Cooling process for inelastic Boltzmann equations for hard spheres, Part I: The Cauchy problem

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Abstract

We develop the Cauchy theory of the spatially homogeneous inelastic Boltzmann equation for hard spheres, for a general form of collision rate which includes in particular variable restitution coefficients depending on the kinetic energy and the relative velocity. It covers physically realistic models for granular materials. We prove (local in time) non-concentration estimates in Orlicz spaces, from which we deduce weak stability and existence theorem. Strong stability together with uniqueness is proved under additional smoothness assumption on the initial datum, for a restricted class of collision rates. Concerning the long-time behaviour, we give conditions for the cooling process to occur or not in finite time.

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1 Introduction and main results

In this paper we address the Cauchy problem for the spatially homogeneous Boltzmann equation modelling the dynamic of a homogeneous system of inelastic hard spheres which interact only through binary collisions. More precisely, describing the gas by the probability density $f(t, v) \geq 0$ of particles with velocity $v \in \mathbb{R}^N$ $(N \geq 2)$ at time $t \geq 0$, we study the existence and the qualitative behaviour of solutions to the Boltzmann equation for inelastic collision

$$\frac{\partial f}{\partial t} = Q(f,f) \quad \text{in} \quad (0,+\infty) \times \mathbb{R}^N, \tag{1.1}$$

$$f(0,\cdot) = f_{\rm in} \quad \text{in} \quad \mathbb{R}^N. \tag{1.2}$$

The use of Boltzmann inelastic hard spheres-like models to describe dilute, rapid flows of granular media started with the seminal physics paper [17], and a huge physics litterature has developped in the last twenty years. The study of granular systems in such regime is motivated by their unexpected physical behavior (with the phenomena of *collapse* –or "*cooling effect*"– at the kinetic level and *clustering* at the hydrodynamical level), their use to derive hydrodynamical equations for granular fluids, and their industrial applications.

From the mathematical viewpoint, works on the Cauchy problem for these models have been first restricted to the so-called *inelastic Maxwell model*, which is an approximation where the collision rate is replaced by a mean value independent on the relative velocity (see [6] for instance). This simplified model is important because of its analytic simplifications allowing to use powerful Fourier transform tools. Nevertheless, although it is possible to modify the collision operator by a multiplication by a function of the kinetic energy in order to restore its dimensional homogeneity (see [6] for this *pseudo-Maxwell molecules model*), fine properties of the distribution (such as overpopulated tails or self-similar solutions) are broken or modified by the approximation. Another simplification which has lead to interesting results is the restriction to one-dimensional models (in space and velocity) (see [2], [27] and [3]), where, on the contrary to the elastic case, the collision operator has a non-trivial outcome. Also the recent papers [13, 7] have studied the case of inelastic hard spheres in various regimes (for instance in a thermal bath, i.e. when a heat source term is added to the equation) in any dimension. Another common major physical simplification is to deal with *constant restitution coefficients*. This choice, while reasonnable from the viewpoint of the mathematical complexity of the model, appears inadequate to describe the whole variety of behaviors of these materials (see the discussion and models in [6] and [27] and the references therein). Lately the work [6] has considered some cases of restitution coefficients possibly depending on the kinetic energy of the solution, and the works [27], [3] have considered some cases of restitution coefficients depending on the relative velocity.

In this work, we shall construct solutions to the freely cooling Boltzmann equation for hard spheres in any dimension $N \geq 2$ and for a general framework of measure-valued inelasticity coefficients which covers in particular *variable restitution coefficients* possibly depending on the relative velocity and the kinetic energy of the solution. Our framework enables to consider interesting physical features, such as elasticity increasing when the relative velocity or the temperature decrease (*"normal" granular media*) or the opposite phenomenon (*"anomalous" granular media*). Let us emphasize that these solutions are new even in the case of a constant restitution coefficient. We also discuss various conditions on the collisions rate for the collapse to occur or not in finite time. A second part of this work [22] will be concerned with the existence of self-similar solutions and the tail behavior of the distribution.

Before we explain our results and methods in detail, let us introduce the problem.

1.1 A general framework for the collision operator

The bilinear collision operator $Q(f, f)$ models the interaction of particles by means of inelastic binary collisions (preserving mass and total momentum but dissipating kinetic energy). We denote by B the rate of occurance of collision of two particles with pre-collisional velocities v and v_* which gives rise to post-collisional velocities v' and v'_*. The collision may be schematically written

$$(1.3) \qquad \{v\} + \{v_*\} \xrightarrow{B} \{v'\} + \{v'_*\} \quad \text{with} \quad \begin{cases} v' + v'_* = v + v_* \\ \\ |v'|^2 + |v'_*|^2 < |v|^2 + |v_*|^2. \end{cases}$$

More precisely, we define the collision operator by its action on test functions (which is related to the *observables* of the probability density). Taking $\varphi = \varphi(v)$ to be some well-suited regular function, we introduce the following weak formulation of the collision operator

$$(1.4) \quad \langle Q(f,f), \varphi \rangle := \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f_* \, f \int_D (\varphi'_* + \varphi' - \varphi - \varphi_*) \, B(\mathcal{E}, v - v_*; dz) \, dv \, dv_*$$

where $D := \{u \in \mathbb{R}^N; \, |u| \leq 1\}$. Here and below we use the shorthand notations $\psi = \psi(v)$, $\psi_* = \psi(v_*)$, $\psi' = \psi(v')$ and $\psi'_* = \psi(v'_*)$ for any function ψ on $\mathbb{R}^N$. For any $z \in D$ and $v, v_* \in \mathbb{R}^N$ we define

$$(1.5) \qquad \begin{cases} v' = (v + v_*)/2 + z \, |v_* - v|/2 \\ \\ v'_* = (v + v_*)/2 - z \, |v_* - v|/2, \end{cases}$$

which is nothing but a parametrization, for any fixed *pre-collisional particles* $\{v, v_*\}$, of all possible resulting *post-collisional particles* $\{v', v'_*\}$ in (1.3). Finally, $\mathcal{E}$ is the *kinetic energy* of the distribution f, defined by

$$\mathcal{E} := \int_{\mathbb{R}^N} f \, |v|^2 \, dv.$$

The *collision rate* B is the product of the norm of the *relative velocity* by the *collisional cross section*, $B = |v - v_*| \, b$, reflecting the fact that we are dealing with *hard spheres* which undergo contact interactions. The *collisional cross section* b is a non-negative measure on D, depending on the kinetic energy $\mathcal{E}$, and on the pre-collisional velocities v, v_*. It depends on the velocity only through $v - v_*$ by Gallilean invariance. The non-negative real $|z|$ is the *restitution coefficient* which measures the loss of energy in the collision, since

$$(1.6) \qquad |v'|^2 + |v'_*|^2 - |v|^2 - |v_*|^2 = -\frac{1}{2}(1 - |z|^2) \, |v_* - v|^2 \leq 0.$$

In the above formula, $|z| = 1$ corresponds to an elastic collision while $z = 0$ corresponds to a completely inelastic collision (or *sticky collision*). In the sequel we shall denote $u = v - v_*$ the *relative velocity*, and for a vector $x \in \mathbb{R}^N \backslash \{0\}$, we shall denote $\hat{x} = x/|x|$.

A first simple consequence of the definition of the operator (1.4) and of the parametrization (1.5) is that mass and momentum are conserved

$$\frac{d}{dt} \int_{\mathbb{R}^N} f \begin{pmatrix} 1 \\ v \end{pmatrix} dv = 0,$$

a fact that we easily derive (at least formally), multiplying the equation (1.1) by $\varphi = 1$ or $\varphi = v$ and integrating in the velocity variable (using (1.4)). In the same way, multiplying equation (1.1) by $\varphi = |v|^2$, integrating and using (1.6), we obtain that the kinetic energy is dissipated

$$\frac{d}{dt}\mathcal{E}(t) = -D(f) \leq 0, \tag{1.7}$$

where we define the energy dissipation functional D and the energy dissipation rate β, which measures the (averaged) inelasticity of collisions, by

$$\begin{aligned} D(f) &:= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f\, f_*\, |u|^3\, \beta(\mathcal{E},u)\; dv\, dv_* \\ \beta(\mathcal{E},u) &:= \frac{1}{4}\int_D (1-|z|^2)\, b(\mathcal{E},u;dz) \geq 0. \end{aligned}$$

Finally, we introduce the *cooling time*, associated to the process of cooling (possibly in finite time) of granular gases:

$$T_c := \inf\{T \geq 0,\ \mathcal{E}(t) = 0\ \forall\, t > T\} = \sup\{S \geq 0,\ \mathcal{E}(t) > 0\ \forall\, t < S\}. \tag{1.8}$$

This cooling effect (or collapse) is one of the main motivations for the physical and mathematical study of granular media.

The Boltzmann equation (1.1) is complemented with an initial condition (1.2) where the initial datum is supposed to satisfy the moment conditions

$$0 \leq f_{\rm in} \in L^1_q(\mathbb{R}^N), \qquad \int_{\mathbb{R}^N} f_{\rm in}\, dv = 1, \qquad \int_{\mathbb{R}^N} f_{\rm in}\, v\, dv = 0 \tag{1.9}$$

for some $q \geq 2$. Notice that we can assume without loss of generality the two last moment conditions in (1.9), since we may always reduce to that case by a scalling and translation argument. Here we denote, for any integer $q \in \mathbb{N}$, the Banach space

$$L^1_q = \left\{ f : \mathbb{R}^N \longrightarrow \mathbb{R} \text{ measurable};\ \ \|f\|_{L^1_q} := \int_{\mathbb{R}^N} |f(v)|\,(1+|v|^q)\, dv < \infty \right\}.$$

We also define the weighted Sobolev spaces $W^{k,1}_q$ ($q \in \mathbb{R}$ and $k \in \mathbb{N}$) by the norm

$$\|f\|_{W^{k,1}_q} = \sum_{|s|\leq k} \|\partial^s f\,(1+|v|^q)\|_{L^1}.$$

We introduce the space of normalized probability measures on $\mathbb{R}^N$, denoted by $M^1(\mathbb{R}^N)$, and the space $BV_q(\mathbb{R}^N)$ ($q \in \mathbb{R}$) of Bounded Variation functions, defined as the set of the weak limits in $\mathcal{D}'(\mathbb{R}^N)$ of sequences of smooth functions which are bounded in $W^{1,1}_q(\mathbb{R}^N)$. Throughout the paper we denote by "C" various constants which do not depend on the collision rate B.

1.2 Mathematical assumptions on the collision rate

Let us state the basic assumptions on the collision rate B:

- B takes the form

  (1.10) $$B = B(\mathcal{E}, u; dz) = |u|\, b(\mathcal{E}, u; dz),$$

  where b is a finite measure on D for any $\mathcal{E}, u$. This measure b satisfies the following properties:

- It satisfies the symmetry property

  (1.11) $$b(\mathcal{E}, u; dz) = b(\mathcal{E}, -u, -dz).$$

- For any $\varphi \in C_c(\mathbb{R}^N)$ the function

  (1.12) $$(v, v_*, \mathcal{E}) \mapsto \int_D \varphi(v')\, b(\mathcal{E}, u; dz)$$

  is continuous.

- There exists a continuous function $\alpha : (0, \infty) \to (0, \infty)$, which measures the intensity of interactions, such that

  (1.13) $$\forall\, u \in \mathbb{R}^N,\ \mathcal{E} > 0 \quad \alpha(\mathcal{E}) = \int_D b(\mathcal{E}, u; dz).$$

For the energy coupled models we will need the following additional assumption:

- The measure b satisfies the following angular spreading property: for any $\mathcal{E} > 0$, there is a function $j_{\mathcal{E}}(\varepsilon) \geq 0$, going to 0 as $\varepsilon \to 0$, such that

  (1.14) $$\forall\, \varepsilon > 0,\ u \in \mathbb{R}^N \qquad \int_{\left\{|\hat{u}\cdot z| \in [-1,1]\setminus[-1+\varepsilon;1-\varepsilon]\right\}} b(\mathcal{E}, u; dz) \leq \alpha(\mathcal{E})\, j_{\mathcal{E}}(\varepsilon).$$

  Moreover we assume that this convergence is uniform according to $\mathcal{E}$ when it is restricted to a compact set of $(0, +\infty)$.

For the uniqueness of the energy coupled models, we shall need the following assumption

H1. The cross-section b reduces to a measure on the sphere

(1.15) $$\mathcal{C}_{u,e} = \frac{1-e}{2}\, \hat{u} + \frac{1+e}{2}\, \mathbb{S}^{N-1},$$

where $e : (0, \infty) \to [0, 1]$, $\mathcal{E} \mapsto e(\mathcal{E})$ depends only on the kinetic energy, and $\alpha = \alpha(\mathcal{E})$ and $e = e(\mathcal{E})$ are locally Lipschitz on $(0, +\infty)$. Morerover, b is assumed to be absolutely continuous according to the Hausdorff measure on $\mathcal{C}_{u,e}$, and thus writes

$$(1.16)\qquad b(\mathcal{E},u;dz) = \delta_{\{z=(1-e)\hat{u}/2+(1+e)\sigma/2\}}\,\tilde{b}(\mathcal{E},|u|,\hat{u}\cdot\sigma)\,d\sigma$$

where $d\sigma$ is the uniform measure on the unit sphere, and $\tilde{b}$ is a non-negative measurable function.

In the study of the cooling process, we always assume:

H2. The energy dissipation rate $\beta(\mathcal{E},u)$ in (1.8) is continuous on $(0,+\infty)\times\mathbb{R}^N$ and satisfies

$$(1.17)\qquad \beta(\mathcal{E},u) > 0 \quad \forall\, u\in\mathbb{R}^N,\ \mathcal{E}>0.$$

We will also need one of the two following additional assumptions:

H3. For any $\mathcal{E}_0,\mathcal{E}_\infty\in(0,\infty)$ (with $\mathcal{E}_0\geq\mathcal{E}_\infty$) there exists ψ such that

$$(1.18)\qquad \beta(\mathcal{E},u)\geq\psi(|u|)\quad \forall\,\mathcal{E}\in(\mathcal{E}_\infty,\mathcal{E}_0),\ \ \forall\, u\in\mathbb{R}^N,$$

with $\psi\in C(\mathbb{R}_+,\mathbb{R}_+)$ and such that for any $R>0$ there exists $\psi_R>0$,

$$(1.19)\qquad \psi(|u|)\geq\psi_R\,|u|^{-1}\quad \forall\, u\in\mathbb{R}^N,\ \ |u|>R/2.$$

This assumption is quite natural. In particular, it holds for a *"normal" granular media.*

H4. The cross-section b reduces to a measure on the sphere $\mathcal{C}_{u,e}$ and it is absolutely continuous according to the Hausdorff measure, where $e:(0,\infty)\times(0,\infty)\to[0,1]$, $(\mathcal{E},|u|)\mapsto e(\mathcal{E},|u|)$ is a continuous function. In particular, (1.16) holds. Moreover we assume that for any given $\mathcal{E}$ and $|u|$, the function $z\mapsto\tilde{b}(\mathcal{E},|u|,z)$ is non-negative, nondecreasing and convex on $(-1,1)$.

The fact that b is a finite measure on D allows to define the splitting $Q=Q^+-Q^-$ where Q^+ and Q^- are defined in dual form by

$$(1.20)\qquad \langle Q^+(g,f),\varphi\rangle := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g_*\,f\int_D\varphi'\,|u|\,b(\mathcal{E},u;dz)\,dv\,dv_*$$

and

$$(1.21)\qquad \langle Q^-(g,f),\varphi\rangle := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g_*\,f\int_D\varphi\,|u|\,b(\mathcal{E},u;dz)\,dv\,dv_*.$$

A straightforward computation shows that it is possible to give a very simple strong form of Q^- as follows

$$(1.22)\qquad Q^-(g,f)=L(g)\,f$$

where L is the convolution operator

$$L(g)(v) := \alpha(\mathcal{E}) \int_{\mathbb{R}^N} g(v_*)\,|v - v_*|\,dv_*. \tag{1.23}$$

Under assumption **H4**, the expression of Q^+ reduces to

$$\langle Q^+(g,f),\varphi\rangle := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} g_*\,f\,|u| \int_{\mathbb{S}^{N-1}} \varphi'\,\tilde{b}(\mathcal{E},|u|,\hat{u}\cdot\sigma)\,d\sigma\,dv\,dv_*. \tag{1.24}$$

We refer to [6] for physical motivations for the case when $e = e(\mathcal{E})$ and to [27] for the case when $e = e(|u|)$. Under assumption **H4** and when one assumes that $\tilde{b}$ only depends on $\hat{u}\cdot\sigma$, the energy dissipation rate just writes

$$\beta(\mathcal{E},u) = C_N\,(1-e^2), \tag{1.25}$$

where C_N is a constant depending on the dimension.

We note that the classical Boltzmann collision operator for inelastic hard spheres with a constant normal restitution coefficient $e \in [0,1]$, as studied for instance in [6] and [13], is included as a particular case of our model, and satisfies all the assumptions above. But the formalism (1.4)–(1.14) is much more general than this case. In particular, we may also consider:

1. Uniformly inelastic collision processes such that

$$\exists\, z_0 \in (0,1) \quad \text{s.t.} \quad \operatorname{supp} B(\mathcal{E},u,.) \subset D(0,z_0) \quad \forall\, u \in \mathbb{R}^N,\ \ \forall\, \mathcal{E} > 0, \tag{1.26}$$

which includes the *sticky particles model* when $z_0 = 0$.

2. The physically important case of a normal restitution coefficient e depending on the relative velocity and the kinetic energy with a cross-section $\tilde{b}$ depending on $\mathcal{E}$, u and $\hat{u}\cdot\sigma$. In particular it covers the kind of models studied in [6] (where e depends on $\mathcal{E}$, and $\tilde{b}$ is independent on $\mathcal{E}$ and u).

3. This formalism also covers multidimensional versions of the kind of models proposed in [27], which corresponds to the case where b is the product of a measure depending on $|u|$, $|z|$ and a measure of $\hat{u}\cdot z$ absolutely continuous according to the Hausdorff measure. One easily checks that our assumptions are quite natural for this kind of models as well.

1.3 Statement of the main results

Let us now define the notion of solutions we deal with in this paper.

Definition 1.1 *Consider an initial datum $f_{\rm in}$ satisfying (1.9) with $q = 2$. A non-negative function f on $[0,T]\times\mathbb{R}^N$ is said to be a solution to the Boltzmann equation (1.1)-(1.2) if*

$$f \in C([0,T]; L^1_2(\mathbb{R}^N)), \tag{1.27}$$

and if (1.1)-(1.2) holds in the sense of distributions, that is,

$$\int_0^T \left\{ \int_{\mathbb{R}^N} f \, \frac{\partial \phi}{\partial t} dv + \langle Q(f,f), \phi \rangle \right\} dt = \int_{\mathbb{R}^N} f_{in} \, \phi(0,.) \, dv \tag{1.28}$$

for any $\phi \in C^1_c([0,T) \times \mathbb{R}^N)$.

It is worth mentioning that (1.27) ensures that the collision term $Q(f,f)$ is well defined as a function of $L^1(\mathbb{R}^N)$. Indeed, on the one hand, we deduce from $f \in C([0,T]; L^1_2(\mathbb{R}^N))$ that $\mathcal{E}(t) \in K_1$ on $[0,T]$ and thus $\alpha(\mathcal{E}(t)) \in K_2$ on $[0,T]$ for some compact sets $K_i \subset (0,\infty)$. On the other hand, from the dual form (1.20) it is immediate that $Q^\pm$ is bounded from $L^1_1 \times L^1_1$ into L^1, with bound $\alpha(\mathcal{E})$ (see also [13, 22] for some *strong forms* of the $Q^+(f,f)$ term). It turns out that a solution f, defined as above, is also a solution of (1.1)-(1.2) in the mild sense:

$$f(t,.) = f_{\rm in} + \int_0^t Q(f(s,.)) \, ds \quad \text{a.e. in} \quad \mathbb{R}^N.$$

Another consequence is that if $f \in L^\infty([0,T), L^1_q)$ then f satisfies the *chain rule*

$$\frac{d}{dt} \int_{\mathbb{R}^N} \beta(f) \, \phi \, dv = \langle Q(f,f), \beta'(f) \, \phi \, \rangle \quad \text{in} \quad \mathcal{D}'([0,T)), \tag{1.29}$$

for any $\beta \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, $\phi \in L^\infty_{1-q}(\mathbb{R}^N)$, see [14, 10, 19].

Let us state the main results of this paper. First, we give a Cauchy Theorem valid when the collision rate B is independent on the kinetic energy.

Theorem 1.2 *Assume that B satisfies the assumptions (1.10)-(1.13) with $b = b(u; dz)$: the cross-section does not depend on the kinetic energy. Take an initial datum $f_{\rm in}$ satisfying (1.9) with $q = 3$. Then*

(i) For all $T > 0$, there exists a unique solution $f \in C([0,T]; L^1_2) \cap L^\infty(0,T; L^1_3)$ to the Boltzmann equation (1.1)-(1.2). This solution conserves mass and momentum,

$$\int_{\mathbb{R}^N} f(t,v) \, dv = 1, \qquad \int_{\mathbb{R}^N} f(t,v) \, v \, dv = 0 \qquad \forall \, t \geq 0, \tag{1.30}$$

and has a decreasing kinetic energy

$$\mathcal{E}(t_2) \leq \mathcal{E}(t_1) \leq \mathcal{E}_{in} = \mathcal{E}(0) \qquad \forall \, t_2 \geq t_1 \geq 0. \tag{1.31}$$

(ii) Its time of life (as introduced in (1.8)) is $T_c = \infty$, in particular, $\mathcal{E}(t) > 0$ for any $t > 0$. Morever, assuming **H2-H3** *or* **H2-H4** *(with e and $\tilde{b}$ independent on the kinetic energy), there holds*

$$\mathcal{E}(t) \to 0 \;\; \text{and} \;\; f(t,.) \;\rightharpoonup\; \delta_{v=0} \;\; \text{in} \;\; M^1(\mathbb{R}^N)\text{-weak}\,* \;\; \text{when} \;\; t \to T_c. \tag{1.32}$$

In other words, the cooling process does not occur in finite time, but asymptotically in large time.

Remarks 1.3 *Let us discuss the assumptions and conclusions of this theorem.*

1. Under assumption **H4** *and when the collision rate is independent on the kinetic energy, one can prove in fact that there exists a unique solution* $f \in C([0,\infty); L^1)$ *satisfying (1.30) and (1.31) for any initial condition* $f_{\rm in}$ *satisfying (1.9) with* $q = 2$. *The proof is quite more technical and we refer to [23] where the result is presented for the true elastic collision Boltzmann equation; nevertheless the proof may be readily adapted to the inelastic collisional framework.*

2. The existence and uniqueness part of Theorem 1.2 (point (i)) extends to a cross-section $B = B(u; dz) \ge 0$ *which satisfies the sole assumptions*

$$\begin{cases} B(-u; -dz) = B(u; dz), \\ \displaystyle\int_D B\, dz \le C_0\,(1 + |v| + |v_*|) \\ \displaystyle (v, v_*) \mapsto \int_D \varphi(v')\, B(u; dz) \in C(\mathbb{R}^N \times \mathbb{R}^N) \qquad \forall\, \varphi \in C_c(\mathbb{R}^N) \end{cases}$$

for some constant $C_0 \in \mathbb{R}_+$. *This corresponds to the so-called cut-off hard potentials (or variable hard spheres) in the context of inelastic gases.*

3. For a uniformly dissipative collision model, i.e. such that

$$\beta(u) \ge \beta_0 \in (0, \infty),$$

a fact which holds under assumption (1.26) or under assumption **H4** *with a restitution coefficient* e *satisfying* $e(|u|) \le e_0 \in [0,1)$ *for any* $u \in \mathbb{R}^N$, *we may prove the additionnal a priori bound*

$$\int_0^\infty \|f(t,.)\|_{L^1_3}\, dt \le C(\|f_{in}\|_{L^1_2}, \beta_0).$$

As a consequence, one can easily adapt the proof of existence and uniqueness in Theorem 1.2 and then one can easily establish that the existence part of Theorem 1.2 holds for any initial datum $f_{\rm in}$ *satisfying (1.9) with* $q = 2$.

4. The existence and uniqueness part of Theorem 1.2 (point (i)) immediately extends to a time dependent collision rate $B = |u|\,\gamma(t)\, b(t, u; dz)$ *where* $b(t, u; \cdot)$ *is a probability measure for any* $u \in \mathbb{R}^N$, $t \in [0, T]$ *such that* $b(t, u; dz) = b(t, -u; -dz)$, *and* $\gamma(t)$ *is a positive function in* $L^\infty(0, T)$.

Now, let us turn to the case where the collision kernel depends on the kinetic energy of the solution.

Theorem 1.4 *Assume now that* B *satisfies the assumptions (1.10)-(1.14) and that the cross-section* $b = b(\mathcal{E}, u; dz)$ *depends also on the kinetic energy* $\mathcal{E}$. *Take an initial datum* $f_{\rm in}$ *satisfying (1.9) with* $q = 3$.

(i) *There exists at least one maximal solution* $f \in C([0,T_c); L^1_2) \cap L^\infty(0,T_c; L^1_3)$ *for some* $T_c \in (0,+\infty]$ *which satisfies the conservation laws (1.30) and the decay of the kinetic energy (1.31).*

(ii) *If the collision rate satisfies the assumption* **H1***, and the initial datum satisfies the additional assumption* $f_{\rm in} \in BV_4 \cap L^1_5$*, then this solution is unique in the class of functions* $C([0,T], L^1_2) \cap L^\infty(0,T; L^1_3)$ *for any* $T \in (0,T_c)$*.*

(iii) *The asymptotic convergence (1.32) holds under the additional assumptions* **H2-H3** *or* **H2-H4***.*

(iv) *If* α *is bounded near* $\mathcal{E} = 0$ *and* $j_{\mathcal{E}}$ *converges to* 0 *as* $\varepsilon \to 0$ *uniformly near* $\mathcal{E} = 0$*, or if* B *satifies* **H4***,* β *is bounded by an increasing function* β_0 *which only depends on the energy, and* $f_{in}\, e^{a_\eta |v|^\eta} \in L^1$ *with* $\eta \in (1,2]$*,* $a_\eta > 0$*, then* $T_c = +\infty$*.*

(v) *If* $\beta(\mathcal{E},u) \geq \beta_0\, \mathcal{E}^\delta$ *with* $\beta_0 > 0$ *and* $\delta < -1/2$*, then* $T_c < \infty$*.*

Remark 1.5 *Let us discuss the assumptions and conclusions of this theorem.*

1. In point (ii), the assumption we make in order to get the uniqueness part of the theorem could most probably be relaxed to a smoothness assumption on b *of the form* b *depends only on* $\mathcal{E}$ *and* z *and* $\mathcal{E} \to b(\mathcal{E}; dz)$ *is locally Lipschitz from* $(0,+\infty)$ *to* $W^{-1,1}(D)$*.*

2. Under the assumptions of point (ii) on the initial datum, by using a bootstrap a posteriori *argument as introduced in [23], one can indeed prove that there exists a unique solution* $f \in C([0,\infty); L^1)$ *satisfying (1.30) and (1.31) for any initial condition* $f_{\rm in}$ *satisfying (1.9) with* $q > 4$ *and* $f_{in} \in BV_4$*.*

1.4 Plan of the paper

We gather in Section 2 some new integrability estimates on the collision operator which can be of independent interest. Concerning the gain term we prove convolution-like estimates in Orlicz spaces. These estimates generalize similar estimates in Lebesgue spaces in the elastic and the inelastic case. Concerning the loss term we give simple bounds from below obtained by convexity. We give then estimates on the *global* operator in Orlicz space, which show essentially that even if the bilinear collision operator is not bounded, its evolution semi-group is bounded in any Orlicz space (with bound depending on time). The proof is based on Young's inequality and only requires elementary tools. In Section 3 we start looking at *solutions* of the Boltzmann equation and we prove Theorem 1.2, on the basis of moments estimates in L^1. In Section 4, we extend the existence result to collision rates depending on the kinetic energy of the solution by proving a weak stability result on the basis of (local in time) non-concentration estimates obtained by the study of Section 2, to obtain the existence part of Theorem 1.4. The uniqueness part of Theorem 1.4 is obtained by proving a strong stability result valid for smooth solution. In Section 5 we study the cooling process and prove the remaining parts of Theorem 1.2 and Theorem 1.4.

2 Estimates in Orlicz spaces

In this section we gather some new functional estimates on the collision operator in Orlicz spaces, that will be used in the sequel to obtain (local in time) non-concentration estimates. Let us introduce the following decomposition $b = b^t_\varepsilon + b^r_\varepsilon$ of the cross-section b for $\varepsilon \in (0,1)$:

$$\begin{cases} b^t_\varepsilon(\mathcal{E}, u; dz) = b(\mathcal{E}, u; dz)\, \mathbf{1}_{\{-1+\varepsilon \le \hat{u}\cdot z \le 1-\varepsilon\}} \\ b^r_\varepsilon(\mathcal{E}, u; dz) = b(\mathcal{E}, u; dz) - b^t_\varepsilon(\mathcal{E}, u; dz) \end{cases}$$

where $\mathbf{1}_{\{-1+\varepsilon \le \hat{u}\cdot z \le 1-\varepsilon\}}$ denotes the usual indicator function of the set $\{-1+\varepsilon \le \hat{u}\cdot z \le 1-\varepsilon\}$. When no confusion is possible the subscript ε shall be omitted.

In the sequel, Λ denotes a function C^2 strictly increasing, convex satisfying the assumptions (A.1), (A.2) and (A.3). This function defines the Orlicz space $L^\Lambda(\mathbb{R}^N)$, which is a Banach space (see the definition in appendix).

2.1 Convolution-like estimates on the gain term

In this subsection we shall prove convolution-like estimates in Orlicz spaces. These estimates extend existing results in Lebesgue spaces: see [15, 16, 24] in the elastic case and [13] in the inelastic case for a constant normal restitution coefficient. The proof relies only on elementary tools, essentially Young's inequality, in the spirit of [9]. Another proof could be given by interpolating between the L^1 and L^∞ theories, as in [15, 16] (using tools of [4]), but this path leads to more technical difficulties. Moreover the proof given here has several advantages: its simplicity, the fact that it handles only the dual form of Q^+ and the fact that it is naturally well-suited to deal with Orlicz spaces, since it is based on Young's inequality.

As shown by the formula for the differential of the Orlicz norm in the appendix, the crucial quantity to estimate is

$$\int_{\mathbb{R}^N} Q^+(f,f)\, \Lambda'\left(\frac{f}{\|f\|_{L^\Lambda}}\right) dv.$$

Most of the difficulty is related to the fact that the bilinear operator Q^+ is not bounded because of the term $|v - v_*|$ in the collision rate. Nevertheless it is possible to prove a compactness-like estimate with respect to this algebraic weight. When combined with the damping effect of the loss term this estimate shall show that the evolution semi-group of the global collision operator is bounded in any Orlicz space.

Let us state the result

Theorem 2.1 *For any function $f \in L^1_1 \cap L^\Lambda$, for any $\varepsilon \in (0,1)$, there is an explicit constant $C^+_{\mathcal{E}}(\varepsilon)$ such that*

$$
\begin{aligned}
\int_{\mathbb{R}^N} Q^+(f,f)\,\Lambda'\left(\frac{f}{\|f\|_{L^\Lambda}}\right)\,dv \le \alpha(\mathcal{E})\,\Bigg[C^+_{\mathcal{E}}(\varepsilon)\,N^{\Lambda^*}\left(\Lambda'\left(\frac{|f|}{\|f\|_{L^\Lambda}}\right)\right)\|f\|_{L^1_1}\|f\|_{L^\Lambda}& \\
+\,(2+2^{N+2})\,j_{\mathcal{E}}(\varepsilon)\,\|f\|_{L^1_1}\int_{\mathbb{R}^N} f\,\Lambda'\left(\frac{f}{\|f\|_{L^\Lambda}}\right)|v|\,dv\Bigg]. & \qquad (2.1)
\end{aligned}
$$

Remark 2.2 *Let us comment on the conclusions of this theorem.*

1. We establish estimates for the quadratic Boltzmann collision operator but similar bilinear estimates could be proved under additional assumption on b, namely that either no frontal collision occurs*, i.e. $b(\mathcal{E},u;dz)$ should vanish for $\hat{u}$ close to z, or* no grazing collision occurs*, i.e. $b(\mathcal{E},;dz)$ should vanish for $\hat{u}$ close to $-z$. For more details on these bilinear estimates and the corresponding assumptions, we refer to [24] where they are proved in Lebesgue spaces in the elastic framework.*

2. Let us emphasize that for $z \sim 0$ (close to sticky collisions), the jacobian of the change of variable $(v,v_*) \to (v',v'_*)$ (both velocities at the same time) is blowing up. However in our method, we only use the changes of variable $v \to v'$ and $v_* \to v'$, keeping the other velocity unchanged, and the jacobians of these changes of variable remain uniformly bounded as $z \to 0$. This explains why our bounds includes the sticky particules model, and are uniform as $z \to 0$.

Proof of Theorem 2.1. Let us denote

$$
\varphi(f) = \Lambda'\left(\frac{f}{\|f\|_{L^\Lambda}}\right).
$$

Using the decomposition $b = b^t + b^r$, we control separately the two terms I^t and I^r in the decomposition

$$
\begin{aligned}
\int_{\mathbb{R}^N} Q^+(f,f)\,\varphi(f)\,dv &= \int_{\mathbb{R}^N\times\mathbb{R}^N\times D} f f_*\varphi(f')\,|u|\,b^t(\mathcal{E},u;dz)\,dv\,dv_* \\
&\quad + \int_{\mathbb{R}^N\times\mathbb{R}^N\times D} f f_*\varphi(f')\,|u|\,b^r(\mathcal{E},u;dz)\,dv\,dv_* =: I^t + I^r.
\end{aligned}
$$

Using the bound

$$
|u| = |v - v_*| \le |v| + |v_*|
$$

we have

$$
\begin{aligned}
I^t &\le \int_{\mathbb{R}^N\times\mathbb{R}^N\times D} (f|v|)f_*\varphi(f')\,b^t(\mathcal{E},u;dz)\,dv\,dv_* \\
&\quad + \int_{\mathbb{R}^N\times\mathbb{R}^N\times D} f(f_*|v_*|)\varphi(f')\,|u|\,b^t(\mathcal{E},u;dz)\,dv\,dv_* =: I^t_1 + I^t_2.
\end{aligned}
$$

Now these two terms are treated similarly: the two changes of variable $\phi_1 : v \to v'$ and $\phi_2 : v_* \to v'$ (while the other integration variables are kept fixed) are allowed thanks to the truncation. Indeed it is straightforward to compute their jacobian:

$$\begin{cases} J_{\phi_1}(v, v_*, z) = 2^N \, (1 + z \cdot \hat{u})^{-1} \\ J_{\phi_2}(v, v_*, z) = 2^N \, (1 - z \cdot \hat{u})^{-1} \end{cases}$$

which yields the bound

$$2^{N-1} \le J_{\phi_1}, \; J_{\phi_2} \le 2^N \, \varepsilon^{-1}. \tag{2.2}$$

Thus, by applying the Young's inequality (A.4)

$$f_* \varphi(f') = \|f\|_{L^\Lambda} \left(\frac{f_*}{\|f\|_{L^\Lambda}} \right) \varphi(f') \le \|f\|_{L^\Lambda} \, \Lambda \left(\frac{f_*}{\|f\|_{L^\Lambda}} \right) + \|f\|_{L^\Lambda} \, \Lambda^*(\varphi(f')),$$

we get for I_1^t the following estimate

$$\begin{aligned} I_1^t \;\; \le \;\; & \|f\|_{L^\Lambda} \int_{\mathbb{R}^N \times \mathbb{R}^N \times D} f|v|\Lambda \left(\frac{f_*}{\|f\|_{L^\Lambda}} \right) b^t(\mathcal{E}, u; dz) \, dv \, dv_* \\ & + \|f\|_{L^\Lambda} \int_{\mathbb{R}^N \times \mathbb{R}^N \times D} f|v|\Lambda^*(\varphi(f')) \, b^t(\mathcal{E}, u; dz) \, dv \, dv_* =: I_{1,1}^t + I_{1,2}^t. \end{aligned}$$

On the one hand, using

$$\forall \, x \in \mathbb{R}_+, \quad \Lambda(x) \le x \, \Lambda'(x),$$

which is a trivial consequence of the fact that $\Lambda(0) = 0$ and Λ' is increasing, we have

$$I_{1,1}^t \le \alpha(\mathcal{E}) \, \|f\|_{L^1_1} \int_{\mathbb{R}^N} f \, \varphi(f) \, dv.$$

The Hölder's inequality in Orlicz spaces (A.5) recalled in the appendix then yields

$$I_{1,1}^t \le \alpha(\mathcal{E}) \, N^{\Lambda^*} \left(\Lambda' \left(\frac{|f|}{\|f\|_{L^\Lambda}} \right) \right) \|f\|_{L^1_1} \|f\|_{L^\Lambda}. \tag{2.3}$$

On the other hand, using that $\Lambda^*(y) = y \, (\Lambda')^{-1}(y) - \Lambda((\Lambda')^{-1}(y))$, we get

$$I_{1,2}^t \le \int_{\mathbb{R}^N \times \mathbb{R}^N \times D} f|v| f' \, \varphi(f') \, b^t(\mathcal{E}, u; dz) \, dv \, dv_*.$$

Since the cross-section b^t is truncated, we can apply the change of variable $v_* \to v'$, with the bound (2.2), and we get

$$I_{1,2}^t \le \alpha(\mathcal{E}) \, 2^N \varepsilon^{-1} \, \|f\|_{L^1_1} \int_{\mathbb{R}^N} f \, \varphi(f) \, dv.$$

The Hölder's inequality (A.5) then yields

$$I^t_{1,2} \le \alpha(\mathcal{E})\, 2^N \varepsilon^{-1}\, \|f\|_{L^1_1}\, N^{\Lambda^*}\left(\Lambda'\left(\frac{|f|}{\|f\|_{L^\Lambda}}\right)\right)\, \|f\|_{L^\Lambda}. \tag{2.4}$$

Next, the term I^t_2 is exactly similar to I^t_1, except that one has to use the change of variable $v \to v'$ instead of $v_* \to v'$ (with the bound (2.2) again). Therefore gathering (2.3), (2.4) and the same estimate for I^t_2, we obtain

$$I^t \le 2\,\alpha(\mathcal{E})\,(1+2^N\varepsilon^{-1})\,\|f\|_{L^1_1}\left[N^{\Lambda^*}\left(\Lambda'\left(\frac{|f|}{\|f\|_{L^\Lambda}}\right)\right)\right]\|f\|_{L^\Lambda}. \tag{2.5}$$

Finally, for the term I^r, we can split it as

$$\begin{aligned} I^r \;&\le\; \int_{\mathbb{R}^N\times\mathbb{R}^N\times D} f\, f_*\, \varphi(f')\, \mathbf{1}_{\{\hat u\cdot z\ge 0\}}\, |u|\, b^r(\mathcal{E},u;dz)\, dv\, dv_* \\ &\quad + \int_{\mathbb{R}^N\times\mathbb{R}^N\times D} f\, f_*\, \varphi(f')\, \mathbf{1}_{\{\hat u\cdot z\le 0\}}\, |u|\, b^r(\mathcal{E},u;dz)\, dv\, dv_* =: I^r_1 + I^r_2. \end{aligned}$$

Then for I^r_1, we use the Young's inequality (A.4) to obtain

$$\begin{aligned} I^r_1 \le & \int_{\mathbb{R}^N\times\mathbb{R}^N\times D} f\, f_*\, \varphi(f_*)\, \mathbf{1}_{\{\hat u\cdot z\ge 0\}}\, |u|\, b^r(\mathcal{E},u;dz)\, dv\, dv_* \\ & + \int_{\mathbb{R}^N\times\mathbb{R}^N\times D} f\, f'\, \varphi(f')\, \mathbf{1}_{\{\hat u\cdot z\ge 0\}}\, |u|\, b^r(\mathcal{E},u;dz)\, dv\, dv_*. \end{aligned}$$

In the second integral we make the change of variable $v \to v'$, whose jacobian is less than 2^N thanks the truncation $\hat u \cdot z \ge 0$ and the formula for the jacobian, and we use that under the truncation

$$|v - v_*| \le 2|v' - v_*| \le 2(1+|v'|)(1+|v_*|).$$

Hence we obtain

$$\begin{aligned} I^r_1 \;&\le\; (1+2^{N+1})\left(\sup_{u\in\mathbb{R}^N}\int_D b^r(\mathcal{E},u;dz)\right)\|f\|_{L^1_1}\int_{\mathbb{R}^N} f\,\varphi(f)\,(1+|v|)\,dv \\ &\le\; (1+2^{N+1})\,\alpha(\mathcal{E})\, j_{\mathcal{E}}(\varepsilon)\,\|f\|_{L^1_1}\int_{\mathbb{R}^N} f\,\varphi(f)\,(1+|v|)\,dv. \end{aligned}$$

The term I^r_2 is treated similarly using Young's inequality and the change of variable $v_* \to v'$, whose jacobian is also less than 2^N under the truncation $\hat u \cdot z \le 0$. It satisfies therefore the same estimate. Thus we obtain the estimate

$$I^r \le (2+2^{N+2})\,\alpha(\mathcal{E})\, j_{\mathcal{E}}(\varepsilon)\,\|f\|_{L^1_1}\int_{\mathbb{R}^N} f\,\varphi(f)\,(1+|v|)\,dv. \tag{2.6}$$

Defining

$$C^+_{\mathcal{E}}(\varepsilon) = 2(1+2^N\varepsilon^{-1}) + (2+2^{N+2})\, j_{\mathcal{E}}(\varepsilon), \tag{2.7}$$

we conclude the proof gathering (2.5) and (2.6). □

2.2 Minoration of the loss term

In this subsection we recall a well-known result about the minoration of the loss term Q^-. Let us recall first the following classical estimate.

Lemma 2.3 *For any non-negative measurable function f such that*

$$f \in L^1_1(\mathbb{R}^N), \qquad \int_{\mathbb{R}^N} f\,dv = 1, \qquad \int_{\mathbb{R}^N} f\,v\,dv = 0, \tag{2.8}$$

we have

$$\forall\, v \in \mathbb{R}^N, \quad \int_{\mathbb{R}^N} f_*\,|v - v_*|\,dv_* \ge |v|.$$

Proof of Lemma 2.3. Using Jensen's inequality

$$\int_{\mathbb{R}^N} \varphi(g_*)\,d\mu_* \ge \varphi\left(\int_{\mathbb{R}^N} g_*\,d\mu_*\right)$$

with the probability measure $d\mu_* = f_*\,dv_*$, the measurable function $v_* \mapsto g_* = v - v_*$ and the convex function $\varphi(s) = |s|$, we deduce the result. □

Then the proof of the following proposition is straightforward:

Proposition 2.4 *For a non-negative function f satisfying (2.8), we have*

$$\int_{\mathbb{R}^N} Q^-(f,f)\,\Lambda'\left(\frac{f}{\|f\|_{L^\Lambda}}\right)\,dv \ge \alpha(\mathcal{E}) \int_{\mathbb{R}^N} f\,\Lambda'\left(\frac{f}{\|f\|_{L^\Lambda}}\right)\,|v|\,dv. \tag{2.9}$$

2.3 Estimate on the global collision operator and *a priori* estimate on the solutions

Combining Theorem 2.1 and Proposition 2.4 we get

Theorem 2.5 *Let us consider a non-negative function f satisfying (2.8). Then there is an explicit constant $C_{\mathcal{E}}$ depending on the collision rate through the functions α and $j_{\mathcal{E}}$ such that*

$$\int_{\mathbb{R}^N} Q(f,f)\,\Lambda'\left(\frac{f}{\|f\|_{L^\Lambda}}\right)\,dv \le C_{\mathcal{E}}\,\left[N^{\Lambda^*}\left(\Lambda'\left(\frac{|f|}{\|f\|_{L^\Lambda}}\right)\right)\right]\,\|f\|_{L^1_1}\|f\|_{L^\Lambda}.$$

More precisely, $C_{\mathcal{E}} = \alpha(\mathcal{E})\,C^+_{\mathcal{E}}(\varepsilon_0)$, with ε_0 such that $j_{\mathcal{E}}(\varepsilon_0) \le (2 + 2^{N+2})^{-1}\,\|f\|^{-1}_{L^1_1}$ and where $C^+_{\mathcal{E}}$ is defined in (2.7).

Proof of Theorem 2.5. One just has to combine (2.1) and (2.9) and pick a ε_0 small enough such that

$$(2 + 2^{N+2})\,\|f\|_{L^1_1}\,j_{\mathcal{E}}(\varepsilon_0) \le 1.$$

□

Corollary 2.6 *Assume that B satisfies (1.10)-(1.11) and (1.13)-(1.14) and let consider a solution $f \in C([0,T]; L^1_2)$ to the Boltzmann equation (1.1)-(1.2) associated to an initial datum $f_{in} \in L^1_2$ and to the collision rate B. Assume moreover that (1.30) holds and there exists a compact set $K \subset (0,+\infty)$ such that*

$$\forall\, t \in [0,T], \quad \mathcal{E}(t) \in K.$$

Then, there exists a C^2, strictly increasing and convex function Λ satisfying the assumptions (A.1), (A.2) and (A.3) (which only depends on f_{in}) and a constant C_T (which depends on K, T and B) such that

$$\sup_{[0,T]} \|f(t,.)\|_{L^\Lambda} \le C_T.$$

Remark 2.7 *Let us emphasize that these non-concentration bounds are valid for the sticky particules model (in this case they provide an exponentially growing bound in L^Λ for all times). As a particular case we deduce some explicit bounds on the entropy when it is finite initially. Moreover, since our bounds are uniform as $b \rightharpoonup \delta_{z=0}$, we also deduce a proof of the sticky particules limit (for a cross-section being a diffuse measure converging to a Dirac mass at $z = 0$) by the Dunford-Pettis Lemma. This shows moreover that this limit is not singular.*

Proof of Corollary 2.6. Since $f_{\text{in}} \in L^1(\mathbb{R}^N)$, as recalled in the appendix, a refined version of the De la Vallée-Poussin theorem [20, Proposition I.1.1] (see also [18, 19]) guarantees that there exists a function Λ satisfying the properties listed in the statement of Corollary 2.6 and such that

$$\int_{\mathbb{R}^N} \Lambda(|f_{\text{in}}|)\, dv < +\infty.$$

Then the L^Λ norm of f satisfies

$$\frac{d}{dt}\|f_t\|_{L^\Lambda} = \left[N^{\Lambda^*}\left(\Lambda'\left(\frac{|f|}{\|f\|_{L^\Lambda}}\right)\right)\right]^{-1} \int_{\mathbb{R}^N} Q(f,f)\, \Lambda'\left(\frac{|f|}{\|f\|_{L^\Lambda}}\right) dv$$

thanks to Theorem A.2, and thus using Theorem 2.5, we get

$$\forall\, t \in [0,T], \quad \frac{d}{dt}\|f_t\|_{L^\Lambda} \le C_{\mathcal{E}(t)}\, \|f_t\|_{L^1_1}\, \|f_t\|_{L^\Lambda}.$$

Thanks to the assumptions (1.13) and (1.14), the constant $C_{\mathcal{E}(t)}$ provided by Theorem 2.5 is uniform when the kinetic energy belongs to a compact set. Thus we deduce

$$\forall\, t \in [0,T], \quad \frac{d}{dt}\|f_t\|_{L^\Lambda} \le C_K\, \|f_t\|_{L^1_1}\, \|f_t\|_{L^\Lambda}. \tag{2.10}$$

for some explicit constant $C_K > 0$ depending on K and the collision rate. We conclude thanks to the Gronwall lemma. $\square$

3 Proof of the Cauchy theorem for non-coupled collision rate

In this section we fix $T_* > 0$ and we assume that the collision rate B satisfies

$$B = B(t, u; dz) = |u| \, \gamma(t) \, b(t, u; dz), \tag{3.1}$$

where b is a probability measure on D for any $t \in [0, T_*]$ and $u \in \mathbb{R}^N$ satisfying

$$\forall \, t \in [0, T_*], \ \forall \, u \in \mathbb{R}^N, \quad b(t, u; dz) = b(t, -u; -dz) \tag{3.2}$$

and where γ satisfies

$$0 \le \gamma(t) \le \gamma_* \quad \text{on} \quad (0, T_*). \tag{3.3}$$

3.1 Propagation of moments

In this subsection we establish several moments estimates which are well known for the Boltzmann equation with elastic collision, see [23, 21, 5] and the references therein, as well as the recent works [13, 7] for the inelastic case. Let us emphasize that these moment estimates are uniform with respect to the normal restitution coefficient e or more generally to the support of $b(t, u; \cdot)$ in D.

First we give a result of propagation of moments valid for general collision rates using a rough version of the Povzner inequality.

Proposition 3.1 *Assume that B satisfies (3.1)–(3.3). For any $0 \le f_{\rm in} \in L^1_q(\mathbb{R}^N)$ with $q > 2$ and $T > 0$, there exists C_T such that any solution f to the inelastic Boltzmann equation (1.1),(1.2) on $[0, T]$ satisfies, at least formally,*

$$\sup_{[0,T]} \|f(t, \cdot)\|_{L^1_k} \le C_T.$$

Proof of Proposition 3.1. We make the proof for the third moment, the general moment estimate being similar. For any function $\Psi : \mathbb{R}^N \to \mathbb{R}_+$ such that $\Psi(v) := \psi(|v|^2)$ for some function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$, the evolution of the associated moment is given by

$$\frac{d}{dt} \int_{\mathbb{R}^N} f \, \Psi \, dv = \int_{\mathbb{R}^N \times \mathbb{R}^N} f \, f_* \, K_\Psi \, dv \, dv_*,$$

where

$$K_\Psi := \frac{1}{2} \int_D (\Psi' + \Psi'_* - \Psi - \Psi_*) \, B(t, u; dz).$$

For $\psi(z) = z^s$, $s > 1$, the function ψ is super-additive, that is $\psi(x) + \psi(y) \le \psi(x+y)$, and it is an increasing function. As a consequence,

$$\begin{aligned} \Psi' + \Psi'_* - \Psi - \Psi_* &\le \psi(|v'|^2) + \psi(|v'_*|^2) - \psi(|v'|^2 + |v'_*|^2) \\ &\quad + \psi(|v|^2 + |v_*|^2) - \psi(|v|^2) - \psi(|v_*|^2) \\ &\le \psi(|v|^2 + |v_*|^2) - \psi(|v|^2) - \psi(|v_*|^2), \end{aligned}$$

which implies

$$K_\Psi \le \frac{\gamma(t)}{2}\,|v-v_*|\,\left[\psi(|v|^2+|v_*|^2)-\psi(|v|^2)-\psi(|v_*|^2)\right].$$

Making the choice $\psi(x) = x^{3/2}$ and using the inequality

$$\begin{aligned}(x^{1/2}+y^{1/2})\,[(x+y)^{3/2}-x^{3/2}-y^{3/2}] &\le C\,(x^{1/2}+y^{1/2})\,\min(x^{1/2}y, xy^{1/2}) \\ &\le C\,(xy+x^{1/2}y^{3/2})\end{aligned} \tag{3.4}$$

for any $x, y > 0$, we get

$$\frac{d}{dt}\int_{\mathbb{R}^N} f\,|v|^3\,dv \le C\,\gamma(t)\int_{\mathbb{R}^N\times\mathbb{R}^N} f\,f_*\,(|v|^2\,|v_*|^2+|v|\,|v_*|^3)\,dv\,dv_*, \tag{3.5}$$

and we conclude thanks to the Gronwall Lemma. □

Finally we give a much more precise result on the evolution moment in the case when assumption **H4** is made. One the one hand, we prove uniform in time propagation of algebraic moment (as introduced in [25, 1, 11]) and exponential moment (which starting reference is [5]). On the other hand we prove appearance of exponential moment (while appearance of algebraic moments where initiated in [8, 29]) using carefully tools developed in [7]. These estimates may be seen as *a priori* bounds, but in fact, by the bootstrap argument introduced in [23], they can be obtained *a posteriori* for any solution given by the existence part of Theorem 1.2 and Theorem 1.4.

Proposition 3.2 *We make assumption* **H4** *on B. A solution f to the inelastic Boltzman equation (1.1),(1.2) on $[0, T_c)$ satisfies the additional moment properties:*

(i) For any $s > 2$, there exists $C_s > 0$ such that

$$\sup_{t\in[0,T_c)} \|f(t,.)\|_{L^1_s} \le \max\{\|f_{in}\|_{L^1_s}, C_s\}. \tag{3.6}$$

(ii) If $f_{\rm in}\,e^{r\,|v|^\eta} \in L^1(\mathbb{R}^N)$ for $r > 0$ and $\eta \in (0,2]$, there exists $C_1, r' > 0$, such that

$$\sup_{t\in[0,T_c)} \int_{\mathbb{R}^N} f(t,v)\,e^{r'\,|v|^\eta}\,dv \le C_1. \tag{3.7}$$

(iii) For any $\eta \in (0,1/2)$ and $\tau \in (0,T_c)$ there exists $a_\eta, C_\eta \in (0,\infty)$ such that

$$\sup_{t\in[\tau,T_c)} \int_{\mathbb{R}^N} f(t,v)\,e^{a_\eta\,|v|^\eta}\,dv \le C_\eta. \tag{3.8}$$

Let us emphasize that all the constants do not depend on the inelasticity coefficient e (so that the estimates are uniform with respect to the inelasticity of the Boltzmann operator) and that the constant C_s, a_η, C_η may depend on f_{in} only through its kinetic energy $\mathcal{E}_{in}$.

Proof of Proposition 3.2. The proof of (i) is classical and we refer for intance to [23, 21, 28] and the references therein. The proofs of (ii) and (iii) are variants of [5, Theorem 3]. Let us define

$$m_p := \int_{\mathbb{R}^N} f\,|v|^{2p}\,dv.$$

Taking $\psi(x) = x^{p/2}$ and B of the above form, there holds

$$\frac{d}{dt} m_p = \int_{\mathbb{R}^N} Q(f,f)\,|v|^{2p}\,dv = \alpha(\mathcal{E}) \int_{\mathbb{R}^N\times\mathbb{R}^N} f\,f_*\,|v-v_*|\,K_p(v,v_*)\,dv\,dv_*, \tag{3.9}$$

where

$$K_p(v,v_*) := \frac{1}{2}\int_{\mathbb{S}^{N-1}} (|v'|^{2p} + |v'_*|^{2p} - |v|^{2p} - |v_*|^{2p})\,\frac{\tilde{b}(\mathcal{E},|u|,\sigma\cdot\hat{u})}{\alpha(\mathcal{E})}\,d\sigma. \tag{3.10}$$

From [7, Lemma 1, Corollary 1] (see also [13, Lemma 3.1 to Lemma 3.4]), there holds

$$K_p(v,v_*) \le \gamma_p\,(|v|^2 + |v_*|^2)^p - |v|^{2p} - |v_*|^{2p} \tag{3.11}$$

where $(\gamma_p)_{p=3/2,2,\dots}$ is a decreasing sequence of real numbers such that

$$0 < \gamma_p < \min(1, \frac{4}{p+1}) \tag{3.12}$$

(notice that the assumptions [7, (2.11)-(2.12)-(2.13)] are satisfied under our assumptions on the collision kernel). Let us emphasize that the estimate (3.11) does not depend on the inelasticity coefficient $e(\mathcal{E},|u|)$. Then, from [7, Lemma 2 and Lemma 3], we have

$$\frac{1}{\alpha(\mathcal{E})}\int_{\mathbb{R}^N} Q(f,f)\,|v|^{2p}\,dv \le \gamma_p\,S_p - (1-\gamma_p)\,m_{p+1/2} \tag{3.13}$$

with

$$S_p := \sum_{k=1}^{k_p} \binom{p}{k} (m_{k+1/2}\,m_{p-k} + m_k\,m_{p-k+1/2}),$$

where $k_p := [(p+1)/2]$ is the integer part of $(p+1)/2$ and $\binom{p}{k}$ stands for the binomial coefficient. Gathering (3.9) and (3.13), we get

$$\frac{d}{dt} m_p \le \alpha(\mathcal{E})\,(\gamma_p\,S_p - (1-\gamma_p)\,m_{p+1/2}) \qquad \forall\, p = 3/2, 2, \dots \tag{3.14}$$

By Hölder's inequality and the conservation of mass,

$$m_p^{1+\frac{1}{2p}} \le m_{p+1/2}$$

and, by [7, Lemma 4], for any $a \ge 1$, there exists $A > 0$ such that

$$S_p \le A\,\Gamma(a\,p + a/2 + 1)\,Z_p$$

with

$$Z_p := \max_{k=1,..,k_p} \{z_{k+1/2}\, z_{p-k},\ z_k\, z_{p-k+1/2}\}, \quad z_p := \frac{m_p}{\Gamma(a\,p+1/2)}.$$

We may then rewrite (3.14) as

$$\frac{dz_p}{dt} \le \alpha(\mathcal{E}) \left(A\,\gamma_p \frac{\Gamma(a\,p+a/2+1)}{\Gamma(ap+1/2)} Z_p - (1-\gamma_p)\,\Gamma(a\,p+1/2)^{1/2p}\, z_p^{1+1/2p} \right) \tag{3.15}$$

for any $p = 3/2, 2, \dots$ On the one hand, from (3.12), there exists A' such that

$$A\,\gamma_p \frac{\Gamma(ap+a/2+1)}{\Gamma(ap+1/2)} \le A'\, p^{a/2-1/2} \qquad \forall\, p = 3/2, 2, \dots \tag{3.16}$$

On the other hand, thanks to the Stirling's formula $n! \sim n^n\, e^{-n}\sqrt{2\pi n}$ when $n \to \infty$ and the estimate (3.12), there exists $A'' > 0$ such that

$$(1-\gamma_p)\,\Gamma(a\,p+1/2)^{1/2p} \ge A''\, p^{a/2} \qquad \forall\, p = 3/2, 2, \dots \tag{3.17}$$

Gathering (3.15), (3.16) and (3.17), we obtain the differential inequality

$$\frac{dz_p}{dt} \le \alpha(\mathcal{E}) \left(A'\, p^{a/2-1/2}\, Z_p - A''\, p^{a/2}\, z_p^{1+1/2p} \right) \tag{3.18}$$

for any $p = 3/2, 2, \dots$

Step 3. Proof of 3.7. On the one hand, we remark, by an induction argument, that taking $p_0 := \max(3/2, (2A'/A'')^2)$ the sequence of functions $z_p := x^p$ is a sequence of supersolutions of (3.18) for any $x > 0$ and for $p \ge p_0$. On the other hand, choosing x_0 large enough, which may depend on p_0, with have from (i) that the sequence of functions $z_p := x^p$ is a sequence of supersolutions of (3.18) for any $x \ge x_0$ and for $p \in \{3/2, \dots, p_0\}$. As a consequence, since z_p for $p = 0, 1/2, 1$ are bounded by $\|f_{\rm in}\|_{L^1_2}$, we have proved that there exists x_0 such that the set

$$\mathcal{C}_x := \left\{ z = (z_p); \quad z_p \le x^p \ \ \forall\, p \in \frac{1}{2}\,\mathbb{N} \right\} \tag{3.19}$$

is invariant under the flow generated by the Boltzmann equation for any $x \ge x_0$: if $f(t_1) \in \mathcal{C}_x$ then $f(t_2) \in \mathcal{C}_x$ for any $t_2 \ge t_1$.

We put $a := 2/\eta \ge 1$. Noticing that

$$\int_{\mathbb{R}^N} f(v)\, e^{r\,|v|^\eta}\, dv = \sum_{k=0}^{\infty} \frac{r^k}{k!}\, m_{k\,\eta/2} \tag{3.20}$$

we get, from the assumption made on $f_{\rm in}$, that

$$m_{k/a}(0) \le C_0\, \frac{k!}{r^k} \quad \forall\, k \in \mathbb{N}.$$

Since we may assume $r \in (0,1]$, the function $y \mapsto C_0 \frac{\Gamma(y+1)}{r^y}$ is increasing, and we deduce by Hölder's inequality that for any p

$$m_p(0) \le C_0 \frac{\ell_p!}{r^{\ell_p}} \le C_0 \frac{\Gamma(ap+2)}{r^{ap+2}} \quad \text{with} \quad \ell_p := [a\,p] + 1.$$

From the definition of z_p we deduce

$$z_p(0) \le C_0 \frac{ap\,(ap+1)}{r^{ap+2}} \le x_1^p \tag{3.21}$$

for any p and for some constant $x_1 \in (0,\infty)$. Choosing $x := \max\{x_0, x_1\}$ we get from (3.19) and (3.21) that for any p

$$z_p(t) \le x^p \quad \forall\, t \in [0, T_c).$$

Therefore, we have

$$m_p(t) \le \Gamma(ap + 1/2)\, x^p \qquad \forall\, p = 3/2, 2, \dots, \quad \forall\, t \in [0, T_c).$$

The function $y \mapsto \Gamma(y+1/2)\, x^y$ being increasing, we deduce from Hölder's inequality that for any $k \in \mathbb{N}^*$

$$m_{k/a}(t) \le \Gamma(ap+1/2)\, x^p \le \Gamma(k + a/2 + 1/2)\, x^{k/a+1/2} \quad \text{with} \quad p := [2k/a]/2 + 1/2.$$

For $r' < 2x^{-1/a}(1+a)^{-1}$ we have

$$\begin{aligned} \forall\, t \in [0,T_c) \quad \int_{\mathbb{R}^N} f(t,v)\, e^{r'\,|v|^\eta}\, dv &\le \sum_{k=0}^{\infty} \frac{\Gamma(k + a/2 + 1/2)}{k!}\, x^{k/a+1/2}\, (r')^k \\ &\le C \sum_{k=0}^{\infty} \left(\frac{a+1}{2} x^{1/a} r' \right) < \infty \end{aligned}$$

from which (3.7) follows.

Step 4. Proof of 3.8. Let fix $\tau \in (0, T_c)$. We claim that there exists x large enough and some increasing sequence of times $(t_p)_{p \ge p_0}$ which are bounded by τ such that for any p

$$\forall\, t \in [t_p, T_c) \quad z_p(t) \le x^p. \tag{3.22}$$

We yet know by classical arguments (see [23, 28]) that for p_0 (defined at the beginning of Step 3) there exists x_1, larger than x_0 defined in (3.19), such that (3.22) holds for any $p \le p_0$ and $t_p = \tau/2$. We then argue by induction, assuming that for $p \ge p_0$ there holds:

$$z_k \le x^k \quad \text{on} \quad [t_{p-1/2}, T_c) \quad \forall\, k \le p - 1/2 \tag{3.23}$$

$$z_p \ge x^p \quad \text{on} \quad [t_{p-1/2}, t_p), \tag{3.24}$$

for some $x \geq x_1$ to be defined. If (3.24) does not hold, there is nothing to prove thanks to Step 3. Gathering (3.23), (3.24) with (3.18) we get from the definition of p_0 and the fact that $\mathcal{E}(t) \in [\mathcal{E}(\tau), \mathcal{E}(0)]$ so that $\alpha(\mathcal{E}) \geq \alpha_0 > 0$

$$\frac{dz_p}{dt} \leq -\alpha_0 \, \frac{A''}{2} \, p^{a/2} \, z_p^{1+1/2p} \quad \text{on} \quad (t_{p-1/2}, t_p). \tag{3.25}$$

Integrating this differential inequality we obtain

$$-z_p^{-\frac{1}{2p}}(t_p) \leq z_p^{-\frac{1}{2p}}(t_{p-1/2}) - z_p^{-\frac{1}{2p}}(t_p) \leq -\frac{1}{2p} \, \frac{A'' \, \alpha_0}{2} \, p^{a/2} \, (t_p - t_{p-1/2}).$$

Defining (t_p) in the following way:

$$t_0 := \frac{\tau}{2}, \quad t_p := t_{p-1/2} + \frac{\tau}{2} \, \frac{p^{1-a/2}}{s_a}, \quad s_a := \sum_{p=0}^{\infty} p^{1-a/2}$$

and defining $x_2 := (8 \, s_a)^2/(A'' \, \alpha_0 \, \tau)^2$ we have then proved $z_p(t_p) \leq x_2^p$ and therefore $z_p(t) \leq x^p$ for any $t \geq (t_p, T_c)$ with $x = \max\{x_1, x_2\}$ thanks to Step 3. Setting $a := 2/\eta > 4$ ($\eta < 1/2$) we have

$$\sum_{k=0}^{\infty} t_{1+k/2} \leq \tau \tag{3.26}$$

and we conclude as in the end of Step 3. $\square$

3.2 Stability estimate in L^1_2 and proof of the uniqueness part of Theorem 1.2

Proposition 3.3 *Assume that B satisfies (3.1)–(3.3). For any two solutions f and g of the inelastic Boltzmann equation (1.1),(1.2) on $[0, T]$ ($T \leq T_*$) we have*

$$\frac{d}{dt} \int_{\mathbb{R}^N} |f-g| \, (1+|v|^2) \, dv \leq C \, \gamma_* \int_{\mathbb{R}^N} (f+g) \, (1+|v|^3) \, dv \int_{\mathbb{R}^N} |f-g| \, (1+|v|^2) \, dv. \tag{3.27}$$

We deduce that there is $C_T > 0$ depending on B and $\sup_{t \in [0,T]} \|f + g\|_{L^1_3}$ such that

$$\forall \, t \in [0, T], \quad \|f_t - g_t\|_{L^1_2} \leq \|f_{\text{in}} - g_{\text{in}}\|_{L^1_2} \, e^{C_T t}.$$

In particular, there exists at most one solution to the Cauchy problem for the inelastic Boltzmann equation in $C([0, T]; L^1_2) \cap L^1(0, T; L^1_3)$.

Proof of Proposition 3.3. We multiply the equation satisfied by $f - g$ by $\phi(t, y) = \text{sgn}(f(t, y) - g(t, y)) \, k$, where $k = (1 + |v|^2)$. Using the chain rule (1.29), we get for all $t \geq 0$

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^N} |f-g|\,k\,dv &= \frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N\times D} [(f-g)g_* + f(f_*-g_*)] \\
&\qquad\qquad (\phi' + \phi'_* - \phi - \phi_*)\,B(t,u;dz)\,dv_*\,dv \\
&= \frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N\times D} (f-g)\,(f_*+g_*) \\
&\qquad\qquad (\phi' + \phi'_* - \phi - \phi_*)\,B(t,u;dz)\,dv_*\,dv \\
&\leq \frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N\times D} |f-g|\,(f_*+g_*) \\
&\qquad\qquad (k' + k'_* - k + k_*)\,B(t,u;dz)\,dv_*\,dv,
\end{aligned}$$

where we have just use the symmetry hypothesis (3.1), (3.2) on B and a change of variable $(v,v_*) \to (v_*,v)$. Then, thanks to the bounds (3.1), (3.3) we deduce

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^N} |f-g|\,k\,dv &\leq \gamma_* \int_{\mathbb{R}^N\times\mathbb{R}^N} |u|\,|f-g|\,(f_*+g_*)\,k_*\,dv_*dv \\
&\leq \gamma_* \int_{\mathbb{R}^N} |f-g|\,k\,dv \int_{\mathbb{R}^N} (f_*+g_*)\,k_*^{3/2}\,dv_*
\end{aligned}$$

which yields the differential inequality (3.27). The end of the proof is straightforward by a Gronwall Lemma. □

The uniqueness in $C([0,T];L^1_2) \cap L^1(0,T;L^1_3)$ as stated in Theorem 1.2 is given by Proposition 3.3.

3.3 Sketch of the proof of the existence part of Theorem 1.2

As for the existence part, we briefly sketch the proof. We follow a method introduced in [23] and developped in [12]. We split the proof in three steps.

Step 1. Let us first consider an initial datum $f_{\rm in}$ satisfying (1.9) with $q = 4$ and let us define the truncated collision rates $B_n = B\,\mathbf{1}_{|u|\leq n}$. The associated collision operators Q_n are bounded in any L^1_q, $q \geq 1$, and are Lipschitz in L^1_2 on any bounded subset of L^1_2. Therefore following a classical argument from Arkeryd, see [1], we can use the Banach fixed point Theorem and obtain the existence of a solution $0 \leq f_n \in C([0,T];L^1_2) \cap L^\infty(0,T;L^1_4)$ for any $T > 0$, to the associated Boltzmann equation (1.1)-(1.2), which satisfies (1.30)-(1.31).

Step 2. From Proposition 3.1, for any $T > 0$, there exists C_T such that

$$\sup_{[0,T]} \|f_n\|_{L^1_4} \leq C_T.$$

Moreover, coming back to the proof of Proposition 3.3 (see also the first step in the proof of [12, Theorem 2.6]), we may establish the differential inequality

$$\frac{d}{dt}\|f_n - f_m\|_{L^1_2} \leq C_1\,\|f_n + f_m\|_{L^1_3}\,\|f_n - f_m\|_{L^1_2} + \frac{C_2}{n}\,\|f_n + f_m\|^2_{L^1_4}$$

for any integers $m \geq n$. Gathering these two informations we easily deduce that (f_n) is a Cauchy sequence in $C([0,T];L^1_2)$ for any $T0$. Denoting by $f \in C([0,T];L^1_2) \cap L^\infty(0,T;L^1_4)$ its limit, we obtain that f is a solution to the Boltzmann equation (1.1)-(1.2) associated to the collision rate B and the initial datum f_{in} by passing to the limit in the weak formulation (1.28) of the Boltzmann equation written for f_n.

Step 3. When the initial datum f_{in} satisfies (1.9) with $q = 3$ we introduce the sequence of initial data $f_{in,\ell} := f_{\text{in}}\, \mathbf{1}_{\{|v| \leq \ell\}}$. Since $f_{in,\ell} \in L^1_4$, the preceding step give the existence of a sequence of solutions $f_\ell \in C([0,T];L^1_2) \cap L^\infty(0,T;L^1_3)$ for any $T > 0$ to the Boltzmann equation (1.1),(1.2) associated to the initial datum $f_{in,\ell}$. From Proposition 3.1, for any $T > 0$, there exists C_T such that

$$\sup_{[0,T]} \|f_\ell\|_{L^1_3} \leq C_T.$$

Thanks to (3.27) we establish that (f_ℓ) is a Cauchy sequence in $C([0,T];L^1_2)$ and we conclude as before. $\square$

Remark 3.4 *Note here that an alternative path to the proof of existence could have been the use of the result of propagation of Orlicz norm which shows here that the solution is uniformly bounded for $t \in [0,T]$ in a certain Orlicz space. Together with the propagation of moments and Dunford-Pettis Lemma, it would yield the existence of a solution by classical approximation arguments and weak stability results as presented below. More generally the propagation of Orlicz norm by the collision operator can be seen as a new tool (as well as a clarification) for the theory of solutions to the spatially homogeneous Boltzmann equation with no entropy bound, as in the inelastic case, or in the elastic case when the initial datum has infinite entropy, see also [1, 23] where other strategies of proof are presented.*

4 Proof of the Cauchy theorem for coupled collision rate

4.1 Weak stability and proof of the existence part of Theorem 1.4

Proposition 4.1 *Consider a sequence $B_n = B_n(t,u;dz)$ of collision rates satisfying the structure conditions (3.1)-(3.2) and the uniform bound*

$$0 \leq \gamma_n(t) \leq \gamma_T \quad \forall\, t \in [0,T], \quad \forall\, n \in \mathbb{N}^*,$$

and let us denote by $f_n \in C([0,T);L^1_2) \cap L^\infty(0,T;L^1_3)$ the solution associated to B_n thanks to the existence result of the preceding section (existence and uniqueness part of Theorem 1.2 and Remark 1.3 4th point). Assume furthermore that (f_n) belongs *to a weak compact set of $L^1((0,T)\times\mathbb{R}^N)$ and that there exists a collision rate B satisfying (3.1)-(3.3) and such that for any $\psi\in C_c(\mathbb{R}^N)$*

$$\gamma_n \;\to\; \gamma \quad \textit{and} \quad \int_D \psi(v')\, b_n(t,u;dz) \;\to\; \int_D \psi(v')\, b(t,u;dz) \quad \textit{a.e.}$$

Then there exists a function $f\in C([0,T);L^1_2)\cap L^\infty(0,T;L^1_3)$ and a subsequence f_{n_k} such that

$$f_{n_k} \rightharpoonup f \quad \textit{weakly in} \quad L^1((0,T)\times\mathbb{R}^N),$$

and f is a solution to the Boltzmann equation (1.1)-(1.2) associated to B.

Such a stability/compactness result is very classical and we refer to [10, 1] for its proof.

Proof of the existence part of Theorem 1.4. We assume without restriction that there exists a decreasing function α_0 such that $\alpha\le\alpha_0$ on $[0,\mathcal{E}_{\rm in}]$. We proceed in three steps.

Step 1. We start with some *a priori* bounds. We set $Y_3 := \|f\|_{L^1_3}$. From the Povner inequality (3.5) (with $\gamma(t)=\alpha(\mathcal{E}(t))$ and the dissipation of energy equation (1.7), we have

$$\frac{d}{dt}Y_3 \le C_1\,\alpha_0(\mathcal{E})\,Y_3, \quad Y_3(0)=Y_3(f_{\rm in}) \tag{4.1}$$

and

$$\frac{d}{dt}\mathcal{E} \ge -C_1\,\alpha_0(\mathcal{E})\,Y_3, \quad \mathcal{E}(0)=\mathcal{E}_{\rm in}, \tag{4.2}$$

for some constant C_1 (which depends on $\mathcal{E}_{\rm in}$). There exists T_* such that any solution $(Y_3,\mathcal{E})$ to the above differential inequalities system is defined on $[0,T_*]$ and satisfies

$$\sup_{[0,T_*]} Y_3(t) \le 2\,Y_3(f_{\rm in}), \quad \inf_{[0,T_*]}\mathcal{E}(t) \ge \mathcal{E}_{\rm in}/2. \tag{4.3}$$

More precisely, we choose T_* such that

$$C_1\,\alpha_0(\mathcal{E}_{\rm in}/2)\,T_* \le Y_3(f_{\rm in}) \quad \text{and} \quad C_1\,\alpha_0(\mathcal{E}_{\rm in}/2) 2\,Y_3(f_{\rm in})\,T_* \le \mathcal{E}_{\rm in}/2,$$

in such a way that if $(Y_3,\mathcal{E})$ satisfies $Y_3\le 2\,Y_3(f_{\rm in})$ and (4.2) on $(0,T_*)$ or if $(Y_3,\mathcal{E})$ satisfies $\mathcal{E}\ge\mathcal{E}_{\rm in}/2$ and (4.1) on $(0,T_*)$ then (4.3) holds. Then we introduce

$$X := \Big\{\mathcal{E}\in C([0,T_*]),\ \mathcal{E}_{\rm in}/2\le\mathcal{E}(t)\le\mathcal{E}_{\rm in} \ \text{ on } \ (0,T_*)\Big\}.$$

Step 2. Let us consider a function $\mathcal{E}_1\in X$ and define $B_2(t,u;dz) := B(\mathcal{E}_1(t),u;dz)$. From assumption (1.13) we may write

$$B_2(t,u;dz) = |u|\,\gamma_2(t)\,b_2(t,u;dz)$$

where b_2 is a probability measure and $\gamma_2(t)$ satisfies

$$\gamma_2(t) = \alpha(\mathcal{E}_1(t)) \le \alpha_0(\mathcal{E}_{\rm in}/2) < +\infty \qquad \forall\, t\in[0,T_*].$$

Thanks to Theorem 1.2 there exists a unique solution $f_2 \in C([0,T_*];L^1_2) \cap L^\infty(0,T_*;L^1_3)$ to the Boltzmann equation (1.1)-(1.2) associated to the collision rate B_2 and we set $\mathcal{E}_2 := \mathcal{E}(f_2)$. In such a way we have defined a map $\Phi : X \to X$, $\Phi(\mathcal{E}_1) = \mathcal{E}_2$.

In order to apply the Schauder fixed point Theorem, we aim to prove that Φ is continuous and compact from X to X. Consider $(\mathcal{E}_1^n)$ a sequence of X which uniformly converges to $\mathcal{E}_1$. Since $(\mathcal{E}_1^n)$ belongs to the compact set $[\mathcal{E}_{\rm in}/2, \mathcal{E}_{\rm in}]$ for any n and any $t \in [0,T_*]$, we deduce by applying Corollary 2.6 to the sequence (f_2^n) associated to $B_2^n(t,u;dz) = B(\mathcal{E}_1^n(t),u;dz)$ that

$$\forall\, n \ge 0, \quad \sup_{[0,T_*]} \int_{\mathbb{R}^N} \Lambda(f_2^n(t,v))\, dv \le C_2, \tag{4.4}$$

for a superlinear function Λ and a constant $C_2 > 0$. Moreover, from Proposition 3.1 we have

$$\forall\, n \ge 0, \quad \sup_{[0,T_*]} \int_{\mathbb{R}^N} f_2^n(t,v)\, |v|^3\, dv \le C_3 \tag{4.5}$$

for some constant $C_3 > 0$.

On the one hand, gathering (4.4), (4.5) and using the Dunford-Pettis Lemma, we obtain that (f_2^n) belongs to a weak compact set of $L^1((0,T_*) \times \mathbb{R}^3)$. Propositon 4.1 then implies that there exists $f_2 \in C([0,T_*];L^1_2) \cap L^\infty(0,T_*;L^1_3)$ such that, up to a subsequence, $f_2^n \rightharpoonup f_2$ weakly in $L^1(0,T;L^1_2)$ and f_2 is the solution to the Boltzmann equation associated to $B_2(t,u;dz) = B(\mathcal{E}_1(t),u;dz)$. Since this limit is unique by the previous study, the whole sequence (f_2^n) converges weakly to f_2, and in particular

$$\mathcal{E}_2^n \rightharpoonup \mathcal{E}_2 \quad \text{weakly in} \quad L^1(0,T) \tag{4.6}$$

where $\mathcal{E}_2$ is the kinetic energy of f_2.

On the other hand, there holds

$$\frac{d}{dt}\mathcal{E}_2^n = -\int_{\mathbb{R}^N \times \mathbb{R}^N} f_2^n\, f_{2*}^n\, |u|^3\, \beta(\mathcal{E}_1^n, u)\, dv dv_* =: -D_2^n.$$

Since $\beta(\mathcal{E}_1^n, u) \le \alpha(\mathcal{E}_1^n)/4 \le \alpha_0(\mathcal{E}_{\rm in}/2)/4$, we deduce from (3.1) that D_2^n is bounded in $L^\infty(0,T)$ which in turn implies

$$\|\mathcal{E}_2^n\|_{W^{1,\infty}(0,T)} \le C_4. \tag{4.7}$$

From the Ascoli Theorem we infer that the sequence $(\mathcal{E}_2^n)$ belongs to a compact set of $C([0,T])$. Since the cluster points for the uniform norm are included in the set of cluster points for the L^1 norm, it then follows from (4.6) that $\Phi(\mathcal{E}_1^n) = \mathcal{E}(f_2^n)$ converges to $\mathcal{E}(f_2) = \Phi(\mathcal{E}_1)$ for the uniform norm on $C([0,T])$, which ends the proof of the continuity of Φ. Of course, the *a priori* bound (4.7) and the Ascoli Theorem also imply that Φ is a compact map on X. We may thus use the Schauder fixed point Theorem to conclude to the existence of at least one $\bar{\mathcal{E}} \in X$ such that $\Phi(\bar{\mathcal{E}}) = \bar{\mathcal{E}}$.

Then, the solution $\bar f \in C([0,T_*];L^1_2) \cap L^\infty(0,T_*;L^1_3)$ to the Boltzmann equation associated to $\bar B(t,u;dz) := B(\bar{\mathcal{E}}(t),u;dz)$ satisfies

$$\int_{\mathbb{R}^N} \bar f(t,v)\,|v|^2\,dv = \Phi(\bar{\mathcal{E}})(t) = \bar{\mathcal{E}}(t)$$

and therefore $\bar f$ is a solution to the Boltzmann equation associated to B in $C([0,T_*];L^1_2) \cap L^\infty(0,T_*;L^1_3)$.

Step 3. We then consider the class of solution $f : (0,T_1) \to L^1_3$ such that $f \in C([0,T];L^1_2) \cap L^\infty(0,T;L^1_3)$ for any $T \in (0,T_1)$, $\mathcal{E}$ is decreasing, f is mass conserving. By Zorn Lemma, there exists a maximal interval $[0,T_c)$ such that

$$(T_c < \infty \text{ and } \mathcal{E}(t) \to 0 \text{ when } t \to T_c) \quad \text{or} \quad T_c = \infty.$$

In order to end the proof, the only thing one has to remark is that if $T_c < \infty$ and $\lim_{t \nearrow T_c} \mathcal{E}(t) = \mathcal{E}_c > 0$, then $\lim_{t \nearrow T_c} Y_3(t) < \infty$ (by (4.1)) so that $f \in C([0,T_c];L^1_2) \cap L^\infty(0,T_c;L^1_3)$ and we may extends the solution f to a larger time interval. □

4.2 Strong stability and uniqueness part of Theorem 1.4

In this subsection we give a quantitative stability result in strong sense, under the additional assumption of some smoothness on the initial datum and the collision rate. Let us first prove a simple result of propagation of the total variation of the gradient of the distribution.

Proposition 4.2 *Let B be a collision rate satisfying assumptions (3.1)-(3.2)-(3.3) and $0 \le f_{\rm in} \in BV_4 \cap L^1_5$ an initial datum. Then there exists C_{T_*}, depending on γ_* and $\|f_{\rm in}\|_{L^1_5}$, such that any solution $f \in C([0,T_*],L^1_2) \cap L^\infty(0,T_*,L^1_3)$ to the Boltzmann equation constructed in the previous step satisfies*

$$\forall\, t \in [0,T_*], \quad \|f_t\|_{BV_4} \le \|f_{\rm in}\|_{BV_4}\, e^{C_{T_*}\, t}$$

Proof of Proposition 4.2. The proof is based on the same kind of Povzner inequality as above. Let us first prove the estimate by *a priori* approach, for the sake of clearness. We have the following formula for the differential of Q:

$$\nabla_v Q(f,f) = Q(\nabla_v f, f) + Q(f, \nabla_v f).$$

This property is proved in the elastic case in [28] but it is strictly related to the invariance property of the collision operator

$$\tau_h Q(f,f) = Q(\tau_h f, \tau_h f)$$

where the translation operator τ_h is defined by

$$\forall\ v\ \in\ \mathbb{R}^N, \quad \tau_h f(v) = f(v-h).$$

It is easily seen that it remains true in the inelastic case under our assumptions. The propagation of the L^1_5 norm has already been established. Then we estimate the time derivative of the L^1_4 norm of the gradient along the flow:

$$\begin{aligned}
\frac{d}{dt}\|\nabla_v f_t\|_{L^1_4} &= \int_{\mathbb{R}^N\times\mathbb{R}^N\times D} f\,(\nabla_v f_*)\Big[(1+|v'|^4)\,\mathrm{sgn}(\nabla_v f)' + (1+|v'_*|^4)\,\mathrm{sgn}(\nabla_v f)'_* \\
&\quad -(1+|v|^4)\,\mathrm{sgn}(\nabla_v f) - (1+|v_*|^4)\,\mathrm{sgn}(\nabla_v f)_*\Big]\, B\,dv\,dv_* \\
&\le \int_{\mathbb{R}^N\times\mathbb{R}^N\times D} f\,|\nabla_v f_*|\Big[(1+|v'|^4) + (1+|v'_*|^4) - (1+|v|^4) \\
&\quad -(1+|v_*|^4)\Big]\, B\,dv\,dv_* + 4\,\gamma_*\,\|f_t(1+|v|^5)\|_{L^1}\,\|\nabla_v f(1+|v|)\|_{L^1} \\
&\le C\,\|f_t\|_{L^1_5}\,\|\nabla_v f\|_{L^1_4}
\end{aligned}$$

using a Povzner inequality as in (3.4). This shows the *a priori* propagation of the BV_4 norm by a Gronwall argument.

Now let us explain how to obtain the same estimate by *a posteriori* approach. First concerning the *a posteriori* propagation of the L^1_5 norm, it is similar to the method in [23] and does not lead to any difficulty. Concerning the propagation of BV_4 norm, we look at some "discretized derivative". Let us denote $k = \mathrm{sgn}(\tau_h f - f)\,(1+|v|^4)$. We can compute by the chain rule the following time derivative (using the invariance property of the collision operator)

$$\begin{aligned}
\frac{d}{dt}\|\tau_h f_t - f_t\|_{L^1_4} &= \int_{\mathbb{R}^N\times\mathbb{R}^N\times D} (\tau_h f \tau_h f_* - f f_*)\,[k'-k]\,B\,dv\,dv_* \\
&= \int_{\mathbb{R}^N\times\mathbb{R}^N\times D} (\tau_h f - f) f_*\,[k'+k'_*-k-k_*]\,B\,dv\,dv_* \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N\times D} (\tau_h f - f)(\tau_h f_* - f_*)\,[k'+k'_*-k-k_*]\,B\,dv\,dv_* \\
&\le \int_{\mathbb{R}^N\times\mathbb{R}^N\times D} |\tau_h f - f| f_*\,\Big[|v'|^4+|v'_*|^4-|v|^4+|v_*|^4\Big]\,B\,dv\,dv_* \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N\times D} |\tau_h f - f||\tau_h f_* - f_*| \\
&\qquad\qquad \Big[|v'|^4+|v'_*|^4+|v|^4+|v_*|^4\Big]\,B\,dv\,dv_*.
\end{aligned}$$

Then using the same rough Povzner inequality as in the proof of Proposition 3.1, we have

$$\Big[|v'|^4+|v'_*|^4-|v|^4+|v_*|^4\Big]|v-v_*| \le C\,(1+|v|^4)(1+|v_*|^5)$$

and

$$\Big[|v'|^4+|v'_*|^4+|v|^4+|v_*|^4\Big] \le C\,\left[(1+|v|^4)(1+|v_*|^5) + (1+|v_*|^4)(1+|v|^5)\right].$$

Hence we deduce that

$$\frac{d}{dt}\|\tau_h f_t - f_t\|_{L^1_4} \le C\,\gamma_*\,\|\tau_h f_t - f_t\|_{L^1_4}\left[\|f\|_{L^1_5} + \|\tau_h f_t - f_t\|_{L^1_5}\right]$$

and for $|h| \le 1$, we deduce

$$\frac{d}{dt}\|\tau_h f_t - f_t\|_{L^1_4} \le C\,\gamma_*\,\|\tau_h f_t - f_t\|_{L^1_4}\|f\|_{L^1_5}.$$

By a Gronwall argument it shows for any $|h| \le 1$ that

$$\forall\, t \in [0, T_*], \quad \|\tau_h f_t - f_t\|_{L^1_4} \le \|\tau_h f_{\rm in} - f_{\rm in}\|_{L^1_4}\, e^{C_{T_*}\, t}$$

for a constant C_{T_*} depending on γ_* and $\sup_{t\in[0,T_*]}\|f_t\|_{L^1_5}$. By dividing by h and letting h goes to 0, we conclude that

$$\forall\, t \in [0, T_*], \quad \|\nabla_v f_t\|_{M^1_4} \le \|\nabla_v f_{\rm in}\|_{M^1_4}\, e^{C_{T_*}\, t}$$

which ends the proof. $\square$

Now let us assume that the collision rate satisfies (1.10)–(1.14) plus the additional assumption **H1**: the measure b reduces to a mesure on the sphere $C_{u,e}$ with $e(\mathcal{E}), \alpha(\mathcal{E}) : (0, +\infty) \to [0, 1]$ locally Lipschitz functions. Let us take $f_{\rm in} \in BV_4 \cap L^1_5$ and let us consider two solutions $f, g \in C([0, T_c]; L^1_2) \cap L^\infty(0, T; L^1_3)$ constructed by the previous steps. For these two solutions the function $e(\mathcal{E})$ is locally Lipschitz, so is the function $\beta(\mathcal{E})$ and the differential equation (1.7) satisfied by $\mathcal{E}(f_t)$ on $[0, T_*]$ implies that it is bounded from below on this interval. Thus thanks to the continuity of α, the assumptions of Proposition 4.2 are satisfied, and thus the BV_4 norm is bounded on any time interval $[0, T_*] \subset [0, T_c)$.

Proposition 4.3 *Let B be a collision rate satisfying (1.10)–(1.14) plus the additionnal assumption* **H1**. *Let $f, g \in C([0, T_*]; L^1_2) \cap L^\infty(0, T_*; L^1_3)$ be two solutions with mass 1 and momentum 0 such that $\mathcal{E}(f_t), \mathcal{E}(g_t) \in K$ on $[0, T_*]$ with K compact of $(0, +\infty)$ and*

$$\forall\, t \in [0, T_*], \quad \|f_t\|_{BV_4}, \|g_t\|_{BV_4} \le C_{T_*}.$$

Then there is a constant C'_{T_*} depending on B, K and C_{T_*} such that

$$\forall\, t \in [0, T_*], \quad \|f_t - g_t\|_{L^1_2} \le \|f_{\rm in} - g_{\rm in}\|_{L^1_2}\, e^{C'_{T_*}\, t}.$$

Proof of Proposition 4.3. Let us denote Q_f (resp. Q_g) the collision operator with collision rate associated with $\mathcal{E} = \mathcal{E}(f_t)$ (resp. $\mathcal{E} = \mathcal{E}(g_t)$). Without restriction we assume by symmetrization that $\tilde{b}$ has its support included in $\hat{u} \cdot \sigma \le 0$.

Let us denote $D = f - g$ and $S = f + g$. The evolution equation on D writes

$$\frac{\partial}{\partial t} D = \frac{1}{2}\left[Q_f(D, S) + Q_f(S, D)\right] + \left[Q_f(g, g) - Q_g(g, g)\right]$$

and thus the time derivative of the L^1_2 norm of D is

$$\begin{aligned}
\frac{d}{dt}\|D\|_{L^1_2} &= \frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}} S D_* \Big[(1+|v|^2)\,\mathrm{sgn}(D') + (1+|v'_*|^2)\,\mathrm{sgn}(D'_*) \\
&\qquad -(1+|v|^2)\,\mathrm{sgn}(D) - (1+|v_*|^2)\,\mathrm{sgn}(D)_*\Big]\,|u|\,\tilde{b}(\mathcal{E}(f_t),\hat{u}\cdot\sigma)\,dv\,dv_*\,d\sigma \\
&\quad + \int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}} g g_* \Big[(1+|v'_{e(f_t)}|^2)\,\mathrm{sgn}(D'_{e(f_t)})\tilde{b}(\mathcal{E}(f_t),\hat{u}\cdot\sigma) \\
&\qquad -(1+|v'_{e(g_t)}|^2)\,\mathrm{sgn}(D'_{e(g_t)})\tilde{b}(\mathcal{E}(g_t),\hat{u}\cdot\sigma)\Big]\,|u|\,dv\,dv_*\,d\sigma \\
&\quad - \int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}} g g_* (1+|v|^2)\,\mathrm{sgn}(D)|u|\,\Big[\tilde{b}(\mathcal{E}(f_t),\hat{u}\cdot\sigma) - \tilde{b}(\mathcal{E}(g_t);\hat{u}\cdot\sigma)\Big]\,dv\,dv_*\,d\sigma \\
&=: I_1 + I_2 + I_3
\end{aligned}$$

(the subscripts recall that the post-collisional velocities depend on the choice of the restitution coefficient e). The first term is easily dealt with by the same arguments as in the non-coupled case:

$$I_1 \le \int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}} S\,|D_*|\,(1+|v|^2)\,|u|\,\tilde{b}(\mathcal{E}(f_t),\hat{u}\cdot\sigma)\,dv\,dv_*\,d\sigma \le \alpha(\mathcal{E}(f_t))\,\|S\|_{L^1_3}\,\|f_t-g_t\|_{L^1_1}.$$

The third term I_3 is controlled by

$$I_3 \le |\alpha(\mathcal{E}(f_t)) - \alpha(\mathcal{E}(g_t))|\,\|g\|_{L^1_3}\,\|g\|_{L^1_1}$$

and using that α is locally Lipschitz on K we get

$$I_3 \le C_K\,|\mathcal{E}(f_t)-\mathcal{E}(g_t)|\,\|g\|_{L^1_3}\,\|g\|_{L^1_1} \le C_K\,\|f_t-g_t\|_{L^1_2}\,\|g\|_{L^1_3}\,\|g\|_{L^1_1}$$

for some constant C_K depending on α and K.

As for the second term I_2, we use the change of variable $v_* \to v'$ with v,σ fixed and e given. This change of variable depends on e and we denote $v_* = \phi_{\sigma,e}(v,v')$. Let us denote the jacobian by J_e. It is computed in [13]:

$$J_e(\cos\theta) = \left(\frac{1+e}{4}\right)^N (1-\cos\theta) \tag{4.8}$$

and thus since by symmetrization we suppose here that $\theta\in[\pi/2,\pi]$, we have

$$\forall\, e\in[0,1],\ \forall\,\theta\in[0,\pi],\quad J_e(\cos\theta) \in \left[\left(\frac{1}{4}\right)^N, 2\left(\frac{1}{2}\right)^N\right]. \tag{4.9}$$

Thus we get

$$\begin{aligned}
I_2 = \int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}} g\,(1+|v'|^2)\,\mathrm{sgn}(D')\,|u|\,\Big[&g(\phi_{\sigma,e(f_t)}(v,v'))\,J_{e(f_t)}(\cos\theta) \\
&-g(\phi_{\sigma,e(g_t)}(v,v'))\,J_{e(g_t)}(\cos\theta)\Big]\,\tilde{b}(\mathcal{E},\cos\theta)\,dv\,dv'\,d\sigma.
\end{aligned}$$

So we can split this term as

$$
\begin{aligned}
I_2 = \int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}} & g\,(1+|v'|^2)\,\mathrm{sgn}(D')\,|u|\,\Big[g(\phi_{\sigma,e(f_t)}(v,v')) \\
& -g(\phi_{\sigma,e(g_t)}(v,v'))\Big]\,J_{e(f_t)}(\cos\theta)\,\tilde b(\mathcal{E},\cos\theta)\,dv\,dv'\,d\sigma \\
& + \int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}} g\,(1+|v'|^2)\,\mathrm{sgn}(D')\,|u|\,\Big[J_{e(f_t)}(\cos\theta) \\
-J_{e(g_t)}(\cos\theta)\Big]\,& g(\phi_{\sigma,e(g_t)}(v,v'))\,\tilde b(\mathcal{E},\cos\theta)\,dv\,dv'\,d\sigma = I_{2,1}+I_{2,2}.
\end{aligned}
$$

For the term $I_{2,2}$ we use that, from the formula (4.8) and the fact that $\mathcal{E}\mapsto e(\mathcal{E})$ is locally Lipschitz,

$$
\Big|J_{e(f_t)}(\cos\theta)-J_{e(g_t)}(\cos\theta)\Big| \le C\,|e(f_t)-e(g_t)| \le C_K\,\|\mathcal{E}(f_t)-\mathcal{E}(g_t)\|_{L^1_2} \le C_K\,\|f_t-g_t\|_{L^1_2}.
$$

Then doing the (elastic) change of variable backward $v'\to v_*$ (whose jacobian is bounded by (4.9)) we get

$$
I_{2,2} \le C_K\,\|f_t-g_t\|_{L^1_2}\,\|g\|_{L^1_3}\,\|g\|_{L^1_1}.
$$

We now aim to prove that for any functions f, g which energies $\mathcal{E}_f$ and $\mathcal{E}_g$ belong to a compact $K\subset(0,\infty)$ there exists a constant C_K such that the following functional inequality holds

$$
I_{2,1} \le C_K\,\|f_t-g_t\|_{L^1_2}\,\|g\|_{L^1_4}\,\|g\|_{BV_4}. \tag{4.10}
$$

Let first assume that f and g are smooth functions, say $f,g\in\mathcal{D}(\mathbb{R}^N)$. We have

$$
\begin{aligned}
\Big|g(\phi_{\sigma,e(f_t)}(v,v'))-g(\phi_{\sigma,e(g_t)}(v,v'))\Big| &\le \|\phi_{\sigma,e(f_t)}(v,v')-\phi_{\sigma,e(g_t)}(v,v')\| \\
&\left(\int_0^1 \Big|\nabla_v g((1-t)\phi_{\sigma,e(f_t)}(v,v')+t\phi_{\sigma,e(g_t)}(v,v'))\Big|\,dt\right).
\end{aligned}
$$

As for the difference $|\phi_{\sigma,e(f_t)}(v,v')-\phi_{\sigma,e(g_t)}(v,v')|$, it is easy to see that for some fixed v,v',σ the corresponding $v_*=\phi_{\sigma,e}(v,v')$ are aligned for any e (on the line determined by the plan defined by v,v',σ and the direction defined by the angle $\theta/2$ between $v'-v$ and v_*-v). Thus it remains to look for the algebraic length of $[\phi_{\sigma,e(f_t)}(v,v'),\phi_{\sigma,e(g_t)}(v,v')]$ on this line, which is given explicitely in [13]:

$$
|\phi_{\sigma,e(f_t)}(v,v')-\phi_{\sigma,e(g_t)}(v,v')| = \frac{|v-v'|}{\cos\theta/2}\,\frac{2|e(f_t)-e(g_t)|}{(1+e(f_t))(1+e(g_t))}.
$$

Thus we get

$$
|\phi_{\sigma,e(f_t)}(v,v')-\phi_{\sigma,e(g_t)}(v,v')| \le C_K\,|u|\,\|f_t-g_t\|_{L^1_2}
$$

and the term $I_{2,1,1}$ is controlled by (using the uniform bound (4.9) on $J_{e(f_t)}(\cos\theta)$)

$$I_{2,1,1} \leq C_K \, \|f_t - g_t\|_{L^1_2} \int_0^1 \int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}} g\,(1+|v'|^2)\,|u|^2$$
$$|\nabla_v g((1-t)\phi_{\sigma,e(f_t)}(v,v') + t\phi_{\sigma,e(g_t)}(v,v'))|\, b(\cos\theta)\, dv\, dv'\, d\sigma\, dt.$$

Finally for any $t \in [0,1]$ we want to perform the change of variable

$$v' \to (1-t)\phi_{\sigma,e(f_t)}(v,v') + t\phi_{\sigma,e(g_t)}(v,v'). \tag{4.11}$$

Some tedious but elementary computations yields

$$\phi_{\sigma,e}(v,v') = v - \frac{4|v-v'|}{1+e}\left[\frac{\sigma}{2\cos\theta/2} + \frac{v-v'}{|v-v'|}\right].$$

We deduce that

$$(1-t)\phi_{\sigma,e_1}(v,v') + t\phi_{\sigma,e_2}(v,v') = \phi_{\sigma,e_0}(v,v')$$

with
$$e_0 = \frac{te_1 + (1-t)e_2 + e_1e_2}{1+(1-t)e_1+te_2} \in [\min\{e_1,e_2\}, \max\{e_1,e_2\}].$$

Thus we deduce that the jacobian of the change of variable (4.11) is given by

$$\left(J_{e(f_t,g_t)}(\cos\theta)\right)^{-1} \quad \text{with} \quad e(f_t,g_t) = \frac{te(f_t)+(1-t)e(g_t)+e(f_t)e(g_t)}{1+(1-t)e(f_t)+te(g_t)}$$

and thus is uniformly bounded thanks to (4.8). Therefore we obtain (4.10) for smooth functions. When $f, g \in BV_4$ we argue by density, introducing two sequences of smooth functions (f_n) and (g_n) which converge respectively to f and g in L^1 and are bounded in BV_4, we pass to the limit $n \to \infty$ in the functionnal inequality (4.10) written for the functions f_n and g_n. We then easily conclude that (4.10) also holds for f and g.

Collecting all the terms we thus get

$$\frac{d}{dt}\|f_t - g_t\|_{L^1_2} \leq C'_{T_*}\, \|f_t - g_t\|_{L^1_2}$$

where C'_{T_*} depends on K, b and some uniform bounds on $\|f\|_{L^1_4}$ and $\|g\|_{BV_4}$. This concludes the proof by a Gronwall argument. □

The uniqueness part of Theorem 1.4 follows straightforwardly from Proposition 4.3 and the discussion made just before its statement.

5 Study of the cooling process

In this section we prove the cooling asymptotic as stated in point (ii) of Theorem 1.2 and points (iii), (iv), (v) of Theorem 1.4. We first prove the collapse of the distribution function in the sense of weak * convergence to the Dirac mass in the set of measures.

Proposition 5.1 *Let $T_c \in (0,+\infty]$ be the time of life of the solution. Under the sole additional assumption* **H2**, *there holds*

$$f(t,.) \underset{t\to T_c}{\rightharpoonup} \delta_{v=0} \ \ \textit{weakly} * \ \ \textit{in } M^1(\mathbb{R}^N). \tag{5.1}$$

Proof of Proposition 5.1. We split the proof in two steps.
Step 1. Assume first that $\mathcal{E} \to 0$ when $t \to T_c$. This includes the case when $T_c < +\infty$ (since the convergence to 0 of the kinetic energy follows from the existence proof in this case) and it will be established under additional assumptions on B when $T_c = +\infty$ but probably holds true under the sole assumption **H2** in this case as well. For any $0 \le \varphi \in \mathcal{D}(\mathbb{R}^N \backslash \{0\})$, there exists $r > 0$ such that $\varphi = 0$ on $D(0,r)$ and then, there exists $C_\varphi = C_\varphi(r, \|\varphi\|_\infty)$ such that $|\varphi(v)| \le C_\varphi\, |v|^2$. As a consequence,

$$\int_{\mathbb{R}^N} f\,\varphi\,dv \le C_\varphi\,\mathcal{E}(t) \to 0,$$

from which we deduce that any weak * limit $\bar\mu$ of f in M^1 satisfies $\operatorname{supp}\bar\mu \subset \{0\}$. Therefore, (5.1) follows using the conservations (1.30) and the energy bound (1.31).
Step 2. Assume next that $\mathcal{E} \to \mathcal{E}_\infty > 0$ (and thus also $T_c = +\infty$). Then for a fixed time $T > 0$ and for any non-negative sequence (t_n) increasing and going to $+\infty$, there exists a subsequence (t_{n_k}) and a measure $\bar\mu \in L^\infty(0,T;M^1_2)$ such that the function $f_k(t,v) := f(t_{n_k}+t, v)$ satisfies

$$f_k \rightharpoonup \bar\mu \ \ \text{weakly} * \ \ \text{in} \ \ L^\infty(0,T;M^1). \tag{5.2}$$

Moreover, for any $\varphi \in C_c(\mathbb{R}^N)$, there holds

$$\frac{d}{dt}\int_{\mathbb{R}^N} f_k\,\varphi\,dv = \langle Q(f_k,f_k),\varphi\rangle \quad \text{on} \quad (0,T),$$

with $\langle Q(f_k,f_k),\varphi\rangle$ bounded in $L^\infty(0,T)$. From Ascoli Theorem, we get

$$\int_{\mathbb{R}^N} f_k\,\varphi\,dv \ \to \ \int_{\mathbb{R}^N} \varphi\,d\bar\mu(v) \quad \text{uniformly on} \quad [0,T].$$

As a consequence, for any given function $\chi_\varepsilon \in C_c(\mathbb{R}^3 \times \mathbb{R}^3)$ such that $0 \le \chi_\varepsilon \le 1$ and $\chi_\varepsilon(v,v_*) = 1$ for every (v,v_*) such that $|v| \le \varepsilon^{-1}$ and $|v_*| \le \varepsilon^{-1}$ we may pass to the limit (using the continuity of $\beta = \beta(\mathcal{E}, u)$ which is uniform on the compact set determined by $[\mathcal{E}_\infty, \mathcal{E}_0]$ and the support of χ_ε)

$$\int_0^T D_\varepsilon(f_k)\,dt \underset{k\to+\infty}{\longrightarrow} \int_0^T \int_{\mathbb{R}^N\times\mathbb{R}^N} |u|^3 \beta(\mathcal{E}_\infty, u)\,\chi_\varepsilon(v, v_*)\,d\bar{\mu}\,d\bar{\mu}_*\,dt, \tag{5.3}$$

where we have defined for any measure (or function) λ:

$$D_\varepsilon(\lambda) := \int_{\mathbb{R}^N\times\mathbb{R}^N} |u|^3\,\beta(\mathcal{E}, u)\,\chi_\varepsilon(v, v_*)\,d\lambda(v)\,d\lambda(v_*).$$

From the dissipation of energy (1.7) and the estimate from below (1.18), there holds

$$\frac{d}{dt}\mathcal{E}(t) \le -D(f) \quad \text{with} \quad D(f) := \int_{\mathbb{R}^N\times\mathbb{R}^N} |u|^3\,\beta(\mathcal{E}, u)\,f\,f_*\,dv\,dv_*,$$

which in turn implies that $t \mapsto D(f(t,.)) \in L^1(0,\infty)$, and then

$$\int_0^T D_\varepsilon(f_k)\,dt \le \int_0^T D(f_k)\,dt = \int_{t_{n_k}}^{t_{n_k}+T} D(f)\,dt \underset{k\to\infty}{\longrightarrow} 0. \tag{5.4}$$

Gathering (5.3) and (5.4), and letting ε goes to 0, we deduce that

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} |u|^3 \beta(\mathcal{E}_\infty, u)\,d\bar{\mu}\,d\bar{\mu}_* = 0 \quad \text{on} \quad (0,T).$$

The positivity (1.17) of $\beta(\mathcal{E}_\infty, u)$ then implies that $\bar{\mu} = \bar{c}\,\delta_{v=\bar{w}}$ for some measurable functions $\bar{w} : (0,T) \to \mathbb{R}^N$ and $\bar{c} : (0,T) \to \mathbb{R}_+$. Moreover, from the conservation of mass and momentum (1.30) and the bound of energy (1.31) we deduce that $\bar{c} = 1$ and $\bar{w} = 0$ a.e. It is then classical to deduce (by the uniqueness of the limit and the fact that it is independent on time) that (5.1) holds. □

To conclude that this weak convergence of the distribution to the Dirac mass as time goes to infinity implies the convergence of the kinetic energy to 0 (i.e. the kinetic energy of the Dirac mass) we have to show that no kinetic energy is "created" at infinify as $t \to T_c$. To this purpose we put stronger assumptions on the collision rate. The first additional assumption **H3** roughly speaking means that the energy dissipation functional is strong enough to forbid it, whereas the second additional assumption **H4** allows to use the uniform propagation of moments of order strictly greater than 2 to forbid it.

Proposition 5.2 *Let $T_c \in (0, +\infty]$ be the time of life of the solution. Then if either $T_c < +\infty$, or $T_c = \infty$ and B satisfies additional assumptions* **H2-H3** *or* **H2-H4***, we have*

$$\mathcal{E}(t) \to 0 \quad \textit{when} \quad t \to T_c. \tag{5.5}$$

Proof of Proposition 5.2. We split the proof in three steps.

Step 1. Assume first $T_c < \infty$. The claim follows from the existence proof.

Step 2. Assume now $T_c = \infty$ and that B satisfies assumption **H3**: (1.18)-(1.19). We argue by contradiction: assume that $\mathcal{E}(t) \not\to 0$, that is, there exists $\mathcal{E}_\infty > 0$ such that $\mathcal{E}(t) \in (\mathcal{E}_\infty, \mathcal{E}_{\rm in})$. Reasoning as in Proposition 5.1, we get, for a fixed time $T > 0$ and for any sequence (t_n) increasing and going to $+\infty$, that there exists a subsequence (t_{n_k}) and a measure $\bar\mu \in L^\infty(0,T; M^1_2)$ such that the function $f_k(t,v) := f(t_{n_k}+t, v)$ satisfies (5.2) and

$$\int_0^T D^0_\varepsilon(f_k)\,dt \to \int_0^T D^0_\varepsilon(\bar\mu)\,dt, \tag{5.6}$$

where we have defined for any measure (or function) λ:

$$D^0_\varepsilon(\lambda) := \int_{\mathbb{R}^N\times\mathbb{R}^N} |u|^3\,\psi(|u|)\,\chi_\varepsilon(v,v_*)\,d\lambda(v)\,d\lambda(v_*).$$

From the dissipation of energy (1.7) and the estimate from below (1.18), there holds

$$\frac{d}{dt}\mathcal{E}(t) \le -D^0(f) \quad \text{with} \qquad D^0(f) := \int_{\mathbb{R}^N\times\mathbb{R}^N} |u|^3\,\psi(|u|)\,f\,f_*\,dvdv_*, \tag{5.7}$$

which in turn implies that $t \mapsto D^0(f(t,.)) \in L^1(0,\infty)$, and then

$$\int_0^T D^0_\varepsilon(f_k)\,dt \le \int_0^T D_0(f_k)\,dt = \int_{t_{n_k}}^{t_{n_k}+T} D_0(f)\,dt \underset{k\to\infty}{\longrightarrow} 0. \tag{5.8}$$

Gathering (5.6) and (5.8), and letting ε goes to 0, we deduce that $D^0(\bar\mu) = 0$ on $(0,T)$. The positivity of ψ implies as in Proposition 5.1 that $\operatorname{supp}\bar\mu \subset \{0\}$ and $\bar\mu = \delta_{v=0}$. As this limit is unique and independent on time we deduce that (5.1) holds.

Now, on the one hand, taking $R = \sqrt{\mathcal{E}_\infty/2}$ there holds

$$\int_{B_R^c} f\,|v|^2\,dv = \int_{\mathbb{R}^N} f\,|v|^2\,dv - \int_{B_R} f\,|v|^2\,dv \ge \mathcal{E}_\infty - R^2 \ge \mathcal{E}_\infty/2 \tag{5.9}$$

for any $t \ge 0$. On the other hand, for T large enough, there holds thanks to (5.1)

$$\int_{B_{R/2}} f\,dv \ge \frac{1}{2} \quad \text{for any} \quad t \ge T. \tag{5.10}$$

Remarking that on $B_{R/2} \times B_R^c$ there holds, thanks to (1.19),

$$|u|^3\,\psi(|u|) \ge \frac{|v_*|^3}{8}\,\psi\left(\frac{|v_*|}{2}\right) \ge \psi_R\,\frac{|v_*|^2}{4}, \tag{5.11}$$

we may put together (5.7)-(5.11) and we get thanks to (5.9) and (5.10)

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}(t) &\leq -\int_{B_{R/2}}\int_{B_R^c} |v-v_*|^3\,\psi(|v-v_*|)\, f\, f_*\, dvdv_* \\
&\leq -\frac{\psi_R}{4}\int_{B_{R/2}} f\, dv \int_{B_R^c} f_*\,|v_*|^2\, dv_* \leq -\frac{\psi_R}{4}\,\frac{1}{2}\,\frac{\mathcal{E}_\infty}{2}
\end{aligned}$$

for any $t \geq T$. This implies that $\mathcal{E}$ becomes negative in finite time and we get a contradiction.

Step 3. Finally, assume that $T_c = +\infty$ and B satisfies assumption **H4**. On the one hand, thanks to (3.6), there holds

$$\sup_{[0,\infty)} \int_{\mathbb{R}^N} f(t,v)\,|v|^3\, dv < \infty.$$

On the other hand, arguing as in Step 2, we obtain (keeping the same notations) that (5.2) and then (from the uniform bound in L^1_3)

$$\mathcal{E}(f_k) \to \bar{\mathcal{E}} = \mathcal{E}(\bar{\mu}) \quad \text{ and } \quad D(\bar{\mu}) = 0.$$

The dissipation of energy vanishing implies that

$$|u|^3\,\mu\,\mu_* \equiv 0 \quad \text{or} \quad \beta(\bar{\mathcal{E}}, u) \text{ is not positive on } (0,T)\times\mathbb{R}^{2N}.$$

In the first case we deduce that $\bar{\mu} = \delta_{v=0}$ as in Step 2 and then $\bar{\mathcal{E}} = \mathcal{E}(\delta_{v=0}) = 0$. In the second case we deduce, from (1.17), that $\bar{\mathcal{E}}$ is not positive. In both case, there exists τ_k such that $\tau_k \to \infty$ and $\mathcal{E}(\tau_k) \to 0$ and therefore (5.2) holds since $\mathcal{E}$ is decreasing. □

Now we turn to some criterions for the cooling process to occur or not in finite time.

Proposition 5.3 *Assume that α is bounded near $\mathcal{E} = 0$, and $j_\mathcal{E}$ converges to 0 as $\varepsilon \to 0$ uniformly near $\mathcal{E} = 0$, then $T_c = +\infty$.*

Proof of Proposition 5.3. It is enough to remark that, thanks to the hypothesis made on α and $j_\mathcal{E}$, the *a priori* bound in Orlicz norm that one deduces from (2.10) as in Corollary 2.6 extends to all times:

$$\forall\, t \geq 0 \qquad \|f_t\|_{L^\Lambda} \leq \|f_{\rm in}\|_{L^\Lambda}\, \exp\Big(C\,\|f_{\rm in}\|_{L^1_2}\, t\Big)$$

for some constant C depending on the collision rate. It shows that the energy cannot vanish in finite time. □

Proposition 5.4 *Assume that B satisfies* **H4***, that for some increasing and positive function β_0 there holds $\beta(\mathcal{E}, u) \leq \beta_0(\mathcal{E})$ for any $u \in \mathbb{R}^N$, $\mathcal{E} \geq 0$, and that $f_{\rm in}\, e^{r\,|v|^\eta} \in L^1$ for some $r > 0$ and $\eta \in (1,2]$, then $T_c = +\infty$.*

Proof of Proposition 5.4. From the dissipation of energy (1.7), the bound on β and the decay of the energy (1.31) we have

$$\frac{d\mathcal{E}}{dt} \ge -\beta_0(\mathcal{E}_{\rm in}) \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f\, f_*\, |u|^3\, dv dv_* =: -\beta_0(\mathcal{E}_{\rm in})\, (I_{1,R} + I_{2,R})$$

where

$$\begin{cases} I_{1,R} := \displaystyle\int_{\mathbb{R}^N\times\mathbb{R}^N} |u|^3\, \mathbf{1}_{\{|u|\le R\}}\, f\, f_*\, dv\, dv_* \\ I_{2,R} := \displaystyle\int_{\mathbb{R}^N\times\mathbb{R}^N} |u|^3\, \mathbf{1}_{\{|u|\ge R\}}\, f\, f_*\, dv\, dv_*. \end{cases}$$

On the one hand, for any $R > 0$, we have using (1.30)

$$I_{1,R} \le R \int_{\mathbb{R}^N\times\mathbb{R}^N} |u|^2\, f\, f_*\, dv\, dv_* = 2\, R\, \mathcal{E}.$$

On the other hand, we infer from Proposition 3.2 (since B satisfies **H4**) that

$$\sup_{t\in[0,T_c)} \int_{\mathbb{R}^N} f(t,v)\, e^{2\, r'\, |v|^\eta}\, dv \le C_1$$

for some $r', C_1 \in (0,\infty)$. Therefore

$$\begin{aligned} I_{2,R} &\le \int_{\mathbb{R}^N\times\mathbb{R}^N} (4\,|v|^3 + 4\,|v_*|^3)\, 2\, \mathbf{1}_{\{|v|>R/2\}}\, f\, f_*\, dv\, dv_* \\ &\le 8\, e^{-r'\, R^\eta} \int_{\mathbb{R}^N} (1+|v|^3)\, e^{r'\,|v|^\eta}\, f\, dv \int_{\mathbb{R}^N} (1+|v_*|^3)\, f_*\, dv_* \le C_2\, e^{-r'\, R^\eta}. \end{aligned}$$

Gathering these three estimates, we deduce

$$\frac{d}{dt}\mathcal{E} \ge -C_3\, R\, \mathcal{E} - C_3\, e^{-r'\, R^\eta},$$

which in turns implies, thanks to the Gronwall Lemma,

$$\forall\, R > 0, \qquad \inf_{t\in[0,T]} \mathcal{E}(t) \ge \mathcal{E}_{\rm in}\, e^{-C_3\, R\, T} - \frac{e^{-r'\, R^\eta}}{R}.$$

We conclude that $\mathcal{E}(t) > 0$ for any $t \in [0,T]$ and any fixed $T > 0$, choosing R large enough (using that $\eta > 1$). $\square$

Proposition 5.5 *Assume $\beta(\mathcal{E},u) \ge \beta_0\, \mathcal{E}^\delta$ with $\beta_0 > 0$ and $\delta < -1/2$, then $T_c < +\infty$.*

Proof of Proposition 5.5. On the one hand, from the dissipation of energy (1.7) and the bound on β, we have

$$\frac{d\mathcal{E}}{dt} \le -\beta_0\, \mathcal{E}^\delta \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f\, f_*\, |u|^3\, dv dv_*.$$

On the other hand, from Jensen inequality and the conservation of mass and momentum, there holds

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f\, f_*\, |u|^3\, dv dv_* \geq \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f\, f_*\, |u|^2\, dv dv_*\right)^{3/2} = (2\,\mathcal{E})^{3/2}.$$

Gathering these two estimates, we get

$$\frac{d}{dt}\mathcal{E} \leq -\beta_0\, \mathcal{E}^{\delta+3/2}$$

and $\mathcal{E}$ vanishes in finite time. □

Appendix: Some facts about Orlicz spaces

The goal of this appendix is to gather some results about Orlicz spaces in order to make this paper as self-contained as possible. The definition and Hölder's inequality are recalls of results which can be found in [26] for instance. We also state and prove a simple formula for the differential of Orlicz norms, which is most probably not new, but for which we were not able to find a reference.

Definition

We recall here the definition of Orlicz spaces on $\mathbb{R}^N$ according to the Lebesgue measure. Let $\Lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be a function C^2 strictly increasing, convex, such that

$$\Lambda(0) = \Lambda'(0) = 0, \tag{A.1}$$

$$\forall\, t \geq 0, \quad \Lambda(2\,t) \leq c_\Lambda\, \Lambda(t), \tag{A.2}$$

for some constant $c_\Lambda > 0$, and which is superlinear, in the sense that

$$\frac{\Lambda(t)}{t} \underset{t\to+\infty}{\longrightarrow} +\infty. \tag{A.3}$$

We define L^Λ the set of measurable functions $f : \mathbb{R}^N \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^N} \Lambda(|f(v)|)\, dv < +\infty.$$

Then L^Λ is a Banach space for the norm

$$\|f\|_{L^\Lambda} = \inf\left\{\lambda > 0 \mid \int_{\mathbb{R}^N} \Lambda\left(\frac{|f(v)|}{\lambda}\right) dv \leq 1\right\}$$

and it is called the *Orlicz space* associated with Λ. The proof of this last point can be found in [26, Chapter III, Theorem 3]. Note that the usual Lebesgue spaces L^p for $1 \leq p < +\infty$ are recovered as particular cases of this definition for $\Lambda(t) = t^p$.

Let us mention that for any $f \in L^1(\mathbb{R}^N)$, a refined version of the De la Vallée-Poussin theorem [20, Proposition I.1.1] (see also [18, 19]) guarantees that there exists a function Λ satisfying all the properties above and

$$\int_{\mathbb{R}^N} \Lambda(|f(v)|)\, dv < +\infty.$$

Hölder's inequality in Orlicz spaces

Let Λ a function C^2 strictly increasing, convex satisfying the assumptions (A.1), (A.2) and (A.3), and Λ^* its *complementary Young function*, given (when Λ is C^1) by

$$\forall\, y \geq 0, \quad \Lambda^*(y) = y(\Lambda')^{-1}(y) - \Lambda((\Lambda')^{-1}(y)).$$

It is straightforward to check that Λ^* satisfies the same assumptions as Λ. Recall the Young's inequality

$$\forall\, x,\, y \geq 0, \qquad x\, y \leq \Lambda(x) + \Lambda^*(y). \tag{A.4}$$

Then one can define the following norm on the Orlicz space L^{Λ^*}:

$$N^{\Lambda^*}(f) = \sup\left\{ \int_{\mathbb{R}^N} |fg|\, dv\,;\ \int_{\mathbb{R}^N} \Lambda(|g|)\, dv \leq 1 \right\}.$$

One can extract from [26, Chapter III, Section 3.4, Propositions 6 and 9] the following result

Theorem A.1 *(i) We have the following Hölder's inequality for any $f \in L^\Lambda$, $g \in L^{\Lambda^*}$:*

$$\int_{\mathbb{R}^N} |fg|\, dv \leq \|f\|_{L^\Lambda}\, N^{\Lambda^*}(g). \tag{A.5}$$

(ii) There is equality in (A.5) if and only if there is a constant $0 < k^* < +\infty$ such that

$$\left(\frac{|f|}{\|f\|_{L^\Lambda}}\right)\left(\frac{k^*|g|}{N^{\Lambda^*}(g)}\right) = \Lambda\left(\frac{|f|}{\|f\|_{L^\Lambda}}\right) + \Lambda^*\left(\frac{k^*|g|}{N^{\Lambda^*}(g)}\right) \tag{A.6}$$

for almost every $v \in \mathbb{R}^N$.

Differential of Orlicz norms

In order to propagate bounds on Orlicz norms along the flow of the Boltzmann equation, we shall need a formula for the time derivative of the Orlicz norm.

Theorem A.2 *Let Λ be a function C^2 strictly increasing, convex satisfying (A.1), (A.2), (A.3), and let $0 \neq f \in C^1([0,T], L^\Lambda)$. Then we have*

$$\frac{d}{dt}\|f_t\|_{L^\Lambda} = \left[N^{\Lambda^*}\left(\Lambda'\left(\frac{|f|}{\|f\|_{L^\Lambda}}\right)\right)\right]^{-1} \int_{\mathbb{R}^N} \partial_t f\, \Lambda'\left(\frac{|f|}{\|f\|_{L^\Lambda}}\right) dv. \tag{A.7}$$

Proof of Theorem A.2. From [26, Chapter III, Proposition 6]), our assumptions on Λ imply that

$$\int_{\mathbb{R}^N} \Lambda\left(\frac{|f|}{\|f\|_{L^\Lambda}}\right) dv = 1 \tag{A.8}$$

for all $0 \neq f \in L^\Lambda$. By differentiating this quantity along t we get:

$$0 = \int_{\mathbb{R}^N} \partial_t f \, \Lambda'\left(\frac{|f|}{\|f\|_{L^\Lambda}}\right) dv - \frac{1}{\|f_t\|_{L^\Lambda}} \frac{d}{dt}\|f_t\|_{L^\Lambda} \int_{\mathbb{R}^N} f \, \Lambda'\left(\frac{|f|}{\|f\|_{L^\Lambda}}\right) dv$$

Now using the case of equality in the Hölder's inequality (A.5) we have

$$\int_{\mathbb{R}^N} f \, \Lambda'\left(\frac{|f|}{\|f\|_{L^\Lambda}}\right) dv = \|f\|_{L^\Lambda} \, N^{\Lambda^*}\left(\Lambda'\left(\frac{|f|}{\|f\|_{L^\Lambda}}\right)\right)$$

since the equality (A.6) is trivially satisfied with

$$g = \Lambda'\left(\frac{|f|}{\|f\|_{L^\Lambda}}\right)$$

and $k^* = N^{\Lambda^*}(g)$, using that

$$xy = \Lambda(x) + \Lambda^*(y)$$

as soon as $y = \Lambda'(x)$. This concludes the proof. □

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