A Boltzmann equation for elastic, inelastic and coalescing collisions

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December 9, 2003

Abstract

Existence, uniqueness and qualitative behavior of the solution to a spatially homogeneous Boltzmann equation for particles undergoing elastic, inelastic and coalescing collisions are studied. Under general assumptions on the collision rates, we prove existence and uniqueness of a L^1 solution. This shows in particular that the cooling effect (due to inelastic collisions) does not occur in finite time. In the long time asymptotic, we prove that the solution converges to a mass-dependent Maxwellian function (when only elastic collisions are considered), to a velocity Dirac mass (when elastic and inelastic collisions are considered) and to 0 (when elastic, inelastic and coalescing collisions are taken into account). We thus show in the latter case that the effect of coalescence is dominating in large time. Our proofs gather deterministic and stochastic arguments.

Key words: Existence, Uniqueness, Long time asymptotic, Povzner inequality, Entropy dissipation method, Stochastic interpretation.

AMS Subject Classification: 82C40, 60J75.

Contents

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1 Introduction and notations

We consider the Cauchy problem for a spatially homogeneous kinetic equation modeling (at a mesoscopic level) the dynamics of a system of particles characterized by their mass and impulsion. These particles are supposed to undergo collisions. Each collision results in an elastic rebound, in an inelastic rebound or in a coalescence. These different kinds of collision are taken into account through a classical Boltzmann collision operator, a Granular collision operator (of inelastic interactions) and a Smoluchowski coalescence operator respectively. More precisely, describing the gas by the concentration density $f(t, m, p) \geq 0$ of particles with mass $m \in (0, +\infty)$ and impulsion $p \in \mathbb{R}^3$ at time $t \geq 0$, we study existence, uniqueness and long time behavior of a solution to the Boltzmann-like equation

$$(1.1) \qquad \left\{ \begin{array}{rcl} \dfrac{\partial f}{\partial t} & = & Q(f) = Q_B(f) + Q_G(f) + Q_S(f) \quad \text{in} \quad (0,\infty) \times (0,\infty) \times \mathbb{R}^3, \\ & & \\ f(0) & = & f_{in} \quad \text{in} \quad (0,\infty) \times \mathbb{R}^3. \end{array} \right.$$

In this introduction, we first describe the collision operators Q_B, Q_G, and Q_S. We then deal with possible assumptions on the rates of collision and on the initial condition. Finally, we give the main ideas of the results, some references, and the plan of the paper.

1.1 Collision operators

Let us introduce some notation that will be of constant use in the sequel. We define the phase space of mass-momentum variable $y := (m, p) \in Y := (0, \infty) \times \mathbb{R}^3$, the velocity variable $v = p/m$, the radius variable $r = m^{1/3}$ and the energy variable $\mathcal{E} = |p|^2/m$. Then, for $y^\sharp \in Y$, we will denote by $m^\sharp$, $p^\sharp$, $v^\sharp$, $r^\sharp$, $\mathcal{E}^\sharp$ the associated mass, momentum, velocity, radius and energy respectively. We also denote by $\{y^\sharp\}$ a particle which is characterized by $y^\sharp \in Y$ and we write $\varphi^\sharp = \varphi(y^\sharp)$ for any function $\varphi : Y \to \mathbb{R}$. Finally, for a pair of particles $\{y, y_*\}$, we define some reduced mass variables, the velocity of the center of mass and the relative velocity by

$$m_{**} = m + m_*, \quad \mu = \frac{m}{m_{**}}, \quad \mu_* = \frac{m_*}{m_{**}}, \quad \bar{\mu} = \frac{m m_*}{m_{**}},$$

$$v_{**} = \mu\, v + \mu_*\, v_* \quad \text{and} \quad w = |v_* - v|.$$

For any function $T : Y^2 \to \mathbb{R}$, we will write $T = T(y, y_*)$ and $T_* = T(y_*, y)$.

We now describe the collision terms which are responsible of the changes in the density function due to creation and annihilation of particles with given phase space variable because of the interaction of particles by binary collisions. First, the Boltzmann collision operator $Q_B(f)$ models *reversible elastic binary collisions*, that is collisions which preserve masses, total momentum and kinetic energy. These collisions occur with *symmetric* rate a_B. In other words, denoting by $\{y, y_*\}$ the *pre-collisional* particles and by $\{y', y'_*\}$ the resulting *post-collisional* particles,

$$\{y\} + \{y_*\} \xrightarrow{a_B} \{y'\} + \{y'_*\} \quad \text{with} \quad \begin{cases} m' = m, \quad m'_* = m_*, \\ p' + p'_* = p + p_*, \\ \mathcal{E}' + \mathcal{E}'_* = \mathcal{E} + \mathcal{E}_*. \end{cases} \tag{1.2}$$

The rate of elastic collision $a_B = a_B(y, y_*; y', y'_*)$ satisfies

$$a_B(y, y_*; y', y'_*) = a_B(y_*, y; y'_*, y') = a_B(y', y'_*; y, y_*) \geq 0. \tag{1.3}$$

The first equality expresses that collisions concern *pairs* of particles. The second one expresses the reversibility of elastic collisions: the inverse collision $\{y', y'_*\} \to \{y, y_*\}$ arises with the same probability than the direct one (1.2). The Boltzmann operator reads

$$Q_B(f)(y) = \int_Y \int_{S^2} a_B \left(f'_* f' - f f_*\right) d\nu dy_*. \tag{1.4}$$

Here, for every pair of *post-collisional* particles $\{y, y_*\}$ and every solid angle $\nu \in S^2$, the pair of *pre-collisional* particles $\{y', y'_*\}$ are given by $y' = (m, mv')$, $y'_* = (m_*, m_* v'_*)$ with

$$\begin{cases} v' &= v + 2\,\mu_* \,\langle v_* - v, \nu\rangle \;\nu, \\ v'_* &= v_* - 2\,\mu \,\langle v_* - v, \nu\rangle \;\nu, \end{cases} \tag{1.5}$$

where $\langle ., . \rangle$ stands for the usual scalar product on $\mathbb{R}^3$. Let us explain the meaning of the Boltzmann term $Q_B(f)(y)$ for any given particle $\{y\}$. The nonnegative part, the so-called *gain term* $Q_B^+(f)$, accounts for all the pairs of particles $\{y', y'_*\}$ which collide and give rise to the particle $\{y\}$ as one of the resulting particles. It is worth mentioning that, for any *post-collisional* particles $\{y, y_*\}$, equations (1.5) is nothing but a parameterization (thanks to the solid angle $\nu \in S^2$) of all possible *pre-collisional* velocities (v', v'_*), that is pairs of velocities which satisfy the conservations (1.2). The nonpositive part, the *loss term* $Q_B^-(f)$, counts all possible collisions of the particle $\{y\}$ with another particle $\{y_*\}$.

Next, the Granular collision operator $Q_G(f)$ models *inelastic binary collisions* (preserving masses and total momentum but dissipating kinetic energy), which occur with rate a_G:

$$\{y\} + \{y_*\} \xrightarrow{a_G} \{y''\} + \{y''_*\} \quad \text{with} \quad \begin{cases} m'' = m, \quad m''_* = m_*, \\ p'' + p''_* = p + p_*, \\ \mathcal{E}'' + \mathcal{E}''_* < \mathcal{E} + \mathcal{E}_*. \end{cases} \tag{1.6}$$

In order to quantify the in-elasticity effect and make precise (1.6), it is convenient to parameterize, for any fixed *pre-collisional* particles $\{y, y_*\}$, the resulting *post-collisional* particles $\{y'', y''_*\}$ in the following way:

$$\begin{cases} v'' &= v + (1+e)\,\mu_* \,\langle v_* - v, \nu\rangle \;\nu, \\ v''_* &= v_* - (1+e)\,\mu \,\langle v_* - v, \nu\rangle \;\nu. \end{cases} \tag{1.7}$$

The *deflection solid angle* ν goes all over S^2 and where the *restitution coefficient* e goes all over $(0, 1)$. The coefficient e measures the loss of normal relative velocity during the collision, since

$$\langle v'' - v''_*, \nu\rangle = e \,\langle v_* - v, \nu\rangle \,. \tag{1.8}$$

The case where $e = 1$ corresponds to an elastic collision while $e = 0$ and $\nu = (v_* - v)/|v_* - v|$ indicate a completely inelastic (or *sticky*) collision. The rate of inelastic collision $a_G = a_G(y, y_*; y'', y''_*)$ satisfies the relation

$$a_G(y, y_*; y'', y''_*) = a_G(y_*, y; y''_*, y'') \geq 0, \tag{1.9}$$

which expresses again the fact that (1.6) is an event concerning a *pair* of particles. The Granular operator reads

$$Q_G(f)(y) = \int_Y \int_{S^2} \int_0^1 \left(\frac{\tilde{a}_G}{e} \, \tilde{f} \, \tilde{f}_* - a_G \, f \, f_* \right) \, ded\nu dy_*. \tag{1.10}$$

For any given particle $\{y\}$, the *gain term* $Q_G^+(f)(y)$ in $Q_G(f)(y)$ accounts for all the pairs of *pre-collisional* particles $\{\tilde{y}, \tilde{y}_*\}$ which collide and give rise to the particle $\{y\}$. Inverting (1.7), the *pre-collisional* particles $\{\tilde{y}, \tilde{y}_*\}$ can be parameterized in the following way: $\tilde{y} = (m, m\tilde{v})$, $\tilde{y}_* = (m_*, m_*\tilde{v}_*)$ with

$$\tilde{v} = v + \frac{1+e}{e} \, \mu_* \, \langle v_* - v, \nu \rangle \, \nu, \quad \tilde{v}_* = v_* - \frac{1+e}{e} \, \mu \, \langle v_* - v, \nu \rangle \, \nu. \tag{1.11}$$

We have set $\tilde{a}_G = a_G(\tilde{y}, \tilde{y}_*; y, y_*)$. Note that $1/e$ stands for the Jacobian function of the substitution $(y, y_*) \mapsto (\tilde{y}, \tilde{y}_*)$. The *loss term* $Q_G^-(f)(y)$ counts again all the possible collisions of the particle $\{y\}$ with another particle $\{y_*\}$.

Finally, the Smoluchowski coalescence operator models the following microscopic collision: two *pre-collision* particles $\{y\}$ and $\{y_*\}$ aggregate and lead to the formation of a single particle $\{y_{**}\}$, the mass and momentum being conserved during the collision. In other words,

$$\{y\} + \{y_*\} \xrightarrow{a_S} \{y_{**}\} \quad \text{with} \quad \left\{ \begin{array}{l} m_{**} = m + m_*, \\ p_{**} = p + p_*. \end{array} \right. \tag{1.12}$$

The coalescence being again a pair of particles event, it results that the coalescence rate $a_S = a_S(y, y_*)$ is symmetric

$$a_S(y, y_*) = a_S(y_*, y) \geq 0. \tag{1.13}$$

The Smoluchowski coalescence operator is thus given by

$$\begin{aligned} Q_S(f)(y) \quad = \quad & \frac{1}{2} \int_{\mathbb{R}^3} \int_0^m a_S(y_*, y - y_*) \; f(y_*) \; f(y - y_*) \; dm_* dp_* \\ & - \int_{\mathbb{R}^3} \int_0^\infty a_S(y, y_*) \; f(y) \; f(y_*) \; dm_* dp_*. \end{aligned} \tag{1.14}$$

The *gain term* $Q_S^+(f)(y)$ accounts for the formation of particles $\{y\}$ by coalescence of smaller ones, the factor $1/2$ avoiding to count twice each pair $\{y_*, y - y_*\}$. The *loss term* $Q_S^-(f)(y)$ describes the depletion of particles $\{y\}$ by coalescence with other particles.

Let us emphasize that the effect of these three kinds of collision are very different as it can be observed comparing (1.2), (1.6) and (1.12). On the one hand elastic and inelastic collisions leave invariant the mass distribution, while coalescing collisions make grow the mean mass. On the other hand, kinetic energy is conserved during a Boltzmann collision while it decreases during a Granular or a Smoluchowski collision. In contrast to the Boltzmann equation for elastic collisions, where each collision is reversible at the microscopic level, inelastic and coalescing collisions are irreversible microscopic processes.

1.2 On the collision rates

We want to address now the question of the assumptions we have to make on the collision rates a_B, a_G and a_S. To that purpose, we need to describe a little the physical background of the collision events. For a more detailed physical discussion we refer to [3, 58, 59].

There exists many physical situations where particles evolve according to (at least one of) the above rules of collisions: ideal gases in kinetic physics for elastic collisions [14], granular materials for inelastic collisions [15], astrophysical bodies for coalescence collisions [12], to quote a few of them. We shall rather consider the case of liquid droplets carried out by a gaseous phase and undergoing collisions where, as we will see, the three above rules of collision arise together. The modeling of such liquid sprays is of major importance because of the numerous industrial processes in which they occur. It includes combustion-reaction in motor chambers and physics of aerosols. It also appears in meteorology science in order to predict the rain drop formation.

The Boltzmann formalism we adopt here (description of droplets by the density function) has been introduced by Williams [60] and then developed in [45, 57, 58, 5, 59]. There have been a lot of fundamental studies to improve the understanding of the complex physical effects that play a role in such a two-phase flow. The essential of this research focused on the gas-droplets interactions (turbulent dispersion, burning rate, secondary break-up, ...) see [45, 49, 55, 16, 39, 59]. But in dense sprays, the effect of droplet collisions is of great importance and has to be taken into account. Experimental and theoretical studies (Brazier-Smith et al. [8], Ashgriz-Poo [3] and Estrade et al. [28]) have shown that the interaction between two drops with moderate value of Weber number We (see (1.19) for the definition of We) may basically result in:
(a) a grazing collision in which they just touch slightly without coalescence,
(b) a permanent coalescence,
(c) a temporary coalescence followed in a separation in which few satellite droplets are created.
Since the dynamics of such collisions are very complicated, the available expressions for predicting their outcomes are at the moment mostly empirical. Anyway, the collisions (a) may be well modeled by an elastic collisions (1.2) or by a *stretching* collision in which velocities are unchanged:

$$\{y\} + \{y_*\} \xrightarrow{a_U} \{y\} + \{y_*\}. \tag{1.15}$$

While stretching collisions are often considered in the physical literature, there is no need to take them into account from the mathematical point of view, since the corresponding operator Q_U vanishes identically. Collisions of type (b) are naturally modeled by a coalescence collision (1.12). It is more delicate to model collisions (c), because of the many situations in which it can result. Nevertheless, it can be roughly modeled by an inelastic collision (1.6), where satellization is responsible of the in-elasticity of the collision. Here, possible transfer of mass between the two particles as well as loss of mass (due to satellization) are neglected.

Therefore, at the level of the distribution function, the dynamics of a spray of droplets may be described by the Boltzmann equation (1.1). Of course, such an equation only takes into account binary collisions and neglects the fragmentation of droplets due to the action of the gas, as well as condensation/evaporation of droplets. It also neglects the fluid interaction, in particular the velocity correlation in the collision (see [59]), as well as collisions giving rise to two or more particles with different masses than the initial ones. Nevertheless, equation (1.1) is the most complete Boltzmann collision model we have found in the literature.

We now split into two parts the discussion about the collision rates. First, we address the question of what is the rate that two particles encounter and do collide. Next, we address the question of what is the outcome of the collision event.

It is well known in the Boltzmann theory, that for two free particles interacting by contact collision (hard spheres), the associated total collision frequency a is given by

$$a(y, y_*) = a_{HS}(y, y_*) := (r + r_*)^2 \, |v - v_*|. \tag{1.16}$$

Roughly speaking, a is the rate that two particles $\{y\}$ and $\{y_*\}$ meet. Such a rate is deduced by solving the *scattering problem* for one free particle in a hard sphere potential.
Here the situation is much more intricate since the droplets are not moving in an empty space, but they are rather surrounded by the flows of an ambient gas. Even if the flow is not explicitly taken

into account in the model (1.1), drops can not be realistically considered as going in straight line between two collisions. When a small droplet $\{y_*\}$ approaches a larger one $\{y\}$, it may be deflected of $\{y\}$ due to its interaction with the surrounding gas. It is thus possible that $\{y_*\}$ circumvent $\{y\}$, so that the collision does not occur. The function a may also take into account the fact that the collision between two droplets does not necessarily result in significant change of trajectory (stretching collision (1.15)).
The effect of the deviation of the trajectory in the collision efficiency has been first studied by Langmuir and addressed then by many researchers both from theoretical, numerical and experimental points of view, in particular, in view to the application to meteorology sciences. Langmuir [35] and Beard-Grover [6] have considered the case when $m/m_* << 1$ or $m_*/m << 1$. The case $m \sim m_*$ is much more complicated, and we refer to Davis, Sartor [18] and Neiburger et al. [43] for an analytic expression. See also Pigeonneaux [46] and the numerous references therein for a recent state of the art on that subject. Let us finally quote the experimental study of Brazier-Smith et al. [8]. In any cases, the total collision rate obtained in those works may be written as a modified hard sphere collision rate

$$a(y,y_*) = E(y,y_*)\, a_{HS}(y,y_*) \quad \text{with} \quad 0 \le E \le 1. \tag{1.17}$$

From a mathematical point of view, we will always assume that the total collision efficiency $a(y,y_*)$, i.e. the rate that two particles $\{y\}$, $\{y_*\}$ do collide, is a measurable function on Y^2 and satisfies

$$\forall\, y, y_* \in Y, \qquad 0 \le a(y,y_*) = a(y_*,y) \le A\,(1+m+m_*)\,(1+|v|+|v_*|), \tag{1.18}$$

for some constant $A > 0$. Note that such an assumption is always satisfied by a total collision efficiency a given by (1.17).

To fix the ideas, one can take for instance the following expression of a given by Beard and Grover in [6]

$$E_{BG}(y,y_*) = E(\Delta, We) = \left(\frac{2}{\pi}\,\arctan\left[\max(\alpha_0 + \alpha_1\, Z - \alpha_2\, Z^2 + \alpha_3\, Z^3, 0)\right]\right)^2,$$

where the Weber number We and the mass quotient Δ are defined by (recall that $w = |v - v_*|$),

$$\Delta := \frac{\min(r,r_*)}{\max(r,r_*)}, \qquad We := \min(r,r_*)\, w, \tag{1.19}$$

and with

$$Z = \ln(\Delta^2\, We/K_0), \quad K_0 = \exp(-\beta_0 - \beta_1\, \ln We + \beta_2\, (\ln We)^2),$$

α_i, β_i being numerical positive real numbers. In contrast to the model of Langmuir [35, 58] for which E vanishes for small value of $\Delta^2\, We$, observe that

$$\forall\, \Delta \in (0,1], \quad E_{BG}(\Delta, We) \to 1 \quad \text{when} \quad We \to 0. \tag{1.20}$$

Once two particles have collided, one has to determine what is the outcome of the collision event. This question has been addressed in several physical works, and we refer to Brazier-Smith et al. [8], Ashgriz-Poo [3] and Estrade et al. [28] to quote few of the most significant works. More precisely these authors have mainly proposed an equation for the border line between the region of coalescing collisions (one output particle) and the region of other type of collisions (more than one output) in the plane of deflection angle $\Theta \in [0, \pi/2]$ (or impact parameter b) - Weber number We (for not too large values, typically $We \le 100$) for different values of $\Delta \in (0,1]$. It is worth mentioning that the authors do not quantify the in-elasticity of the rebound (when it occurs) that is the value of the restitution parameter $e \in (0,1]$. As a consequence, we have not been able to find in the physical literature explicit values of the kinetic coefficients a_B, a_G and a_S. Moreover, they show that the number of particles after the collision increases when the Weber number increases and that for large of We satellization really occurs. From this point of view, the validity of the Boltzmann model (1.1) is very contestable since satellization is not taken into account and that particles with large velocity (and therefore pairs of particles with large Weber number) will be created by elastic and inelastic rebounds even if we start with compactly supported initial datum. Once again we refer to [8, 3, 28] for more precise physical description and to the survey articles by Villedieu-Simon [59] and by Post-Abraham [47] and the references therein.

Therefore, the kinetic coefficients a_B, a_G and a_S take into account both the rate of occurrence of collision and the probability that this one results in an elastic, inelastic, or coalescing collision. Abusing notations, we assume that

$$\begin{aligned} a_B &= a_B(y, y_*, \nu) = E_B(y, y_*, \cos\Theta)\, a(y, y_*), \\ a_G &= a_G(y, y_*, \nu, e) = E_G(y, y_*, \cos\Theta, e)\, a(y, y_*), \\ a_S &= a_S(y, y_*) = E_S(y, y_*)\, a(y, y_*), \end{aligned} \tag{1.21}$$

where $\Theta \in [0, \pi/2]$ is the deflection angle (of v' or v'' with respect to v) defined by

$$\Theta \in [0, \pi/2] \quad \cos\Theta := \left| \left\langle \frac{v - v_*}{|v - v_*|}\, ,\, \nu \right\rangle \right|. \tag{1.22}$$

The probability of elastic collision $E_B \geq 0$, of inelastic collision $E_G \geq 0$ and of coalescing collision $E_S \geq 0$ are measurable functions of their arguments, they are symmetric in y and y_*, and they satisfy, for all y, y_* in Y,

$$\bar{E}_B + \bar{E}_G + E_S \leq 1 \quad \text{with} \quad \bar{E}_B := \int_{S^2} E_B\, d\nu, \quad \bar{E}_G := \int_{S^2} \int_0^1 E_G\, de d\nu. \tag{1.23}$$

With such a structure assumption, the symmetry conditions (1.3), (1.9) and (1.13) clearly hold. For future references we also define the total elastic and inelastic collision rates (for all y, y_* in Y)

$$\bar{a}_B = \int_{S^2} a_B\, d\nu = a\bar{E}_B, \qquad \bar{a}_G = \int_0^1 \int_{S^2} a_G\, d\nu de = a\bar{E}_G. \tag{1.24}$$

At last, thanks to the first inequality in (1.23), the collision efficiencies $\bar{a}_B$, $\bar{a}_G$, a_S and a satisfy, for all y, y_* in Y,

$$\bar{a}_B + \bar{a}_G + a_S \leq a. \tag{1.25}$$

Let us finally emphasize that we implicitly take into account the stretching collisions (1.15), setting $a_U := a - \bar{a}_B - \bar{a}_G - a_S \geq 0$.

Let us give an idea of possible shapes for E_B, E_G, E_S. As discussed in [59], the efficiency coefficients depend only on We, Δ (see (1.19)) and on the impact parameter $b = (r + r_*)\cos\Theta$. The collision efficiencies are then given by

$$E_B(y, y_*, \nu) = \kappa \cos\Theta\, \mathbf{1}_{\Theta \in \Lambda_B}, \quad E_G(y, y_*, \nu, e) = \kappa \cos\Theta\, \mathbf{1}_{\Theta \in \Lambda_G}, \quad E_S(y, y_*) = \kappa \int_{S^2} \cos\Theta\, \mathbf{1}_{\Theta \in \Lambda_S}\, d\nu,$$

where Λ_B, Λ_S and Λ_G are disjoint subsets (unions of intervals) of $[0, \pi/2)$ which are continuously depending of $y, y_* \in Y$, $e \in [0, 1]$, and $\kappa^{-1} := \int_{S^2} \cos\Theta\, d\nu$. For example, Brazier-Smith et al. [8] propose $\Lambda_B = \Lambda_G = \emptyset$ and $\Lambda_S = [0, \Theta_{cr})$ where the critical impact parameter b_{cr} (and thus the corresponding Θ_{cr}) is defined by

$$b_{cr} = b_{cr}(We, \Delta) := \min\left(1, \frac{\beta(\Delta)}{\sqrt{We}}\right), \tag{1.26}$$

for some continuous and decreasing function $\beta : (0,1] \to (0,\infty)$. Other examples are due to Ashgriz-Poo [3] and Estrade et al. [28]. In any of them, coalescence collisions are dominating for small value of the Weber number when $\Delta = 1$, and this preponderance increases when Δ decrease. Therefore the following bound holds: there exists $We_0 > 0$ and $\kappa_0 > 0$ such that

$$\forall\, \Delta > 0,\ \forall\, We \in [0, We_0] \qquad E_S(y, y_*) \geq \kappa_0. \tag{1.27}$$

In these models, the coalescence efficiency may vanish for large values of the Weber number.

Acknowledgements. We gratefully acknowledge the partial support of the European Research Training Network *HYKE* HPRN-CT-2002-00282 during this work. We sincerly thank Philippe Villedieu for the many helpful comments and advises he made during the preparation of this work.

1.3 On the initial condition

The initial datum f_{in} is supposed to satisfy

$$0 \leq f_{in} \in L^1_{k^2}(Y), \qquad \int_Y f_{in}\, m\, dy = 1, \qquad \int_Y f_{in}\, p\, dy = 0, \tag{1.28}$$

for the weight functions $k : Y \to \mathbb{R}_+$ defined by

$$k = k_S := 1 + m + |p| + |v| \quad \text{or} \quad k = k_B := \frac{1}{m} + m + \mathcal{E}. \tag{1.29}$$

Here and below, we denote, for any nonnegative measurable function ℓ on Y, the Banach space

$$L^1_\ell = \left\{ f : Y \mapsto \mathbb{R} \text{ measurable;}\ \ \|f\|_{L^1_\ell} := \int_Y |f(y)|\, \ell(y)\, dy < \infty \right\}. \tag{1.30}$$

Let us notice that we do not loose generality assuming the two last moment conditions in (1.28), since we may always reduce to that case by a scaling and translation argument.

1.4 Aims and references

Our main aim in the present paper is to give results about existence, uniqueness, and long time behavior of a solution to (1.1). Roughly speaking, we shall establish the following two results.

Existence and uniqueness. Under the structure assumption (1.21), (1.23) on the collision rates a_B, a_G and a_S and the boundness assumption (1.18) on the total collision rate a, there exists a unique solution $f \in C([0,\infty); L^1(Y))$ to the Boltzmann equation (1.1) with initial condition f_{in} satisfying (1.28).

Long time behavior. Under further suitable assumptions of positivity on a_B, a_G and a_S the long time behavior is the following

$$f(t) \;\to\; \Gamma \quad \text{when} \quad t \to \infty,$$

where

- Γ is a centered mass-dependent Maxwellian with same mass distribution and temperature than f_{in} when $a_G = a_S = 0$ and $a_B > 0$;

- Γ is a centered degenerated mass-dependent Maxwellian (Dirac mass) with same mass distribution than the initial datum when $a_S = 0$, $a_B \geq 0$ and $a_G > 0$;

- $\Gamma = 0$ when $a_G \geq 0$, $a_B \geq 0$ and $a_S > 0$.

This last result, the main of the paper, establishes that each particle's mass tends to infinity in large time. We will give two proofs of it: a deterministic one (based on moment arguments) which allows us to deal only with the pure coalescence equation and a probabilistic one (based on a stochastic interpretation of (1.1)) which is valid in the general case.

Concerning the existence theory, the main difficulty is that when we are concerned by physical unbounded rates, the kernels Q_B, Q_G and Q_S do not map L^1 into L^1. Thus a classical Banach fixed point theorem fails. Basically there are two strategies to overcome this difficulty. The robustness one (which some time extends to spatially non homogeneous context) is to argue by compactness/stability. See the pioneer work by Arkeryd [2] for the case of elastic collisions. In order to apply this method, one has to prove super-linear estimates on the density function f. For the elastic Boltzmann equation this key information is given by the so-called H-Theorem of Boltzmann, which in particular implies that the entropy is bounded. For the kinetic coalescence equation one may do more or less the same, but under a structure hypothesis on the coalescence rate [41, 25].
The second strategy, which we will adopt here, is based on a suitable modification of the proof of a uniqueness result as introduced in [40] for the Boltzmann equation and then taken up again in [26, 27] for a Boltzmann equation for a gas of Bose particles. This method makes possible to prove existence of L^1 solution when no estimate of super linear functional of the density is available. Concerning the uniqueness, we refer, for instance, to [20, 40] for elastic collisions and to [4, 50, 44, 48] for the Smoluchowski equation and for coalescing collisions. Finally, our long time asymptotic behavior result is based on an entropy dissipation method (as introduced in [22]) and also on a stochastic interpretation of the solution.

The most studied operator and equation is undoubtedly the Boltzmann equation for elastic collision since the pioneer works of Carleman [13] and the famous contribution of DiPerna-Lions [21]. For a mathematical and physical presentation of the Boltzmann equation we refer to [14, 56] and the references therein.

The mathematical study of Granular media, which involves inelastic collisions, has received much attention very recently. We refer to [15, 10, 54] and the references therein for further discussions about modeling and physical meaning of that operator, see also [7, 9] for related models. Let us emphasize that more or less stochasticity can be introduce in the inelastic collision. One may assume that the restitution coefficient is determined by the other parameters $e = \bar{e}(y, y_*, \nu)$. In this case, the rate of inelastic collision writes $a_G = \gamma_G(y, y_*, \nu)\, \delta_{e=\bar{e}}$. Here for commodity and simplicity we make the opposite assumption that a_G has a density in the e variable. To our knowledge, existence proofs have been handled only in two cases: the one-dimensional case, see [54], and the case of Maxwellian rate and fixed restitution coefficient, that is, $a_G = \delta_{e=e_0}$ for some $e_0 \in (0, 1)$, see [9, 10]. See also [11, 31] for recent results on modified Boltzmann equations with inelastic collision and [42] for some extensions of the present work to the Boltzmann equation for Granular media. Let us emphasize that it was conjectured that finite time collapse occurs for a class of collisional rates. Our existence result shows that it is not the case.

Least has been done concerning the kinetic coalescence equation (i.e. equation (1.1) with $a_B = a_S = 0$). We may only quote the recent works [48, 25]. See also the paper by Slemrod for coagulation models with discrete velocities [51]. It is however closely related to the Smoluchowski coagulation equation encountered in colloid chemistry, physics of the atmosphere or astrophysics (see for example [23]), where only the mass is taken into consideration. In fact, the Smoluchowski coagulation model may be seen and obtained as a simplified model of the coalescence model (1.1)-(1.14) eliminating the v variable if one knows the shape of the velocity distribution. In many applications involving dense sprays of droplets, no information is known *a priori* on the shape of the velocity distribution and therefore the dependency on v must be kept in the model. A lot of mathematical work has been devoted to the coagulation equation such as existence, uniqueness, conservation of mass and gelation phenomena, long time behavior including convergence to a equilibrium state or self-similarity asymptotic. For further references and results on the coagulation model we refer to [23] and the monograph of Dubowskii [24], as well as the recent surveys [1, 37].

1.5 Plan of the paper

In Section 2, we first give the main physical properties of the collision operators, and we state our main results. Then we study the three operators separately. Section 3 is devoted to the study (existence, uniqueness, *a priori* estimates, long time behavior) of the kinetic Smoluchowski equation $\partial_t f = Q_S(f)$. We study the mass-dependent Boltzmann equation $\partial_t f = Q_B(f)$ in Section 4, while Section 5 concerns the Granular media equation $\partial_t f = Q_G(f)$. Gathering all the arguments, we give an existence and uniqueness proof for the full equation (1.1) in Section 6. Introducing a stochastic interpretation of the solution, we also study the long time behavior of the solution. We finally present some more or less explicit solutions concerning specific rates in Section 7.

2 Main results

In this section we first describe the main physical properties of the collision operators. We then give the definition of solutions we will deal with in this paper. We will finally list the main results concerning the Boltzmann equation (1.1) when some or all the rules of collisions are considered.

2.1 Some properties of collision operators

We want to address now a very simple discussion about the weak and strong representation of the collision kernels. In the whole subsection, g and φ stand for sufficiently integrable functions on Y, and g is supposed to be nonnegative.

First, using the substitution $(y', y'_*) \mapsto (y, y_*)$ (resp. $(\tilde{y}, \tilde{y}_*) \mapsto (y, y_*)$) in the gain term of Q_B (resp. Q_G), we deduce, using the symmetry of collisions (1.3) and (1.9), that

$$\int_Y Q_B(g)\,\varphi\,dy = \frac{1}{2}\int_Y\int_Y\int_{S^2} a_B\,g\,g_*\,(\varphi' + \varphi'_* - \varphi - \varphi_*)\,dydy_*d\nu, \tag{2.1}$$

$$\int_Y Q_G(g)\,\varphi\,dy = \frac{1}{2}\int_Y\int_Y\int_{S^2}\int_0^1 a_G\,g\,g_*\,(\varphi'' + \varphi''_* - \varphi - \varphi_*)\,dydy_*d\nu de. \tag{2.2}$$

The reversibility condition on the rate a_B (second equality in (1.3)) makes possible to perform one more substitution $(y, y_*) \to (y', y'_*)$ to obtain

$$\int_Y Q_B(g)\,\varphi\,dy = \frac{1}{4}\int_Y\int_Y\int_{S^2} a_B\,(g\,g_* - g'\,g'_*)\,(\varphi' + \varphi'_* - \varphi - \varphi_*)\,dydy_*d\nu. \tag{2.3}$$

Performing the substitution $(y - y_*, y_*) \mapsto (y, y_*)$ in the gain term of $Q_S(g)$, we also deduce that

$$\int_Y Q_S(g)\,\varphi\,dy = \frac{1}{2}\int_Y\int_Y a_S\,g\,g_*\,(\varphi_{**} - \varphi - \varphi_*)\,dydy_*. \tag{2.4}$$

These identities provide fundamental physical informations on the operators, well-choosing the test function φ in (2.1), (2.2), (2.3) and (2.4). We want to list some of them now. First, mass and momentum are collisional invariant for the three operators:

$$\int_Y Q_B(g)\begin{pmatrix} m \\ p \end{pmatrix} dy = \int_Y Q_G(g)\begin{pmatrix} m \\ p \end{pmatrix} dy = \int_Y Q_S(g)\begin{pmatrix} m \\ p \end{pmatrix} dy = 0. \tag{2.5}$$

In fact, since elastic and inelastic collisions are mass preserving, we also have, for all $\psi : (0,\infty) \mapsto \mathbb{R}$

$$\int_Y Q_B(g)\,\psi(m)\,dy = \int_Y Q_G(g)\,\psi(m)\,dy = 0. \tag{2.6}$$

Coalescence makes decrease the number of particles: for instance,

$$D_{1,S}(g) := -\int_Y Q_S(g)\,dy = \frac{1}{2}\int_Y\int_Y a_S\,g\,g_*\,dydy_* \geq 0. \tag{2.7}$$

The Boltzmann operator conserves energy

$$\int_Y Q_B(g)\,\mathcal{E}\,dy = 0, \tag{2.8}$$

while Granular and Smoluchowski operators satisfy

$$D_{\mathcal{E},G}(g) := -\int_Y Q_G(g)\,\mathcal{E}\,dy = \frac{1}{2}\int_Y\int_Y\int_{S^2}\int_0^1 a_G\,g\,g_*\,\delta_{\mathcal{E},G}\,dydy_*d\nu de \geq 0, \tag{2.9}$$

$$D_{\mathcal{E},S}(g) := -\int_Y Q_S(g)\,\mathcal{E}\,dy = \frac{1}{2}\int_Y\int_Y a_S\,g\,g_*\,\delta_{\mathcal{E},S}\,dydy_* \geq 0, \tag{2.10}$$

with

$$\delta_{\mathcal{E},G} := \mathcal{E}+\mathcal{E}_*-\mathcal{E}''-\mathcal{E}''_* = (1-e^2)\,\bar\mu\,\langle v-v_*,\nu\rangle^2 \quad \text{and} \quad \delta_{\mathcal{E},S} := \mathcal{E}+\mathcal{E}_*-\mathcal{E}_{**} = \bar\mu\,w^2. \tag{2.11}$$

Observe that coalescence has a stronger *cooling effect* than inelastic collisions, since $\delta_{\mathcal{E},G} < \delta_{\mathcal{E},S}$. Finally, defining $h(g) = g\log g$ and using (2.3), we get

$$D_{h,B}(g) := -\int_Y Q_B(g)\,h'(g)\,dy = \frac{1}{4}\int_Y\int_Y\int_{S^2} a_B\,(g'\,g'_* - g\,g_*)\,\log\frac{g'\,g'_*}{g\,g_*}\,dydy_*d\nu \geq 0, \tag{2.12}$$

which is the key information for the H-Theorem: the irreversibility of Boltzmann equation.

2.2 Definition of solutions

Let us now define the notion of solutions we deal with in this paper.

Definition 2.1 *Assume (1.18), (1.21), (1.23). Recall that k_S is defined in (1.29). Consider an initial condition satisfying (1.28) with $k = k_S$. A nonnegative function f on $[0,\infty)\times Y$ is said to be a solution to the Boltzmann equation (1.1) if*

$$f \in C([0,\infty); L^1_{k_S}(Y)), \tag{2.13}$$

and if (1.1) holds in the sense of distributions, that is,

$$\int_0^T\int_Y \left\{ f\,\frac{\partial\phi}{\partial t} + Q(f)\,\phi \right\} dydt = \int_Y f_{in}\,\phi(0,.)\,dy, \tag{2.14}$$

for any $t > 0$ and any $\phi \in C^1_c([0,T)\times Y)$.

It is worth mentioning that (2.13) and (1.18) ensure that the collision term $Q(f)$ is well defined as a function of $L^1(Y)$, so that (2.14) always makes sense. It turns out that a solution f, defined as above, is also a solution of (1.1) in the mild sense:

$$f(t,.) = f_{in} + \int_0^t Q(f(s,.))\,ds \quad \text{a.e. in} \quad Y. \tag{2.15}$$

Another consequence is that if $f \in L^\infty([0,T), L^1_{k^2})$ and if the total collision efficiency satisfies $a \leq k\,k_*$ for some weight function $k : Y \to \mathbb{R}_+$, then f satisfies the *chain rule*

$$\frac{d}{dt}\int_Y \beta(f)\,\phi\,dy = \int_Y Q(f)\,\beta'(f)\,\phi\,dy \quad \text{in} \quad \mathcal{D}'([0,T)), \tag{2.16}$$

for any $\beta \in C^1(\mathbb{R})\cap W^{1,\infty}(\mathbb{R})$ and any measurable function ϕ such that $\phi/k \in L^\infty(Y)$, see [32, 36].

2.3 Elastic collisions

We first consider the mass-dependent Boltzmann equation for elastic collisions.

Theorem 2.2 *Assume (1.18), (1.21), (1.23), with $a_G = a_S = 0$. Consider an initial condition satisfying (1.28) with $k = k_B$ defined in (1.29). Then there exists a unique solution f to (1.1) such that for all $T \geq 0$, $f \in C([0,T), L^1_{k_B}) \cap L^\infty([0,T), L^1_{k_B^2})$. This solution conserves momentum, mass distribution, and kinetic energy: for all bounded measurable maps $\phi : Y \mapsto \mathbb{R}$ and for all $t \geq 0$,*

$$\text{(2.17)} \qquad \int_Y p f(t,y) dy = \int_Y p f_{in}(y) dy; \qquad \int_Y \phi(m) f(t,y) dy = \int_Y \phi(m) f_{in}(y) dy;$$

$$\text{(2.18)} \qquad \int_Y \mathcal{E} f(t,y) dy = \int_Y \mathcal{E} f_{in}(y) dy.$$

Concerning the long time behavior of the solution, we have the following result.

Theorem 2.3 *In addition to the assumptions of Theorem 2.2, suppose that for some $\delta \in [0, 1/2]$, some function $\psi : [0, \pi/2] \mapsto \mathbb{R}_+$, and some $m_0 > 0$,*

$$\text{(2.19)} \qquad a_B \geq (m m_*)^\delta \, |v_* - v| \, \psi(\Theta) > 0 \quad a.e. \ on \quad Y^2 \times S^2,$$

$$\text{(2.20)} \qquad f_{in} \, (\log f_{in} + m^{6-4\delta}) \in L^1(Y),$$

$$\text{(2.21)} \qquad f_{in} = 0 \quad for \ a.e. \quad p \in \mathbb{R}^3, \ m \in (0, m_0).$$

Then the solution f to (1.1) satisfies the following weak version of the H-Theorem

$$\text{(2.22)} \qquad \forall\, t \geq 0, \qquad H(f(t,.)) + \int_0^t D_{h,B}(f(s,.)) \, ds \leq H(f_{in}),$$

with $D_{h,B}$ defined by (2.12) and $H(f) := \int_Y f \, \log f \, dy$. Furthermore, there holds, as t tends to infinity,

$$\text{(2.23)} \qquad f(t,.) \rightharpoonup M \quad in \quad L^1(Y) - weak,$$

where M is the unique mass-dependent Maxwellian function defined by

$$\text{(2.24)} \qquad M(y) := \frac{\rho(m)}{(2\,\pi\, m\, \Sigma)^{3/2}} \exp\left(-\frac{\mathcal{E}}{2\,\Sigma}\right),$$

with same mass distribution, momentum and kinetic energy as f_{in}, that is,

$$\text{(2.25)} \qquad \rho(m) := \int_{\mathbb{R}^3} f_{in}(m,p) \, dp, \quad \Sigma := \left(3 \int_{(0,\infty)} \rho \, dm\right)^{-1} \left(\int_Y f_{in} \, \mathcal{E} \, dy\right).$$

These theorems just extend some previous known results on the classical (without mass dependence) Boltzmann equation, see [56]. As for the classical Boltzmann equation, they are based on a Povzner inequality (which makes possible to bound weight L^1 norms of the solution) and on the H-Theorem (which expresses the mesoscopic irreversibility of microscopic reversible elastic collisions). Following classical stability/compactness methods, one may also prove existence of solution for initial data satisfying $f_{in} \, k_B + f_{in} \, |\log f_{in}| \in L^1(Y)$. We thus believe that the strong weight L^1 bound $f_{in} \, k_B^2 \in L^1$ as well as (2.20) and (2.21) are technical hypothesis, but we have not be able to prove uniqueness and to study the long time asymptotic without these assumptions.

2.4 Elastic and inelastic collisions

We next consider the Boltzmann equation with elastic and inelastic collisions.

Theorem 2.4 *Assume (1.18), (1.21), (1.23) with $a_S = 0$. Consider an initial condition satisfying (1.28) with $k = k_B$ defined in (1.29). Then there exists a unique solution f to (1.1) such that for all $T \geq 0$, $f \in C([0,T), L^1_{k_B}) \cap L^\infty([0,T), L^1_{k_B^2})$. This solution furthermore conserves momentum and mass distribution (i.e. (2.17) holds), while the total kinetic energy satisfies*

$$\frac{d}{dt} \int_Y f \, \mathcal{E} \, dy = -D_{\mathcal{E},G}(f) \geq 0, \tag{2.26}$$

where the term of dissipation $D_{\mathcal{E},G}$ is defined by (2.9). In particular, $t \mapsto \int_Y f(t,y) \, \mathcal{E} \, dy$ is nonincreasing and $t \mapsto D_{\mathcal{E},G}(f(t,.)) \in L^1([0,\infty))$.

Under a suitable lower-bound of the inelastic collision rate a_G, we also have the following result.

Theorem 2.5 *In addition to the assumptions of Theorem 2.4, assume that the total inelastic collision rate $\bar{a}_G$ (see (1.24)) is continuous and satisfies $\bar{a}_G(y, y_*) > 0$ for all $(y, y_*) \in Y^2$ such that $v \neq v_*$. Then the kinetic energy is strictly decreasing, and, as t tends to infinity,*

$$f(t,.) \rightharpoonup \rho(m) \, \delta_{p=0} \quad in \quad M^1(Y) - weak. \tag{2.27}$$

where ρ is defined by (2.25). The velocity distribution

$$j(t,v) := \int_0^\infty f(t, m, m\,v) \, m^4 \, dm \tag{2.28}$$

satisfies, as t tends to infinity,

$$j(t,.) \rightharpoonup \delta_{v=0} \quad in \quad M^1(\mathbb{R}^3) - weak. \tag{2.29}$$

If furthermore there exist some constants $\kappa > 0$ and $\delta \in [0, 1/2]$ such that, for all y, y_* in Y,

$$\hat{a}_G := \int_{S^2} \int_0^1 (1 - e^2) \, \langle v - v_*, \nu \rangle^2 \, a_G(y, y_*, \cos\Theta, e) d\nu de \geq \kappa (m m_*)^\delta |v - v_*|^3, \tag{2.30}$$

the following rate of convergence holds, for some constant $C \in (0, \infty)$,

$$\forall \, t \geq 1, \quad \int_{\mathbb{R}^3} |v|^2 \, j(t,v) \, dv = \int_Y f(t,y) \, \mathcal{E} \, dy \leq \frac{C}{t^2}. \tag{2.31}$$

For the mass independent inelastic Boltzmann equation, existence of L^1 solutions is known in dimension 1 [54] and in all dimensions for the pseudo Maxwell molecules cross-section [9]. To our knowledge, Theorem 2.4 is thus the first existence (and uniqueness) result of L^1 solutions to the inelastic Boltzmann equation for the hard spheres cross-section in dimension $N > 1$. It also answers by the negative to the question of finite time cooling, see [54]. Theorem 2.5 shows that the cooling effect occurs asymptotically in large time and thus extends to that context previous known results.

2.5 Elastic, inelastic and coalescing collisions

We finally treat the case of the full Boltzmann equation (1.1).

Theorem 2.6 *Assume (1.18), (1.21), (1.23), and consider an initial condition satisfying (1.28) with $k = k_B$ defined in (1.29). Then there exists a unique solution f to (1.1) such that for all $T \geq 0$, $f \in C([0,T), L^1_{k_B}) \cap L^\infty([0,T), L^1_{k_B^2})$. This solution furthermore conserves mass and momentum*

$$\int_Y f(t,y) \, m \, dy = \int_Y f_{in}(y) \, m \, dy, \qquad \int_Y p f(t,y) dy = \int_Y p f_{in}(y) dy, \tag{2.32}$$

while kinetic energy and total number of particles density decrease, more precisely

$$(2.33)\qquad \frac{d}{dt}\int_Y f\,dy = -D_{1,S}(f), \qquad \frac{d}{dt}\int_Y f\,\mathcal{E}\,dy = -D_{\mathcal{E},G}(f) - D_{\mathcal{E},S}(f),$$

where $D_{1,S}$, $D_{\mathcal{E},G}$ *and* $D_{\mathcal{E},S}$ *were defined by (2.7), (2.9) and (2.10). In particular,*

$$(2.34)\qquad D_{1,S}(f),\ D_{\mathcal{E},G}(f),\ D_{\mathcal{E},S}(f) \in L^1([0,\infty)).$$

When $a_B = a_G = 0$ *the same results holds replacing* k_B *by* k_S *defined in (1.29).*

Existence results for the pure kinetic coalescence equation (that is $a_B = a_G = 0$) have been previously obtained in [48, 25]. In [48], measure solutions have been built for general kernels, but L^1 solution have been obtained for more restrictive kernels. The authors have also proved a stabilization result to a family of stationary solutions but they were not able to identify that limit to be 0. In [25], an additional structure assumption (see (3.30)) on the coalescence kernel has been made, which permits to prove that any L^p norm is a Lyapunov function. This assumption is satisfied by the hard sphere collisional efficiency a_{HS} defined by (1.16) but not for any general coalescence rate a_S of the form (1.17). The method used in [25] does anyway not extend to the case where $a_B \not\equiv 0$ or $a_G \not\equiv 0$ but it applies to spatially inhomogeneous model.

The next result shows that coalescence dominates other phenomena for large times.

Theorem 2.7 *In addition to the assumptions of Theorem 2.6, suppose that for any* $m_0 \in (0,\infty)$, *there exists* $A_0 > 0$ *such that,*

$$(2.35)\qquad m_*\,\mathbf{1}_{m_*\le m_0}\,[\bar E_B + \bar E_G] \le A_0\,\tilde E_{inel} \quad \text{on } Y^2,$$

where

$$(2.36)\qquad \tilde E_{inel}(y,y_*) := (1+\bar\mu w^2)E_S(y,y_*) + \bar\mu \int_{S^2}\int_0^1 (1-e^2)\,\langle v - v_*, \nu\rangle^2\, E_G(y,y_*,\nu,e)\,d\nu de.$$

Also assume that for any $y \in Y$, *there exists* $\varepsilon > 0$ *such that*

$$(2.37)\qquad a_S(y,y_*) > 0 \quad \text{for a.e.} \quad y_* \in B_Y(y,\varepsilon).$$

Then

$$(2.38)\qquad f(t,.) \;\to\; 0 \quad \text{in} \quad L^1(Y) \quad \text{when} \quad t\to\infty.$$

Here the hypothesis (2.37) on a_S seems to be very general, and we believe that it is not restrictive for a physical application. It is in particular achieved for a coalescence rate a_S given by (1.21) with a and E_S satisfying (1.17), (1.20) and (1.27). It is satisfied by the collision kernel proposed in Brazier-Smith et al. (1.26). Of course, hypothesis (2.37) is fundamental in order to coalescence process dominate, not making that assumption (taking for instance $a_S(y,y_*) = 0$ for any y, y_* with $|y - y_*| \le 1$ and $\bar a_G > 0$) the asymptotic behavior should be driven by the inelastic Granular operator and (2.27) should hold again.
The hypothesis (2.35) on a_B and a_G are less obviously satisfied by collision kernels discussed in the physical literature, mainly because the collision rates a_B and a_G are not explicitly written. Notice that (2.35) automatically holds when collision are not elastic, quasi-elastic nor grazing, that is when $a_B = 0$, $E_G = E_G(\Delta, We, \Theta, e) = 0$ for any $\Theta \in (\Theta_0, \pi/2]$ and $e \in (e_0, 1]$ with $e_0 \in (0,1)$ and $\Theta_0 \in (0,\pi/2)$ and E_S satisfying (1.27). Indeed, in that case, condition (2.35) may be reduced to

$$(2.39)\qquad m_*\,\mathbf{1}_{m_*\le m_0}\,\bar E_G \le A_0\,\{E_S + \frac{\bar\mu}{\min(m,m_*)^2}\,We^2\,\bar E_G\}.$$

Then condition (2.39) holds for $We < We_0$ because of (1.27) and for $We \geq We_0$ because of

$$\frac{\bar{\mu}}{\min(m, m_*)^2} \geq \frac{1}{2\,\min(m, m_*)} \geq \frac{1}{2\,m_0^2}\, m_*\, \mathbf{1}_{m_* \leq m_0}.$$

Therefore, assumption (2.35) contains two conditions. On the one hand, it says that for moderate values of the Weber number (says $We \leq 1$) elastic and inelastic collisions do not dominate coalescence, and that always holds when E_S satisfies (1.27). On the other hand, it says that for large values of Weber number (says $We \geq 1$) elastic and quasi-elastic collisions do not dominate (strong) inelastic collisions. We believe that this second condition is technical and should be removed. Finally, the assumption on f_{in} is not the most general (a condition such as $f_{in}\,(1 + m + |p|) \in L^1(Y)$ would be more natural), but this is not really restrictive from a physical point of view.

The convergence result (2.38) means exactly that the *total concentration* $\int_Y f(t, y)dy$ tends to 0 as time tends to infinity. In other words, the mass of each particle tends to infinity: coalescence is the dominating phenomenon in large time. The convergence (2.38) is not *a priori* obvious because, when the collision rate a vanishes on $v = v_*$ (which is the case for a collision rate given by (1.16)-(1.17)), the density function $S(m, p) = \lambda(dm)\,\delta_{p=mv_0}$ is a stationary solution to (1.1) for any bounded measure $\lambda \in M^1(0, +\infty)$ and any vector $v_0 \in \mathbb{R}^3$. In particular, Theorem 2.7 implies that the zero solution is the only stationary state which is reached in large time when starting from an L^1 initial data. It also means that the cooling process (due to coalescing and inelastic collisions) is dominated (under assumption (2.35)) by the mass growth process (due to coalescence). We thus identify more accurately the asymptotic state than in [48], and we do it without any structure condition as introduced in [25]. We extend to more realistic kernels the result presented in [30].

In the pure coalescence case, we may give another asymptotic behavior for solutions which are O-symmetric. We say that $g \in L^1(Y)$ is O-symmetric if g is symmetric with respect to the origin 0 in the impulsion variable $p \in \mathbb{R}^3$, that is

$$g(m, -p) = g(m, p) \quad \text{for a.e.} \quad (m, p) \in Y. \tag{2.40}$$

Theorem 2.8 *In addition to the assumptions of Theorem 2.6, suppose that f_{in} is O-symmetric and that (2.37) holds. Assume also that $a_B = a_G = 0$, and that a_S satisfies the natural conditions*

$$a_S(m, -p, m_*, -p_*) = a_S(m, p, m_*, p_*) \quad \textit{for a.e.} \quad y, y_* \in Y, \tag{2.41}$$

and

$$a_S(m, p, m_*, p_*) \leq a_S(m, p, m_*, -p_*) \quad \textit{for a.e.} \quad y, y_* \in Y \quad \textit{such that} \quad \langle p, p_* \rangle > 0. \tag{2.42}$$

Then the solution f to (1.1) given by Theorem 2.6 is also O-symmetric and satisfies

$$\int_{\mathbb{R}^3} f(t, .)\,|p|^2\,dy \leq \int_{\mathbb{R}^3} f_{in}\,|p|^2\,dy \quad \forall\, t \geq 0. \tag{2.43}$$

Moreover, the velocity distribution j defined by (2.28) satisfies

$$\int_{\mathbb{R}^3} |v|\,j(t, v)\,dv = \int_Y |p|\,f(t, y)\,dy \to 0 \quad \textit{when} \quad t \to \infty, \tag{2.44}$$

and therefore, (2.29) holds.

Note that under the very stringent (and not physical) condition that the coalescence rate satisfies $a_S \geq (m + m_*)\,|v - v_*|$ (and $a_B = a_G = 0$) we may also show that (2.31) holds.

3 The kinetic coalescence equation

In this section we focus on the sole kinetic Smoluchowski equation

$$\frac{\partial f}{\partial t} = Q_S(f) \ \text{ on } \ (0,\infty)\times Y, \quad f(0,.) = f_{in} \ \text{ on } \ Y, \tag{3.1}$$

where Q_S is given by (1.14). Our aim is to prove Theorems 2.6, 2.7 and 2.8 in this particular situation. We first present a simple computation leading to a uniqueness result, then we gather some *a priori* estimates. Next we prove an existence result, and we conclude with some proofs concerning the long time asymptotic.

We assume in the whole section that $a_B \equiv 0$, $a_G \equiv 0$, (1.18), (1.21), (1.23), and consider an initial condition satisfying (1.28) with $k = k_S$ or $k = k_B$ defined in (1.29).

3.1 Uniqueness

We start with an abstract uniqueness lemma.

Lemma 3.1 *Let us assume that a_S and k are two measurable nonnegative functions on Y^2 and Y respectively such that for any $y, y_* \in Y$ there holds*

$$0 \le a_S(y,y_*) = a_S(y_*,y) \le k\, k_* \quad \textit{and} \quad k_{**} \le k + k_*. \tag{3.2}$$

Then there exists at most one weak solution to the kinetic Smoluchowski equation (3.1) such that for all $T \ge 0$,

$$f \in C([0,T); L^1_k) \cap L^\infty([0,T); L^1_{k^2}). \tag{3.3}$$

Note that under (1.18), one may choose $k = C_A k_S$ or $k = C_A k_B$ (for C_A a constant). The uniqueness part of Theorem 2.6 thus follows immediately when ($a_B \equiv 0$ and $a_G \equiv 0$).

Proof of Lemma 3.1. We consider two weak solutions f and g associated to the same initial datum f_{in} and which satisfy (3.3). We write the equation satisfied by $f - g$ that we multiply by $\phi(t,y) = \text{sign}(f(t,y) - g(t,y))\, k$. Using the chain rule (2.16) and the weak formulation (2.4) of the kinetic Smoluchowski operator, we get for all $t \ge 0$,

$$\begin{aligned}
\frac{d}{dt}\int_Y |f-g| k\, dy &= \frac{1}{2}\int_Y\int_Y a_S(y,y_*)\,((f-g)g_* + f(f_*-g_*))\,(\phi_{**} - \phi - \phi_*)\,dy_* dy \\
&= \frac{1}{2}\int_Y\int_Y a_S(y,y_*)\,(f-g)\,(f_*+g_*)\,(\phi_{**} - \phi - \phi_*)\,dy_* dy \\
&\le \frac{1}{2}\int_Y\int_Y a_S(y,y_*)\,|f-g|\,(f_*+g_*)\,(k_{**} - k + k_*) dy_* dy,
\end{aligned}$$

where we have just used the symmetry hypothesis (3.2) on a_S. Then, thanks to the bounds (3.2), we deduce

$$\frac{d}{dt}\int_Y |f-g| k\, dy \le \int_Y\int_Y k\, k_*\, |f-g|\,(f_*+g_*)\,k_*\, dy_* dy = \|f+g\|_{L^1_{k^2}} \|f-g\|_{L^1_k}.$$

One easily concludes, by using the Gronwall Lemma, that for all $T \ge 0$,

$$\sup_{[0,T]} \|f(t,.) - g(t,.)\|_{L^1_k} \le \|f(0,.) - g(0,.)\|_{L^1_k} \exp\left(\sup_{[0,T]} \|f(t,.) + g(t,.)\|_{L^1_{k^2}} T\right) \tag{3.4}$$

which is identically null, since $f(0,.) = g(0,.)$. ◻

We deduce that O-symmetry propagates.

Corollary 3.2 *Assume that a_S and k satisfy the assumptions of Lemma 3.1. Suppose also that f_{in} satisfies the O-symmetry condition (2.40) while a_S meets (2.41). Then a solution f to (3.1) satisfying (3.3) is also O-symmetric.*

Proof of Corollary 3.2. Introduce the notation $f^\sharp = f^\sharp(m,p) := f(m,-p)$. Clearly $f^\sharp$ also satisfies (3.3). A simple computation shows that

$$\begin{aligned} Q_S^+(f)(m,-p) &= \int_0^m \int_{\mathbb{R}^3} a_S(m-m_*,-p-p_*,m_*,p_*)\, f(m-m_*,-p-p_*)\, f(m_*,p_*)\, dm_* dp_* \\ &= \int_0^m \int_{\mathbb{R}^3} a_S(m-m_*,-p+q_*,m_*,-q_*)\, f(m-m_*,-p+q_*)\, f(m_*,-q_*)\, dm_* dq_* \\ &= \int_0^m \int_{\mathbb{R}^3} a_S(m-m_*,p-q_*,m_*,q_*)\, f^\sharp(m-m_*,p-q_*)\, f^\sharp(m_*,q_*)\, dm_* dq_* \\ &= Q_S^+(f^\sharp)(m,p). \end{aligned}$$

We have made the substitution $q_* = -p_*$ and then used the symmetry (2.41) of a_S. By the same way, one may prove $Q_S^-(f)(m,-p) = Q_S^-(f^\sharp)(m,p)$ for any $(m,p) \in Y$. In other words, the function $f^\sharp(t,m,p) := f(t,m,-p)$ is a solution to the Smoluchowski equation, and by hypothesis, $f^\sharp(0,.) = f_{in}$. Lemma 3.1 ensures that $f^\sharp = f$ and the claim is proved. $\square$

3.2 A priori estimates

We begin by some physical and formal a priori estimates.

Lemma 3.3 *Let f be a solution to the kinetic Smoluchowski equation (3.1). Then mass and momentum conservation (2.32) hold (at least formally), and the dissipation of energy and of number of particles (2.33) also hold (with $D_{\mathcal{E},G} \equiv 0$). As a matter of fact, for any sub-additive function $\psi : Y \to (0,\infty)$, that is $\psi_{**} \le \psi + \psi_*$, the map*

$$t \mapsto \int_Y f(t,y)\psi(y)\, dy \tag{3.5}$$

is nonincreasing. As an illustration of this fact, there holds

$$t \mapsto \int_Y f(t,y)\, |p|\, dy \quad \textit{is nonincreasing,} \tag{3.6}$$

$$t \mapsto \int_Y f(t,y)\, \zeta(m)\, dy \quad \textit{is nonincreasing,} \tag{3.7}$$

$$t \mapsto \int_Y f(t,y)\, \xi(|v-v_0|)\, dy \quad \textit{is nonincreasing,} \tag{3.8}$$

for any nonincreasing function ζ on $(0,\infty)$, any nondecreasing function ξ on $(0,\infty)$ and any $v_0 \in \mathbb{R}^3$. Another consequence is

$$\int_Y f(T,y)\, m^\alpha\, dy + \frac{1}{2}\int_0^T \int_Y\int_Y a_S\, m^\alpha\, f\, f_*\, dy dy_* dt \le \int_Y m^\alpha\, f_{in}\, dy, \tag{3.9}$$

for any $\alpha \in (-\infty,0]$ and $T > 0$.

Proof of Lemma 3.3. These results may be formal when the solution only satisfy (2.16) for bounded functions ϕ. It become rigorous when f satisfies an extra moment condition or when we deal with approximated solutions (to equations with cutoff rates). We assume in this proof that we may apply (2.16) and (2.4) without questioning.
First note that (2.32) is an immediate consequence of (2.16) applied with $\phi(y) = m$ and $\phi(y) = p$

(and $\beta(x) = x$) thanks to (2.5) (this last following from (2.4)). Next, the map defined by (3.5) is nonincreasing thanks to (2.16) and (2.4) applied with $\phi = \psi$ (and $\beta(x) = x$). We next deduce (3.6) choosing $\psi = |p|$ in (3.5), (3.7) choosing $\psi = \zeta(m)$ and (3.8) choosing $\psi = \xi(|v - v_0|)$ (note that $|v_{**} - v_0| \le \max(|v - v_0|, |v_* - v_0|)$). We finally obtain (3.9) applying (2.16) and (2.4) with $\phi = m^\alpha$ (and $\beta(x) = x$) and remarking that $m^\alpha + m_*^\alpha - m_{**}^\alpha \ge m^\alpha$ when $\alpha \le 0$. $\square$

The next lemma gives some estimations on L^1 norms with weight of the Smoluchowski term Q_S.

Lemma 3.4 *There exists a constant C_A, depending only on A (see (1.18)), such that for any measurable function $g : Y \mapsto (0, \infty)$ and any $z \in (1, 2]$,*

$$\int_Y Q_S(g)\,(m^z + |p|^z)\,dy \le C_A \int_Y (1 + m + |v| + |p| + \mathcal{E})\,g\,dy \int_Y (1 + m^z + |p|^z)\,g\,dy, \tag{3.10}$$

$$\int_Y Q_S(g)\,(m^2 + \mathcal{E}^2)\,dy \le C_A \int_Y (m^{-1} + m + \mathcal{E})\,g\,dy \int_Y (m^{-2} + m^2 + \mathcal{E}^2)\,g\,dy, \tag{3.11}$$

$$\begin{aligned}\int_Y Q_S(g)\,(m^3 + |p|^3)dy \le C_A &\int_Y (m + m^2 + |p|^2 + \mathcal{E})\,g\,dy \\ &\times \int_Y (m^2 + |p|^2 + \mathcal{E}^2 + m^3 + |p|^3)\,g\,dy,\end{aligned} \tag{3.12}$$

$$\int_Y Q_S(g)\,(m^3 + \mathcal{E}^3)\,dy \le C_A \int_Y (m^{-1} + m^2 + \mathcal{E}^2)\,g\,dy \int_Y (m^{-2} + m^3 + \mathcal{E}^3)\,g\,dy. \tag{3.13}$$

Proof of Lemma 3.4. These results follow from tedious but straightforward computations. We only show the two first inequalities, the two last ones being proved similarly.
Proof of (3.10). Defining $\Phi(\zeta, \zeta_*) := (\zeta + \zeta_*)^z - \zeta^z - (\zeta_*)^z$, we observe that

$$\Phi(\zeta, \zeta_*) \le 2\,\min(\zeta\,\zeta_*^{z-1}, \zeta^{z-1}\,\zeta_*), \quad (\zeta + \zeta_*)\,\Phi(\zeta, \zeta_*) \le 4\,[\zeta\,\zeta_*^z + \zeta^z\,\zeta_*]. \tag{3.14}$$

First, using (1.18) and the obvious fact that $\Phi(m, m_*) \le 2m^z + 2m_*^z$,

$$\begin{aligned} a_S \Phi(m, m_*) &\le A(1 + m + m_*)(1 + |v| + |v_*|)\Phi(m, m_*) \\ &\le A(1 + |v| + |v_*|)[(m + m_*)\Phi(m, m_*) + \Phi(m, m_*)] \\ &\le C_A(1 + |v| + |v_*|)[m m_*^z + m^z m_* + m^z + m_*^z] \\ &\le C_A(T + T_*) \end{aligned} \tag{3.15}$$

where $T = T(y, y_*) = (1 + m + |v| + |p|)(m_*^z + m_*^z|v_*|)$ and $T_* = T(y_*, y)$.
Next, using the first inequality in (3.14),

$$\begin{aligned} a_S \Phi(|p|, |p_*|) &\le A(1 + m + m_*)(1 + |v| + |v_*|)\Phi(|p|, |p_*|) \\ &\le C_A(1 + m + |v_*| + |p| + m|v_*|)|p|^{z-1}|p_*| \\ &\quad + C_A(1 + m_* + |v| + |p_*| + m_*|v|)|p_*|^{z-1}|p| \\ &\le C_A(S + S_*) \end{aligned} \tag{3.16}$$

where $S = (|p| + \mathcal{E})(|p_*|^{z-1} + m_*|p_*|^{z-1} + |p_*|^z)$. Since furthermore $z \in (1, 2]$, we deduce that $|p_*|^{z-1} \le 1 + |p_*|^z$, that $m_*|p_*|^{z-1} = m_*^z|v_*|^{z-1} \le m_*^z + |p_*|^z$ and that $m_*^z|v_*| \le m_*^z + |p_*|^z$. Hence, for some numerical constant C,

$$S + T \le C(1 + m + |v| + |p| + \mathcal{E})(1 + m_*^z + |p_*|^z). \tag{3.17}$$

Applying finally (2.4) with $\varphi = m^z + |p|^z$, we obtain

$$\begin{aligned} \int_Y Q_S(g)(m^z + |p|^z)dy &= \int_Y \int_Y a_S[\Phi(m, m_*) + \Phi(|p|, |p_*|)]g g_* dy dy_* \\ &\le C_A \int_Y \int_Y a_S[S + T + S_* + T_*]g g_* dy dy_*, \end{aligned} \tag{3.18}$$

which leads to (3.10).
Proof of (3.11). Observe now that

$$\begin{aligned}
\mathcal{E}_{**}^2 - \mathcal{E}^2 - \mathcal{E}_*^2 &= m_{**}^{-2}|mv + m_* v_*|^4 - m^2|v|^4 - m_*^2|v_*|^4 \\
&= m_{**}^{-2}\{m^3 m_*(4|v|^3|v_*| - 2|v|^4) + m^2 m_*^2(6|v|^2|v_*|^2 - |v|^4 - |v_*|^4) \\
&\qquad + m m_*^3(4|v||v_*|^3 - 2|v_*|^4)\} \\
&\le m_{**}^{-2}\{4m^3|v|^3 m_*|v_*|\mathbf{1}_{\{|v|\le 2|v_*|\}} + 6m^2|v|^2 m_*^2|v_*|^2\mathbf{1}_{\{|v|\le 3|v_*|\le 9|v|\}} \\
&\qquad + 4m|v|m_*^3|v_*|^3\mathbf{1}_{\{|v_*|\le 2|v|\}}\}.
\end{aligned}$$

Therefore, we get, after some tedious but straightforward computations,

$$a_S[\mathcal{E}_{**}^2 - \mathcal{E}^2 - \mathcal{E}^2] \le C_A(U + U_*) \tag{3.19}$$

with $U = [m^{-1}+m+\mathcal{E}][m_*^{-2}+m_*^2+\mathcal{E}_*+\mathcal{E}_*^2]$. We have used here the inequalities $m|v|^3 \le m^{-2}+\mathcal{E}^2$, $m^2|v|^3 \le m^{-2} + \mathcal{E}^2$, $|p| \le m + \mathcal{E}$, and $|v| \le m^{-1} + \mathcal{E}$.
Applying (2.4) with the choice $\phi = \mathcal{E}^2$, and using (3.15) with $z = 2$, we obtain

$$\begin{aligned}
\int_Y Q_S(g)(m^2 + \mathcal{E}^2)dy &\le C_A \int_Y \int_Y (T + T_* + U + U_*) g g_* dy dy_* \\
&\le C_A \int_Y (1 + m + |v| + |p| + m^{-1} + \mathcal{E}) g dy \\
&\qquad \times \int_Y (|p_*| + m_*|p_*| + m_*^{-2} + \mathcal{E}_* + \mathcal{E}_*^2 + m_*^2) g_* dy \\
&\le C_A \int_Y (m^{-1} + m + \mathcal{E}) g dy \int_Y (m^{-2} + m^2 + \mathcal{E}^2) g dy.
\end{aligned} \tag{3.20}$$

For the last inequality, we used $1 \le m^{-1}+m$, $1 \le m^{-2}+m^2$, and $\mathcal{E}+\mathcal{E}^2 \le 1+2\mathcal{E}^2 \le m^{-2}+m^2+2\mathcal{E}^2$. This concludes the proof of (3.11). □

An immediate and fundamental consequence of Lemmas 3.3 and 3.4 is the following.

Corollary 3.5 *A solution f to (3.1) satisfies, at least formally, for any T,*

$$\text{for } z \in (1,2], \quad f_{in}\,(k_S^z + \mathcal{E}) \in L^1(Y) \quad \text{implies} \quad \sup_{[0,T]} \int_Y f(t,y)\,(k_S^z + \mathcal{E})\,dy \le C_T, \tag{3.21}$$

$$f_{in}\,(k_S^3 + m^{-2}) \in L^1(Y) \quad \text{implies} \quad \sup_{[0,T]} \int_Y f(t,y)\,(k_S^3 + m^{-2})\,dy \le C_T, \tag{3.22}$$

$$\text{for } z = 2 \text{ and } 3, \quad f_{in}\,k_B^z \in L^1(Y) \quad \text{implies} \quad \forall\, T > 0 \quad \sup_{[0,T]} \int_Y f(t,y)\,k_B^z\,dy \le C_T, \tag{3.23}$$

where the constant C_T depends on T, f_{in} and A (see (1.18)).

Proof of Corollary 3.5. We only show (3.21), the other claims being proved similarly. Assume thus that $(k_S^z+\mathcal{E})f_{in} \in L^1$ with z fixed in $(1,2]$. This assumption is equivalent to $(1+m^z+|p|^z+|v|^z+\mathcal{E})\,f_{in} \in L^1$. First, from Lemma 3.3 (or more precisely from (2.32), (2.33), (3.6), (3.8)), we have $(1+m+|p|+|v|^z+\mathcal{E})\,f \in L^\infty([0,\infty), L^1)$. Next, applying (2.16) with $\phi = m^z + |p|^z$ (and $\beta(x) = x$) and using (3.10) in Lemma 3.3 we conclude (3.21) thanks to the Gronwall Lemma. □

3.3 Existence

We shall deduce from the previous estimates in Corollary 3.5 and a modification of the proof of the uniqueness lemma 3.1 the existence part of Theorem 2.6 (in the case where $a_B = a_G = 0$ and replacing k_B by k_S). First of all note that

$$0 \le a_S \le A\,(1 + m + m_*)\,(1 + |v| + |v_*|) \le A\,k_S(y)\,k_S(y_*), \tag{3.24}$$

so that (3.2) holds with the choice $k = k_S$. We split the proof in several steps.
First Step. We will first assume in this step that

$$\int_Y f_{in}[k_S^3 + m^{-2}]dy < \infty, \tag{3.25}$$

and we introduce the coalescence equation with cutoff

$$\frac{\partial g^n}{\partial t} = Q_{S,n}(g^n) \text{ on } (0,\infty)\times Y, \quad g^n(0,.) = f_{in} \text{ on } Y, \tag{3.26}$$

where $Q_{S,n}$ is the coalescence kernel associated to the coalescence rate $a_{S,n}(y,y_*) := a_S(y,y_*)\wedge n$. The coalescence rate being bounded it is a classical application of Banach fixed point Theorem to prove that there exists a unique solution $0 \le g^n \in C([0,\infty); L^1_{k_S^3+m^{-2}}(Y))$ to (3.26) associated to the initial datum f_{in} satisfying (3.25). We refer to [25] section 6 where we may consider the Banach space $X := L^1_{k_S^3+m^{-2}}$. Let us point out that we use here, in a fundamental way, the fact that for a nonnegative measurable function h, $Q_{S,n}(h)$ is also a measurable function, which is a direct consequence of the strong representation (1.14) of the coalescence operator. Because of the estimates on g^n, it is possible to establish rigorously that g^n satisfies for each $n \in \mathbb{N}$ the (a priori formal) properties stated in Lemma 3.3 and Corollary 3.5. In particular, for any $T > 0$, there holds

$$\sup_{[0,T]} \|g^n(t,.)\|_{L^1_{k_S^3}} \le C_T, \tag{3.27}$$

where $C_T \in (0,\infty)$ may depends of T and f_{in}, but not on the truncation parameter $n \in \mathbb{N}^*$.

We now repeat the proof of the uniqueness Lemma 3.1. For $l \ge n$, we write the equation satisfied by $g^n - g^l$, we multiply it by $\phi = \mathrm{sign}(g^n - g^l)k_S$, and we use (2.16) and (2.4) to obtain

$$\begin{aligned}
\frac{d}{dt}\int_Y |g^n - g^l| k_S dy &= \frac{1}{2}\int_Y\int_Y a_{S,n}(g^n g^n_* - g^l g^l_*)(\phi_{**} - \phi - \phi_*)dy_* dy \\
&\quad + \frac{1}{2}\int_Y\int_Y (a_{S,n} - a_{S,l}) g^l g^l_* (\phi_{**} - \phi - \phi_*)dy_* dy \\
&\le A\int_Y\int_Y (g^n + g^l) k_S^2 |g^n_* - g^l_*| k_{S*} dy_* dy \\
&\quad + \int_Y\int_Y a_S \mathbf{1}_{\{a_S \ge n\}} g^l g^l_* (k_S + k_{S*}) dy_* dy \\
&\le A\|g^n + g^l\|_{L^1_{k_S^2}} \|g^n - g^l\|_{L^1_{k_S}} \\
&\quad + 2A\int_Y g^l(k_S + k_S^2)dy \left(\int_Y g^l (k_S + k_S^2)\, \mathbf{1}_{k_S \ge \sqrt{n}/\sqrt{A}} dy\right)
\end{aligned}$$

where we have used the fact $a_S \mathbf{1}_{\{a_S \ge n\}} \le A k_S k_{S*}(\mathbf{1}_{\{k_S \ge \sqrt{n}/\sqrt{A}\}} + \mathbf{1}_{\{k_{S*} \ge \sqrt{n}/\sqrt{A}\}})$. With the notation

$$u_{l,n} = \|g^n - g^l\|_{L^1_{k_S}}, \qquad B_T = \sup_{n\in\mathbb{N}^*}\sup_{t\in[0,T]} \|g^n(t)\|_{L^1_{k_S^3}}, \tag{3.28}$$

we end up with the differential inequality

$$\frac{d}{dt} u_{l,n} \le (2AB_T)\, u_{l,n} + (8AB_T)\frac{B_T\sqrt{A}}{\sqrt{n}}. \tag{3.29}$$

The Gronwall Lemma implies $\sup_{[0,T]} u_{l,n}(t) \to 0$ when $n, l \to \infty$. Hence (f_n) is a Cauchy sequence in $C([0,\infty), L^1_{k_S})$, and there exists $f \in C([0,\infty), L^1_{k_S})$ such that $g^n \to f$ in $C([0,T), L^1_{k_S})$ for any $T > 0$. Of course, (3.27) allows to deduce that $f \in L^\infty([0,T), L^1_{k_S^2})$. There is no difficulty to pass to the limit in the weak formulation (2.14) of (3.26), and that proves the existence of a solution f to (3.1) with initial datum f_{in} satisfying (3.25).

Second Step. When just assuming that $f_{in} \in L^1_{k_S^2}$ we consider the sequence of solutions (f_n) to (3.1) associated to the rate a_S and the initial data $f_n(0,.) = f_{in}\mathbf{1}_{\{k_S \le n\}}\mathbf{1}_{\{m \ge 1/n\}}$ (which satisfies (3.25)) for which existence has been established just above. Then one easily deduces from Corollary 3.5 that for each $T \ge 0$,

$$\sup_n \sup_{[0,T]} \|f_n\|_{L^1_{k_S^2}} < \infty.$$

We may use directly the estimate (3.4) for the difference $f_m - f_n$, and prove that (f_n) is a Cauchy sequence in $C([0,T); L^1_{k_S}) \cap L^\infty([0,T); L^1_{k_S^2})$ for any $T > 0$. We conclude just like before. Let us emphasize that the information $f \in L^\infty([0,T); L^1_{k_S + k_S^2})$ for any $T \ge 0$ is sufficient to deduce that the statements of Lemma 3.3 rigorously hold. The uniqueness of the solution has yet been shown in Lemma 3.1. $\square$

We conclude this subsection by a slight improvement of the existence result established in [25].

Proposition 3.6 *Assume that a_S satisfies the following structure assumption*

$$a_S(y,y_*) \le a_S(y,y_{**}) + a_S(y_*,y_{**}) \quad \forall\, y, y_* \in Y. \tag{3.30}$$

For any f_{in} such that there exists $z \in (1,2]$,

$$0 \le f_{in}\,\{(1+m+|p|+|v|)^z + \mathcal{E}\} \in L^1(Y), \tag{3.31}$$

there exists at least a solution $f = f(t,y) \in C([0,\infty); L^1(Y))$ to the kinetic Smoluchowski equation which satisfies (3.6), (3.9), (3.8) and the solution conserves mass and momentum (2.32).

Proof of Proposition 3.6. The only new claims with respect to the existence result in [25] are the conservations (2.32). As usually, it is a straightforward consequence of (3.21) since $z > 1$. $\square$

3.4 Long time behavior

The aim of this subsection is to give a first (and deterministic) proof to Theorem 2.7 which is only valid when $a_B = a_G = 0$ and under the additionnal condition

$$a_S \ge \underline{a}_S, \text{ with } \underline{a}_S \text{ continuous on } Y^2 \text{ and satisfying (2.37).} \tag{3.32}$$

We will give the general proof in Section 6.

Proof of Theorem 2.7 under strong assumptions. We split the proof into three parts. In the first one we prove that f stabilizes around a solution of the shape $\lambda(t,dm)\,\delta_{p=mv_t}$ using a *dissipative entropy* argument. In the second step, using a rich enough class of Liapunov functionals, we establish that λ and v are unique and not time-depending. We actually prove there exists $\lambda \in M^1(\mathbb{R}_+)$ and $v_0 \in \mathbb{R}^3$ such that

$$f(t,y) \;\rightharpoonup\; \lambda(dm)\,\delta_{p=m\,v_0} \quad \text{in} \quad \mathcal{D}'(Y) \quad \text{when} \quad t \to \infty. \tag{3.33}$$

This was proved (by a different method) in [48] under less general assumptions on a_S and f_{in}. In the last step we prove, arguing by contradiction, that $\lambda = 0$.

Step 1. Let us consider an increasing sequence $(t_n)_{n\ge1}$, $t_n \to \infty$ and put $f_n(t,.) := f(t+t_n,.)$ for $t \in [0,T]$ and $n \ge 1$. We realize, thanks to Lemma 3.3 (recall the expression of k_S), that

$$f_n \text{ is bounded in } L^\infty([0,T); L^1_{k_S}) \quad \text{and} \quad \int_Y f_n(t,y)\,\psi(y)\,dy \text{ is bounded in } BV(0,T) \tag{3.34}$$

for any $\psi \in L^\infty(Y)$. Therefore, up to the extraction of a subsequence, there exists $\Gamma \in C([0,T); M^1(Y)-weak)$ such that

$$(3.35) \qquad f_n \rightharpoonup \Gamma \quad \mathcal{D}'([0,T) \times Y), \qquad \int_Y \psi(y)\, f_n(t,y)\, dy \to \int_Y \psi(y)\, \Gamma(t,dy) \quad C([0,T)),$$

for any $\psi \in C_b(Y)$. Then, for any $\chi \in C_b(Y^2)$, the above convergence is strong enough in order to pass to the semi-inferior limit as follows

$$(3.36) \qquad \int_0^T \int_Y \int_Y \chi(y,y_*)\, \Gamma(t,dy)\, \Gamma(t,dy_*)\, dt \le \liminf_{n\to\infty} \int_0^T \int_Y \int_Y \chi(y,y_*)\, f_n\, f_{n*}\, dy dy_* dt.$$

Using standard truncation arguments (see [25]) and the fact that $\underline{a}_S$ is continuous (see (3.32)), we deduce from (3.36)

$$(3.37) \qquad \int_0^T \underline{D}_{1,S}(\Gamma(t,.))\, dt \le \liminf_{n\to\infty} \int_{t_n}^\infty D_{1,S}(f(t,.))\, dt,$$

where $\underline{D}_{1,S}$ is defined by (2.7) with $\underline{a}_S$ instead of a_S. Using now (2.34), we deduce that the RHS of (3.37) vanishes. Thus

$$(3.38) \qquad \forall t \in [0,T] \quad \underline{a}_S(y,y_*)\, \Gamma(t,dy)\, \Gamma(t,dy_*) = 0 \quad \text{in} \quad \mathcal{D}'(Y^2).$$

Thanks to (2.37), we deduce that for all t,

$$\mathbf{1}_{\{v\neq v_*\}} \Gamma(t,dy)\, \Gamma(t,dy_*) = 0 \quad \text{in} \quad \mathcal{D}'(Y^2).$$

We finally deduce that for any $t \in [0,T]$ fixed, the support of $\Gamma(t,.)$ is contained in the subset $\{(m, m\, v_t),\ m \in \mathbb{R}_+\}$ for some $v_t \in \mathbb{R}^3$. Thus, we have $\Gamma(t,dy) = \lambda(t,dm)\, \delta_{p=mv_t}$ with $\lambda \in C([0,T), M^1(\mathbb{R}_+) - weak)$.

Step 2. We now prove that $\lambda(t,dm)$ and v_t are unique and not depending on time, nor on the sequence (t_n), so that (3.33) holds. First, we claim that for any $R \in (0,\infty)$, there exists a real number $\alpha(R) \ge 0$, which does not depend on the sequence (t_n), such that

$$(3.39) \qquad \forall t \in [0,T] \qquad \int_0^R \lambda(t,dm) = \alpha(R).$$

That allows to identify for any $t \in [0,T]$ the measure $\lambda(t,.)$ which is therefore not a function of time: $\lambda(t,dm) = \lambda(dm)$. In order to prove (3.39) we argue as follows. We fix $R, \varepsilon > 0$ and we define $\zeta_\varepsilon \in C_c(\mathbb{R}_+)$ by $\zeta_\varepsilon = 1$ on $[0,R]$, $\zeta_\varepsilon(m) = 1 - \varepsilon^{-1}(m-R)$ for any $m \in [R, R+\varepsilon)$ and $\zeta_\varepsilon = 0$ on $[R+\varepsilon, \infty)$. Gathering (3.35) and (3.7), there exists a real number $\alpha(R,\varepsilon) \ge 0$ such that for any $t \in [0,T]$ there holds

$$(3.40) \qquad \begin{aligned} \int_{\mathbb{R}_+} \zeta_\varepsilon(m)\, \lambda(t,dm) &= \int_Y \zeta_\varepsilon(m)\, \Gamma(t,dy) = \lim_{n\to\infty} \int_Y \zeta_\varepsilon(m)\, f_n(t,y)\, dy \\ &= \lim_{s\to\infty} \int_Y \zeta_\varepsilon(m)\, f(s,y)\, dy =: \alpha(R,\varepsilon). \end{aligned}$$

But $\zeta_\varepsilon(m) \searrow \mathbf{1}_{[0,R]}(m)$ for any $m > 0$. Hence $(\alpha(R,\varepsilon))$ is decreasing when $\varepsilon \searrow 0$. We then may pass to the limit $\varepsilon \to 0$ in (3.40) and we obtain (3.39) with $\alpha(R) := \lim_{\varepsilon\to 0} \alpha(R,\varepsilon)$. This convergence holds only $\lambda(t,.)$-almost everywhere on $\mathbb{R}_+$, but it suffices to characterize the measure λ.

Next, we claim that for any $u \in \mathbb{R}^3$, there exists $\beta(u) \in \{0,1\}$, which does not depend again on the sequence (t_n), such that

$$(3.41) \qquad \forall t \in [0,T] \qquad \mathbf{1}_{\{v_t \neq u\}} = \beta(u).$$

This of course uniquely determines $v_0 \in \mathbb{R}^3$ such that $v_t = v_0$ on $[0,T]$ and then (3.33) holds. Let us establish (3.41). We fix $u \in \mathbb{R}^3$ and, for any $\varepsilon > 0$, we define $\phi_\varepsilon = 0$ on $[0,\varepsilon/2]$, $\phi_\varepsilon(s) = 2s/\varepsilon - 1$ for any $s \in [\varepsilon/2,\varepsilon]$ and $\phi_\varepsilon = 1$ on $[\varepsilon,\infty)$. Then, we set $\zeta_\varepsilon(m) = \phi_\varepsilon(m)$ and $\xi_\varepsilon(v) = \phi_\varepsilon(|v-u|)$. From (3.7) we have, for any $t \in [0,T]$,

$$\left| \int_Y \xi_\varepsilon \, f_n \, dy - \int_Y \xi_\varepsilon \, \zeta_\varepsilon \, f_n \, dy \right| = \int_Y \xi_\varepsilon \, (1-\zeta_\varepsilon) \, f_n \, dy \le \int_Y \mathbf{1}_{m\in[0,\varepsilon]} \, f_n \, dy \le \int_Y \mathbf{1}_{m\in[0,\varepsilon]} \, f_{in} \, dy. \tag{3.42}$$

From (3.8) and since ϕ_ε is increasing, there exists a real number $\gamma_\varepsilon(u) \ge 0$ such that

$$\gamma_\varepsilon(u) := \lim_{s\to\infty} \int_Y \xi_\varepsilon(v) \, f(s,y) \, dy = \lim_{n\to\infty} \int_Y \xi_\varepsilon(v) \, f_n(t,y) \, dy \qquad \forall \, t \in [0,T].$$

From (3.35) and since $y \to \zeta_\varepsilon(m) \, \xi_\varepsilon(v)$ belongs to $C_b(Y)$, we obtain

$$\int_0^\infty \zeta_\varepsilon(m) \, \lambda(dm) \; \xi_\varepsilon(v_t) = \int_Y \zeta_\varepsilon \, \xi_\varepsilon \, \Gamma(t,dy) = \lim_{n\to\infty} \int_Y \zeta_\varepsilon \, \xi_\varepsilon \, f_n(t,y) \, dy \qquad \forall \, t \in [0,T].$$

Therefore, passing first to the limit $n \to \infty$ in (3.42), we have for any $\varepsilon > 0$

$$\left| \gamma_\varepsilon(u) - \int_0^\infty \zeta_\varepsilon(m) \, \lambda(dm) \; \xi_\varepsilon(v_t) \right| \le \int_Y \mathbf{1}_{m\in[0,\varepsilon]} \, f_{in} \, dy \qquad \forall \, t \in [0,T]. \tag{3.43}$$

In the limit $\varepsilon \searrow 0$ we have $\zeta_\varepsilon \nearrow 1$ pointwise on $(0,\infty)$ and $\xi_\varepsilon(v) \nearrow \mathbf{1}_{v\neq u}$ for any $v \in \mathbb{R}^3$. In particular $\gamma_\varepsilon(u)$ is increasing and thus converges as $\varepsilon \searrow 0$. We deduce from (3.43), since $f_{in} \in L^1(Y)$, that

$$\int_0^\infty \lambda(dm) \; \mathbf{1}_{v_t\neq u} = \lim_{\varepsilon\to 0} \gamma_\varepsilon(u) \qquad \forall \, t \in [0,T],$$

from which (3.41) follows.

Step 3. We now prove that $\lambda \equiv 0$. We argue by contradiction assuming that $\lambda \not\equiv 0$. For any $R \in (0,\infty]$, we define (thanks to (3.7))

$$\alpha(R) := \lim_{t\to\infty} \int_0^R \int_{\mathbb{R}^3} f(t,y) \; dy.$$

We first remark that there exists $R > 0$ such that $\alpha(R/2) < \alpha(R)$. If not, we would have for any $R > 0$

$$\alpha(R) = \alpha(R/2) = ... = \alpha(R/2^n) \le \int_0^{R/2^n} \int_{\mathbb{R}^3} f_{in}(y) \, dy \;\to\; 0,$$

and that contradicts with the fact that $\|\lambda\| = \lim_{R\to\infty} \alpha(R) > 0$.
Let us thus fix $R > 0$ and $\varepsilon > 0$ such that $\alpha(R/2) + 2\varepsilon < \alpha(R)$. Thanks to (3.33) there exists $T > 0$ and then $\delta > 0$ such that

$$\int_0^{R/2} \int_{\mathbb{R}^3} f(T,y) \, dy \le \alpha(R/2) + \varepsilon, \qquad \int_{R/2}^R \int_{|v-v_0|\le\delta} f(T,y) \, dy \le \varepsilon, \tag{3.44}$$

since $f(T,.) \in L^1(Y)$. We define $\Lambda := \{y; (m \le R/2) \text{ or } (m \in [R/2,R], \, |v-v_0| \le \delta)\}$ and we observe that $y_{**} \in \Lambda$ implies $y \in \Lambda$ or $y_* \in \Lambda$, so that $y \mapsto \mathbf{1}_\Lambda(y)$ is a sub-additive function. On the one hand, thanks to (3.5) and (3.44), we have for any $t \ge T$

$$\int_Y f(t,y) \mathbf{1}_{y\in\Lambda} \, dy \le \int_Y f(T,y) \mathbf{1}_{y\in\Lambda} \, dy \le \alpha(R/2) + 2\varepsilon. \tag{3.45}$$

On the other hand, thanks to (3.33), we have

$$\alpha(R) \le \lim_{t\to\infty} \int_Y f(t,y) \mathbf{1}_{y\in\Lambda}\, dy. \tag{3.46}$$

Therefore, gathering (3.45) and (3.46) we obtain $\alpha(R) \le \alpha(R/2) + 2\varepsilon$. This contradicts our choice for R and ε. □

Let us remark that we can not extend this deterministic proof to the full Boltzmann equation (1.1), because of the less rich class of Lyapunov functionals available in that general case. For the full Boltzmann equation (1.1) we then could only prove the following result: there exists $\lambda \in M^1(0,\infty)$ such that for any increasing sequence (t_n) which converges to infinity, there exists a subsequence $(t_{n'})$ and $u \in \mathbb{R}^3$ such that

$$f(t_{n'} + .,.) \rightharpoonup \lambda(dm)\, \delta_{p=m\,u} \quad \text{weakly in} \quad C([0,T]; M^1(Y)),$$

where λ does not depend of the subsequence $(t_{n'})$ but $u \in \mathbb{R}^3$ may depend on it.

Proof of Theorem 2.8. From Corollary 3.2 we already know that f satisfies the symmetry property (2.40). Then, we just compute, using (2.4), (2.41) and (2.42),

$$\begin{aligned} \frac{d}{dt}\int_Y f\,|p|^2\,dy &= \int_Y\int_Y a_S\, f\, f_*\, \langle p, p_*\rangle \left(\mathbf{1}_{\{\langle p,p_*\rangle>0\}} + \mathbf{1}_{\{\langle p,p_*\rangle<0\}}\right) dydy_* \\ &= \int_Y\int_Y f\, f_*\, \langle p,p_*\rangle\, (a_S(m,p,m_*,p_*) - a_S(m,p,m_*,-p_*))\, \mathbf{1}_{\{\langle p,p_*\rangle>0\}}\, dydy_* \le 0, \end{aligned}$$

and we obtain (2.43). Now, on the one hand, by definition (2.28) of j, the moment condition (1.28), and the mass conservation (2.32), there holds

$$\int_{\mathbb{R}^3} j\, dv = \int_Y f(t,y)\, m\, dy \equiv 1 \tag{3.47}$$

and

$$\int_{\mathbb{R}^3} j\,|v|\,dv = \int_Y f(t,y)\,|p|\,dy \le \left(\int_Y f(t,y)dy\right)^{1/2} \left(\int_Y |p|^2 f(t,y) dy\right)^{1/2}.$$

Thanks to (2.38) and (2.43), we deduce from the above estimate

$$\int_{\mathbb{R}^3} j(t,v)\,|v|\,dv \to 0 \quad \text{when} \quad t\to\infty. \tag{3.48}$$

Gathering (3.48) and (3.47) allows us to conclude that $j(t,v)dv$ tends to $\delta_{v=0}$ in $M^1(Y)$. □

4 The mass-dependent Boltzmann equation

In this section we focus on the sole Boltzmann equation for elastic collisions

$$\frac{\partial f}{\partial t} = Q_B(f) \;\text{ on }\; (0,\infty)\times Y, \quad f(0,.) = f_{in} \;\text{ on }\; Y, \tag{4.1}$$

where Q_B is given by (1.4). We assume in the whole section (1.18), (1.21), (1.23), that $a_G \equiv 0$, $a_S \equiv 0$, and we consider an initial condition satisfying (1.28) with $k = k_B$ defined in (1.29). We recall that $\bar{a}_B$ and $\bar{E}_B$ were defined in (1.24) and (1.23).

4.1 Uniqueness

We begin with a uniqueness result.

Lemma 4.1 *Let us just assume that a_B and k are two nonnegative measurable functions on $Y^2 \times S^2$ and Y respectively, such that the first symmetry condition on a_B in (1.3) holds and such that for any $y, y_* \in Y$,*

$$0 \le \bar{a}_B(y, y_*) = \int_{S_2} a_B(y, y_*, \nu) d\nu \le k k_* \quad \text{and} \quad k' + k'_* - k - k_* \le 0. \tag{4.2}$$

Then there exists at most one solution f to the mass dependent Boltzmann equation (4.1) such that for all $T > 0$, $f \in C([0,T); L^1_k) \cap L^\infty([0,T); L^1_{k^2})$.

Remark that under (1.18), one may choose $k = C_A k_B$ (with C_A a constant) in the above lemma The uniqueness part of Theorem 2.2 immediately follows.

Proof of Lemma 4.1. We repeat the proof of Lemma 3.1. We multiply by $\phi(t,y) = \mathrm{sign}(f(t,y) - g(t,y))\, k$ the equation satisfied by $f - g$. Using the weak formulations (2.16) and (2.1) of the Boltzmann equation and operator, we get for all $t \ge 0$,

$$\begin{aligned} \frac{d}{dt} \int_Y |f - g| k dy &= \frac{1}{2} \int_Y \int_Y \int_{S^2} a_B \left((f-g) g_* + f(f_* - g_*)\right) (\phi' + \phi'_* - \phi - \phi_*)\, d\nu dy_* dy \\ &= \frac{1}{2} \int_Y \int_Y \int_{S^2} a_B \, (f-g)\, (f_* + g_*)\, (\phi' + \phi'_* - \phi - \phi_*)\, d\nu dy_* dy \\ &\le \frac{1}{2} \int_Y \int_Y \int_{S^2} a_B \, |f-g|\, (f_* + g_*)\, (k' + k'_* - k + k_*)\, d\nu dy_* dy, \end{aligned}$$

where we have just used the symmetry hypothesis (1.3) on a_B and the substitution $(y, y_*) \to (y_*, y)$. Then, thanks to the bounds (4.2), we deduce

$$\frac{d}{dt} \int_Y |f - g| k dy \le \int_Y \int_Y k\, k_* \, |f - g|\, (f_* + g_*)\, k_* \, dy_* dy = \|f + g\|_{L^1_{k^2}} \|f - g\|_{L^1_k}, \tag{4.3}$$

and we conclude as in the proof of Lemma 3.1. □

4.2 A priori estimates and existence

We begin by gathering some information satisfied (at least formally) by a solution to (4.1).

Lemma 4.2 *A solution f to the mass-dependent Boltzmann equation (4.1) conserves, at least formally, momentum, mass distribution and energy, (2.17), (2.18). In particular, if f_{in} satisfies (2.21), then*

$$\forall\, t \ge 0, \quad f(t,y) = 0 \quad \text{for a.e.} \quad p \in \mathbb{R}^3, \; m \in (0, m_0). \tag{4.4}$$

Proof of Lemma 4.2. This is an immediate consequence of (2.16) and (2.1) with the choices $\phi(y) = \phi(m)$, $\phi(y) = p$, $\phi(y) = \mathcal{E}$ (and $\beta(x) = x$). Moreover, we prove (4.4) making the choice $\phi(m) = \mathbf{1}_{0 \le m \le m_0}$ in the second identity of (2.17). □

We next give some estimates on L^1 norms with weight of the Boltzmann term $Q_B(f)$. It is based on a Povzner lemma, adapted to the mass-dependent case. We use here (and only here) the structure condition (1.21) on a_B.

Lemma 4.3 *There exists a constant C_A, depending only on A (see (1.18)), such that for any nonnegative measurable function h on Y,*

$$\int_Y Q_B(h)\,\mathcal{E}^2\,dy \le C_A \int_Y (m^{-1}+m+\mathcal{E})\,h\,dy \int_Y (m^{-2}+m^2+\mathcal{E}^2)\,h\,dy, \tag{4.5}$$

$$\int_Y Q_B(h)\,\mathcal{E}^3\,dy \le C_A \int_Y (m^{-1}+m^2+\mathcal{E}^2)\,h\,dy \int_Y (m^{-3}+m^3+\mathcal{E}^3)\,h\,dy. \tag{4.6}$$

Proof of Lemma 4.3. We split the proof into several steps.

Step 1. Preliminaries. Writing the fundamental identity (2.1) with $\varphi = \mathcal{E}^n$ for $n = 2$ or 3, we get

$$\int_Y Q_B(h)\,\mathcal{E}^n dy = \int_Y\int_Y h\,h_*\,\mathcal{K}_n\,dydy_*, \tag{4.7}$$

where

$$\mathcal{K}_n := \int_{S^2} a_B(y,y_*,\cos\Theta)\,\{(\mathcal{E}')^n + (\mathcal{E}'_*)^n - \mathcal{E}^n - \mathcal{E}_*^n\}\,d\nu. \tag{4.8}$$

Here v' and v'_* are defined from v, v_* and ν with the help of (1.5) and Θ has been defined by (1.22). It is convenient to introduce another parameterization of post collisional velocities in order to make the computation more tractable. One easily deduces from (1.5) that, for any $\nu \in S^2$,

$$|v' - v_{**}| = \mu_*|v - v_*|; \quad |v'_* - v_{**}| = \mu|v - v_*|. \tag{4.9}$$

We can then define the following alternative parameterization of v', v'_*,

$$\left\{\begin{array}{lcl} v' &=& v_{**} + \mu_*\,[v - v_* + 2\,\langle v_* - v, \nu\rangle\,\nu] = v_{**} + \mu_*\,w\,\sigma, \\ v'_* &=& v_{**} - \mu\,[v - v_* + 2\,\langle v_* - v, \nu\rangle\,\nu] = v_{**} - \mu\,w\,\sigma, \end{array}\right. \tag{4.10}$$

with $\sigma \in S^2$. In other words, for any $\nu \in S^2$, we set

$$\sigma = \frac{v - v_*}{w} + 2\left\langle \frac{v_* - v}{w}, \nu\right\rangle \nu = (\overrightarrow{\iota_1}\,\cos\phi + \overrightarrow{\iota_2}\,\sin\phi)\,\sin\theta + \frac{v - v_*}{w}\,\cos\theta, \tag{4.11}$$

and that indeed defines $\sigma \in S^2$ and next $\phi \in [0, 2\pi]$, $\theta \in [0, \pi]$. Here, $(\overrightarrow{\iota_1}, \overrightarrow{\iota_2}, \frac{v-v_*}{w})$ is the direct orthonormal basis of $\mathbb{R}^3$ such that $\langle v_{**}, \overrightarrow{\iota_2}\rangle = 0$.
Note also that $\cos\Theta = \sin(\theta/2)$. Indeed, on one hand $\Theta \in [0, \pi/2]$ is the angle between $v' - v$ and $v - v_{**}$ (or $v'_* - v'$ and $v_* - v_{**}$). On the other hand, $\theta \in [0, \pi]$ is that between $v - v_{**}$ and $v' - v_{**}$ (or between $v_* - v_{**}$ and $v'_* - v_{**}$).
We now perform first the change of variables $\nu \to \sigma$ in the integral expression (4.8) of $\mathcal{K}_n$, observing that $d\sigma = 2\,\cos\Theta\,d\nu$ and next, the substitution $\sigma \to (\theta, \phi)$, observing that $d\sigma = \sin\theta\,d\theta\,d\phi$, and we obtain

$$\begin{aligned} \mathcal{K}_n(y,y_*) &= \int_{S^2} \frac{a_B(y,y_*;\cos\Theta)}{2\,\cos\Theta}\,\{(\mathcal{E}')^n + (\mathcal{E}'_*)^n - \mathcal{E}^n - \mathcal{E}_*^n\}\,d\sigma \\ &= \int_0^\pi a_B(y,y_*;\sin(\theta/2))\,\cos(\theta/2)\left[\int_0^{2\pi}\{(\mathcal{E}')^n + (\mathcal{E}'_*)^n - \mathcal{E}^n - \mathcal{E}_*^n\}\,d\phi\right] d\theta \end{aligned} \tag{4.12}$$

where now v' and v'_* are defined with the help of the new parameterization (4.10), (4.11).

Step 2. The Povzner Lemma. Our aim is now to check that for any $y, y_* \in Y$ and $\theta \in [0, \pi]$,

$$\begin{aligned} \frac{1}{4\pi}\int_0^{2\pi}\{(\mathcal{E}')^2 + (\mathcal{E}'_*)^2 - \mathcal{E}^2 - \mathcal{E}_*^2\}d\varphi \quad &\le \quad -\mu\mu_*\sin^2\theta\,(\mathcal{E}^2 + \mathcal{E}_*^2) \\ &\quad +8\bar{\mu}\,|v|\,|v_*|\,(\mathcal{E} + \mathcal{E}_*) + 26\,\mu\,\mu_*\,\mathcal{E}\,\mathcal{E}_*, \end{aligned} \tag{4.13}$$

while

$$\frac{1}{12\pi}\int_0^{2\pi}\{(\mathcal{E}')^3+(\mathcal{E}'_*)^3-\mathcal{E}^3-\mathcal{E}_*^3\}d\varphi \quad\le\quad 4m^2\bar\mu|v|^5|v_*|+15m\bar\mu^2|v|^4|v_*|^2 \\ +15m_*\bar\mu^2|v_*|^4|v|^2+4m_*^2\bar\mu|v_*|^5|v|. \tag{4.14}$$

We will only prove (4.13), because the proof of (4.14) uses exactly the same arguments. In this whole step, we fix y and y_* and $\theta\in[0,\pi]$, and we define α to be the angle between the vectors v and v_*. We also introduce the coordinates (ξ,η,ζ) in the orthonormal basis $(\overrightarrow{\iota_1},\overrightarrow{\iota_2},(v-v_*)/w)$ of $\mathbb{R}^3$. Hence the coordinates of v, v_* and v_{**} are

$$v_{**}=:(\xi_0,0,\zeta_0),\quad v=(\xi_0,0,\zeta_0+\mu_*w),\quad v_*=(\xi_0,0,\zeta_0-\mu w),$$

so that

$$|v|^2=|v_{**}|^2+(\mu_*w)^2+2\zeta_0\mu_*w,\quad |v_*|^2=|v_{**}|^2+(\mu w)^2-2\zeta_0\mu w. \tag{4.15}$$

Since $\sigma=(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta)$, we deduce from (4.10), (4.11)

$$\begin{aligned} |v'|^2&=|v_{**}|^2+(\mu_*w)^2+2\mu_*w(\zeta_0\cos\theta+\xi_0\sin\theta\cos\varphi),\\ |v'_*|^2&=|v_{**}|^2+(\mu w)^2-2\mu w(\zeta_0\cos\theta+\xi_0\sin\theta\cos\varphi).\end{aligned} \tag{4.16}$$

Gathering (4.15) and (4.16), we get

$$\begin{aligned} |v'|^2&=|v|^2+2\mu_*(w\zeta_0)(\cos\theta-1)+2\mu_*(w\xi_0)\sin\theta\cos\varphi,\\ |v'_*|^2&=|v_*|^2-2\mu(w\zeta_0)(\cos\theta-1)-2\mu(w\xi_0)\sin\theta\cos\varphi,\end{aligned}$$

and then

$$\begin{aligned} \mathcal{E}'&=\mathcal{E}+\bar\mu(2w\zeta_0)(\cos\theta-1)+2\bar\mu(w\xi_0)\sin\theta\cos\varphi=:\mathcal{E}+A,\\ \mathcal{E}'_*&=\mathcal{E}_*-\bar\mu(2w\zeta_0)(\cos\theta-1)-2\bar\mu(w\xi_0)\sin\theta\cos\varphi=:\mathcal{E}_*-A.\end{aligned} \tag{4.17}$$

We remark that

$$w\xi_0=|v||v_*|\sin\alpha, \tag{4.18}$$

since both quantities equal twice the area of the triangle (Ovv_*), and

$$w\zeta_0=\frac12(|v|^2-|v_*|^2+(\mu^2-\mu_*^2)\,w^2)=\mu|v|^2-\mu_*|v_*|^2-(\mu-\mu_*)|v||v_*|\cos\alpha, \tag{4.19}$$

since $\mu^2-\mu_*^2=(\mu+\mu_*)(\mu-\mu_*)=\mu-\mu_*$, $1+\mu-\mu_*=2\mu$, $1+\mu_*-\mu=2\mu_*$, and $\langle v,v_*\rangle=|v||v_*|\cos\alpha$. We also remark that, with the notations introduced in (4.17),

$$(\mathcal{E}'_*)^2+(\mathcal{E}')^2-\mathcal{E}^2-\mathcal{E}_*^2=2A^2+2(\mathcal{E}-\mathcal{E}_*)A. \tag{4.20}$$

Using that $\int_0^{2\pi}2\,\cos^2\varphi\,d\varphi=\int_0^{2\pi}d\varphi=2\pi$ while $\int_0^{2\pi}\cos\varphi\,d\varphi=0$, we obtain

$$\begin{aligned}&\int_0^{2\pi}\{(\mathcal{E}')^2+(\mathcal{E}'_*)^2-\mathcal{E}^2-\mathcal{E}_*^2\}\frac{d\varphi}{4\pi}\\ &=4(\bar\mu)^2(w\zeta_0)^2(1-\cos\theta)^2-2\bar\mu(w\zeta_0)(\mathcal{E}-\mathcal{E}_*)(1-\cos\theta)+2\bar\mu^2(w\xi_0)^2\sin^2\theta.\end{aligned} \tag{4.21}$$

Using now (4.18) and (4.19), we deduce that

$$\int_0^{2\pi}\{(\mathcal{E}')^2+(\mathcal{E}'_*)^2-\mathcal{E}^2-\mathcal{E}_*^2\}\frac{d\varphi}{4\pi}=S(y,y_*,\theta)+S(y_*,y,\theta) \tag{4.22}$$

where

$$S(y,y_*,\theta) = -2|v|^4 m\mu\bar{\mu}(1-\cos\theta)[1-2\mu\mu_*(1-\cos\theta)] \tag{4.23}$$
$$+2|v|^3|v_*|m\bar{\mu}(\mu-\mu_*)(1-\cos\theta)\cos\alpha[1-4\mu\mu_*(1-\cos\theta)]$$
$$+|v|^2|v_*|^2\bar{\mu}^2[2(1-\cos\theta)+\sin^2\theta\sin^2\alpha+2(1-\cos\theta)^2\{(\mu-\mu_*)^2\cos\alpha-2\mu\mu_*\}].$$

We observe that $4\mu\mu_* \le 1$, so that

$$\Big[1-2\mu\mu_*(1-\cos\theta)\Big] \ge 1-\frac{1}{2}(1-\cos\theta) = \frac{1}{2}(1+\cos\theta). \tag{4.24}$$

We thus deduce that

$$\begin{aligned} S(y,y_*,\theta) \quad &\le \quad -|v|^4 m\mu\bar{\mu}(1-\cos\theta)(1+\cos\theta) \\ &\qquad +8|v|^3|v_*|m\bar{\mu}+13|v|^2|v_*|^2\bar{\mu}^2. \end{aligned} \tag{4.25}$$

Finally, (4.13) follows from (4.25) and (4.22).

Step 3. Conclusion. First note that with our new parameterization (see Step 1),

$$\begin{aligned} \bar{a}_B(y,y_*) \quad &= \quad \int_{S_2} a_B(y,y_*,\cos\Theta)d\nu \\ &= \quad \int_0^\pi d\theta \int_0^{2\pi} d\varphi a_B(y,y_*,\sin(\theta/2))\cos(\theta/2) \\ &= \quad 2\pi\int_0^\pi a_B(y,y_*,\sin(\theta/2))\cos(\theta/2)d\theta. \end{aligned}$$

Gathering now (4.13) (where we neglect the nonpositive term), (4.12), and using the bound (1.18), we deduce that

$$\begin{aligned} \mathcal{K}_2(y,y_*) \quad &\le \quad C_A(1+m+m_*)(1+|v|+|v_*|) \\ &\qquad \times\left\{\frac{m^2m_*^2}{(m+m_*)^2}|v|^2|v_*|^2+\frac{m^2m_*}{m+m_*}|v|^3|v_*|+\frac{mm_*^2}{m+m_*}|v||v_*|^3\right\}, \end{aligned} \tag{4.26}$$

for some the constant C_A depending only on A. Recalling (4.7), a tedious but straightforward computation allows us to conclude that (4.5) holds, using essentially some symmetry arguments and the facts that $mm_*/(m+m_*) \le m$ and $mm_*/(m+m_*) \le m_*$.
We omit the proof of (4.6), since it uses the same arguments, making use of (4.14) instead of (4.13). □

As an immediate consequence of Lemmas 4.2 and 4.3, we obtain the following a priori bounds for the solutions of (4.1). We omit the proof since it follows the same line as that of Corollary 3.5.

Corollary 4.4 *A solution f to (4.1) satisfies, at least formally, for any T,*

$$\text{for } z=2 \text{ and } 3 \qquad f_{in}\,k_B^z \in L^1(Y) \quad \text{implies} \quad \sup_{[0,T]}\int_Y f(t,y)\,k_B^z\,dy \le C_{T,z}, \tag{4.27}$$

where the constant $C_{T,z}$ depends only on T, $\|f_{in}\|_{L^1_{k_B^z}}$, and on A.

Proof of Theorem 2.2. It follows line by line Subsection 3.3. It suffices to use of Corollary 4.4 instead of Corollary 3.5, and to use the computation of Lemma 4.1 instead of that of Lemma 3.1. □

4.3 Long time behavior

The aim of this subsection is to prove Theorem 2.3. We start giving a more accurate version of the first estimate on the weight integral of the collision term Q_B stated in Lemma 4.2.

Lemma 4.5 *In addition to the current assumptions on a_B, suppose the structure hypothesis (2.19), and fix $m_0 > 0$ and $m_1 > 0$. For any measurable function $h : Y \to \mathbb{R}_+$ such that $h = 0$ for a.e. $m \in (0, m_0)$, $p \in \mathbb{R}^3$ and $\int_Y \mathbf{1}_{\{m_0<m<m_1\}} h(y)dy \geq \kappa_1 > 0$, there holds*

$$\int_Y Q_B(h)\,\mathcal{E}^2\,dy \leq C_1 - C_2 \int_Y h\,\mathcal{E}^2\,dy, \tag{4.28}$$

where C_1, C_2 are positive constants depending only on a_B, on $\int_Y h\,(1 + m^2 + m^{6-4\delta} + \mathcal{E})\,dy$ and on κ_1, m_0, m_1.

Proof of Lemma 4.5. In the whole proof, an *adapted* constant is a constant depending only on the quantities allowed in the statement. Its value of it may change from one line to another.
We come back to the proof of Lemma 4.3. Using (4.7) with $n = 2$, taking into account the negative contribution in the Povzner inequality (4.13), and using finally (2.19), we easily obtain, for some positive constant C_A depending only on A (see (1.18)),

$$\begin{aligned} \int_Y Q_B(h)\,\mathcal{E}^2\,dy \quad &\leq \quad C_A \int_Y (m^{-1} + m + \mathcal{E})hdy \int_Y (m^{-2} + m^2 + \mathcal{E}^2)hdy \\ &\quad - \int_0^{\pi/2} \sin^2\theta \cos(\theta/2)\psi(\theta)d\theta \int_Y \int_Y \mu\mu_* \mathcal{E}^2 (mm_*)^\delta |v - v_*| hh_* dydy_* \\ &=: \quad I_1 - I_2. \end{aligned} \tag{4.29}$$

First, one easily obtains the existence of an adapted constant C such that

$$I_1 \leq C \left\{ 1 + \int_Y \mathcal{E}^2 hdy \right\}. \tag{4.30}$$

Next, since $|v - v_*| \geq |v| - |v_*|$, we get, for some adapted constants $C > 0$, $c > 0$,

$$\begin{aligned} I_2 \quad &\geq \quad c \int_Y \frac{m^{3+\delta} m_*^{1+\delta}}{(m + m_*)^2} |v|^5 hh_* dydy_* - C \int_Y \int_Y \frac{m^{3+\delta} m_*^{1+\delta}}{(m + m_*)^2} |v|^4 |v_*| hh_* dydy_* \\ &=: \quad J_1 - J_2. \end{aligned} \tag{4.31}$$

Since h vanishes for $m < m_0$, and since $m^{2\delta}|v| \leq m^{-1} + m + \mathcal{E}$,

$$\begin{aligned} J_2 \quad &\leq \quad C \int_Y \mathcal{E}^2 hdy \int_Y m_*^{2\delta} |v_*| h_* dy_* \\ &\leq \quad C \int_Y (m_0^{-1} + m + \mathcal{E}) hdy \int_Y \mathcal{E}^2 hdy = C \int_Y \mathcal{E}^2 hdy. \end{aligned} \tag{4.32}$$

Since for all $m > m_0$, all $m_0 < m_* < m_1$, $m + m_* \leq m(1 + m_1/m_0)$,

$$\begin{aligned} J_1 \quad &\geq \quad \frac{c}{(1 + m_1/m_0)^2} \int_Y m^{1+\delta} |v|^5 hdy \int_Y m_*^{1+\delta} \mathbf{1}_{\{m_0<m_*<m_1\}} h_* dy_* \\ &\geq \quad c \int_Y m^{1+\delta} |v|^5 hdy. \end{aligned} \tag{4.33}$$

Finally, by the Young inequality, we deduce that for all $\varepsilon > 0$,

$$m^2\,|v|^4 \leq \varepsilon^{4/5} m^{1+\delta}\,|v|^5 + \frac{1}{\varepsilon^5}\,m^{6-4\delta}, \tag{4.34}$$

so that

$$J_1 \geq \frac{c}{\varepsilon^{4/5}} \int_Y \mathcal{E}^2 h dy - \frac{c}{\varepsilon^{5+4/5}} \int_Y m^{6-4\delta} h dy. \tag{4.35}$$

Gathering all the above inequalities and choosing ε small enough allows us to conclude the proof. ◻

In the following statement, we gather all the estimates we are able to obtain for the solution f to the Boltzmann equation and which are relevant to study the long time asymptotic.

Lemma 4.6 *Under the hypothesis of Theorem 2.3, the solution f to the Boltzmann equation (4.1) associated to f_{in} satisfies (2.22) and*

$$\sup_{t\geq 0} \int_Y f(t,y)\,(m^{-1} + m + \mathcal{E} + \mathcal{E}^2)\,dy < \infty, \tag{4.36}$$

from which we deduce

$$\int_0^\infty D_{h,B}(f(t,.))\,dt < \infty. \tag{4.37}$$

Proof of Lemma 4.6. We start proving that (2.22) holds. We just sketch the proof, and we refer to [2] for details. We first consider the solution f_ℓ to the Boltzmann equation (4.1) with rate $a_B^\ell = a_B \wedge \ell$ and initial condition $f_{in}^\ell = f_{in} + \ell^{-1} M$ where M is the Maxwellian (2.24). On the truncated equation we may prove that for some constant C_l, $f_\ell(t,.) \geq \ell^{-1} M \exp(-C_\ell T)$ for any $t \in (0,T)$. Thus $|\log f_\ell| \leq C_{\ell,T}\,\mathcal{E}$ on $[0,T]$, and we may choose $\beta(x) = h(x) = x \log x$ in (2.16), and deduce the following strong H-Theorem

$$H(f_\ell(t,.)) + \int_0^t \left\{ \int_Y\int_Y\int_{S^2} \frac{a_B \wedge \ell}{4}\,(f'_\ell\, f'_{\ell,*} - f_\ell\, f_{\ell,*})\,\log\frac{f'_\ell\, f'_{\ell,*}}{f_\ell\, f_{\ell,*}}\,dydy_*d\nu \right\} ds = H(f_{in}^\ell). \tag{4.38}$$

Passing to the limit when $\ell \to \infty$ we get that the resulting limit f satisfies the weak version (2.22) of the H-Theorem.
Next, we recall that from Lemma 4.2 we have yet $(m^{-1} + m^2 + \mathcal{E})\,f \in L^\infty([0,\infty), L^1)$ and that f satisfies (4.4). Using the conservation of mass distribution and of energy (see (2.17) and (2.18)), we deduce that one may apply Lemma 4.6 to $h = f(t,.)$, the constants $C_1 > 0$ and $C_2 > 0$ being time-independent. Applying (2.16) with $\phi = \mathcal{E}^2$ (and $\beta(x) = x$) and using (4.28), we deduce that

$$\frac{d}{dt}\int_Y \mathcal{E}^2 f dy \leq C_1 - C_2 \int_Y \mathcal{E}^2 f dy, \tag{4.39}$$

from which (4.36) follows. We finally prove (4.37). From the weak H-Theorem (2.22), we have for any $T > 0$

$$\int_0^T D_{h,B}(f(t,.))\,dt \leq H(f_{in}) - H(f(T,.)) \leq H(f_{in}) - \int_Y f(T,y)\,\ln f(T,y)\,\mathbf{1}_{\{f(T,y)\leq 1\}}\,dy. \tag{4.40}$$

It thus suffices to check that $-\int_Y f(T,y)\,\ln f(T,y)\,\mathbf{1}_{\{f(T,y)\leq 1\}}\,dy$ is bounded by a constant not depending on T. The set $\{f(T,y) \leq 1\}$ may be decomposed into two parts, namely $\{f(T,y) \leq 1\} = \{f(T,y) \leq \exp(-2/m - 2m - 2\mathcal{E})\} \cup \{\exp(-2/m - 2m - 2\mathcal{E}) \leq f(T,y) \leq 1\}$. Using the elementary inequality $-s \ln s \leq 4\sqrt{s}$ on $[0,1]$ for the first subset and just that $s \mapsto -\ln s$ is a decreasing function for the second subset, we obtain

$$\begin{aligned} & - \int_Y f(T,y)\,\ln f(T,y)\,\mathbf{1}_{\{f(T,y)\leq 1\}}\,dy \\ & \leq -\int_Y f(T,y)\,\ln f(T,y)\,\mathbf{1}_{\{f(T,y)\leq e^{-2/m-2m-2\mathcal{E}}\}}\,dy \\ & \quad - \int_Y f(T,y)\,\ln f(T,y)\,\mathbf{1}_{\{e^{-2/m-2m-2\mathcal{E}}\leq f(T,y)\leq 1\}}\,dy \\ & \leq 4\int_Y e^{-1/m-m-\mathcal{E}}\,dy + \int_Y f(T,y)\,\left(\frac{2}{m} + 2\,m + 2\,\mathcal{E}\right)\,dy. \end{aligned} \tag{4.41}$$

We conclude gathering (4.36), (4.40) and (4.41). □

We end the preliminary steps for the proof of Theorem 2.3 by the following functional characterization of Maxwellian functions..

Lemma 4.7 *1. Consider a nonnegative function $g \in L^1(\mathbb{R}^3; (1+|v|^2)\, dv)$ such that*

$$g'\, g'_* = g\, g_* \quad \text{for a.e.} \quad v, v_* \in \mathbb{R}^3,\ \nu \in S^2, \tag{4.42}$$

where v' and v'_* are defined by (1.5) with $m = m_*$. Then g is a Maxwellian, i.e. there exist some constants $v_0 \in \mathbb{R}^3$, $\sigma \in (0,\infty)$, and $\gamma \in [0,\infty)$ such that for a.e. $v \in \mathbb{R}^3$,

$$g(v) = \frac{\gamma}{(2\pi\sigma)^{3/2}} e^{-\frac{|v-v_0|^2}{2\sigma}}. \tag{4.43}$$

2. Consider a nonnegative function $f \in L^1(Y; (1+\mathcal{E})\, dy)$ such that

$$f'\, f'_* = f\, f_* \quad \text{for a.e.} \quad y, y_* \in Y,\ \nu \in S^2 \tag{4.44}$$

where $y' = (m, mv')$ and $y'_* = (m_*, m_* v'_*)$ with v' and v'_* defined by (1.5). Then f is a mass-dependent Maxwellian, i.e. there exist a function $0 \le \gamma \in L^1((0,\infty))$, a constant $v_0 \in \mathbb{R}^3$, and a constant $\sigma \in (0,\infty)$ such that for a.e. $y \in Y$,

$$f(y) = \frac{\gamma(m)}{(2\pi m\sigma)^{3/2}} e^{-\frac{|p-mv_0|^2}{2m\sigma}}. \tag{4.45}$$

Proof Lemma 4.7-1. Although the proof of this result has been yet established, see [17, 38, 2], we present here the sketch of an alternative (but very similar) proof, that we split into four steps. Of course, we may assume that $\int_{\mathbb{R}^3} g dv > 0$, otherwise, we just set $\gamma = 0$.
Step 1. Let us define for any $\varepsilon > 0$

$$\rho_\varepsilon(z) := \frac{1}{\varepsilon^3}\rho\left(\frac{z}{\varepsilon}\right), \quad \rho(z) = \frac{1}{(2\pi)^{3/2}}\, e^{-|z|^2/2},$$

so that

$$\int_{\mathbb{R}^3} \rho_\varepsilon(z)\, dz = 1, \quad \int_{\mathbb{R}^3} z\rho_\varepsilon(z)\, dz = 0, \quad \int_{\mathbb{R}^3} |z|^2\, \rho_\varepsilon(z)\, dz = 3\varepsilon^2.$$

We then define $g_\varepsilon := g \star \rho_\varepsilon \in C^\infty(\mathbb{R}^3)$ and we realize that

$$\begin{aligned} &\int_{\mathbb{R}^3} g_\varepsilon(z)\, dz = \int_{\mathbb{R}^3} g(z)\, dz, \quad \int_{\mathbb{R}^3} z g_\varepsilon(z)\, dz = \int_{\mathbb{R}^3} z\, g(z)\, dz, \\ &\int_{\mathbb{R}^3} |z|^2\, g_\varepsilon(z)\, dz = \int_{\mathbb{R}^3} |z|^2\, g(z)\, dz + 3\varepsilon^2. \end{aligned} \tag{4.46}$$

Furthermore, g_ε still satisfies (4.42). Indeed, using (4.42) (for g), and then the substitution $(w', w'_*) \mapsto (w, w_*)$,

$$\begin{aligned} g_\varepsilon\, g_{\varepsilon *} &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} g(w)\, g(w_*)\, \exp\{-(|w-v|^2 + |w_* - v_*|^2)\}\, dw dw_* \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} g(w')\, g(w'_*)\, \exp\{-(|w-v|^2 + |w_* - v_*|^2)\}\, dw dw_* \\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} g(w)\, g(w_*)\, \exp\{-(|w'-v|^2 + |w'_* - v_*|^2)\}\, dw dw_* = g'_\varepsilon\, g'_{\varepsilon *}, \end{aligned}$$

since, a straightforward computation leads to $|w'-v|^2 + |w'_* - v_*|^2 = |w - v'|^2 + |w_* - v'_*|^2$.

Step 2. Let v, v_* be fixed. Recall the parameterization (4.10), (4.11) of v', v'_* (with here $m = m_*$). Note that for all $\phi \in [0, 2\pi]$, $\nu_0 = (\overrightarrow{\iota_1} \cos\phi + \overrightarrow{\iota_2} \sin\phi) \perp (v - v_*)$, and that $\phi \in [0, 2\pi] \mapsto \nu_0 \in \{z \in \mathbb{R}^3,\ |z| = 1,\ z \perp v - v_*\}$ is a bijection. We thus deduce, since g_ε satisfies (4.42), that for any ν_0, any $\theta \in [0, \pi]$,

$$
\begin{aligned}
F(\theta) := g_\varepsilon &\left(\frac{v+v_*}{2} + \frac{v-v_*}{2} \cos\theta + \frac{|v-v_*|}{2} \nu_0 \sin\theta \right) \\
&\times g_\varepsilon \left(\frac{v+v_*}{2} - \frac{v-v_*}{2} \cos\theta - \frac{|v-v_*|}{2} \nu_0 \sin\theta \right) = g_\varepsilon(v) g_\varepsilon(v_*).
\end{aligned}
$$

Since furthermore g_ε is smooth, we deduce that $F'(0) = 0$ and thus

$$
\forall\, \nu_0 \in S^2,\ \nu_0 \perp v - v_*, \quad \langle g_\varepsilon(v_*) \nabla g_\varepsilon(v) - g_\varepsilon(v) \nabla g_\varepsilon(v_*), \nu_0 \rangle = 0. \tag{4.47}
$$

Step 3. First note that g_ε does not vanish since that $\mathrm{supp}\rho_\varepsilon = \mathbb{R}^3$ and $g \not\equiv 0$. Thus, one may define $h_\varepsilon := \log g_\varepsilon$, and (4.47) becomes, for all v, v_*,

$$
\forall\, \nu_0 \in S^2,\ \nu_0 \perp v - v_*, \quad \langle \nabla h_\varepsilon(v) - \nabla h_\varepsilon(v_*), \nu_0 \rangle = 0. \tag{4.48}
$$

We may deduce by an elementary differential calculus (see [14]) that there exists $a, c \in \mathbb{R}$, $b \in \mathbb{R}^3$ such that $h_\varepsilon(v) = a|v|^2 + \langle b, v \rangle + c$. Therefore $g_\varepsilon = \exp(a|v|^2 + \langle b, v \rangle + c)$ on $\mathbb{R}^3$.

Step 4. Identifying the constants a, b and c, there holds

$$
g_\varepsilon(v) = \frac{\gamma_\varepsilon}{(2\pi\sigma_\varepsilon)^{3/2}} \, e^{-\frac{|v-v_\varepsilon|^2}{2\sigma_\varepsilon}}, \tag{4.49}
$$

with

$$
\gamma_\varepsilon = \int_{\mathbb{R}^3} g_\varepsilon dv, \quad v_\varepsilon = \int_{\mathbb{R}^3} g_\varepsilon v dv, \quad \sigma_\varepsilon = \frac{1}{3} \int_{\mathbb{R}^3} g_\varepsilon |v|^2 dv. \tag{4.50}
$$

On the one hand, we know that $g_\varepsilon = g \star \rho_\varepsilon$ tends to g in L^1. On the other hand, it is clear from (4.49) and (4.46) that g_ε tends to $\frac{\gamma}{(2\pi\sigma)^{3/2}} \, e^{-\frac{|v-v_0|}{2\sigma}}$ a.e., where σ, ρ, v_0 are defined by (4.50) with $\varepsilon = 0$. This concludes the proof. ◻

Proof of Lemma 4.7-2. Let us set $\mathcal{O} = \{m, \int f(m,p) dp > 0\}$. Note that for $m \notin \mathcal{O}$, (4.45) holds, choosing $\gamma(m) = 0$. The functional equation (4.44) with $m = m_* \in \mathcal{O}$ and (4.43) imply that for any $m \in \mathcal{O}$, there exist $\tilde\gamma(m)$, $\tilde\sigma(m)$ and $\tilde v_0(m)$ such that

$$
f(m, mv) = \frac{\tilde\gamma(m)}{(2\pi\tilde\sigma(m))^{3/2}} \, e^{-\frac{|v-\tilde v_0(m)|^2}{2\tilde\sigma(m)}}.
$$

which can be written, using other functions $\gamma(m)$, $\sigma(m)$ and $v_0(m)$,

$$
f(m, p) = \frac{\gamma(m)}{(2\pi m \sigma(m))^{3/2}} \, e^{-\frac{|p - m v_0(m)|^2}{2m\sigma(m)}}.
$$

Using (4.44) with $v_* = 0$ and using the parameterization (1.5) for v', v'_*, we deduce that for all $\nu \in S^2$,

$$
\frac{|p - m v_0(m)|^2}{m\sigma(m)} + \frac{|m_* v_0(m_*)|^2}{m_*\sigma(m_*)} = \frac{|p - m v_0(m) - 2\bar\mu \langle v, \nu \rangle \nu|^2}{m\sigma(m)} + \frac{|2\bar\mu \langle v, \nu \rangle \nu - m_* v_0(m_*)|^2}{m_*\sigma(m_*)}.
$$

A straightforward computation shows that

$$
4 \langle v, \nu \rangle^2 \mu\bar\mu \left(-\frac{1}{\sigma(m)} + \frac{1}{\sigma(m_*)} \right) + 4 \langle v, \nu \rangle \bar\mu \left\langle \frac{v_0(m)}{\sigma(m)} - \frac{v_0(m_*)}{\sigma(m_*)}, \nu \right\rangle = 0.
$$

We thus deduce that $\sigma(m) = \sigma(m_*)$ and then that $v_0(m) = v_0(m_*)$ for a.e. $m, m_* \in \mathcal{O}$. This implies that v_0 and σ are constant on $\mathcal{O}$. $\square$

We are now able to present the

Proof of Theorem 2.3. Let us consider an increasing sequence $(t_n)_{n\geq 1}$, $t_n \to \infty$ and put $f_n(t,.) := f(t+t_n,.)$ for $t \in [0,T]$ and $n \geq 1$. We realize, from (2.17), (2.22) and (4.36), that for all T,

$$f_n \ \text{ is bounded in } \ L^\infty([0,T]; L^1_{k_B^2} \cap L^1 \log L^1)$$

and then, using the fact that f solves the Boltzmann equation, that (4.1),

$$\int_Y f_n(t,y)\,\psi(y)\,dy \ \text{ is bounded in } \ BV(0,T)$$

for any $\psi \in L^\infty(Y)$. Therefore, up to the extraction of a subsequence, there exists $\Gamma \in C([0,T]; L^1_{k_B})$ such that

$$f_n \rightharpoonup \Gamma \text{ weakly in } L^1((0,T)\times Y) \text{ and } \int_Y f_n(t,y)\,\psi(y)\,dy \to \int_Y \Gamma(t,y)\,\psi(y)\,dy \text{ in } C([0,T]), \tag{4.51}$$

for any function ψ on Y such that $|\psi|\,k_B^{-1} \in L^\infty(Y)$. On the one hand, the above convergence is strong enough in order to pass to the semi-inferior limit as follows

$$\int_0^T D_{h,B}(\Gamma(t,.))\,dt \leq \liminf_{n\to\infty} \int_0^T D_{h,B}(f_n(t,.))\,dt \leq \lim_{n\to\infty} \int_{t_n}^\infty D_{h,B}(f(s,.))\,ds = 0,$$

where, for the last equality, we have used (4.37). We deduce that for each t, $\Gamma(t,.)$ satisfies the functional relation (4.44) and then, Lemma 4.7 implies that $\Gamma(t,.)$ is a mass-dependent Maxwellian function.
On the other hand, from (4.51) and (2.17) we have

$$\int_Y \Gamma(t,y)\,\phi(m)\,dy = \int_Y f_{in}\,\phi(m)\,dy,$$

for any bounded measurable function $\phi : (0,\infty) \mapsto \mathbb{R}$, so that, with the notations of (2.25),

$$\int_{\mathbb{R}^3} \Gamma(t,y)\,dp = \rho(m) \quad \forall\, m > 0 \text{ and } \forall\, t \in (0,T).$$

We also have, from (4.51), (2.17) and (2.18), that for any $t \in (0,T)$

$$\int_Y \Gamma(t,y)\,p\,dy = \int_Y f_{in}\,p\,dy = 0, \quad \int_Y \Gamma(t,y)\,\mathcal{E}\,dy = \int_Y f_{in}\,\mathcal{E}\,dy = 3\left(\int_0^\infty \rho(m)\,dm\right)\Sigma.$$

One easily concludes that for all t, $\Gamma(t) = M$, where M is the Maxwellian defined in (2.24). Hence the Γ does not depend on t nor on the sequence (t_n), so that we may conclude (2.23). $\square$

5 The mass-dependent Granular media equation

In this section we sketch the proof of Theorems 2.4 and 2.5. The main difficulty, compared to the Boltzmann elastic equation, is to extend the Povzner inequality to this inelastic context. We thus consider the inelastic Boltzmann equation

$$\frac{\partial f}{\partial t} = Q_G(f) \ \text{ on } \ (0,\infty)\times Y, \quad f(0,.) = f_{in} \ \text{ on } \ Y, \tag{5.1}$$

where Q_G is given by (1.10). We assume in the whole section that $a_S \equiv 0$, $a_B \equiv 0$, (1.18), (1.21), (1.23), and we consider an initial condition satisfying (1.28) with $k = k_B$ defined in (1.29). We recall that $\bar{a}_G$ and $\bar{E}_G$ were defined in (1.24) and (1.23).

All the statements we present below also hold replacing $Q_G(f)$ by $Q_B(f) + Q_G(f)$ in the RHS of (5.1). This follows without difficulty gathering the arguments of the preceding section with those introduced below.

5.1 Uniqueness

We start with uniqueness.

Lemma 5.1 *Assume that a_G and k are two nonnegative measurable functions on $Y^2 \times S^2 \times (0,1)$ and Y respectively, such that the symmetry condition (1.9) holds, and such that for any $y, y_* \in Y$, any $\nu \in S^2$, $e \in (0,1)$,*

$$0 \leq \bar{a}_G = \int_{S^2} \int_0^1 a_G \, ded\nu \leq k k_* \quad \text{and} \quad k'' + k''_* - k - k_* \leq 0. \tag{5.2}$$

Then there exists at most one solution f to (5.1) such that for all $T > 0$, $f \in C([0,T); L^1_k) \cap L^\infty([0,T); L^1_{k^2})$.

The proof is a fair copy of that of Lemma 4.1. Note that here again, one may choose $k = k_B$ defined in (1.29) under (1.18), so that the uniqueness part of Theorem 2.4 follows.

5.2 A priori estimates and existence

We next state the conservations for such an equation.

Lemma 5.2 *A solution f to the mass-dependent Granular equation (5.1) conserves, at least formally, momentum, mass distribution (2.17) while kinetic energy decreases (2.26).*

The proof is again identical to the corresponding result (Lemma 4.2) for equation (4.1).
We next present some a priori bounds for the Granular operator, which rely on a Povzner lemma.

Lemma 5.3 *There exists a constant C_A, depending only on A (see (1.18)), such that for any nonnegative measurable function h on Y,*

$$\int_Y Q_G(h) \, \mathcal{E}^2 \, dy \leq C_A \int_Y (m^{-1} + m + \mathcal{E}) \, h \, dy \int_Y (m^{-2} + m^2 + \mathcal{E}^2) \, h \, dy, \tag{5.3}$$

$$\int_Y Q_G(h) \, \mathcal{E}^3 \, dy \leq C_A \int_Y (m^{-1} + m^2 + \mathcal{E}^2) \, h \, dy \int_Y (m^{-3} + m^3 + \mathcal{E}^3) \, h \, dy. \tag{5.4}$$

Proof of Lemma 5.3. We follow here the line of the proof of Lemma 4.3. We will only check (5.3), the other case being treated similarly. We split the proof in several steps.

Step 1. Preliminaries. Using (2.2) with $\phi(y) = \mathcal{E}^2$, we deduce that

$$\int_Y Q_G(h) \, \mathcal{E}^2 dy = \int_Y \int_Y h \, h_* \int_0^1 \mathcal{K}_2 \, dy dy_* de \tag{5.5}$$

where

$$\mathcal{K}_2(y, y_*, e) := \int_{S^2} a_G(y, y_*, \cos\Theta, e) \, \{(\mathcal{E}'')^2 + (\mathcal{E}''_*)^2 - \mathcal{E}^2 - \mathcal{E}_*^2\} \, d\nu. \tag{5.6}$$

Here v'' and v''_* are defined from v, v_* and ν, e with the help of (1.7) and Θ has been defined by (1.22). We introduce a new parameterization of post collisional velocities, in the same spirit as in the elastic case. An easy computation shows that one may write

$$v'' = v_{**} + \lambda\mu_* w\sigma, \quad v''_* = v_{**} - \lambda\mu w\sigma, \tag{5.7}$$

with

$$\lambda = \left(1 - (1-e^2)\cos^2\Theta\right)^{1/2} \in (0,1), \quad \sigma = \lambda^{-1}\left(\frac{v - v_*}{w} + (1+e)\cos\Theta\nu\right) \in S^2. \tag{5.8}$$

Considering the direct orthonormal basis $(\vec{\imath_1}, \vec{\imath_2}, \frac{v-v_*}{w})$ of $\mathbb{R}^3$ such that $\langle v_{**}, \vec{\imath_2}\rangle = 0$, we may parameterize σ in this basis, writing

$$\sigma = (\vec{\imath_1}\cos\phi + \vec{\imath_2}\sin\phi)\sin\theta + \frac{v - v_*}{w}\cos\theta, \tag{5.9}$$

Performing the substitution $\nu \mapsto (\theta, \phi)$, we get

$$\mathcal{K}_2(y, y_*, e) = \int_0^\pi a_G(y, y_*, \cos\Theta, e)\frac{\lambda\sin\theta d\theta}{(1+e)\cos\Theta}\int_0^{2\pi} d\phi\{(\mathcal{E}'')^2 + (\mathcal{E}''_*)^2 - \mathcal{E}^2 - \mathcal{E}_*^2\} \tag{5.10}$$

where now v'' and v''_* are defined by (5.7).

Step 2. The Povzner Lemma. As in the proof of Lemma 4.3, we denote by $v_{**} = (\xi_0, 0, \zeta_0)$ the coordinates of v_{**} in the basis $(\vec{\imath_1}, \vec{\imath_2}, \frac{v-v_*}{w})$. Then one easily checks that

$$\begin{aligned} v = (\xi_0, 0, \zeta_0 + \mu_* w), \quad v_* = (\xi_0, 0, \zeta_0 - \mu w), \\ v'' = (\xi_0 + \lambda\mu_* w\cos\phi\sin\theta, \lambda\mu_* w\sin\phi\sin\theta, \zeta_0 + \lambda\mu_* w\cos\theta), \\ v''_* = (\xi_0 - \lambda\mu_* w\cos\phi\sin\theta, -\lambda\mu_* w\sin\phi\sin\theta, \zeta_0 - \lambda\mu_* w\cos\theta). \end{aligned} \tag{5.11}$$

Then, a straightforward computation shows that

$$\mathcal{E}'' = \mathcal{E} - \mu_* B + A, \qquad \mathcal{E}''_* = \mathcal{E}_* - \mu B - A,$$

where, α standing the angle between v and v_*, and noting that (4.18) and (4.19) still hold,

$$B := (1-\lambda^2)\,\bar{\mu}w^2 = (1-\lambda^2)\,\bar{\mu}\,[|v|^2 + |v_*|^2 - 2|v||v_*|\cos\alpha], \tag{5.12}$$

and

$$\begin{aligned} A &:= -2\bar{\mu}(1 - \lambda\cos\theta)(w\zeta_0) + 2\bar{\mu}\lambda(w\xi_0)\cos\phi\sin\theta \\ &= -2\,\bar{\mu}\,[\mu|v|^2 - \mu_*|v_*|^2 - (\mu - \mu_*)|v||v_*|\cos\alpha]\,(1 - \lambda\,\cos\theta) \\ &\quad +2\lambda\bar{\mu}|v||v_*|\sin\alpha\sin\theta\cos\phi. \end{aligned} \tag{5.13}$$

We deduce that

$$\begin{aligned} &(\mathcal{E}'')^2 + (\mathcal{E}''_*)^2 - \mathcal{E}^2 - \mathcal{E}_*^2 \\ &\quad = B^2\,(\mu^2 + \mu_*^2) - 2\,B(\mu_*\mathcal{E} + \mu\mathcal{E}_*) + 2\,A^2 + 2\,A\,(\mathcal{E} - \mu_* B - \mathcal{E}_* + \mu B). \end{aligned}$$

A tedious computation using that $\int_0^{2\pi}\cos\phi d\phi = 0$, while $\int_0^{2\pi}\cos^2\phi d\phi = \pi$ shows that

$$\frac{1}{2\pi}\int_0^{2\pi}((\mathcal{E}'')^2 + (\mathcal{E}''_*)^2 - \mathcal{E}^2 - \mathcal{E}_*^2)d\phi = S(y, y_*, \theta, \lambda) + S(y_*, y, \theta, \lambda) \tag{5.14}$$

where

$$S(y, y_*, \theta, \lambda) = \alpha_1|v|^4 + \alpha_2|v|^3|v_*| + \alpha_3|v|^2|v_*|^2, \tag{5.15}$$

with

$$\begin{aligned}(5.16) \qquad \alpha_1 &= \bar\mu^2(\mu^2+\mu_*^2)(1-\lambda^2)^2 + 8\bar\mu^2(1-\lambda\cos\theta)^2\mu^2 - 2\bar\mu^2(1-\lambda^2)\\ &\quad -4\bar\mu m\mu(1-\lambda\cos\theta) - 4\bar\mu^2\mu(1-\lambda\cos\theta)(\mu-\mu_*)(1-\lambda^2),\end{aligned}$$

$$\begin{aligned}(5.17) \qquad \alpha_2 &= -4\bar\mu^2(\mu^2+\mu_*^2)(1-\lambda^2)^2\cos\alpha - 16\bar\mu^2(1-\lambda\cos\theta)^2\mu(\mu-\mu_*)\cos\alpha\\ &\quad +4\bar\mu^2(1-\lambda^2)\cos\alpha + 8\bar\mu^2(1-\lambda\cos\theta)\mu(\mu-\mu_*)(1-\lambda^2)\cos\alpha\\ &\quad +4\bar\mu m(1-\lambda\cos\theta)(\mu-\mu_*)\cos\alpha + 4\bar\mu^2(1-\lambda\cos\theta)(\mu-\mu_*)^2(1-\lambda^2),\end{aligned}$$

$$\begin{aligned}(5.18) \qquad \alpha_3 &= \bar\mu^2(\mu^2+\mu_*^2)(1-\lambda^2)^2[1+2\cos^2\alpha] + 2\bar\mu^2\lambda^2\sin^2\alpha\sin^2\theta\\ &\quad +4\bar\mu^2(1-\lambda\cos\theta)^2[-2\mu\mu_* + (\mu-\mu_*)^2\cos^2\alpha] - 2\bar\mu^2(1-\lambda^2)\\ &\quad -2\bar\mu(1-\lambda\cos\theta)\big[-2\bar\mu + \mu\bar\mu(\mu-\mu_*)(1-\lambda^2) + \mu_*\bar\mu(\mu-\mu_*)(1-\lambda^2)\\ &\qquad\qquad +2\bar\mu(\mu-\mu_*)^2\cos^2\alpha(1-\lambda^2)\big].\end{aligned}$$

First, a straightforward computation shows that

$$(5.19) \qquad \alpha_2 \le 104 m\bar\mu \text{ and } \alpha_3 \le 57\bar\mu^2.$$

Next, using that $(1-\lambda^2)^2 - 2(1-\lambda^2) \le 0$, and that $|\cos\theta| \le 1$, we get

$$(5.20) \qquad \alpha_1 \le 4\frac{\mu\bar\mu m}{(m+m_*)^2}(1-\lambda\cos\theta)P(\lambda),$$

with $P(\lambda) := m_*\lambda^2(m-m_*) + 2\lambda m m_* - m(m+m_*)$. But P is nonpositive on $[0,1]$. Indeed, $P'(\lambda)=0$ only for $\lambda=\lambda_0 := m/(m_*-m)$. Thus if $m_* < 2m$, P' does not vanish on $[0,1]$, so that $P(\lambda) \le \max[P(0),P(1)] = \max[-m(m+m_*), -(m-m_*)^2] \le 0$. Next if $m_* \ge 2m$, then $P(\lambda) \le P(\lambda_0) = \frac{mm_*^2}{m_*-m}[(m/m_*)^2 + (m/m_*) - 1] \le 0$ since $m/m_* \in [0,1/2]$.

We finally deduce that α_1 is nonpositive, so that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}((\mathcal{E}'')^2 + (\mathcal{E}_*'')^2 - \mathcal{E}^2 - \mathcal{E}_*^2)d\phi &\le 104 m\bar\mu|v|^3|v_*| + 114\bar\mu^2|v|^2|v_*|^2 + 104 m_*\bar\mu|v_*|^3|v|\\ (5.21) \qquad &\le 104\bar\mu|v||v_*|(\mathcal{E}+\mathcal{E}_*) + 114\mu\mu_*\mathcal{E}\mathcal{E}_*.\end{aligned}$$

Step 3. Conclusion. First note that, with the notations of Step 1,

$$(5.22) \qquad \bar a_G(y,y_*) = \int_0^1 de \int_0^\pi a_G(y,y_*,\cos\Theta,e)\frac{\lambda\sin\theta d\theta}{(1+e)\cos\Theta}\int_0^{2\pi} d\phi.$$

Thus, gathering (1.18), (5.21), and (5.10), we get

$$(5.23) \quad \int_0^1 \mathcal{K}_2(y,y_*,e)de \le C_A(1+m+m_*)(1+|v|+|v_*|)(\bar\mu|v||v_*|(\mathcal{E}+\mathcal{E}_*) + \mu\mu_*\mathcal{E}\mathcal{E}_*).$$

As in the proof of Lemma 4.3, a tedious computation using finally (5.5) allows us to conclude that (5.3) holds. □

As an immediate consequence of Lemmas 5.2 and 5.3, we obtain the following a priori bounds on the solutions of (5.1). We omit the proof since it follows the same line as that of Corollary 3.5.

Corollary 5.4 *A solution f to (5.1) satisfies, at least formally, for any T,*

$$(5.24) \qquad \text{for } z=2 \text{ and } 3 \qquad f_{in}\,k_B^z \in L^1(Y) \quad \text{implies} \quad \sup_{[0,T]}\int_Y f(t,y)\,k_B^z\,dy \le C_{T,z},$$

where the constant $C_{T,z}$ depends only on T, $\|f_{in}\|_{L^1_{k_B^z}}$ and on A.

Proof of Theorem 2.4. It follows line by line Subsection 3.3. It suffices to use of Corollary 5.4 instead of Corollary 3.5, and to use the computation of Lemma 5.1 instead of that of Lemma 3.1. □

5.3 Long time behavior

We finally give the

Proof of Theorem 2.5. We split the proof into three parts.
Proof of (2.27). It is based on the dissipation of kinetic energy. Let us consider an increasing sequence $(t_n)_{n\geq 1}$, $t_n \to \infty$ and put $f_n(t,.) := f(t+t_n,.)$ for $t \in [0,T]$ and $n \geq 1$. We then proceed along the line of the proof of Theorem 2.7 to which we refer for details and notations. From (2.17) and (2.26), there holds

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \int_Y f_n\,(1+m^2+\mathcal{E})\,dy < \infty. \tag{5.25}$$

On the one hand, we deduce of (5.25) that, up to the extraction of a subsequence, there exists $\Gamma \in C([0,T), M^1(Y) - weak)$ such that (3.35) holds. On the other hand, we know from Theorem 2.4 that $t \mapsto D_{\mathcal{E},G}(f(t,.)) \in L^1([0,\infty))$, so that

$$\int_0^T D_{\mathcal{E},G}(\Gamma(t,.))\,dt \leq \liminf_{n\to\infty} \int_0^T D_{\mathcal{E},G}(f_n(t,.))\,dt \leq \liminf_{n\to\infty} \int_{t_n}^\infty D_{\mathcal{E},G}(f(s,.))\,ds = 0. \tag{5.26}$$

Gathering (3.35) and (5.26) we get that for all $t \in [0,T]$, all y, y_* in Y,

$$\bar\mu^2 \left(\int_0^1 \int_{S^2} a_G(y,y_*,\nu,e)\,(1-e^2)\,\langle v - v_*,\nu\rangle^2\,d\nu de\right) \Gamma(t,dy_*)\,\Gamma(t,dy) = 0.$$

Since $\bar a_G(y,y_*) > 0$ as soon as $v \neq v_*$ by assumption, we deduce that

$$\mathbf{1}_{\{v\neq v_*\}}\Gamma(t,dy_*)\,\Gamma(t,dy) = 0,$$

and therefore $\Gamma(t,dy)$ is of the shape $\Gamma(t,dy) = \lambda(t,dm)\,\delta_{p=m\,v_t}$, for some $v_t \in \mathbb{R}^3$. Next, observing that $|p|^{4/3} \leq \mathcal{E} + m^2$ by the Young inequality, we may pass to the limit in the conservation laws (2.18) thanks to (5.25), and we obtain for any $t \in [0,T]$,

$$v_t \int_0^\infty m\,\lambda(t,dm) = \int_Y p\,\Gamma(t,dy) = \lim_{n\to\infty} \int_Y p\,f_n(t,y)\,dy = 0 \tag{5.27}$$

and

$$\int_0^\infty \phi(m)\,\lambda(t,dm) = \int_Y \phi(m)\,\Gamma(t,dy) = \lim_{n\to\infty} \int_Y \phi(m)\,f(t_n,y)\,dy = \int_0^\infty \phi(m)\,\rho(m)\,dm, \tag{5.28}$$

for any $\phi \in C_c(0,\infty)$, where ρ is defined in (2.25). We first deduce of (5.28) that $\lambda(t,dm) = \rho(m)$ for any $t \in [0,T]$ and then from (5.27), since $\int_0^\infty m\,\rho(m)\,dm > 0$, that $v_t \equiv 0$ for any $t \in [0,T]$. We then easily conclude the proof of (2.27).

Proof of (2.29). For any $\varphi \in C_b(\mathbb{R}^3)$, we get from (2.28) and (2.27)

$$\int_{\mathbb{R}^3} j(t,v)\,\varphi(v)\,dv = \int_Y \varphi\left(\frac{p}{m}\right)\,m\,f(t,y)\,dy. \longrightarrow \varphi(0)\int_0^\infty m\rho(m)\,dm,$$

and we conclude recalling that $\int_0^\infty m\rho(m)\,dm = 1$.

Proof of (2.31). Using (2.30), the dissipation of kinetic energy (2.26) reads

$$\frac{d}{dt}\int_Y f\,\mathcal{E}\,dy = -\frac{\kappa}{2}\int_Y\int_Y \frac{(m\,m_*)^{1+\delta}}{m+m_*}\,|v-v_*|^3\,f\,f_*\,dydy_*. \tag{5.29}$$

Using that $m\,m_*\,|v-v_*|^2 = m|v|^2 m_* + m m_* |v_*|^2 - 2\,\langle p, p_*\rangle$, and that for all t, $\int_Y m f dy = 1$ while $\int_Y p f dy = 0$, we observe that

$$2\int_Y f\,\mathcal{E}\,dy = \int_Y\int_Y m\,m_*\,|v-v_*|^2\,f\,f_*\,dy dy_*$$
$$\le \left(\int_Y\int_Y (m+m_*)^2\,(m\,m_*)^{1-2\delta}\,f\,f_*\,dy dy_*\right)^{1/3}\left(\int_Y\int_Y \frac{(m\,m_*)^{1+\delta}}{m+m_*}\,|v-v_*|^3\,f\,f_*\,dy dy_*\right)^{2/3}.$$

Thanks to the conservation of mass distribution,

$$C = \left(\int_Y\int_Y (m+m_*)^2\,(m\,m_*)^{1-2\delta}\,f\,f_*\,dy dy_*\right)^{1/3}$$

does not depend on time. We finally deduce that for some $K>0$,

$$\frac{d}{dt}\int_Y f\,\mathcal{E}\,dy \quad\le\quad -K\left(\int_Y f\,\mathcal{E}\,dy\right)^{3/2},$$

and we easily conclude. □

6 The full Boltzmann equation

We now study the full equation (1.1). We thus assume in the whole section that (1.18), (1.21) and (1.23) hold. We consider an initial condition satisfying (1.28) with $k = k_B$ defined in (1.29).

6.1 Existence and uniqueness

All the lemmas below are obtained by gathering the arguments concerning the kinetic Smoluchowski equation, the mass dependent Boltzmann equation and the mass-dependent Granular equation.

Lemma 6.1 *Assume that a_B, a_G and a_S satisfy the symmetry conditions (1.3), (1.9), and (1.13). Let k be a measurable map on Y such that for all y, y_* in Y, all $\nu \in S^2$ and all $e \in (0,1)$,*

$$\bar a_B(y,y_*) + \bar a_G(y,y_*) + a_S(y,y_*) \le k k_*, \tag{6.1}$$
$$k' + k'_* - k - k_* \le 0, \quad k'' + k''_* - k - k_* \le 0 \quad \text{and} \quad k_{**} - k - k_* \le 0. \tag{6.2}$$

Then there exists at most one solution f to the Boltzmann equation (1.1) such that for all $T \ge 0$, $f \in C([0,T); L^1_k) \cap L^\infty([0,T); L^1_{k^2})$.

The proof is immediate using the arguments of Lemmas 3.1, 4.1, and 5.1. Since $C_A k_B$ satisfies all the required properties (for some constant C_A depending on A), the uniqueness part of Theorem 2.6 follows.

Lemma 6.2 *A solution f to the Boltzmann equation (1.1) conserves (at least formally) mass and momentum (2.32). Moreover, the dissipation of total concentration and of kinetic energy (2.33) hold. Finally, there holds,*

$$\frac{d}{dt}\int_Y \psi(m) f dy \le 0 \tag{6.3}$$

for any sub-additive function $\psi : (0,\infty) \mapsto (0,\infty)$, that is $\psi(m_{**}) \le \psi(m) + \psi(m_*)$.

The proof follows the line of that of Lemma 3.3, and relies on the use of (2.16) and (2.1), (2.2), and (2.4) with $\beta(x) = x$, and with suitable choices for ϕ.
Gathering the estimates proved in Lemmas 3.4, 4.3 and 5.3 with those of Lemma 6.2, we obtain the following estimates.

Corollary 6.3 *Recall that k_B was defined in (1.29). A solution f to (1.1) satisfies, at least formally, for any T,*

$$\text{(6.4)} \qquad \textit{for } z = 2 \textit{ and } 3 \qquad f_{in}\, k_B^z \in L^1 \quad \textit{implies} \quad \sup_{[0,T]} \int_Y f(t,y)\, k_B^z\, dy \leq C_{T,z},$$

where the constant $C_{T,z}$ depends only on T, $\|f_{in}\|_{L^1_{k_B^z}}$, and on A (see (1.18)).

Proof of Theorem 2.6. It follows the line of Subsection 3.3, with the help of the bounds stated in Corollary 6.3 and a convenient modification of the proof of Lemma 6.1. □

6.2 A stochastic interpretation

We now introduce a stochastic version of equation (1.1), that contains more information about the particles, which will be useful to study the long time behavior of solutions.
Since it is more convenient here to work with the couple of variables (m, v) rather than (m, p). We introduce the phase space $Z := (0, \infty) \times \mathbb{R}^3$ of (mass, velocity) variables.

Definition 6.4 *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a (sufficiently large) probability space. Consider a càdlàg Z-valued adapted stochastic process $(Z_t)_{t\geq 0} = (M_t, V_t)_{t\geq 0}$. Denote, for each $t \geq 0$, by $Y_t = (M_t, M_t V_t)$, and by Q_t the law of Y_t, which is a probability measure on Y.*
Then $(Z_t)_{t\geq 0}$ is said to solve (SDE) if the following conditions hold.
(i) M is a.s. nondecreasing, and is $(0, \infty)$-valued, while V is $\mathbb{R}^3$-valued.
(ii) The law Q_0 of Y_0 is given by $m f_{in}(y) dy$.
(iii) For any $T < \infty$,

$$\text{(6.5)} \qquad E\left[\sup_{[0,T]} (M_t + |V_t|)\right] < \infty \quad \textit{and} \quad \sup_{[0,T]} E[|V_t|^2] < \infty.$$

(iv) There exists three independent $(\mathcal{F}_t)_{t\geq 0}$-adapted Poisson measures

$$\text{(6.6)} \qquad N_S(ds, dy, du), \quad N_B(ds, dy, d\nu, du), \quad N_G(ds, dy, d\nu, de, du),$$

on $[0, \infty) \times Y \times [0, \infty)$, $[0, \infty) \times Y \times S^2 \times [0, \infty)$, $[0, \infty) \times Y \times S^2 \times (0, 1) \times [0, \infty)$ respectively, with intensity measures

$$\text{(6.7)} \qquad ds Q_s(dy) du, \quad ds Q_s(dy) d\nu du, \quad ds Q_s(dy) d\nu de du$$

respectively, such that a.s., for all $t \geq 0$,

$$\begin{aligned}
\text{(6.8)} \qquad M_t &= M_0 + \int_0^t \int_Y \int_0^\infty m \mathbf{1}_{\left\{u \leq \frac{a_S(Y_{s-}, y)}{m}\right\}} N_S(ds, dy, du),\\
V_t &= V_0 + \int_0^t \int_Y \int_0^\infty \frac{m(v - V_{s-})}{m + M_{s-}} \mathbf{1}_{\left\{u \leq \frac{a_S(Y_{s-}, y)}{m}\right\}} N_S(ds, dy, du)\\
&+ \int_0^t \int_Y \int_{S_2} \int_0^\infty \frac{2m}{m + M_{s-}} \langle v - V_{s-}, \nu\rangle\, \nu \mathbf{1}_{\left\{u \leq \frac{a_B(Y_{s-}, y, \nu)}{m}\right\}} N_B(ds, dy, d\nu, du)\\
&+ \int_0^t \int_Y \int_{S_2} \int_0^1 \int_0^\infty \frac{(1+e)m}{m + M_{s-}} \langle v - V_{s-}, \nu\rangle\, \nu \mathbf{1}_{\left\{u \leq \frac{a_G(Y_{s-}, y, \nu, e)}{m}\right\}} N_G(ds, dy, d\nu, de, du).
\end{aligned}$$

This process $(Z_t)_{t\geq 0}$ represents the evolution of the couple of characteristics (mass, velocity) of a *typical* particle. Of course, $(Y_t)_{t\geq 0}$ represents the evolution of the couple of characteristics (mass, momentum) of the same *typical* particle. We refer to Tanaka [53], Sznitman [52], Graham-Méléard [33] for similar stochastic interpretations of the Boltzmann equation for elastic collisions, and to Deaconu et al. [19] and Fournier-Giet [29] for the Smoluchowski coagulation equation.

Theorem 6.5 *Assume that the conditions of Theorem 2.6 hold. Then there exists a solution* $(Z_t)_{t\geq 0} = (M_t, V_t)_{t\geq 0}$ *to (SDE). This solution furthermore satisfies that for each* t*, the law* Q_t *of* $Y_t = (M_t, M_t V_t)$ *has a density* $h(t,m,p)$*. Then* $f(t,y) = h(t,m,p)/m$ *is the unique solution to (1.1) such that* $f \in C([0,T), L^1_{k_B}) \cap L^\infty([0,T), L^1_{k_B^2})$*. In other words, for all bounded measurable functions* $\psi : Y \mapsto \mathbb{R}$*,* $\phi : Y \mapsto \mathbb{R}$*, any* $t \geq 0$*,*

$$E[\psi(Y_t)] = \int_Y \psi(y)\, mf(t,y)\, dy \quad \text{and} \quad E[\phi(Z_t)] = \int_Y \phi(m,v)\, mf(t,y)\, dy. \tag{6.9}$$

This result is completely unsurprising since existence holds for equation (1.1). We will only give the main steps of the proof, since it is quite standard and tedious. We refer to [53, 52, 33, 19, 29] for detailed proofs of similar results.

Sketch of proof of Theorem 6.5. We first assume in this proof that

$$f_{in} \in L^1_{k_B^3}. \tag{6.10}$$

Step 1. The result of Theorem 6.5 holds if the rates a_B, a_G, and a_S are bounded. Indeed, the existence of a solution $(Z_t)_{t\geq 0}$ can be obtained immediately by using the *exact simulation* technic of Fournier-Giet [29]. The obtained solution clearly satisfies the moment properties that for all $T \geq 0$, $E\left[\sup_{[0,T]}(M_t^{-2} + M_t^2|V_t|^2)\right] < \infty$, since (6.10) ensures that $E\left[(M_0^{-2} + M_0^2|V_0|^2)\right] < \infty$. Using such inequalities, one may prove that for each $t \geq 0$, the law of Y_t has a density $h(t,y)$. Setting $f(t,y) = h(t,y)/m$, the above finite expectation ensures that $f \in C([0,T), L^1_{k_B}) \cap L^\infty([0,T), L^1_{k_B^2})$. Finally, the fact that f solves (1.1) (or rather its weak form (2.14)) follows from a fair computation involving the Itô formula for jump processes.

Step 2. We thus consider a sequence of solutions $(Z_t^l)_{t\geq 0}$ associated with the rates $a_B^l = a_B \wedge l$, $a_G^l = a_G \wedge l$, and $a_S^l = a_S \wedge l$, and with an initial condition f_{in} satisfying (6.10). We also denote by g^l the corresponding solution to (1.1) (that is, the law of Y_t^l is given, for each t, by $mg^l(t,y)dmdp$). Using stochastic versions of the estimates obtained in Lemmas 3.4, 4.3, 5.3 and 6.2, one can check that the sequence $(Z_t^l)_{t\geq 0}$ satisfies the Aldous criterion for tightness (see Jacod Shiryaev [34]). Hence one may find a limiting process $(Z_t)_{t\geq 0}$. Martingale technic allow to show that process $(Z_t)_{t\geq 0}$ solves (SDE) with the rates (without cutoff) a_B, a_G, a_S.

Step 3. The fact that for each $t \geq 0$, the law Q_t of Z_t has a density can be obtained from the proof of Theorem 2.6. Indeed, we have built $(Z_t)_{t\geq 0}$ as the limit of $(Z_t^l)_{t\geq 0}$. Recall that for each t, the density of Z_t^l is given by $mg^l(t,y)$. Following the proof of Theorem 2.6, we realize that the sequence $g^l(t,y)$ is Cauchy in $C([0,T), L^1_{k_B}) \cap L^\infty([0,T), L^1_{k_B^2})$. Hence its limit $g(t,.)$ is still a function. The law of Z_t is thus $mg(t,y)dy$, g being the unique solution to (1.1).

Step 4. Finally, the extension to initial conditions f_{in} satisfying only (1.28) can be obtained by using some approximations, as in the proof of Theorem 2.6. ◻

6.3 Long time behavior

We are finally able to prove that the solution f to (1.1) built in Theorem 2.6 tends to 0 in L^1 under the assumptions of Theorem 2.7. To this aim, we will in fact prove that M_t tends a.s. to infinity where $(Z_t)_{t\geq 0} = (M_t, V_t)_{t\geq 0}$ is a solution to (SDE) associated to f thanks to Theorem 6.5.

The main tools of the proof are the dissipations of the total concentration and kinetic energy (2.34) which can be written, recalling (2.36) and the expressions (2.7), (2.9), (2.10) of $D_{1,S}(f) + D_{\mathcal{E},S}(f) + D_{\mathcal{E},G}(f)$,

$$\int_0^\infty dt \int_Y \int_Y f f_* a_S dy dy_* < \infty, \qquad \int_0^\infty dt \int_Y \int_Y f f_* \bar{\alpha}_{inel} dy dy_* < \infty. \tag{6.11}$$

where $\bar{\alpha}_{inel} = a\tilde{E}_{inel}$. We next remark the following fact.

Lemma 6.6 *Almost surely, $M_\infty = \lim_{t\to\infty} M_t$ exists as an element of $(0,\infty)\cup\{\infty\}$.*

The proof is obvious, since M is a nondecreasing process.

We now introduce some notation. We denote by J_t^S (resp. J_t^B and J_t^G) the number of coalescing (resp. elastic and inelastic) collisions endured by our typical particle before t. In other words,

$$\begin{aligned}
J_t^S &= \int_0^t \int_Y \int_0^\infty \mathbf{1}_{\{u\le a_S(y,Y_{s-})/m\}} N_S(ds,dy,du),\\
J_t^B &= \int_0^t \int_Y \int_{S^2} \int_0^\infty \mathbf{1}_{\{u\le a_B(y,Y_{s-},\nu))/m\}} N_B(ds,dy,d\nu,du),\\
J_t^G &= \int_0^t \int_Y \int_{S_2} \int_0^1 \int_0^\infty \mathbf{1}_{\{u\le a_G(y,Y_{s-},\nu,e))/m\}} N_G(ds,dy,d\nu,de,du).
\end{aligned}$$

Note that $J^S + J^B + J^G$ counts the number of jumps of $\{Z_t\}_{t\ge0}$, that is, $J_t^S + J_t^B + J_t^G = \sum_{s\le t} \mathbf{1}_{\{\Delta Z_s\neq 0\}}$.

Lemma 6.7 *The following estimates on the number of collisions hold: for any $m_0\in(0,\infty)$,*

$$E\left[\mathbf{1}_{\{M_\infty\le m_0\}}\{J_\infty^S + J_\infty^B + J_\infty^G\}\right] < \infty. \tag{6.12}$$

Consequently, $\{M_\infty \le m_0\} \subset \{J_\infty^S + J_\infty^B + J_\infty^G < \infty\}$ a.s. for any $m_0\in(0,\infty)$, and then

$$P[J_\infty^S + J_\infty^B + J_\infty^G < \infty] \ge P[M_\infty \le m_0]. \tag{6.13}$$

Proof of Lemma 6.7. Since M is nonincreasing, and since the intensity measure of N_S is given by $m\, f(s,y)\, dudyds$,

$$\begin{aligned}
E\left[\mathbf{1}_{\{M_\infty\le m_0\}} J_\infty^S\right] &= E\left[\mathbf{1}_{\{M_\infty\le m_0\}} \int_0^\infty\int_Y\int_0^\infty \mathbf{1}_{\{u\le a_S(Y_{s-},y)/m\}} N_S(ds,dy,du)\right]\\
&\le E\left[\int_0^\infty\int_Y\int_0^\infty \mathbf{1}_{\{M_{s-}\le m_0\}}\mathbf{1}_{\{u\le a_S(Y_{s-},y)/m\}} N_S(ds,dy,du)\right]\\
&= \int_0^\infty\int_Y\int_0^\infty E\left[\mathbf{1}_{\{M_s\le m_0\}}\mathbf{1}_{\{u\le a_S(Y_s,y)/m\}}\right]\, mf(s,y)\, dudyds\\
&= \int_0^\infty\int_Y E\left[\mathbf{1}_{\{M_s\le m_0\}}\, a_S(Y_s,y)\right]\, f(s,y)\, dyds\\
&\le m_0 E\left[\int_0^\infty ds \int_Y \frac{a_S(Y_s,y)}{M_s} f(s,y)dy\right]\\
&= m_0 \int_0^\infty ds \int_Y\int_Y \frac{a_S(y_*,y)}{m_*} m_* f(s,y_*) f(s,y) dydy_* < \infty.
\end{aligned}$$

We used here that the law of Y_t is $mf(t,y)dy$, and the first dissipation inequality in (6.11). Next, using the same arguments,

$$\begin{aligned}
E\left[\mathbf{1}_{\{M_\infty\le m_0\}} J_\infty^B\right] &\le E\left[\int_0^\infty ds \int_Y \mathbf{1}_{\{M_s\le m_0\}} \bar{a}_B(Y_s,y) f(s,y)dy\right]\\
&= \int_0^\infty ds\int_Y\int_Y \mathbf{1}_{\{m_*\le m_0\}} m_*\bar{a}_B(y_*,y) f(s,y) f(s,y_*) dydy_*\\
&\le A_0 \int_0^\infty ds \int_Y\int_Y \bar{\alpha}_{inel}(y_*,y) f(s,y) f(s,y_*) dydy_* < \infty,
\end{aligned}$$

where we used (2.35) and (6.11). The same computation allows us to obtain the same bound for $E\left[\mathbf{1}_{\{M_\infty \le m_0\}} J^G_\infty\right]$, and that concludes the proof of (6.12). Inequality (6.13) then directly follows from (6.12). $\square$

We are finally able to prove our main result.

Proof of Theorem 2.7. We argue by contradiction and we thus assume

$$P[M_\infty < \infty] > 0. \tag{6.14}$$

Step 1. The assumption (6.14) ensures that there exists $m_0 \in (0,\infty)$ such that $P[M_\infty \le m_0] > 0$. Denoting by τ the last time of jump of $(Y_t)_{t\ge 0}$, we deduce from (6.13) that

$$P[\tau < \infty] = P[J^S_\infty + J^B_\infty + J^G_\infty < \infty] \ge P[M_\infty \le m_0] > 0.$$

Therefore, we have proved that under assumption (6.14), there exists a time t_0, such that

$$P\left[\text{for all } t \ge t_0,\ Y_t = Y_{t_0}\right] > 0. \tag{6.15}$$

Step 2. We now deduce from (6.15) that there exists a nonnegative function g_0 on Y such that

$$f(t,y) \ge g_0(y) \quad \forall\, t \ge t_0,\ \text{a.e. } y \in Y \quad \text{and} \quad \int_Y g_0(y)\, m\, dy > 0. \tag{6.16}$$

Let consider the nonnegative measure $\Gamma(dy)$ on Y defined by

$$\Gamma(A) = P\left[Y_{t_0} \in A \text{ and } Y_t = Y_{t_0} \ \ t \ge t_0\right]. \tag{6.17}$$

On the one hand, $\Gamma(A) \le P(Y_{t_0} \in A) = \int_A m\, f(t_0,y)\, dy$ for any measurable set $A \subset Y$, which means $\Gamma << m\, f(t_0,y)\, dy$, and the Radon-Nykodim Theorem ensures that $\Gamma(dy) = m\, g_0(y)\, dy$ for some $g_0 \in L^1(Y; m\, dy)$. On the other hand, $\int_Y m\, g_0\, dy = \Gamma(Y) = P(Y_t = Y_{t_0}$ for all $t \ge t_0) > 0$ from (6.15). Finally, for any measurable set $A \subset Y$ and any $t \ge t_0$, there holds

$$\int_A m\, f(t,y)\, dy = P(Y_t \in A) \ge \Gamma(A) = \int_A m\, g_0(y)\, dy$$

and (6.16) follows.

Step 3. The lower bound (6.16) ensures that

$$\forall\, t \ge t_0 \qquad D_{1,S}(f(t,.)) \ge D_{1,S}(g_0) > 0, \tag{6.18}$$

the last strict inequality following from the fact that g_0 does not identically vanish and from the positivity condition (2.37). The lower bound (6.18) obviously contradicts the fact that $D_{1,S}(f) \in L^1([0,\infty))$. We then conclude that (6.14) does not hold and therefore $M_\infty = \infty$ a.s or, equivalently, $1/M_t \to 0$ a.s. when t goes to the infinity. Finally, since M is a nondecreasing process and since $E[1/M_0] = \int_Y f_{in} dy < \infty$, we deduce from the Lebesgue Theorem that

$$\int_Y f(t,y)\, dy = E[1/M_t] \underset{t\to\infty}{\longrightarrow} 0.$$

$\square$

7 On explicit solutions

We present in this section a class of more or less explicit solutions to the Boltzmann equation for elastic and coalescing collisions.

Proposition 7.8 *Assume that $a_G \equiv 0$, that a_B and a_S meet (1.18), (1.21), (1.23). Assume also that a_S depends only on the mass variables*

$$a_S(y, y_*) = a_S(m, m_*). \tag{7.19}$$

Consider a solution $c(t,m) : [0,\infty) \times (0,\infty) \mapsto (0,\infty)$ to the classical Smoluchowski equation

$$\begin{aligned} \frac{\partial}{\partial t} c(t,m) &= \frac{1}{2}\int_0^m a_S(m_*, m-m_*) c(t,m_*) c(t, m-m_*) dm_* \\ &\quad - \int_0^\infty a_S(m,m_*) c(t,m) c(t,m_*) dm_*, \quad (t,m) \in (0,\infty)\times(0,\infty). \end{aligned} \tag{7.20}$$

Then the function $f(t,m,p) : [0,\infty)\times(0,\infty)\times\mathbb{R}^3 \mapsto (0,\infty)$ defined by

$$f(t,m,p) = c(t,m) \frac{e^{-|p|^2/2m}}{(2\pi m)^{3/2}} \tag{7.21}$$

solves the Boltzmann equation for elastic and coalescing collisions

$$\frac{\partial}{\partial t} f = Q_B(f) + Q_S(f), \quad (t,m,p) \in (0,\infty)\times(0,\infty)\times\mathbb{R}^3. \tag{7.22}$$

Since in specific cases, explicit solutions to the Smoluchowski equations are known (see e.g. Aldous, [1]), we obtain in the next corollary some particular examples.

Corollary 7.9 *Assume that $a_G \equiv 0$ and that a_B and meets (1.18), (1.21), (1.23). Then*
(i) if $a_S(y,y_*) \equiv 1$, then

$$f(t,m,p) = \frac{4}{t^2} e^{-2m/t} \frac{e^{-|p|^2/2m}}{(2\pi m)^{3/2}} \tag{7.23}$$

solves the Boltzmann equation for elastic and coalescing collisions (7.22).
(ii) if $a_S(y,y_*) = m + m_*$, then

$$f(t,m,p) = \frac{1}{\sqrt{2\pi}} e^{-t} m^{-3/2} e^{-e^{-2t} m/2} \frac{e^{-|p|^2/2m}}{(2\pi m)^{3/2}} \tag{7.24}$$

solves the Boltzmann equation for elastic and coalescing collisions (7.22).

Besides its own interest, Proposition 7.8 might allow to go one step further in the long time behavior study of general solutions to (7.22). One would expect that any solution to (7.22) behaves as (7.21) for large times. We are far from being able to show such a result.
Note that expression (7.21) is quite unsurprising: since a_S does not depend on the velocity variables and since elastic collisions do not act on masses, it is clear that a solution f to (7.22) satisfies $\int_{\mathbb{R}^3} f(t,m,p)dp = c(t,m)$. Then the fact that given its mass m, a particle has a Gaussian (or Maxwellian) momentum with variance m is reasonable. On one hand, Gaussian distributions are stationary along elastic collisions. On the other hand, Gaussian distributions are stable under coalescence: adding two Gaussian random variables with variances m and m_* produces a new Gaussian random variable with variance $m + m_*$.

Proof of Proposition 7.8. Let thus f be defined by (7.21). First of all recall that Maxwellian functions are steady states for the Boltzmann equations for elastic collisions. In other words, for all t, all y, y_* in Y and all ν in S^2, $ff_* = f'f'_*$. This implies that for all $t \geq 0$, (see (1.4))

$$Q_B(f(t,.)) \equiv 0. \tag{7.25}$$

Next, a fair computation using the fact that c solves (7.20) shows that for all $t \geq 0$, all $y \in Y$,

$$\begin{aligned}\frac{\partial}{\partial t} f(t,m,p) &= \frac{e^{-|p|^2/2m}}{(2\pi m)^{3/2}} \frac{\partial}{\partial t} c(t,m) \\ &= \frac{e^{-|p|^2/2m}}{(2\pi m)^{3/2}} \frac{1}{2} \int_0^m a_S(m_*, m-m_*) c(t,m_*) c(t,m-m_*) dm_* \\ &\quad - \frac{e^{-|p|^2/2m}}{(2\pi m)^{3/2}} \int_0^\infty a_S(m,m_*) c(t,m) c(t,m_*) dm_*.\end{aligned} \tag{7.26}$$

But well-known facts about convolution of Gauusian distributions show that one may write, for $m_* < m$

$$\frac{e^{-|p|^2/2m}}{(2\pi m)^{3/2}} = \int_{\mathbb{R}^3} \frac{e^{-|p_*|^2/2m_*}}{(2\pi m_*)^{3/2}} \frac{e^{-|p-p_*|^2/2(m-m_*)}}{(2\pi(m-m_*))^{3/2}} dp_*, \tag{7.27}$$

and, for $m_* > 0$,

$$1 = \int_{\mathbb{R}^3} \frac{e^{-|p_*|^2/2m_*}}{(2\pi m_*)^{3/2}} dp_*. \tag{7.28}$$

We thus get

$$\begin{aligned}\frac{\partial}{\partial t} f(t,m,p) &= \frac{1}{2} \int_0^m \int_{\mathbb{R}^3} a_S(m_*, m-m_*) c(t,m_*) \frac{e^{-|p_*|^2/2m_*}}{(2\pi m_*)^{3/2}} \\ &\qquad\qquad c(t,m-m_*) \frac{e^{-|p-p_*|^2/2(m-m_*)}}{(2\pi(m-m_*))^{3/2}} dm_* dp_* \\ &\quad - \int_0^\infty \int_{\mathbb{R}^3} a_S(m,m_*) c(t,m) \frac{e^{-|p|^2/2m}}{(2\pi m)^{3/2}} c(t,m_*) \frac{e^{-|p_*|^2/2m_*}}{(2\pi m_*)^{3/2}} dm_* dp_* \\ &= \frac{1}{2} \int_0^m \int_{\mathbb{R}^3} a_S(m_*, m-m_*) f(t,m_*,p_*) f(t,m-m_*,p-p_*) dm_* dp_* \\ &\quad - \int_0^\infty \int_{\mathbb{R}^3} a_S(m,m_*) f(t,m,p) f(t,m_*,p_*) dm_* dp_* \\ &= Q_S(f(t,.))(m,v),\end{aligned} \tag{7.29}$$

see (1.14). Gathering (7.29) and (7.25) allows to conclude that (7.22) holds. $\square$

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