FAST REACTION LIMIT OF THE DISCRETE DIFFUSIVE COAGULATION-FRAGMENTATION EQUATION

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Abstract

The local mass of weak solutions to the discrete diffusive coagulation-fragmentation equation is proved to converge, in the fast reaction limit, to the solution of a nonlinear diffusion equation, the coagulation and fragmentation rates enjoying a detailed balance condition.

1 Introduction

The diffusive coagulation-fragmentation equation describes the dynamics of a system of a large number of clusters undergoing binary coagulation and fragmentation events and moving in space by brownian movement. Assuming that the size of the clusters ranges in the set of positive integers (discrete case) and denoting by $f(t,x) = (f_i(t,x))_{i\geq 1}$, $f_i \geq 0$, the size distribution function at time t and position x, the discrete diffusive coagulation-fragmentation equation reads [?]

$$\frac{\partial f}{\partial t} - d\,\Delta_x f = Q(f) \quad \text{in } (0,+\infty)\times\Omega\times\mathbb{N}^*; \tag{1.1}$$

$$\frac{\partial f}{\partial n} = 0 \quad \text{on } (0,+\infty)\times\partial\Omega\times\mathbb{N}^*; \tag{1.2}$$

$$f(0) = f^{in} \quad \text{in } \Omega\times\mathbb{N}^*. \tag{1.3}$$

Here, Ω is an open bounded subset of $\mathbb{R}^D$, $D \geq 1$, with smooth boundary $\partial\Omega$, $\mathbb{N}^* := \mathbb{N}\setminus\{0\}$ is the set of positive integers, $\partial f/\partial n = (\partial f_i/\partial n)_{i\geq 1}$ denotes the outward normal derivative of f, $d\,\Delta_x f = (d_i\,\Delta_x f_i)_{i\geq 1}$ and $d = (d_i)_{i\geq 1}$, d_i being the diffusion coefficient of the clusters of size i (or i-clusters) which is assumed to be positive and depend

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only on the size. Finally, the reaction term $Q(f) = (Q_i(f))_{i\geq 1}$ accounts for the binary coagulation and fragmentation reactions and is given by

$$Q_i(f) = \frac{1}{2}\sum_{j=1}^{i-1} a_{j,i-j}\, f_j\, f_{i-j} - f_i \sum_{j=1}^{\infty} a_{i,j} f_j - \frac{1}{2}\sum_{j=1}^{i-1} b_{j,i-j}\, f_i + \sum_{j=1}^{\infty} b_{i,j} f_{i+j}\,; \quad i \geq 1.$$

The coagulation and fragmentation rates $(a_{i,j})$ and $(b_{i,j})$ are assumed to depend only on the sizes of the clusters involved in the reactions. Let us recall the meaning of the different terms in $Q_i(f)$: the first term accounts for the formation of i-clusters by coalescence of two smaller ones and the second for the coagulation of i-clusters with other clusters. The third and fourth terms describe, respectively, the breakage of i-clusters into two smaller pieces and the appearance of i-clusters resulting of the fragmentation of larger ones.

The main goal of this paper is to investigate the fast reaction limit of (??)-(??), that is the behavior of solutions to (??)-(??) when $Q(f)$ is scaled by a factor $1/\varepsilon$ and $\varepsilon \to 0$.

As it is well known in gas dynamics, such a proceeding allows one to (formally) deduce from kinetic equations, which describe a system of particles at a microscopic level, hydrodynamical equations which give a macroscopic description of it. From a physical point of view, this new description is valid when the macroscopic properties of the set of particles (temperature, density, velocity, ...) vary sufficiently slowly through its volume (see, e.g. [?, Chapter 1, §5]). During the last two decades several mathematical works have been made in order to rigorously justify the "hydrodynamical limits" of kinetic equations. Our result is much more in the spirit of the relaxation limit from BGK type equations to scalar conservation laws. We refer to [?] and the references therein for more information on this subject.

For performing that fast reaction limit we need to make several assumptions on the data which we state now. Throughout the paper, we make the following assumptions on the coagulation and fragmentation rates.

(H1) Symmetry and growth assumption: there is a positive real number A_0 such that

$$0 \leq a_{i,j} = a_{j,i} \leq A_0\, ij\,; \quad 0 \leq b_{i,j} = b_{j,i} \leq A_0\, ij \quad \forall i,j \geq 1. \qquad (1.4)$$

A further structural assumption on the coagulation and fragmentation rates is made which guarantees the existence of stationary solutions to (??),(??).

(H2) Detailed balance condition: there exists a sequence $F = (F_i)_{i\geq 1} \in X$ such that

$$F_1 = 1\,; \quad F_i > 0\,; \quad a_{i,j} F_i F_j = b_{i,j} F_{i+j} \quad \forall i,j \geq 1; \qquad (1.5)$$

where X is the Banach space

$$X := \left\{ y = (y_i)_{i \geq 1} \in \ell^1(\mathbb{N}^*) \text{ such that } \|y\|_X := \sum_{i=1}^{\infty} i|y_i| < +\infty \right\}.$$

Clearly, (??) implies that $Q(F) = 0$, so that F is a stationary and spatially homogeneous solution to (??), (??). In fact, the sequence $(F_i z^i)_{i \geq 1}$ satisfies (??) for each $z \geq 0$ and is thus also a stationary and spatially homogeneous solution to (??), (??), usually called an equilibrium. However, physically relevant equilibria are those for which the total mass is finite, that is, those belonging to X. Introducing

$$\varphi(z) := \sum_{i=1}^{\infty} i F_i z^i \in [0, +\infty]$$

for $z \geq 0$, a physically relevant equilibrium $(F_i z^i)_{i \geq 1}$ thus satisfies $\varphi(z) < +\infty$. Our next assumption requires that this property is enjoyed by all equilibria. More precisely, we assume that:

(H3) Existence of an equilibrium for any mass:

$$\vartheta_i := (-\ln F_i)_+ \quad \text{satisfies} \quad \frac{\vartheta_i}{i} \underset{i \to \infty}{\longrightarrow} +\infty. \tag{1.6}$$

Consequently, $\vartheta_i = -\ln F_i$ for i large enough and

$$\varphi(z) < +\infty \quad \text{for each } z \geq 0. \tag{1.7}$$

Since φ is an increasing function from $[0, +\infty)$ onto $[0, +\infty)$, we may parametrize the equilibria by their total mass: for a given mass $\varrho \geq 0$, we define the equilibrium $F[\varrho]$ by

$$F_i[\varrho] := F_i z[\varrho]^i, \quad i \geq 1, \quad \text{where} \quad z[\varrho] := \varphi^{-1}(\varrho). \tag{1.8}$$

At last, we require the coagulation and fragmentation rates to fulfil the following technical condition:

(H4) Strict positivity:

$$a_{1,i} > 0, \quad b_{1,i} > 0, \qquad i \geq 1. \tag{1.9}$$

As for the diffusion coefficients, we assume that they depend only on the size and satisfy

$$0 < d_i \leq d^*, \qquad i \geq 1, \tag{1.10}$$

for some $d^* > 0$.

Let us now explain our main result. On a very formal point of view, if we assume that the solution f to (??), (??) is a local equilibrium, that is, $f_i(t,x) = F_i[\varrho(t,x)]$ for each $i \geq 1$, where $\varrho(t,x) = \|f(t,x)\|_X$ denotes the local mass at time t and position x, then $F[\varrho]$ solves

$$\frac{\partial F[\varrho]}{\partial t} - d\Delta_x F[\varrho] = 0 \quad \text{in } (0,+\infty)\times\Omega\times\mathbb{N}^{\uparrow}, \tag{1.11}$$

since $Q(F[\varrho]) = 0$. In order to write the above equation as an equation involving only ϱ, we introduce the functions

$$\Phi(z) := \sum_{i=1}^{\infty} i d_i F_i z^i, \quad z \geq 0, \quad \text{and} \quad \Psi := \Phi \circ \varphi^{-1}. \tag{1.12}$$

Multiplying the i-th equation of (??) by i and summing the resulting identities over $i \geq 1$, we realize that ϱ (formally) satisfies

$$\frac{\partial \varrho}{\partial t} - \Delta_x \Psi(\varrho) = 0 \quad \text{in } (0,+\infty)\times\Omega, \tag{1.13}$$

$$\frac{\partial \varrho}{\partial n} = 0 \quad \text{on } (0,+\infty)\times\partial\Omega, \tag{1.14}$$

$$\varrho(0) = \varrho^{in} \quad \text{in } \Omega, \tag{1.15}$$

with

$$\varrho^{in} := \sum_{i=1}^{\infty} i f_i^{in}.$$

Now, one way to ensure that a solution to (??),(??) is close to a local equilibrium is to penalize the reaction term $Q(f)$ by a factor $1/\varepsilon$ and this is the approach undertaken below. More precisely, for $\varepsilon \in (0,1)$, we consider a solution f^ε to (??),(??) with $Q(f)/\varepsilon$ instead of $Q(f)$ and study the convergence of the local mass $\varrho^\varepsilon(t,x) := \|f^\varepsilon(t,x)\|_X$ as $\varepsilon \to 0$. Before stating the convergence result, we recall the existence of f^ε together with some useful properties.

Theorem 1.1 Let $(a_{i,j})$, $(b_{i,j})$ and $d = (d_i)$ be such that (??), (??), (??), (??) and (??) are fulfilled and consider an initial datum $f^{in} = (f_i^{in})$ such that

$$\int_\Omega \sum_{i=1}^{\infty} f_i^{in}(x) \left(1 + \vartheta_i + |\ln(f_i^{in}(x))|\right) dx < +\infty. \tag{1.16}$$

For any $\varepsilon > 0$, there exists a mild solution

$$f^\varepsilon \in C([0,+\infty); L^1(\Omega; X)), \quad f^\varepsilon \geq 0,$$

to the initial-boundary value problem

$$\frac{\partial f^\varepsilon}{\partial t} - d\Delta_x f^\varepsilon = \frac{1}{\varepsilon} Q(f^\varepsilon) \quad \text{in } (0,+\infty)\times\Omega\times\mathbb{N}^*, \tag{1.17}$$

$$\frac{\partial f^\varepsilon}{\partial n} = 0 \quad \text{on } (0,+\infty)\times\partial\Omega\times\mathbb{N}^*, \tag{1.18}$$

$$f^\varepsilon(0) = f^{in} \quad \text{in } \Omega\times\mathbb{N}^*, \tag{1.19}$$

which satisfies the total mass conservation

$$M_1(f^\varepsilon(t)) = M_1(f^{in}), \quad t \geq 0, \tag{1.20}$$

and the natural bound

$$\sup_{t\geq 0} H(f^\varepsilon(t)) + \int_0^\infty I(f^\varepsilon(t))\,dt + \frac{1}{\varepsilon}\int_0^\infty D(f^\varepsilon(t))\,dt \leq H(f^{in}), \tag{1.21}$$

where the total mass M_1, the entropy H, the Fisher energy I and the entropy dissipation D are given by:

$$M_1(f) := \sum_{i=1}^\infty i f_i, \quad M_1(f) := \int_\Omega M_1(f)\,dx,$$

$$H(f) := \sum_{i=1}^\infty h\left(\frac{f_i}{F_i}\right) F_i, \quad H(f) := \int_\Omega H(f)\,dx,$$

with $h(s) = s\ln s - s + 1 \geq 0$ for $s \geq 0$,

$$I(f) := 4\sum_{i=1}^\infty d_i \left|\nabla_x f_i^{1/2}\right|^2, \quad I(f) := \int_\Omega I(f)\,dx,$$

and

$$D(f) := \frac{1}{2}\sum_{i,j\geq 1} e(a_{i,j} f_i f_j, b_{i,j} f_{i+j}), \quad D(f) := \int_\Omega D(f)\,dx,$$

with $e(r,s) = (r-s)(\ln r - \ln s) \geq 0$ for $r,s \geq 0$. In addition, for any $\varepsilon > 0$, the local mass ρ^ε satisfies the continuity equation

$$\int_\Omega \left(\rho^\varepsilon(t,x) - \rho^{in}(x)\right)\phi(x)\,dx = \int_0^t\int_\Omega \eta^\varepsilon(\tau,x)\,\Delta_x\phi(x)\,dx\,d\tau \tag{1.22}$$

for $t \geq 0$ and $\phi \in W$, where

$$\rho^\varepsilon := \sum_{i=1}^\infty i f_i^\varepsilon, \qquad \eta^\varepsilon := \sum_{i=1}^\infty i d_i f_i^\varepsilon. \tag{1.23}$$

and

$$W := \left\{ \varphi \in C^2(\bar\Omega) \ \text{satisfying} \ \frac{\partial \varphi}{\partial n} = 0 \ \text{on} \ \partial\Omega \right\} .$$

Also, there is a sequence of nonnegative functions

$$(\rho^{\varepsilon;N})_{N\geq 3}, (\nu^{\varepsilon;N})_{N\geq 3} \in C([0,T];L^1(\Omega)) \cap L^1((0,T)\times\Omega) \qquad (1.24)$$

such that

$$(\rho^{\varepsilon;N}, \nu^{\varepsilon;N}) \longrightarrow (\rho^{\varepsilon}, \nu^{\varepsilon}) \ \text{in} \ L^1((0,T)\times\Omega); \qquad (1.25)$$

$$\rho^{\varepsilon;N}(0;\cdot) \longrightarrow \rho^{in}(\cdot) \ \text{in} \ L^1(\Omega); \qquad (1.26)$$

and

$$\int_\Omega \left(\rho^{\varepsilon;N}(T)\varphi(T) - \rho^{\varepsilon;N}(0)\varphi(0) \right) dx = \int_0^T \int_\Omega \left(\rho^{\varepsilon;N} \partial_t \varphi + \nu^{\varepsilon;N} \nabla_x \varphi \right) dx d\tau \qquad (1.27)$$

for $N \geq 3$, $T > 0$, $t \in [0,T]$ and $\varphi \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$.

We are now in a position to state our main result (convergence of (f^ε) and identification of the limit).

Theorem 1.2 Under the notations and assumptions of Theorem ??:

(i) there is a subsequence (f^{ε_k}) of (f^ε), $\varepsilon_k \to 0$, such that

$$f^{\varepsilon_k} \longrightarrow F[\rho] \ \text{in} \ L^1((0,T)\times\Omega;X) \qquad (1.28)$$

for each $T > 0$, where ρ is a (nonnegative) weak solution to the nonlinear diffusive equation (??)-(??), that is,

$$(\rho, \Phi(\rho)) \in L^1(0,T;L^1(\Omega)); \qquad (1.29)$$

and

$$\int_\Omega \left(\rho(T;x) - \rho^{in}(x) \right) \varphi(x)\, dx = \int_0^T \int_\Omega \Phi(\rho)(\tau;x) \nabla_x \varphi(x)\, dx d\tau \qquad (1.30)$$

for each $T > 0$ and $\varphi \in W$. In particular, there holds

$$\rho^{\varepsilon_k} \longrightarrow \rho; \quad \nu^{\varepsilon_k} \longrightarrow \Phi(\rho) \quad \text{in} \ L^1((0,T)\times\Omega): \qquad (1.31)$$

(ii) if $\rho^{in} \in L^2(\Omega)$, the whole family (f^ε) converges:

$$f^{\varepsilon} \longrightarrow F[\rho] \ \text{in} \ L^1((0,T)\times\Omega;X) \qquad (1.32)$$

for each $T > 0$, and ρ is the unique weak solution to (??)-(??) satisfying

$$\hat\Phi(\rho) \in L^1(0,T;L^1(\Omega)); \quad \Phi(\rho) \in L^2(0,T;H^1(\Omega)) \qquad (1.33)$$

for each $T > 0$, where

$$\hat\Phi(r) := \int_0^r \Phi(s)\, ds; \quad s \in \mathbb{R} :$$

Of course, the convergence (??) holds true for the whole family (f^ε) as soon as there is a unique solution to (??), (??). Such a result does not seem to be obvious as (??) needs not be uniformly parabolic (see Lemma ?? below). To our knowledge, the available uniqueness results require either that $(\varrho, \Phi(\varrho)) \in L^2((0,T)\times\Omega)$ [?], $\varrho \in L^1((\tau,T)\times\Omega)$ for each $\tau > 0$ [?], or the regularity (??) [?], both properties being not fulfilled by the limit ϱ obtained in the first assertion of Theorem ??. Fortunately, when $\varrho^{in} \in L^2(\Omega)$, we can use the last assertion of Theorem ?? and adapt the proof in [?] to conclude that the limit ϱ obtained in (??) actually coincides with the unique weak solution to (??)-(??) satisfying (??) [?].

Remark 1.3 The second assertion of Theorem ?? is still valid if we replace the assumption $\varrho^{in} \in L^2(\Omega)$ by

$$\hat{\Phi}(\varrho^{in}) \in L^1(\Omega) \quad \text{and} \quad \varrho^{in} \in \begin{cases} L^{2D/(D+2)}(\Omega) & \text{if } D \geq 3; \\ L^{1+\alpha}(\Omega) & \text{if } D = 2 \ (\alpha > 0); \\ L^1(\Omega) & \text{if } D = 1; \end{cases}$$

which is weaker. Indeed, $\hat{\Phi}$ is subquadratic by (??).

The convergence (??) is the main contribution of this paper and will be proved in Section ??, while the second assertion of Theorem ?? is proved in Section ??. The existence of a solution f^ε to (??)-(??) enjoying the properties stated in Theorem ?? is not new and follows by gathering arguments from [?, ?]. We will nevertheless recall some of them in the next section together with their consequences. In particular, it follows from (??) and (??) that

$$\sum_{i=1}^{\infty} \#_i f_i^\varepsilon \text{ is bounded in } L^1(0,T; L^1(\Omega));$$

which provides a uniform control of f_i^ε for large sizes by (??), and thus allows to control the tail of the series in f_i^ε. In particular, an example to which Theorem ?? applies are the diffusive Becker-Döring equations [?]. For that model, we have $a_{i,j} = b_{i,j} = 0$ whenever $\min\{i,j\} \geq 2$ and (??) is fulfilled as soon as $b_{1,i}/a_{1,i} \to +\infty$ as $i \to +\infty$ (case 1 in [?]).

Let us also mention at this point that, since the pioneering works [?, ?], several papers have been devoted to the study of diffusive coagulation-fragmentation equations (existence of solutions, see [?, ?, ?, ?, ?, ?] and the references therein, large time behaviour [?, ?, ?], stochastic approximation [?, ?, ?]).

Remark 1.4 When the assumption (??) is not fulfilled, the equilibria $(F_i z^i)_{i \geq 1}$ need not belong to X for each $z \geq 0$ and we define

$$z_s := \inf\left\{ z \geq 0; \ (F_i z^i)_{i \geq 1} \in X \right\} \quad \text{and} \quad \varrho_s := \sup_{z \in [0,z_s)} \varphi(z).$$

The convergence (??) is conjectured to be true when

$$M_1\left(f^{in}\right) \le |\Omega|\varrho_s .$$

Indeed, when $\varrho_s < +\infty$, there is no equilibrium corresponding to a total mass above $|\Omega|\varrho_s$.

2 On the existence result

We first state the fundamental a priori estimates enjoyed by solutions f^ε to (??)-(??) under the assumptions of Theorem ??. To simplify notations, we omit the superscript ε throughout this section. Multiplying the i-th equation of (??) by i and summing up the resulting identities yield, after integration over Ω, the (formal) total mass conservation

$$\frac{d}{dt}M_1(f) = 0,$$

from which we deduce that

$$M_1(f(T)) \le M_1(f^{in}), \quad T \ge 0. \tag{2.1}$$

We next multiply the i-th equation of (??) by $\ln(f_i/F_i)$ and sum up the resulting identities to obtain, after integration over $(0,T)\times\Omega$, the following (formal) H-Theorem

$$H(f(T)) + \int_0^T I(f(t))\,dt + \frac{1}{\varepsilon}\int_0^T D(f(t))\,dt = H(f^{in}), \quad T \ge 0,$$

whence (??), since the right-hand side of the above inequality is finite by (??). On the one hand, thanks to the elementary inequality

$$s(\ln s)_+ \le s\ln s + s(-\ln s)\mathbf{1}_{[e^{-y},1]}(s) + (-s\ln s)\mathbf{1}_{[0,e^{-y}]}(s) \le s\ln s + ys + ye^{-y}$$

for $(s,y)\in(0,+\infty)\times[1,+\infty)$, we have

$$\sum_{i=1}^{\infty} f_i(\ln f_i)_+ \le \sum_{i=1}^{\infty} f_i\ln f_i + \sum_{i=1}^{\infty} i f_i + \sum_{i=1}^{\infty} i e^{-i}.$$

On the other hand,

$$\sum_{i=1}^{\infty} \vartheta_i f_i = \sum_{i=1}^{\infty} f_i(-\ln F_i) + \sum_{i=1}^{\infty} f_i(-\ln F_i)_- ,$$

and the second series of the right-hand side of the above identity is actually a finite sum since $(-\ln F_i)_-$ has a compact support by (??). Gathering all these estimates, there exists a constant C_F depending only on F such that

$$E(f) := \sum_{i=1}^{\infty}\left(1 + \vartheta_i + (\ln f_i)_+\right) f_i \le C_F\,(1 + M_1(f) + H(f)).$$

Therefore, we conclude from (??) and (??) that

$$E(f(t)) := \int_\Omega E(f(t,x))\,dx \le C_F \left(1 + M_1(f^{in}) + H(f^{in})\right), \quad t \ge 0. \qquad (2.2)$$

In addition, the Hölder inequality yields that

$$\int_\Omega \sum_{i=1}^\infty (i d_i)^{1/2} |\nabla_x f_i|\,dx \le I(f)^{1/2} M_1(f)^{1/2},$$

whence

$$\int_0^\infty \left(\int_\Omega \sum_{i=1}^\infty (i d_i)^{1/2} |\nabla_x f_i|\,dx \right)^2 dt \le M_1(f^{in})\, H(f^{in}) \qquad (2.3)$$

by (??) and (??).

A rigorous justification of the above computations may be performed along the lines of [?, Section 5] on a sequence of approximations of (??)-(??). More precisely, for $N \ge 3$, we put $Q^N(f) := (Q_i^N(f))_{i\ge1}$, where

$$Q_i^N(f) := \left(\frac{1}{2} \sum_{j=1}^{i-1} \left(a_{j,i-j} f_j f_{i-j} - b_{j,i-j} f_i\right) - \sum_{j=1}^{N-i} \left(a_{i,j} f_i f_j - b_{i,j} f_{i+j}\right) \right) \times \left(1 + \frac{1}{N} \sum_{j=1}^N f_j \right)^{-1}$$

for $i \in \{1,\ldots,N\}$ and $Q_i^N(f) := 0$ for $i \ge N+1$. We also define $f_i^{in,N} := \min\{N, f_i^{in}\}$ for $i \in \{1,\ldots,N\}$ and $f_i^{in,N} := 0$ for $i \ge N+1$. Proceeding as in [?, Section 5] there is a solution $f^{\varepsilon,N}$ to (??)-(??) with $Q(f)$ replaced by $Q^N(f)$ and initial datum $f^{in,N} := (f_i^{in,N})_{i\ge1}$. Then, $f_i^{\varepsilon,N} = 0$ for $i \ge N+1$ and we may argue as in [?] to realize that $f_i^{\varepsilon,N} \in L^1((0,T)\times\Omega)$ for each $i \in \{1,\ldots,N\}$. Keeping ε fixed, we may pass to the limit as $N \to +\infty$ as in [?], [?] with the help of (??) and (??) to obtain a solution f^ε to (??)-(??) with the properties claimed in Theorem ??. Furthermore, using once more (??) and (??) allows us to control uniformly the behaviour for large i of the approximating sequence and improve (??) to (??), see e.g. [?, Section 5.3]. Finally, the functions

$$\rho^{\varepsilon,N} := \sum_{i=1}^N i f_i^{\varepsilon,N} \quad \text{and} \quad \mu^{\varepsilon,N} := \sum_{i=1}^N i d_i f_i^{\varepsilon,N}$$

are bounded as finite sums of bounded functions and enjoy the properties (??), (??), (??) and (??) as a consequence of the convergence of $(f^{\varepsilon,N})$ towards f^ε.

3 The fast reaction limit

We fix an arbitrary positive time T and put $\Omega_T := (0,T) \times \Omega$. Throughout this section, we denote by C any positive constant depending only on F, d, d_1, f^{in} and T. Further dependence of C upon additional parameters will be indicated explicitly.

Step 1. We first deduce from (??) and (??) that

$$\int_\Omega \sum_{i=\ell+1}^\infty i f_i^\varepsilon(t,x)\, dx \le \sup_{i \ge \ell+1} \left\{ \frac{i}{\vartheta_i} \right\} \int_\Omega \sum_{i=1}^\infty \vartheta_i f_i^\varepsilon(t,x)\, dx \le C \sup_{i \ge \ell+1} \left\{ \frac{i}{\vartheta_i} \right\} \underset{\ell \to +\infty}{\longrightarrow} 0;$$

uniformly with respect to $t \ge 0$ and $\varepsilon > 0$. Consequently

$$\omega(\ell) := \sup_{t \ge 0;\, \varepsilon > 0} \int_\Omega \sum_{i=\ell+1}^\infty i f_i^\varepsilon(t,x)\, dx \underset{\ell \to +\infty}{\longrightarrow} 0. \tag{3.1}$$

Next, the bound (??) and the Dunford–Pettis theorem imply that (f^ε) is weakly sequentially compact in $L^1(\Omega_T; X)$. Consequently there exist $f \in L^1(\Omega_T; X)$ and a subsequence (f^{ε_k}) of (f^ε), $\varepsilon_k \to 0$, such that

$$f^{\varepsilon_k} \rightharpoonup f \quad \text{weakly in } L^1(\Omega_T; X). \tag{3.2}$$

Owing to (??), an immediate consequence of (??) is the convergence of the sequence $(\varrho^{\varepsilon_k}, \eta^{\varepsilon_k})$ defined by (??), namely

$$\varrho^{\varepsilon_k} = \sum_{i=1}^\infty i f_i^{\varepsilon_k} \rightharpoonup \varrho := \sum_{i=1}^\infty i f_i \quad \text{weakly in } L^1(\Omega_T);$$

$$\eta^{\varepsilon_k} = \sum_{i=1}^\infty i d_i f_i^{\varepsilon_k} \rightharpoonup \eta := \sum_{i=1}^\infty i d_i f_i \quad \text{weakly in } L^1(\Omega_T).$$

A slightly stronger convergence result for ϱ^{ε_k} can actually be derived from (??). Indeed, this bound being uniform with respect to $t \ge 0$, we infer from (??) and the Dunford–Pettis theorem that there is a weakly compact subset K of $L^1(\Omega)$ such that

$$\varrho^{\varepsilon_k}(t) \in K \quad \text{for every } t \ge 0 \text{ and } k \ge 1. \tag{3.3}$$

It next follows from (??), (??) and (??) that, for $\zeta \in C_0^1(\Omega)$,

$$\left| \int_\Omega (\varrho^{\varepsilon_k}(t+h) - \varrho^{\varepsilon_k}(t))\, \zeta\, dx \right| = \left| \int_t^{t+h} \int_\Omega \eta^{\varepsilon_k} \cdot \nabla_x \zeta\, dx ds \right| \le C(\|\zeta\|_{C^2})\, |h| \underset{h \to 0}{\longrightarrow} 0.$$

Thanks to (??), the above result extends to any $\zeta \in L^\infty(\Omega)$ by a density argument. A variant of the Ascoli–Arzelà theorem then entails that $(\varrho^{\varepsilon_k})$ is relatively compact in $C([0,T]; \text{weak}-L^1(\Omega))$. Consequently

$$\varrho^{\varepsilon_k} \longrightarrow \varrho \quad \text{in } C([0,T]; \text{weak}-L^1(\Omega)); \tag{3.4}$$

and this convergence, together with (??), implies that (??) holds true. Finally, passing to the limit in the continuity equation (??), we have

$$\int_\Omega \big(\rho(t,x) - \rho^{in}(x)\big)\,\xi(x)\,dx = \int_0^t\int_\Omega j(\tau,x)\cdot\nabla_x\xi(x)\,dx\,d\tau, \qquad (3.5)$$

for $t\in(0,T)$ and $\xi\in W$, the space W being defined in Theorem ??.

Step 2. We now claim that

$$\rho^{\varepsilon_k} \longrightarrow \rho \quad \text{strongly in } L^1(\Omega_T). \qquad (3.6)$$

Indeed, we introduce the notations

$$v_\ell^{\varepsilon_k} := \sum_{i=1}^{\ell} i f_i^{\varepsilon_k} \quad\text{and}\quad w_\ell^{\varepsilon_k} := \sum_{i=\ell+1}^{\infty} i f_i^{\varepsilon_k} = \rho^{\varepsilon_k} - v_\ell^{\varepsilon_k}, \quad \ell\ge 1,$$

and let $(\varrho_n)_{n\ge 1}$ be a sequence of mollifiers in $\mathbb{R}^N$ with $\mathrm{supp}\,\varrho_n\subset\{x\in\mathbb{R}^N : |x|\le 1/n\}$. Also, let Ω' be an open subset of Ω such that $Cl(\Omega')\subset\Omega$ and put $\Omega_T' := (0,T)\times\Omega'$. For $\ell\ge 1$ and n large enough, we write

$$\begin{aligned}
\|\rho^{\varepsilon_k} - \rho\|_{L^1(\Omega_T')} &\le \|\rho^{\varepsilon_k} - (\rho^{\varepsilon_k} *_x \varrho_n)\|_{L^1(\Omega_T')} + \|(\rho^{\varepsilon_k}-\rho) *_x \varrho_n\|_{L^1(\Omega_T')} \\
&\quad + \|(\rho *_x \varrho_n) - \rho\|_{L^1(\Omega_T')} \\
&\le \|v_\ell^{\varepsilon_k} - (v_\ell^{\varepsilon_k} *_x \varrho_n)\|_{L^1(\Omega_T')} + 2\|w_\ell^{\varepsilon_k}\|_{L^1(\Omega_T')} \\
&\quad + \|(\rho^{\varepsilon_k}-\rho) *_x \varrho_n\|_{L^1(\Omega_T')} + \|(\rho *_x \varrho_n) - \rho\|_{L^1(\Omega_T')}. \qquad (3.7)
\end{aligned}$$

On the one hand, it follows from (??) and (??) that, for $\ell\ge 1$,

$$\int_0^T\int_\Omega |\nabla_x v_\ell^{\varepsilon_k}|\,dx\,dt \le \left(\frac{1}{\min_{1\le i\le\ell}(i d_i)^{1/2}}\right)\int_0^T\int_\Omega \sum_{i=1}^{\ell}(i d_i)^{1/2}\,|\nabla_x f_i^{\varepsilon_k}|\,dx\,dt \le C(\ell).$$

The above estimate being uniform with respect to $k\ge 1$, classical arguments and the weak compactness of $(v_\ell^{\varepsilon_k})$ entail that

$$\lim_{n\to+\infty}\sup_{k\ge 1}\|v_\ell^{\varepsilon_k} - (v_\ell^{\varepsilon_k} *_x \varrho_n)\|_{L^1(\Omega_T')} = 0 \quad \text{for each } \ell\ge 1.$$

On the other hand, we infer from (??) that, for each fixed $n\ge 1$, the sequence $(\rho^{\varepsilon_k} *_x \varrho_n)$ converges strongly towards $\rho *_x \varrho_n$ in $L^1(\Omega_T')$. Recalling that the estimate (??) controls the behaviour of $w_\ell^{\varepsilon_k}$ uniformly with respect to $k\ge 1$, we may let $k\to+\infty$ in (??) and obtain

$$\begin{aligned}
\limsup_{k\to+\infty}\|\rho^{\varepsilon_k} - \rho\|_{L^1(\Omega_T')} &\le \sup_{k\ge 1}\|v_\ell^{\varepsilon_k} - (v_\ell^{\varepsilon_k} *_x \varrho_n)\|_{L^1(\Omega_T')} + 2T\,\omega(\ell) \\
&\quad + \|(\rho *_x \varrho_n) - \rho\|_{L^1(\Omega_T')}
\end{aligned}$$

for any n ≥ 1 and ‘ ≥ 1. We next let n ! +1 and then ‘ ! +1 to obtain that (%"k) converges strongly towards % in L1(›0T). Since 0 is arbitrary, we conclude that there is a subsequence of (%"k) (not relabeled) such that (%"k) converges towards % almost everywhere in ›T. The claim (??) follows from this last fact and (??) by the Vitali theorem.

Step 3. Pointwise convergence and local boundedness properties of (%"k) may be deduced from (??) and are stated in the next lemma.

Lemma 3.1 There exists a subsequence of (f"k) (not relabeled) which satisfies: for any fi > 0, there exist Ufi ‰ ›T and Rfi > 0 such that

$$meas(›T \setminus Ufi) • fi ; \quad (3.8)$$

$$%"k ! %; \quad D(f"k) ! 0 \quad \text{uniformly in } Ufi ; \quad (3.9)$$

$$%"k • Rfi \text{ in } Ufi \text{ for } k ≥ 1 : \quad (3.10)$$

Proof. We fix fi > 0. Extracting a subsequence if necessary, we deduce from (??) and (??) that (%"k; D(f"k)) converges almost everywhere to (%; 0) in ›T. The Egorov theorem then ensures the existence of a measurable subset Vfi of ›T such that

$$meas(›T \setminus Vfi) • \frac{fi}{2} \quad \text{and} \quad (%"k; D(f"k)) ! (%; 0) \quad \text{uniformly in } Vfi :$$

Next, since % 2 L1(›T), we have

$$meas\, f(t,x) 2 ›T ; %(t,x) ≥ ,g • \frac{k%k_{L1(›T)}}{,}$$

for , > 0. The first two assertions of Lemma ?? then follow with

$$Ufi := Vfi \cap ‰ (t,x) 2 ›T ; %(t,x) • \frac{2\, k%k_{L1(›T)}}{fi} ¾ :$$

Finally, since % is bounded in Ufi, the bound (??) is a straightforward consequence of the uniform convergence (??). □

Step 4. We now show that the entropy dissipation term D(f"k) \measures" a distance between f"k_i and Fi(f"k_1)i.

Lemma 3.2 Let y = (yi)i≥1 be a nonnegative sequence in X such that D(y) < +1. Then, for any ‘ ≥ 1, we have

$$\sup_{1 • i • ‘} \left| yi ¡ Fi\, y1^i \right| • C(‘)\, (1 + kyk_X)^{‘+1}\, D(y)^{1=2} : \quad (3.11)$$

Proof. Observe that the nonnegativity of $e(r,s)$ and (??) imply that

$$D(y) = \sum_{i,j\geq 1} a_{i,j} F_i F_j \, e(z_i z_j, z_{i+j}) \geq \sum_{i=1}^{\ell} a_{1,i} F_i \, e(z_1 z_i, z_{i+1})$$

with $z_i = y_i/F_i$, $i \geq 1$. Therefore, by assumptions (??) and (??),

$$D(y) \geq C(\ell) \sum_{i=1}^{\ell} e(z_1 z_i, z_{i+1}) .$$

Since $|r-s|^2 \leq \max\{r,s\}\, e(r,s)$ for $r,s \geq 0$, we deduce that

$$\begin{aligned}\sum_{i=1}^{\ell} |z_1 z_i - z_{i+1}| &\leq \ell^{1/2} \left(\sum_{i=1}^{\ell} |z_1 z_i - z_{i+1}|^2 \right)^{1/2} \\ &\leq \ell^{1/2} \max_{1\leq i\leq \ell} \{\max\{z_1 z_i, z_{i+1}\}\}^{1/2} D(y)^{1/2} .\end{aligned}$$

Using again (??), we realize that

$$z_1 z_i \leq C(\ell)\, y_1 y_i \leq C(\ell) \|y\|_X^2 \quad \text{and} \quad z_{i+1} \leq C(\ell) \|y\|_X$$

for $i \in \{1,\ldots,\ell\}$. Consequently, we end up with

$$\sum_{i=1}^{\ell} |z_1 z_i - z_{i+1}| \leq C(\ell)\, (1+\|y\|_X)\, D(y)^{1/2} . \qquad (3.12)$$

Observe now that, for $i \in \{1,\ldots,\ell\}$,

$$\left| z_i - z_1^i \right| = \left| \sum_{j=1}^{i-1} z_1^{i-j-1} (z_{j+1} - z_1 z_j) \right| \leq (1+y_1)^{\ell} \sum_{j=1}^{\ell} |z_{j+1} - z_1 z_j| ,$$

whence

$$\sup_{1\leq i\leq \ell} \left| z_i - z_1^i \right| \leq (1+\|y\|_X)^{\ell} \sum_{j=1}^{\ell} |z_{j+1} - z_1 z_j| .$$

Inserting the above inequality in (??) and using once more (??) yield (??). □

Step 5. We claim that, for every $\delta > 0$,

$$f = F[\varpi] \quad \text{a.e. in} \quad U_\delta , \qquad (3.13)$$

the set U_δ being defined in Lemma ??. Indeed, thanks to (??), (??) and (??), we have, for any $\ell \geq 1$ and $k \geq 1$,

$$\sum_{i=1}^{\infty} i|f_i^{\varepsilon_k} - F_i(f_1^{\varepsilon_k})^i| \le \ell C(\ell)(1+R_\beta)^{\ell+1} D(f^{\varepsilon_k})^{1/2} + \sum_{i=\ell+1}^{\infty} i f_i^{\varepsilon_k} + \sum_{i=\ell+1}^{\infty} i F_i (f_1^{\varepsilon_k})^i$$

$$\le C(\ell,\beta) D(f^{\varepsilon_k})^{1/2} + \omega(\ell) + \sum_{i=\ell+1}^{\infty} i F_i R_\beta^i \quad \text{a.e. in } U_\beta .$$

We now first pass to the limit as $k \to +\infty$ with the help of (??) and then let $\ell \to +\infty$ with the help of (??) and (??) to conclude that

$$\sum_{i=1}^{\infty} i|f_i^{\varepsilon_k} - F_i(f_1^{\varepsilon_k})^i| \underset{k\to+\infty}{\longrightarrow} 0 \quad \text{a.e. in } U_\beta . \qquad (3.14)$$

Consequently

$$|\varrho^{\varepsilon_k} - \varphi(f_1^{\varepsilon_k})| \le \sum_{i=1}^{\infty} i|f_i^{\varepsilon_k} - F_i(f_1^{\varepsilon_k})^i| \underset{k\to+\infty}{\longrightarrow} 0 \quad \text{a.e. in } U_\beta ; \qquad (3.15)$$

and the continuity of φ^{-1} along with (??) imply that

$$\lim_{\varepsilon\to 0} f_1^{\varepsilon_k} = \lim_{\varepsilon\to 0} \varphi^{-1}(\varrho^{\varepsilon_k}) = \varphi^{-1}(\varrho) = z[\varrho] \quad \text{a.e. in } U_\beta . \qquad (3.16)$$

Next, coming back to (??) and using (??), we obtain

$$\|f^{\varepsilon_k} - F[\varrho]\|_X \le \|f^{\varepsilon_k} - F[\varphi(f_1^{\varepsilon_k})]\|_X + \|F[\varphi(f_1^{\varepsilon_k})] - F[\varrho]\|_X \underset{k\to+\infty}{\longrightarrow} 0 \qquad (3.17)$$

a.e. in U_β. Recalling (??), we get (??).

Since β is arbitrary, we deduce from (??) that (??) and (??) actually hold a.e. in Ω_T. Since (f^{ε_k}) converges towards f for the weak topology of $L^1(\Omega_T; X)$ and a.e., the Vitali theorem implies that (f^{ε_k}) converges towards f for the strong topology of $L^1(\Omega_T; X)$, whence (??). It also readily follows from (??) and (??) that

$$\varrho = \sum_{i=1}^{\infty} i d_i F_i[\varrho] = \psi(z[\varrho]) = \Sigma(\varrho);$$

and the proof of assertion (ii) of Theorem ?? is complete. □

We end up this section by some remarks on the diffusion equation satisfied by ϱ. The functions φ and ψ defined in (??) and (??), respectively, are smooth, nonnegative, and increasing functions satisfying

$$\lim_{z\to 0} \varphi(z) = \lim_{z\to 0} \psi(z) = 0; \qquad \lim_{z\to+\infty} \varphi(z) = \lim_{z\to+\infty} \psi(z) = +\infty .$$

Consequently, ψ is a smooth and increasing function and it readily follows from (??), (??) and (??) that

$$0 \le \psi(z) \le \bar d z\,; \quad z \ge 0\,. \tag{3.18}$$

Further properties of ψ are gathered in the next lemma.

Lemma 3.3 We have $\psi'(0) = d_1 > 0$ and

$$\psi(z) = d_1 z + 2 (d_2 - d_1) F_2 z^2 + O(z^3)$$

in a neighbourhood of $z = 0$. In addition, if $d_i \to 0$ as $i \to +\infty$, then

$$\lim_{z\to+\infty} \psi'(z) = 0\,.$$

Proof. For $z \approx 0$, we have

$$\varphi^{-1}(z) = z - 2F_2 z^2 + O(z^3)\,; \quad z \to 0\,.$$

The behaviour of the function $\psi = \chi \circ \varphi^{-1}$ near $z = 0$ is then

$$\psi(z) = d_1 z + 2 (d_2 - d_1) F_2 z^2 + O(z^3)\,; \tag{3.19}$$

with $d_1 > 0$ by hypothesis (??), whence the first assertion of Lemma ??. Next, let $\varepsilon \in (0,1)$. There is $i_0 \ge 1$ such that $d_i \le \varepsilon/2$ for $i \ge i_0$. For $w \ge 1$ it follows from (??) that

$$\begin{aligned}
\chi'(w) &\le \bar d \sum_{i=1}^{i_0} i^2 F_i w^{i-1} + \frac{\varepsilon}{2} \sum_{i=i_0+1}^{\infty} i^2 F_i w^{i-1} \\
&\le \frac{\bar d}{F_{i_0+1}} \left(\sum_{i=1}^{i_0} F_i w^{i-1-i_0} \right) (i_0+1)^2 F_{i_0+1} w^{i_0} + \frac{\varepsilon}{2} \varphi'(w) \\
&\le \left(\frac{\bar d}{F_{i_0+1} w} \sum_{i=1}^{i_0} F_i + \frac{\varepsilon}{2} \right) \varphi'(w) \\
&\le \left(\frac{\bar d\, \varphi(1)}{F_{i_0+1} w} + \frac{\varepsilon}{2} \right) \varphi'(w) \\
&\le \varepsilon \varphi'(w)
\end{aligned}$$

for w large enough, say $w \ge w_\varepsilon$. Therefore, if $z \ge z_\varepsilon := \varphi(w_\varepsilon)$, we infer from the above inequality with $w = \varphi^{-1}(z)$ that

$$(\chi' \circ \varphi^{-1})(z) \le \varepsilon (\varphi' \circ \varphi^{-1})(z)\,;$$

that is $\psi'(z) \le \varepsilon$ for $z \ge z_\varepsilon$, and the proof of Lemma ?? is complete. ∎

As a consequence of Lemma ??, we realize that $\psi' > 0$ on $[0,R]$ for each $R > 0$ but might converge to zero as $z \to +\infty$. The equation (??) is then not necessarily uniformly parabolic, in particular when $d_i \to 0$ as $i \to +\infty$. This last condition is often satisfied, as it is generally assumed that d_i behaves as $i^{-\gamma}$ for large i, with $\gamma > 0$.

4 Improved convergence

The proof of the second assertion of Theorem ?? is actually a consequence of the following result.

Proposition 4.1 Let $T > 0$ and recall that $\Omega_T = (0,T) \times \Omega$. Consider a sequence of nonnegative functions

$$(u_k, v_k)_{k \ge 1} \in C([0,T]; L^1(\Omega)) \cap L^1(\Omega_T); \qquad (4.1)$$

and a nonnegative function $u^{in} \in L^2(\Omega)$ such that

$$(u_k, v_k) \longrightarrow (u, \Sigma(u)) \text{ in } L^1(\Omega_T); \qquad (4.2)$$

$$u_k(0) \longrightarrow u^{in} \text{ in } L^2(\Omega); \qquad (4.3)$$

and

$$\int_\Omega (u_k(T)\varphi(T) - u_k(0)\varphi(0))\, dx = \int_0^T \int_\Omega (u_k\, \partial_t \varphi + v_k\, \nabla_x \varphi)\, dx\, d\tau \qquad (4.4)$$

for any $\varphi \in L^2(0,T; H^2(\Omega)) \cap H^1(0,T; L^2(\Omega))$.

Then $u \le \hat{U}$, where $\hat{U}$ denotes the unique weak solution to (??), (??) with $\hat{U}(0) = u^{in}$ satisfying

$$\hat{\Sigma}(\hat{U}) \in L^\infty(0,T; L^1(\Omega)); \quad \Sigma(\hat{U}) \in L^2(0,T; H^1(\Omega)):$$

Proof. The proof is adapted from that of [?, Proposition 1]. We first notice that (??) implies that

$$\int_\Omega u_k(t,x)\, dx = \int_\Omega u_k(0,x)\, dx; \quad t \in [0,T]: \qquad (4.5)$$

Next, let U^{in} be a nonnegative function in $L^1(\Omega)$. By [?], there is a unique weak solution U to (??), (??) with $U(0) = U^{in}$ such that

$$\hat{\Sigma}(U) \in L^\infty(0,T; L^1(\Omega)); \quad \Sigma(U) \in L^2(0,T; H^1(\Omega)):$$

In addition, U belongs to $L^\infty(\Omega_T)$ by the maximum principle, and satisfies

$$\int_\Omega \left(U(T)\varphi(T) - U^{in}\varphi(0)\right) dx = \int_0^T \int_\Omega (U\, \partial_t \varphi + \Sigma(U)\, \nabla_x \varphi)\, dx\, d\tau \qquad (4.6)$$

for $\varphi \in L^2(0,T; H^2(\Omega)) \cap H^1(0,T; L^2(\Omega))$. In particular, we deduce from (??) that

$$\int_\Omega U(t,x)\, dx = \int_\Omega U^{in}(x)\, dx; \quad t \in [0,T]: \qquad (4.7)$$

For $k \geq 1$, we put

$$X_k := u_k - \frac{1}{|\Omega|}\int_\Omega u_k(0,x)\,dx - U + \frac{1}{|\Omega|}\int_\Omega U^{in}(x)\,dx;$$
$$Y_k := v_k - \varphi(U):$$

We infer from (??), (??), (??) and (??) that

$$\int_\Omega \big(X_k(T)\,\xi(T) - X_k(0)\,\xi(0)\big)\,dx = \int_0^T\int_\Omega \left(X_k\,\partial_t\xi + Y_k\,\Delta_x\xi\right)\,dx\,d\tau, \qquad (4.8)$$

for any $\xi \in L^2(0,T;H^2(\Omega)) \cap H^1(0,T;L^2(\Omega))$. By (??) and (??), we have

$$\int_\Omega X_k(t,x)\,dx = 0 \quad \text{for } t \in [0,T]:$$

Consequently, for each $t \in [0,T]$, there is a unique $Z_k(t) \in H^2(\Omega)$ such that

$$-\Delta_x Z_k(t) = X_k(t) \quad \text{in } \Omega; \qquad (4.9)$$
$$\frac{\partial Z_k(t)}{\partial n} = 0 \quad \text{on } \partial\Omega; \qquad (4.10)$$
$$\int_\Omega Z_k(t,x)\,dx = 0: \qquad (4.11)$$

Our goal is now to take Z_k as a test function in (??). Owing to (??) and the regularity of U, we have $X_k \in L^2(\Omega_T)$ and classical elliptic estimates entail that $Z_k \in L^2(0,T;H^2(\Omega))$. We next infer from (??) that

$$\partial_t Z_k = \frac{1}{|\Omega|}\int_\Omega Y_k\,dx - Y_k;$$

and the right-hand side of the above identity belongs to $L^2(\Omega_T)$ by (??) and the regularity of U. Consequently, Z_k also belongs to $H^1(0,T;L^2(\Omega))$. We may thus take $\xi = Z_k$ in (??) and obtain, thanks to (??),

$$\int_\Omega |\nabla_x Z_k(T)|^2\,dx + 2\int_{\Omega_T} X_k\,Y_k\,dx\,d\tau = \int_\Omega |\nabla_x Z_k(0)|^2\,dx:$$

By the Poincaré-Wirtinger inequality, there is a constant C_Ω depending only on Ω such that

$$\int_\Omega |\nabla_x Z_k(0)|^2\,dx \leq C_\Omega \int_\Omega |X_k(0)|^2\,dx:$$

We thus end up with

$$2\int_{\Omega_T} X_k\,Y_k\,dx\,d\tau \leq C_\Omega \int_\Omega |X_k(0)|^2\,dx;$$

which also reads

$$\int_{\Omega_T} u_k v_k \,dxdt \le \int_{\Omega_T} (u_k \beta(U) + v_k U - U\beta(U))\,dxdt + \frac{1}{|\Omega|}\left(\int_\Omega (u_k(0) - U^{in})\,dx\right)\left(\int_{\Omega_T} (v_k - \beta(U))\,dxdt\right) + \frac{C_\Omega}{2}\int_\Omega |X_k(0)|^2\,dx.$$

Since U and $\beta(U)$ are bounded, we may pass to the limit as $k \to +\infty$ in the above inequality and use (??), (??) and the Fatou lemma to obtain

$$\int_{\Omega_T} u\beta(u)\,dxdt \le \int_{\Omega_T} (u\beta(U) + \beta(u)U - U\beta(U))\,dxdt + \frac{1}{|\Omega|}\left(\int_\Omega (u^{in} - U^{in})\,dx\right)\left(\int_{\Omega_T} (\beta(u) - \beta(U))\,dxdt\right) + \frac{C_\Omega}{2}\int_\Omega |\Lambda(U^{in})|^2\,dx;$$

with

$$\Lambda(U^{in}) := u^{in} - U^{in} - \frac{1}{|\Omega|}\int_\Omega (u^{in} - U^{in})(x)\,dx;$$

whence

$$\int_{\Omega_T} (u - U)(\beta(u) - \beta(U))\,dxdt \le \frac{1}{|\Omega|}\left(\int_\Omega (u^{in} - U^{in})\,dx\right)\left(\int_{\Omega_T} (\beta(u) - \beta(U))\,dxdt\right) + \frac{C_\Omega}{2}\int_\Omega |\Lambda(U^{in})|^2\,dx. \quad (4.12)$$

We now consider a sequence $(U_p^{in})_{p\ge 1}$ of nonnegative functions in $L^1(\Omega)$ such that

$$\lim_{p\to+\infty} \|U_p^{in} - u^{in}\|_{L^2(\Omega)} = 0;$$

and denote by U_p the unique weak solution to (??), (??) with $U_p(0) = U_p^{in}$ given by [?]. On the one hand, it follows from (??) that

$$\int_{\Omega_T} (u - U_p)(\beta(u) - \beta(U_p))\,dxdt \le \frac{1}{|\Omega|}\left(\int_\Omega (u^{in} - U_p^{in})\,dx\right)\left(\int_{\Omega_T} (\beta(u) - \beta(U_p))\,dxdt\right) + \frac{C_\Omega}{2}\int_\Omega |\Lambda(U_p^{in})|^2\,dx.$$

On the other hand, by the L^1-contraction property of weak solutions to (??), (??) [?], we have

$$\lim_{p\to+\infty} \sup_{t\in[0,T]} \|U_p(t) - u(t)\|_{L^1(\Omega)} = 0 .$$

Since Φ is increasing, we may pass to the limit as $p \to +\infty$ with the help of the Fatou lemma to conclude that

$$\int_{\Omega_T} (u - \hat{u})\, (\Phi(u) - \Phi(\hat{u}))\, dx d\tau \le 0 .$$

The function Φ being increasing, Proposition ?? readily follows. $\square$

Proposition ?? is actually still valid if we replace the assumption on u^{in} and (??) by

$$\hat{\Phi}(u^{in}) \in L^1(\Omega) \quad \text{and} \quad u^{in} \in \begin{cases} L^{2D/(D+2)}(\Omega) & \text{if } D \ge 3; \\ L^{1+\alpha}(\Omega) & \text{if } D = 2 \quad (\alpha > 0); \\ L^1(\Omega) & \text{if } D = 1 . \end{cases}$$

Indeed, the only modification to be made is the estimate of $\|\nabla_x Z_k(0)\|_{L^2(\Omega)}$. For instance, if $D \ge 3$, the space $W^{1,2D/(D+2)}(\Omega)$ is continuously embedded in $L^2(\Omega)$. Classical elliptic regularity results and the Poincaré inequality then imply that

$$\|\nabla_x Z_k(0)\|_{L^2(\Omega)} \le C_\Omega \|Z_k(0)\|_{W^{2,2D/(D+2)}(\Omega)} \le C_\Omega \|X_k(0)\|_{L^{2D/(D+2)}(\Omega)} .$$

We then argue as in the previous proof, replacing the L^2-norms of $X_k(0)$ and $\rho(U^{in})$ by their $L^{2D/(D+2)}$-norms.

Proof of Theorem ?? (ii.) It is actually a straightforward consequence of Proposition ??. Indeed, we infer from (??) that, for each $k \ge 1$, there is $N_k \ge 3$ such that

$$\|\rho^{\varepsilon_k;N_k} - \rho^{\varepsilon_k}\|_{L^1(\Omega_T)} + \|\cdot^{\varepsilon_k;N_k} - \cdot^{\varepsilon_k}\|_{L^1(\Omega_T)} \le \frac{1}{k} .$$

Therefore,

$$(\rho^{\varepsilon_k;N_k}, \cdot^{\varepsilon_k;N_k}) \longrightarrow (\rho, \Phi(\rho)) \text{ in } L^1(\Omega_T)$$

by (??) and the properties of Φ. Owing to (??), (??), (??) and (??), we may apply Proposition ?? to conclude that assertion (iii) of Theorem ?? holds true for (f^{ε_k}). The uniqueness of the limit then implies the convergence of the whole family. $\square$

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