys. **31** (1983), 519–563.
- [11] I. Jeon, *Existence of gelling solutions for coagulation-fragmentation equations*, Comm. Math. Phys. **194** (1998), 541–567.
- [12] Ph. Laurençot, *On a class of continuous coagulation-fragmentation models*, J. Differential Equations **167** (2000), 145–174.
- [13] Ph. Laurençot, *The Lifshitz-Slyozov equation with encounters*, Math. Models Methods Appl. Sci. **11** (2001), 731–748.
- [14] Ph. Laurençot and S. Mischler, *From the discrete to the continuous coagulation-fragmentation equations*, Proc. Roy. Soc. Edinburgh Sect. A, to appear.
- [15] Lê Châu-Hoàn, *Etude de la classe des opérateurs  $m$ -accrétifs de  $L^1(\Omega)$  et accrétifs dans  $L^\infty(\Omega)$* , Thèse de 3<sup>ème</sup> cycle, Université de Paris VI, 1977.
- [16] F. Leyvraz, *Existence and properties of post-gel solutions for the kinetic equations of coagulation*, J. Phys. A **16** (1983), 2861–2873.
- [17] F. Leyvraz and H.R. Tschudi, *Singularities in the kinetics of coagulation processes*, J. Phys. A **14** (1981), 3389–3405.
- [18] P. Meakin and M. H. Ernst, *Scaling in aggregation with breakup simulations and mean-field theory*, Phys. Rev. Lett. **60** (1988), 2503–2506.
- [19] Z.A. Melzak, *A scalar transport equation*, Trans. Amer. Math. Soc. **85** (1957), 547–560.

- [20] C.M. Sorensen, H.X. Zhang and T.W. Taylor, *Cluster-size evolution in a coagulation-fragmentation system*, Phys. Rev. Lett. **59** (1987), 363–366.
- [21] I.W. Stewart, *A global existence theorem for the general coagulation-fragmentation equation with unbounded kernels*, Math. Methods Appl. Sci. **11** (1989), 627–648.
- [22] I.W. Stewart, *On the coagulation-fragmentation equation*, Z. Angew. Math. Phys. **41** (1990), 917–924.
- [23] W.H. Stockmayer, *Theory of molecular size distribution and gel formation in branched-chain polymers*, J. Chem. Phys. **11** (1943), 45–55.
- [24] I.I. Vrabie, *Compactness methods for nonlinear evolutions*, Second edition, Pitman Monogr. Surveys Pure Appl. Math. **75**, Longman, Harlow, 1995.
- [25] B. Wennberg, *An example of nonuniqueness for solutions to the homogeneous Boltzmann equation*, J. Statist. Phys. **95** (1999), 469–477.
- [26] R.M. Ziff and G. Stell, *Kinetics of polymer gelation*, J. Chem. Phys. **73** (1980), 3492–3499.