Équations aux dérivées partielles / Partial Differential Equations

Convergence to the equilibrium for the Pauli equation without detailed balance condition.

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Résumé. Nous montrons que pour tout $\rho \in (0,1)$, il existe un unique état stationaire de masse ρ pour l'équation de Boltzmann-Pauli homogène, sans hypothèse d'équilibre en détail sur la section efficace. Nous montrons ensuite la convergence en temps grand vers cet équilibre des solutions du problème de Cauchy.

Abstract. We prove that for $\rho \in (0,1)$, the homogeneous Boltzmann-Pauli equation, without detailed balance condition on the cross-section, has a unique steady state of mass ρ. Moreover, we show that the solutions to the Cauchy problem converge to this steady state, as t tends to infinity.

Version Française abrégée.

Nous étudions le problème de Cauchy associé à l'équation de Boltzmann-Pauli homogène. Celle-ci est un modèle simplifié décrivant la dynamique d'un gaz de particules (ions ou electrons) dans un semiconducteur. Nous dénotons $f = f(t,k)$ la densité de particules de vecteur d'onde $k \in B$ à l'instant $t \geq 0$, où B represente la première zone de Brillouin (le tore normalisé de $\mathbb{R}^N$, $N \geq 1$, $|B| = 1$). L'équation de Boltzmann-Pauli homogène s'écrit:

$$\text{(1.1)} \qquad \frac{\partial f}{\partial t} = \mathrm{div}_k(E\,f) + Q(f) \quad \text{in} \quad (0,+\infty)\times B, \qquad f(0,k) = f^{in}(k) \quad \text{on} \quad B.$$

Le champ électrique E satisfait $E \in W^{1,1}(B)$, $\mathrm{div}_k\, E = 0$ et l'operateur de Boltzmann-Pauli $Q(f)$, est donné par:

$$\text{(1.2)} \qquad Q(f)(k) = \int_B \left(\sigma\,(1-f)\,f' - \sigma'\,(1-f')\,f\right)\,dk',$$

où l'on a posé $f = f(k)$, $f' = f(k')$, $\sigma = \sigma(k,k')$ et $\sigma' = \sigma(k',k)$. Nous insistons sur le fait que nous ne faisons pas l'hypothèse d'équilibre en détail.
L'équation (1.1) possède deux propriétées importantes. La première est que le nombre total de particules, ou masse, $N(f) = \int_0^\infty f(k,t)dk$ est indépendant du temps. D'autre

part l'équation (1.1) définit une contraction stricte dans les sous ensembles de fonctions de $L^1(B)$ de masse donnée. Grâce à ces deux propriétées, à l'hypothèse (H1) de positivité ci-dessous et au théorème du point fixe de Tykhonov, nous démontrons que pour tout $\rho \in (0,1)$ il existe une unique solution stationnaire F_ρ dans l'ensemble X_ρ défini ci dessous (cf (1.5)).
Nous considérons ensuite le problème d'évolution. Nous montrons tout d'abord, toujours sous l'hypothèse (H1), que pour toute donnée initiale $f_{in} \in X_\rho$ il existe une unique solution $f \in C([0,\infty), L^1(B))$ qui conserve la masse. Nous démontrons ensuite, sous une hypothèse plus restrictive (H2), que $f(t)$ converge vers F_ρ lorsque t tends vers $+\infty$. Nous utilisons pour ceci la propriétée de contraction stricte à masse fixée et le principe d'invariance de Lasalle. Finalement, sous une hypothèse (H3) encore plus forte, nous démontrons la convergence avec taux exponentiel.

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1. Introduction.

We study the Cauchy problem for the spatially homogeneous Boltzmann-Pauli equation considered as a simplified model for the dynamics of a cloud of particles (electrons or ions) in a semiconductor device. We denote $f = f(t,k) \geq 0$ the density of particles with wave vector $k \in B$ at time $t \geq 0$, being B the first Brillouin zone (the normalized torus of $\mathbb{R}^N$, $N \geq 1$, so that $|B| = 1$). The Boltzmann-Pauli equation reads:

$$(1.1) \qquad \frac{\partial f}{\partial t} = \mathrm{div}_k(E\, f) + Q(f) \quad \text{in} \quad (0,+\infty) \times B, \qquad f(0,k) = f^{in}(k) \quad \text{on} \quad B,$$

where the electric field $E \in W^{1,1}(B)$ satisfies $\mathrm{div}_k E = 0$ and the Boltzmann-Pauli operator $Q(f)$ is:

$$(1.2) \qquad Q(f)(k) = \int_B \left(\sigma\,(1-f)\,f' - \sigma'\,(1-f')\,f\right)\,dk',$$

and where we have set $f = f(k)$, $f' = f(k')$, $\sigma = \sigma(k,k')$ and $\sigma' = \sigma(k',k)$.

We emphasize that we do not make the so-called detailed balance condition hypothesis which usually reads as follows: $\sigma\, M = \sigma'\, M'$ on B^2 where $M(k) = \exp(-\mathcal{E}(k))$ is the Maxwellian associated to the energy $\mathcal{E}$. We introduce instead the following assumptions on the cross section $0 \leq \sigma \in L^1(B^2)$, defining $\underline{\sigma} = \min(\sigma, \sigma')$,

$$(H1) \qquad \sigma > 0 \ \ a.e.,$$

$$(H2) \qquad \sigma_1 := \int_B \operatorname*{infess}_k \underline{\sigma}\, dk' > 0, \quad \text{and} \quad \sigma \in W^{1,1}(B^2),$$

$$(H3) \qquad \operatorname*{infess}_{k'} \sigma + \operatorname*{infess}_{k'} \sigma' \not\equiv 0 \text{ on } B \times B.$$

For any integrable density f the total number of particles, or mass, is given by

$$(1.3) \qquad N(f) = \int_0^\infty f(t,k)dk.$$

An important property of the equation (1.1) is that, at least formally, the total number of particles $N(f)$ is constant in time. We introduce then the sets:

$$X := \{f \in L^\infty(B),\ 0 \le f \le 1\} \quad \text{and} \quad X_\rho := \{f \in X,\ N(f) = \rho\}\,. \tag{1.4}$$

2. The main result.

The main result of this note is the following.

Theorem 1. *1. Assume (H1). For all initial data $f_{in} \in X_\rho$, $\rho \in (0,1)$, there exists a unique solution $f \in C([0,\infty); L^1)$ of (1.1), (1.2). Moreover it preserves the mass, i. e. $f(t) \in X_\rho$ for all $t \ge 0$.*

2. Assume (H1). For any $\rho \in (0,1)$ there exists a unique stationary solution $F_\rho \in X_\rho$. Moreover, if $r < \rho$ then, $F_r < F_\rho$ almost everywhere in B. In particular, $0 < F_\rho < 1$.

3. Assume (H1), (H2). For any initial data $f_{in} \in X_\rho$, the associated solution satisfies

$$H_\rho(f(t,.)) := \int_B |f(t,k) - F_\rho(k)|\, dk \quad \to \quad 0 \quad \text{when} \quad t \to \infty.$$

4. Assume (H1), (H3). For all initial data $f_{in} \in X_\rho$, the solution f converges exponentially fast towards F_ρ in $L^1(B)$ as $t \to +\infty$: $\exists\, \lambda_\rho > 0$

$$H_\rho(f(t,.)) \le H_\rho(f_{in})\, e^{-\lambda_\rho\, t} \qquad \forall\, t \ge 0.$$

Remark 1. The existence result of a stationary solution was first proved in [1] under the hypothesis of detailed balance and for a regular corss section σ. It was then adapted in [6] and [7] without the detailed balance and for a vanishing force E. The result presented here is an improvement of the above results under the general and natural condition on σ (condition (H1)). The proof uses the Tykhonov fixed point theorem, introduced in the field of kinetic equations in [5]. This method is developed in [4,9] to prove the existence of stationary states and self similar solutions for coagulation, fragmentation and inelastic collisions models.
On the other hand, we prove two convergence results for the solutions to the Cauchy problem towards the corresponding stationary solution. Our proof relies on an L^1 contraction property in the spirit of [8,2].
The proof of Theorem 1 uses the following auxiliary result.

Lemma 1. *Let us define*

$$D_j(f,g) := \int_B (Q(f) - Q(g))\ \xi_j'(f-g)\, dk, \quad j = 0, 1$$

where $\xi_0(s) = s_+$ (and then $\xi_0'(s) = \mathbf{1}_{s>0}$) and $\xi_1(s) = |s|$ (and then $\xi_1'(s) = \mathbf{1}_{s>0} - \mathbf{1}_{s<0}$).
(i) Suppose (H1). For all $f \in X_\rho$, $g \in X_r$, $r, \rho \in (0,1)$, there holds

$$D_j(f,g) \ge 0 \qquad \text{and} \qquad (D_j(f,g) = 0,\ \rho \ge r \quad \Longrightarrow f \ge g \ \text{a.e.}).$$

(ii) Assume (H3). For all $f, g \in X_\rho$ with $0 < g < 1$ a.e. there exists $\lambda_g > 0$ such that

$$D_1(f,g) \geq \lambda_g \, \|f-g\|_{L^1}.$$

Proof of Lemma 1. Let us start with the point (i). We notice that

$$\begin{aligned} D_j(f,g) &= \iint \sigma\left[(1-f)\,(f'-g') - (f-g)\,g'\right](\xi_j'(f'-g') - \xi_j'(f-g))\,dkdk' \\ &= \iint \sigma\,(1-f)\,\{\xi_j(f'-g') - (f'-g')\,\xi_j'(f-g)\,\}\,dkdk' \\ &\quad + \iint \sigma\,g'\,\{\xi_j(f-g) - (f-g)\,\xi_j'(f'-g')\,\}\,dkdk'. \end{aligned}$$

This implies $D_j(f,g) \geq 0$, since the functions under the integral signs in the right hand side of the second equality are non negative. Since $D_j(f,g) = D_j(g,f)$, we may interchange the role played by f and g, and we get

$$\begin{aligned} D_j(f,g) &= \frac{1}{2}\iint \left[\sigma\,(1-f+1-g) + \sigma'\,(f+g)\right] \\ &\qquad\qquad \{\xi_j(f'-g') - (f'-g')\,\xi_j'(f-g)\,\}\,dkdk' \\ &\geq \iint \underline{\sigma}\,\{\xi_j(f'-g') - (f'-g')\,\xi_j'(f-g)\,\}\,dkdk', \end{aligned} \tag{1.5}$$

from where we deduce that $\xi_j(f'-g') - (f'-g')\,\xi_j'(f-g) = 0$ a.e. on $B \times B$, so that $\mathrm{sign}(f-g)$ is a constant. The mass condition allows to conclude that $f \geq g$.

In order to prove (ii) under Hypothesis (H3), we proceed like in [2] where the detailed balance case is treated. Indeed, since f and g have the same mass, we deduce from (1.5):

$$\begin{aligned} D_1(f,g) &\geq \frac{1}{2}\iint \left[\operatorname*{infess}_{k'} \sigma\,(1-g) + \operatorname*{infess}_{k'} \sigma'\,g\right]\left[|f'-g'| - (f'-g')\,\mathrm{sign}(f-g)\right]dkdk' \\ &\geq \frac{1}{2}\int \left[\operatorname*{infess}_{k'} \sigma + \operatorname*{infess}_{k'} \sigma'\right]\,\min(g,1-g)\,dk \int |f'-g'|\,dk'. \end{aligned}$$

Proof of Theorem 1, 1^{st} point. Uniqueness is an immediate consequence of Lemma 1 since, if f and g are two solutions of solutions of (1.1), (1.2) with same mass:

$$\frac{d}{dt}\int_B |f-g|\,dk = -D_1(f,g) \leq 0. \tag{1.6}$$

To prove the existence of solutions, we introduce a sequence of regularised cross sections σ_n, satisfying for instance (H1)–(H3), and such that $\sigma_n \to \sigma$ a.e. in B^2. Then, for all initial data $f_{in} \in X_\rho$, there exists a solution $f_n \in C([0,\infty); L^1) \cap L^\infty(0,\infty; X_\rho)$ to the corresponding Boltzmann-Pauli equation. This is done in [10,11] where, the nonhomogeneous case is treated and where the force field E is coupled to the distribution function via the Poisson equation. To pass to the limit in the regularized problem, we make use of the following stability result, whose proof is classical.

Lemma 2 (Stability Principle). *Let (f_n) be a sequence of solutions to the Boltzmann-Pauli equation with cross sections (σ_n) uniformly bounded in $L^\infty(0,\infty;X_\rho)$ and suppose that $\sigma_n \to \sigma$ a.e. in B^2. Then, there exists $f \in C([0,\infty);L^1) \cap L^\infty(0,\infty;X_\rho)$ and a subsequence $(f_{n'})$ such that $f_{n'} \rightharpoonup f$ in $\sigma(L^\infty, L^1)$ and f is a solution to the Boltzmann-Pauli equation with the cross section σ.*

Proof of Theorem 1, 2^{nd} point. By the stability principle above and the uniqueness of solutions, for any n, the map $S_{2^{-n}} : X_\rho \to X_\rho$, $f_{in} \mapsto f(2^{-n})$ is continuous and compact for the weak topology $\sigma(L^\infty, L^1)$. Then, by the Tykhonov's fixed point Theorem, there exists at least one 2^{-n} periodic solution g_n of the Boltzmann-Pauli equation: $g_n \in X_\rho$ and $S_{2^{-n}}\, g_n = g_n$. Since the sequence (g_n) is bounded, by the stability principle, $g_n \rightharpoonup g$ where $g \in X_\rho$ is a solution satisfying $S_t\, g = g$ for all dyadic time $t \geq 0$. We deduce $S_t g = g$ for all $t > 0$ and $F_\rho := g \in X_\rho$ is a stationary solution. □

The property $0 \leq F_\rho \leq 1$ is satisfied by construction. By Lemma 1 the map $\rho \to F_\rho$ is non decreasing. Let consider $\rho > r$ and define $g := F_\rho - F_r \geq 0$. We easily check that g satisfies

$$E \cdot \nabla_k g - \lambda\, g = -S$$

with $\lambda := \displaystyle\int [\sigma\, F'_\rho + \sigma'\,(1 - F'_r)] \in L^1$, $S := \displaystyle\int [\sigma\,(1 - F_r) + \sigma'\, F_\rho]\, g'$. We deduce that

$$\forall \varepsilon > 0 \qquad \int \lambda\,(\varepsilon - g)_+ \, dk = \int (\lambda\,\varepsilon - S)\, \mathbf{1}_{\varepsilon - g \geq 0}.$$

Passing to the limit $\varepsilon \to 0$, we obtain $\displaystyle\int S\,\mathbf{1}_A = 0$, where $A := \{g = 0\}$. Since $S > 0$ a.e. we deduce that $\mathrm{mes}(A) = 0$ and then $\rho \mapsto F_\rho$ is strictly increasing. □

The following result is needed to prove the asymptotic convergence of the solutions to the stationary states as $t \to +\infty$.

Lemma 3. *Assume (H2). If $f_{in} \in X_\rho$ is such that $\partial_k f_{in} \in L^1$, then,*

$$\sup_{t>0} \int_B |\partial_k f(t,k)|\, dk < \infty. \tag{1.7}$$

Proof of Lemma 3. Since

$$\partial_k Q(f) = \int_B \left((\partial_1\sigma)\,(1-f)\,f' - (\partial_2\sigma)'\,(1-f')\,f\right)\, dk' - \int_B \left(\sigma\, f' + \sigma'\,(1-f')\right)\, dk'\, \partial_k f,$$

and $\sigma\, f' + \sigma'\,(1 - f') \geq \underline{\sigma}\, f' + \underline{\sigma}\,(1-f') = \underline{\sigma}$, we obtain thanks to (H2)

$$\frac{d}{dt}\|\partial_k f\|_{L^1} = \int_B \partial_k Q(f)\, \mathrm{sign}(\partial_k f)\, dk \leq \|\partial_1\sigma\|_{L^1} + \|\partial_2\sigma\|_{L^1} - \sigma_1\, \|\partial_k f\|_{L^1}.$$

The estimate (1.7) follows from this differential inequality.

Proof of Theorem 1, 3^{rd} point. Consider $f_{in} \in X_\rho$ and suppose also for the momment that $\partial_k f_{in} \in L^1$. Let us show that $H_\rho(f(t))$ is a strict Lyapunov functional on $L^1(B)$. Since $\frac{d}{dt} H_\rho(f) = -D_1(f, F_\rho)$, we deduce, by the uniqueness result, that the map $t \to H_\rho(f(t))$ is strictly decreasing as long as $f \neq F_\rho$. Moreover, by Lemma 3, $(f(t))_{t>0}$ belongs to a compact subset of $L^1(B)$. Then, the Lasalle's invariance principle implies that $H_\rho(f(t)) \to 0$ when $t \to +\infty$. The result for a general initial data $f_{in} \in X_\rho$ follows by density using Lemma 1.

Proof of Theorem 1, 4^{th} point. It is an immediate consequence of Lemma 1 and (1.6).

Remark 2. For the uniqueness in points (i) and (ii) of Theorem 1, condition (H1) may be weakened. For instance, these uniqueness results still hold assuming $\sigma + \sigma' > 0$ a.e. on $B \times B$ or assuming $\underline{\sigma} > 0$ a.e. on $A \times B$ for some $A \subset B$ such that meas$(A) > 0$. As an example, we may choose $\sigma = \mathbf{1}_{k,k';k_N \ge k'_N}$ which satisfies the first condition but not the second one nor (H1) and $\sigma = \mathbf{1}_{A\times B} + \mathbf{1}_{B\times A}$ (for some $A \subset B$ such that meas$(A) > 0$) which satisfies the second condition but not the first one nor (H1). Let us also emphasize that, at least when $E \equiv 0$, the strict monotonicity of the map $\rho \mapsto F_\rho$ also holds for the cross section $\sigma := \mathbf{1}_{k,k';k_N+\varepsilon \ge k'_N}$, $\varepsilon > 0$, which does not satisfy the assumption (H1).

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