Cooling process for inelastic Boltzmann equations for hard spheres, Part II: Self-similar solutions and tail behavior

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Abstract

We consider the spatially homogeneous Boltzmann equation for inelastic hard spheres, in the framework of so-called *constant normal restitution coefficients*. We prove the existence of self-similar solutions, and we give pointwise estimates on their tail. We also give more general estimates on the tail and the regularity of generic solutions. In particular we prove Haff's law on the rate of decay of temperature, as well as the algebraic decay of singularities. The proofs are based on the regularity study of a rescaled problem, with the help of the regularity properties of the gain part of the Boltzmann collision integral, well-known in the elastic case, and which are extended here in the context of granular gases.

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1 Introduction and main results

1.1 The model

We consider the asymptotic behavior of inelastic hard spheres described by the spatially homogeneous Boltzmann equation with a constant normal restitution coefficient (see [29]). More precisely, the gas is described by the probability density of particles $f(t,v) \geq 0$ with velocity $v \in \mathbb{R}^N$ $(N \geq 2)$ at time $t \geq 0$, which undergoes the evolution equation

$$\frac{\partial f}{\partial t} = Q(f,f) \quad \text{in} \quad (0,+\infty) \times \mathbb{R}^N, \tag{1.1}$$

$$f(0) = f_{\text{in}} \quad \text{in} \quad \mathbb{R}^N. \tag{1.2}$$

The bilinear collision operator $Q(f,f)$ models the interaction of particles by means of inelastic binary collisions (preserving mass and momentum but dissipating kinetic energy). Denoting by $e \in (0,1)$ the *normal restitution coefficient*, we define the collision operator in strong form as

$$Q(g,f)(v) := \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \left(\frac{'f\, 'g_*}{e^2} - f g_* \right) |u| \, b(\hat{u} \cdot \sigma) \, d\sigma \, dv_*, \tag{1.3}$$

where we use notations from [19]. Here $u = v - v_*$ denotes the relative velocity, $\hat{u}$ stands for $u/|u|$, and $'v$, $'v_*$ denotes the possible pre-collisional velocities leading to post-collisional velocities v, v_*. They are defined by

$$'v = \frac{v+v_*}{2} + \frac{'u}{2}, \qquad 'v_* = \frac{v+v_*}{2} - \frac{'u}{2},$$

with $'u = (1-\beta)u + \beta|u|\sigma$ and $\beta = (e+1)/(2e)$ ($\beta \in (1,\infty)$ since $e \in (0,1)$). The function b in (1.3) is (up to a multiplicative factor) the differential collisional cross-section. In the sequel we assume that there exists $b_0, b_1 \in (0,\infty)$ such that

$$\forall\, x \in [-1,1], \quad b_0 \le b(x) \le b_1, \tag{1.4}$$

and that

$$b \text{ is nondecreasing and convex on } (-1,1). \tag{1.5}$$

Note that the "physical" cross-section for hard spheres is given by (see [19, 12])

$$b(x) = \mathrm{cst}\, (1-x)^{-\frac{N-3}{2}},$$

so that it fulfills hypothesis (1.4) and (1.5) when $N = 3$. The Boltzmann equation (1.1) is complemented with an initial datum (1.2) which satisfies (for some $k \ge 2$)

$$0 \le f_{\rm in} \in L^1_k(\mathbb{R}^N), \qquad \int_{\mathbb{R}^N} f_{\rm in}\, dv = 1, \qquad \int_{\mathbb{R}^N} f_{\rm in}\, v\, dv = 0. \tag{1.6}$$

Notice that, without loss of generality, we can assume the two last moment conditions in (1.6), since we may always reduce to that case by a scaling and translation argument.

As explained in [29], the operator (1.3) preserves mass and momentum:

$$\frac{d}{dt} \int_{\mathbb{R}^N} f \begin{pmatrix} 1 \\ v \end{pmatrix} dv = 0, \tag{1.7}$$

while kinetic energy is dissipated

$$\frac{d}{dt} \mathcal{E}(f(t,\cdot)) = -D(f(t,\cdot)), \quad \text{with} \quad \mathcal{E}(f) = \int_{\mathbb{R}^N} f(v)\, |v|^2 \, dv. \tag{1.8}$$

The dissipation functional is given by

$$D(f) := \tau \int_{\mathbb{R}^N \times \mathbb{R}^N} f\, f_*\, |u|^3 \, dv\, dv_*, \quad \tau := m_b \left(\frac{1-e^2}{4} \right),$$

where m_b is the angular momentum defined by

$$m_b := \int_{\mathbb{S}^{N-1}} \left(\frac{1-(\hat{u}\cdot\sigma)}{2}\right) b(\hat{u}\cdot\sigma)\,d\sigma = |\mathbb{S}^{N-2}| \int_0^\pi b(\cos\theta)\,\sin^2\theta/2\,\sin^{N-2}\theta\,d\theta$$

(in the second formula, we have set $\cos\theta = \hat{u}\cdot\sigma$).

The study of the Cauchy theory and the cooling process of (1.1)-(1.2) was done in [29] (where more general models were considered). The equation is well-posed for instance in L^1_2: for $0 \le f_{\rm in} \in L^1_2$, there is a unique solution in $C(\mathbb{R}_+; L^1_2) \cap L^1(\mathbb{R}_+; L^1_3)$ (see Subsection 1.4 for the notations of functional spaces). This solution is defined for all times. It preserves mass, momentum and has a decreasing kinetic energy. The cooling process does not occur in finite time, but asymptotically in large time, i.e. the kinetic energy is strictly positive for all times and the solution satisfies

$$\mathcal{E}(t) \to 0 \quad \text{and} \quad f(t,\cdot) \rightharpoonup \delta_{v=0} \;\text{ in }\; M^1(\mathbb{R}^N)\text{-weak } * \quad \text{when} \quad t \to +\infty,$$

where $M^1(\mathbb{R}^N)$ denotes the space of probability measures on $\mathbb{R}^N$. We refer to [29] for the proofs of these results and for the study of other physically relevant models for which the cooling process occurs in finite time.

1.2 Introduction of rescaled variables

Let us introduce some rescaled variables, in order to study more precisely the asymptotic behavior of the solution. This usual rescaling can be found in [9] and [15] for instance. Roughly speaking it adds an anti-drift to the equation. As we shall show in the sequel, this additional term prevents concentration and thus forces the kinetic energy to remain bounded from below.

We search for a rescaled solution g of the form

$$f(t,v) = K(t)\,g(T(t), V(t,v)),$$

where K, T, V are the scaling functions to be determined. Imposing the conservation of mass and the cancellations of the multiplicative term in front of g in the evolution equation, and using the following homogeneity property

$$Q(g(\lambda\cdot), g(\lambda\cdot))(v) = \lambda^{-(N+1)}\,Q(g,g)(\lambda v) \tag{1.9}$$

(which is obtained by a homothetic change of variable), we obtain by some classical computations the natural choice

$$K(t) = (c_0 + c_1 t)^N, \quad T(t) = \frac{1}{c_0}\ln\left(1 + \frac{c_1}{c_0}t\right) \quad V(t,v) = (c_0 + c_1 t)v$$

for some constants $c_0, c_1 > 0$. A solution f associated to some function g independent of T in this new variables is called a *self-similar solution*, with *profil* g. It is obvious that changing c_0 in this scaling only amounts to a translation of time of the self-similar solution, and changing c_1 only amounts to the multiplication of the self-similar solution by a constant. Hence in the following we fix without restriction $c_0 = c_1 = 1$. Then it is straightforward that g satisfies the evolution equation

$$\frac{\partial g}{\partial t} = Q(g,g) - \nabla_v \cdot (vg). \tag{1.10}$$

This evolution problem preserves the L^1 norm. Any steady state $G(v)$ of (1.10) translates into a self-similar solution

$$F(t,v) = (1+t)^N \, G((1+t)v)$$

of the original equation (1.1). More generally, for any solution g to the Boltzmann equation in self-similar variables (1.10), we associate a solution f to the evolution problem (1.1), defining f by the relation

$$f(t,v) = (1+t)^N \, g(\ln(1+t), (1+t)v). \tag{1.11}$$

Reciprocally, for any solution f to the Boltzmann equation (1.1), we associate a solution g to the evolution problem (1.10), defining g by the relation

$$g(t,v) = e^{-Nt} \, f(e^t - 1, e^{-t} v). \tag{1.12}$$

Given an initial datum $f_{\text{in}} = g_{\text{in}} \in L^1_2$, we know from [29] that there exists a unique solution of (1.1) in $C(\mathbb{R}_+, L^1_2) \cap L^1(\mathbb{R}_+, L^1_3)$. Therefore, thanks to the changes of variables (1.11), (1.12), we deduce that there exists a unique solution g to (1.10) in $C(\mathbb{R}_+, L^1_2) \cap L^1(\mathbb{R}_+, L^1_3)$. Moreover we have the following relations between the moments of f and g:

$$\forall\, t \geq 0, \quad \left\{ \begin{array}{l} \|g(t,\cdot)\,|\cdot|^k\|_{L^1} = e^{k\,t} \, \|f(e^t-1,\cdot)\,|\cdot|^k\|_{L^1} \\[2mm] \|f(t,\cdot)\,|\cdot|^k\|_{L^1} = (1+t)^{-k} \, \|g(\ln(1+t),\cdot)\,|\cdot|^k\|_{L^1}. \end{array} \right. \tag{1.13}$$

1.3 Motivation

On the basis of the study of the non-physical simplified case of Maxwell molecules, Ernst and Brito conjectured that self-similar solutions, when they exist, should attract any solution, in the sense of convergence of the rescaled solution. Thus existence of and informations on these self-similar solutions is expected to yield informations on the asymptotic behavior of the generic solutions. And, as our study shows (as well as the study of diffusively excited inelastic hard spheres in [19]), the over-populated high energy tails for the self-similar solutions precisely indicate the tail behavior of the generic solutions.

Moreover, the kinetic energy of the self-similar solutions behaves like

$$\mathcal{E}(t) \sim_{t \to \infty} \frac{C}{t^2}$$

and it is natural to expect a similar behavior for the rate of decay of the temperature for the generic solutions. This conjecture was made twenty years ago in the pioneering paper [24], where the model of inelastic hard spheres was introduced, and this rate of decay for the temperature is therefore known as *Haff's law*. This law is a typical physical feature of inelastic hard spheres which does not hold for the simplified model of Maxwell molecules. Indeed, in this case, one can derive a closed equation for the kinetic energy, which decreases exponentially fast. More generally, the tail behavior of the solutions are different for Maxwell molecules and hard spheres.

However, until now, mathematical analysis of (spatially homogeneous) inelastic Boltzmann equations essentially dealt with Maxwell molecules because of the strong analytic simplications it provides (see for instance [6, 7, 8]), or with some simplified non-linear friction models (see [26] for instance). In [6], a pseudo-Maxwell approximation of hard spheres was considered, preserving Haff's law and still most of the nice simplications of Maxwell molecules, but leading to different self-similar solutions and tail behaviors. Recently the works [19, 9] laid the first steps for a mathematical analysis of the more realistic inelastic hard spheres model: the former proved the existence of steady states and and gave estimates showing the presence of overpopulated tails for diffusively excited inelastic hard spheres, and the latter proved *a priori* integral estimates on the tail of the steady state (assuming its existence) for the spatially homogeneous inelastic Boltzmann equation with various additionnal terms, such as a diffusion, or an anti-drift as in (1.10). These two papers are the starting point of our study.

In this work, we prove, for spatially homogeneous inelastic hard spheres, the existence of smooth self-similar solutions, and we improve the estimates on their tails of [9] into pontwise ones. We also give a complete regularity study of the generic solutions in the rescaled variables, as well as estimates on their tails. In particular, we give the first mathematical proof of Haff's law and we show the algebraic decay of singularities.

1.4 Notation

Throughout the paper we shall use the notation $\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}$. We denote, for any integer $k \in \mathbb{N}$, the Banach space

$$L^1_k = \left\{ f : \mathbb{R}^N \mapsto \mathbb{R} \text{ measurable}; \;\; \|f\|_{L^1_k} := \int_{\mathbb{R}^N} |f(v)| \, \langle v \rangle^k \, dv < +\infty \right\}.$$

More generally we define the weighted Lebesgue space $L^p_q(\mathbb{R}^N)$ ($p \in [1, +\infty]$, $q \in \mathbb{R}$) by the norm

$$\|f\|_{L^p_q(\mathbb{R}^N)} = \left[\int_{\mathbb{R}^N} |f(v)|^p \, \langle v \rangle^{pq} \, dv \right]^{1/p}$$

when $p < +\infty$ and

$$\|f\|_{L^\infty_q(\mathbb{R}^N)} = \sup_{v \in \mathbb{R}^N} |f(v)| \, \langle v \rangle^q$$

when $p = +\infty$. The weighted Sobolev space $W^{k,p}_q(\mathbb{R}^N)$ ($p \in [1, +\infty]$, $q \in \mathbb{R}$ and $k \in \mathbb{N}$) is defined by the norm

$$\|f\|_{W^{k,p}_q(\mathbb{R}^N)} = \left[\sum_{|s| \le k} \int_{\mathbb{R}^N} |\partial^s f(v)|^p \, \langle v \rangle^{pq} \, dv \right]^{1/p}.$$

Finally, for $h \in \mathbb{R}^N$, we define the translation operator τ_h by

$$\forall \, v \; \in \; \mathbb{R}^N, \quad \tau_h f(v) = f(v - h).$$

We shall denote by "C" various constants which do not depend on the collision kernel B.

1.5 Main results

First we state a result of existence of self-similar solutions.

Theorem 1.1 *For any mass $\rho > 0$, there exists a self-similar profil G with mass ρ and momentum 0:*

$$0 \le G \in L^1_2, \qquad Q(G,G) = \nabla_v \cdot (v\, G), \qquad \int_{\mathbb{R}^N} G \begin{pmatrix} 1 \\ v \end{pmatrix} dv = \begin{pmatrix} \rho \\ 0 \end{pmatrix},$$

which moreover can be built in such a way that G is radially symmetric, $G \in C^\infty$ and

$$\forall \, v \in \mathbb{R}^N, \quad a_1 e^{-a_2 |v|} \le G(v) \le A_1 e^{-A_2 |v|}$$

for some explicit constants $a_1, a_2, A_1, A_2 > 0$.

Second we give a result on the asymptotic behavior of the solution in the rescaled variables.

Theorem 1.2 *For an initial datum*

$$0 \le g_{\rm in} \in L^1_2 \quad \int_{\mathbb{R}^N} g_{\rm in} \begin{pmatrix} 1 \\ v \end{pmatrix} dv = \begin{pmatrix} \rho \\ 0 \end{pmatrix},$$

the unique solution g of (1.10) with initial datum $g_{\rm in}$ in $C(\mathbb{R}_+; L^1_2) \cap L^1(\mathbb{R}_+; L^1_3)$ satisfies:

(i) There exists $\lambda > 0$ such that the solution g can be written $g = g^S + g^R$ in such a way that $g_S \ge 0$ and

$$\begin{cases} \sup_{t \ge 0} \left\| g^S_t \right\|_{H^s_q} < +\infty, & \textit{for any arbitrarily large } s \ge 0, \ q \ge 0, \\ \left\| g^R_t \right\|_{L^1_2} = O\left(e^{-\lambda t}\right). \end{cases}$$

(ii) For any $\tau > 0$ and $s \in [0, 1/2)$, there are some explicit constants $a_1, a_2, A_1, A_2 > 0$ such that

$$\forall\, v \in \mathbb{R}^N, \quad \liminf_{t\to\infty} g(t,v) \geq a_1 e^{-a_2|v|}$$

and

$$\forall\, t \geq \tau, \quad \int_{\mathbb{R}^N} g(t,v)\, e^{-A_1|v|^s}\, dv \leq A_2.$$

All the constants in this theorem can be computed in terms of the mass and kinetic energy of $f_{\rm in}$ or $g_{\rm in}$, and τ.

Finally we state as a separate theorem a consequence of Theorem 1.2 in order to emphasize it.

Theorem 1.3 *As a consequence, for an initial datum*

$$0 \leq f_{\rm in} \in L^1_2 \quad \int_{\mathbb{R}^N} f_{\rm in} \begin{pmatrix} 1 \\ v \end{pmatrix} dv = \begin{pmatrix} \rho \\ 0 \end{pmatrix},$$

the associated solution of the Boltzmann equation (1.1,1.2) satisfies Haff's law in the sense:

$$\forall\, t \geq 0, \quad m\,(1+t)^{-2} \leq \mathcal{E}(t) \leq M\,(1+t)^{-2} \tag{1.14}$$

for some explicit constants $m, M > 0$ depending on the collision kernel and the mass and kinetic energy of f_{in}.

1.6 Method of proof

The main tool is the regularity theory of the collision operator. We show that the gain part satisfies similar regularity properties as in the elastic case, and we use them to study the regularity of the solution in the rescaled variables, in a similar way to the elastic case (see [33]). We show uniform propagation of Lebesgue and Sobolev norms as well as exponential decay of singularities for solutions of (1.10).

These uniform non-concentration estimates immediately show that the temperature is uniformly bounded from below by some positive number in the rescaled variables, which enables to prove Haff's law. The existence of self-similar solutions is proved by the use of a consequence of Tykhonov's fixed point Theorem (see Theorem 4.1), which is an infinite dimensional (rough) version of Poincaré-Bendixon Theorem on dynamical systems, see for instance [3, Théorème 7.4] or [19, 17]. It says that a semi-group on a Banach space $\mathcal{Y}$ with suitable continuity properties, and which stabilizes a nonempty convex weakly compact subset, has a steady state.

Essentially this result reduces the question of proving the existence of a steady state to the one of finding suitable *a priori* estimates on the evolution equation. We apply it to the evolution semi-group of (1.10) in the Banach space $\mathcal{Y} = L^1_2 \cap L^p$, $p > 1$. The existence and continuity properties of the semi-group were proved in [29] and the nonempty convex weakly compact subset of functions with bounded moments and L^p norm (for some $1 < p < +\infty$ and a bound big enough) is stable along the flow thanks to the uniform L^p bounds obtained in the rescaled variables. Then, regularity of the profil is obtained by the previous regularity study, and pointwise estimates on the tail are obtained using results and methods of [9] on the study of moments, together with maximum principles arguments inspired from [19].

The regularity study in the rescaled variables translates in the original variables and yields the algebraic decay of singularities for the Cauchy problem (1.1,1.2). Then the tail of the solution is studied by classical techniques. For the lower bound on one hand we use the spreading effect of the evolution semi-group of (1.10) (in the spirit of [11, 34, 32]). For the upper bound on the other hand we use moments estimates as in [9] and elementary o.d.e. arguments.

1.7 Weak and strong forms of the collision operator

Under our assumptions on b, the function $\sigma \mapsto b(\hat{u} \cdot \sigma)$ is integrable on the sphere $\mathbb{S}^{N-1}$, and we can set without restriction

$$\int_{\mathbb{S}^{N-1}} b(\hat{u} \cdot \sigma)\, d\sigma = |\mathbb{S}^{N-2}| \int_0^\pi b(\cos\theta)\, \sin^{N-2}\theta\, d\theta = 1.$$

Thus we can write the classical splitting $Q = Q^+ - Q^-$ between gain part and loss part. The loss part Q^- is

$$Q^-(g,f)(v) := \left(\int_{\mathbb{R}^N} g(v_*)\, |v - v_*|\, dv_* \right) f(v) = (g * \Phi) f, \tag{1.15}$$

where Φ denotes $\Phi(z) = |z|$. For any distribution g satifying the moment conditions $\int_{\mathbb{R}^N} g\, dv = 1$, $\int_{\mathbb{R}^N} g\, v\, dv = 0$, we have (see for instance [29, Lemma 2.2])

$$(g * \Phi) \geq |v|. \tag{1.16}$$

The gain part Q^+ is defined by

$$Q^+(g,f)(v) := \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \frac{'f\, 'g_*}{e^2}\, |u|\, b(\hat{u} \cdot \sigma)\, d\sigma\, dv_*. \tag{1.17}$$

In the sequel, we shall need two other representations. On the one hand from [12], there holds: for any $\psi \in L^\infty_1$,

$$\int_{\mathbb{R}^N} Q^+(g,f)(v)\, \psi(v)\, dv = \int_{\mathbb{R}^N \times \mathbb{R}^N \times \mathbb{S}^{N-1}} f\, g_*\, |u|\, b(\hat{u} \cdot \sigma)\, \psi(v')\, d\sigma\, dv_*\, dv, \tag{1.18}$$

where v' denotes the post-collisional velocity defined by

$$v' = \frac{v + v_*}{2} + \frac{u'}{2}, \qquad u' = \frac{1-e}{2}\, u + \frac{1+e}{2}\, |u|\, \sigma. \tag{1.19}$$

On the other hand, we shall establish a Carleman type representation for granular gases:

Proposition 1.4 *Let $E^e_{v,'v}$ be the hyperplan orthogonal to the vector $v - \,'v$ and passing through the point $\Omega(v,\,'v)$, defined by*

$$\Omega(v,\,'v) := v + (1-\beta^{-1})\,(v - \,'v) = \big(2-\beta^{-1}\big)\,v + \big(\beta^{-1}-1\big)\,\,'v.$$

(recall that $\beta = (1+e)/(2e)$). Then we have the following representation of the gain term

$$(1.20)\quad Q^+(g,f)(v) = \frac{2^{N-1}}{\beta^{N-1}e^2}\int_{'v\in\mathbb{R}^N}\int_{'v_*\in E^e_{v,'v}} |v-v_*|^{2-N}\,B\,|'v-v|^{-1}\,\,'g_*\,\,'f,$$

where

$$B := B(u,\sigma) = |u|\,b(\hat{u}\cdot\sigma).$$

Proof of Proposition 1.4. We start from the basic identity

$$(1.21)\qquad \frac{1}{2}\int_{\mathbb{S}^{N-1}} F(|u|\sigma - u)\,d\sigma = \frac{1}{|u|^{N-2}}\int_{\mathbb{R}^N}\delta(2\,x\cdot u + |x|^2)\,F(x)\,dx,$$

which can be verified easily by completing the square in the Dirac function, taking the spherical coordinate $x+u = r\,\sigma$ and performing the change of variable $r^2 = s$. We have the following relations

$$\begin{cases} 'v = v + (\beta/2)\ (|u|\sigma - u) \\ 'v_* = v_* - (\beta/2)\ (|u|\sigma - u) \end{cases}$$

and thus starting from the strong form of Q^+ we get

$$Q^+(g,f) = e^{-2}\int_{\mathbb{R}^N\times\mathbb{S}^{N-1}} B\,f\Big(v+(\beta/2)\ (|u|\sigma-u)\Big)g\Big(v_*-(\beta/2)\ (|u|\sigma-u)\Big)\,dv_*\,d\sigma.$$

Applying (1.21) yields

$$Q^+(g,f) = 2e^{-2}\int_{\mathbb{R}^N\times\mathbb{R}^N} |u|^{2-N}\,B\,\delta(2\,x\cdot u+|x|^2)\,f(v+(\beta/2)x)g(v_*-(\beta/2)x)\,dv_*\,dx.$$

We do the change of variable $x \to \,'v = v + (\beta/2)x$ (with jacobian $(\beta/2)^N$). Then, keeping $'v$ fixed, we do the change of variable $v_* \to \,'v_*$ (with jacobian 1). This gives

$$Q^+(g,f) = \frac{2^{N+1}}{\beta^N e^2}\int_{\mathbb{R}^N\times\mathbb{R}^N} |u|^{2-N}\,B\,\delta\,f('v)g('v_*)\,d\,'v_*\,d\,'v.$$

Finally, keeping $'v$ fixed, we decompose orthogonally the variable $'v_*$ as $v+V_1\,n+V_2$ with $V_1 = ('v_*-v)\cdot n$, $n = ('v-v)/|'v-v|$ and V_2 orthogonal to $'v-v$. This gives after computing the Dirac function in the new coordinates

$$Q^+(g,f) = \frac{2^{N+1}}{\beta^N e^2}\int_{\mathbb{R}\times\mathbb{R}^{N-1}\times\mathbb{R}^N} |u|^{2-N}\,B$$
$$\delta\left(\frac{4|'v-v|}{\beta}\Big[(\beta^{-1}-1)|'v-v| - V_1\Big]\right) f('v)g(v+V_1\,n+V_2)\,dV_1\,dV_2\,d\,'v.$$

Removing the Dirac mass leads to (1.20). $\square$

The parametrization by the Carleman representation means that for v and $'v$ fixed, the point $'v_*$ describes the hyperplan orthogonal to $('v-v)$ and passing through the point $\Omega(v,'v)$ on the line determined by v and $'v$. Note that in the elastic case, $\Omega(v,'v) = v$, whereas here $\Omega(v,'v)$ is outside the segment $[v,'v]$, which reflects the fact that for the pre-collisional velocities, the modulus of the relative velocity is bigger than $|v - v_*|$. The geometrical picture (in a plane section) is summerized in Figure 1.

Figure 1: Carleman representation for granular gases

From this proposition we immediately deduce the following representation, which is closer to the classical Carleman representation for the elastic Boltzmann collision operator:

$$Q^+(f,g)(v) = C_e \int_{\mathbb{R}^N} \frac{'f}{|v - \, 'v|^{N-2}} \left\{ \int_{E^e_{v,'v}} {'g_*} \, \tilde{b}(\hat{u} \cdot \sigma) \, d\, 'v_* \right\} d\, 'v,$$

with

$$C_e = \frac{2^{\frac{3N-5}{2}}}{\beta^{2N-4} e^2}$$

and $\tilde{b}(x) = (1-x)^{-(N-3)} \, b(x)$.

2 Regularity properties of the collision operator

In this section the final goal is to estimate quantities such as

$$\int_{\mathbb{R}^N} Q(f,f) \, f^{p-1} \, dv$$

for $p > 1$, i.e. the action of the collision operator on the evolution of the L^p norm (to the power p) of the solution along the flow. We shall use minoration estimates on Q^- deduced from (1.15)-(1.16), together with regularity estimates on Q^+. The latters seem to be new in the inelastic framework but they are an extension of similar estimates in the elastic case $e = 1$, which originated in the works of Lions [27], Bouchut and Desvillettes [10], Lu [28], Mischler and Wennberg [31], Abrahamsson [1] (and are reminiscent of the work of Grad on the linearized collision operator [23]). The main tool is the Carleman representation for granular gases of Proposition 1.4. Before turning to the regularity study of Q^+, we recall convolution-like estimates.

2.1 Convolution-like estimates

In the elastic case $e = 1$, convolution-like estimates for the gain part of the collision operator were first proved in [21, 22]. This proof was simplified by a duality argument in [33], where also a more precise statement was given. These estimates were extended to the inelastic case, for a constant normal restitution coefficient $e \in (0,1]$, in [19] (in a form slightly less precise than in [33]). Also a result weaker in one aspect (less precise for the treatment of the algebraic weight) but more general in another (valid in any Orlicz spaces, and valid for more general collision kernels) was proved in [29]. Here we only state the precise result we shall need, whose proof is straightforward from the arguments in [19, Proof of Lemma 4.1] and [33, Proof of Theorem 2.1].

We make the following assumption on the cross-section: no *frontal collision* should occur, i.e. $b(\cos\theta)$ should vanish for θ close to π:

$$\exists\, \theta_b > 0 \ ; \quad \mathrm{support}\, b\,(\cos\theta) \subset \{\theta \ / \ \ 0 \le \theta \le \pi - \theta_b\} \tag{2.1}$$

This additional assumption will not be needed, however, for the quadratic estimates, i.e. the estimates on $Q^+(f,f)$. Indeed, $Q^+(f,g) = \bar{Q}^+(g,f)$ if $\bar{Q}^+$ is the gain term associated to the cross-section $\bar{b}(\cos\theta) = b(\cos(\pi - \theta))$. In particular, $b(\cos\theta)$ and $[b(\cos\theta) + b(\cos(\pi-\theta))]\, \mathbf{1}_{\cos\theta \ge 0}$ define the same quadratic operator Q^+, and the latter satisfies (2.1) automatically (with $\theta_b = \pi/2$). We note that $Q^+(g,f)$ and $Q^+(f,g)$ will not necessarily satisfy the same estimates, since assumption (2.1) is not symmetric. To exchange the roles of f and g, we will therefore be led to introduce the assumption that no grazing collision should occur, i.e.

$$\exists\, \theta_b > 0 \ ; \quad \mathrm{support}\, b\,(\cos\theta) \subset \{\theta \ / \ \ \theta_b \le \theta \le \pi\}\,. \tag{2.2}$$

Theorem 2.1 *Let $k, \eta \in \mathbb{R}$, $p \in [1, +\infty]$, and let $B = \Phi\, b$ be a collision kernel with b satisfying the assumption (2.1). Then, we have the estimates*

$$\left\| Q^+(g,f) \right\|_{L^p_\eta} \le C_{k,\eta,p}(B)\, \|g\|_{L^1_{|k+\eta|+|\eta|}}\, \|f\|_{L^p_{k+\eta}}\,,$$

where

$$C_{k,\eta,p}(B) = C\ \left(\sin(\theta_b/2)\right)^{\min(\eta,0)-2/p'}\ \ \|b\|_{L^1(\mathbb{S}^{N-1})}\, \|\Phi\|_{L^\infty_{-k}}$$

If on the other hand assumption (2.1) is replaced by assumption (2.2), then the same estimates hold with $Q^+(g,f)$ replaced by $Q^+(f,g)$.

2.2 Lions Theorem for Q^+

In this subsection we assume that the collision kernel $B = \Phi\, b$ satisfies

$$\Phi \in C_0^\infty(\mathbb{R}^N \setminus \{0\}), \quad b \in C_0^\infty(-1,1). \tag{2.3}$$

Then we have the

Theorem 2.2 *Let B be a collision kernel satisfying (2.3). Then the gain part Q^+ satisfies for all $s \in \mathbb{R}_+$ and $\eta \in \mathbb{R}_+$*

$$\|Q^+(g,f)\|_{H^{s+(N-1)/2}_\eta} \leq C(s,B)\, \|g\|_{H^s_\eta} \|f\|_{L^1_{2\eta}}$$

$$\|Q^+(g,f)\|_{H^{s+(N-1)/2}_\eta} \leq C(s,B)\, \|g\|_{H^s_\eta} \|f\|_{L^1_{2\eta}}$$

for some explicit constants $C(s,B) > 0$ depending only on s and the collision kernel.

Proof of Proposition 2.2. We follow closely the proof of [33], inspired from the works of Lions [27] and Wennberg [37]. Indeed the Carleman representation proved above in Proposition 1.4 allows essentially to reduce to the study of the elastic case.

We assume first that $\eta = 0$. We denote

$$\mathcal{B}(|'v -' v_*|, |'v - v|) = \frac{B(|v - v_*|, \cos\theta)}{|v - v_*|^{N-2} |'v - v|}$$

which belongs to $C_0^\infty((\mathbb{R}_+ \setminus \{0\})^2)$ under assumption (2.3). We define the following (Radon transform type) functional: for g smooth enough, Tg is defined by

$$Tg(y) = \int_{\mu y + y^\perp} \mathcal{B}(z,y)\, g(z)\, dz$$

with $\mu = 2 - \beta^{-1}$. Then a straightforward computation from the Carleman representation (1.20) yields

$$Q^+(g,f)(v) = 2^{N-1} \beta^{1-N} e^{-2} \int_{\mathbb{R}^N} f('v)\ (\tau_{'v} \circ T \circ \tau_{-'v})\,(g)(v)\, d\,'v.$$

Thus if one has a bound on T of the form

$$\|Tg\|_{H^{s+(N-1)/2}} \leq C_T\, \|g\|_{H^s}, \quad C_T > 0, \tag{2.4}$$

then by using Fubini's and Jensen's theorems one gets

$$\begin{aligned}
\|Q^+(g,f)\|^2_{H^{s+(N-1)/2}} &\leq C\, \|f\|_{L^1} \int_{\mathbb{R}^N} f('v)\ \|(\tau_{'v} \circ T \circ \tau_{-'v})\,(g)\|^2_{H^{s+(N-1)/2}}\ d'v \\
&\leq C\, \|f\|_{L^1} \int_{\mathbb{R}^N} f('v)\ \|(T \circ \tau_{-'v})\,(g)\|^2_{H^{s+(N-1)/2}}\ d'v \\
&\leq C\, C_T\, \|f\|_{L^1} \int_{\mathbb{R}^N} f('v)\ \|\tau_{-'v} g\|^2_{H^s}\ d'v \\
&\leq C\, C_T\ \|g\|^2_{H^s}\ \|f\|_{L^1} \int_{\mathbb{R}^N} f('v)\, d'v \leq C\, C_T\ \|g\|^2_{H^s}\ \|f\|^2_{L^1},
\end{aligned}$$

which concludes the proof. Thus we are reduced to prove (2.4). But, up to an homothetic factor, T is exactly the operator which was studied in detail in [37] and [33]. More precisely,

$$Tg(y) = \tilde{T}g(\mu y)$$

where $\tilde{T}$ is the Radon transform

$$\tilde{T}g(y) = \int_{y+y^\perp} \tilde{\mathcal{B}}(z,y)\, g(z)\, dz$$

introduced in the elastic case in [37], with

$$\tilde{\mathcal{B}}(z,y) = \mathcal{B}(z, \mu^{-1}\, y).$$

It was proved in [33, Proof of Theorem 3.1] that

$$\|\tilde{T}g\|_{H^{s+(N-1)/2}} \le C\, \|g\|_{H^s}$$

for an explicit bound C depending on $\tilde{\mathcal{B}}$. Coming back to T, we obtain (2.4). This ends the proof when $\eta = 0$. The extension to $\eta > 0$ is straightforward (and exactly similar to [33, Proof of Theorem 3.1]). $\square$

As a Corollary we deduce from Theorem 2.2 the following estimates in Lebesgue spaces by Sobolev embeddings (the proof is exactly similar to [33, Proof of Corollary 3.2]).

Corollary 2.3 *Let B be a collision kernel satisfying (2.3). Then, for all $p \in (1,+\infty)$, $\eta \in \mathbb{R}$, we have*

$$\begin{aligned}
\left\|Q^+(g,f)\right\|_{L^q_\eta} &\le C(p,\eta,B)\, \|g\|_{L^p_\eta}\, \|f\|_{L^1_{2|\eta|}} \\
\left\|Q^+(f,g)\right\|_{L^q_\eta} &\le C(p,\eta,B)\, \|g\|_{L^p_\eta}\, \|f\|_{L^1_{2|\eta|}}
\end{aligned}$$

where the constant $C(p,\eta,B) > 0$ only depends on the collision kernel, p and η, and $q > p$ is given by

$$q = \begin{cases} \dfrac{p}{2 - \frac{1}{N} + p\left(\frac{1}{N} - 1\right)} & \text{if } p \in (1;2] \\ pN & \text{if } p \in [2;+\infty). \end{cases}$$

2.3 Bouchut-Desvillettes-Lu Theorem on Q^+

Now we turn to a slightly different regularity estimate on Q^+, which is a straightforward extension of the works [10, 28] in the elastic case $e = 1$. This class of estimate is weaker than Lions's Theorem 2.2 since the Sobolev norm of Q^+ is controlled by the square of the Sobolev norm of the solution with smaller order, which does not allow to take advantage of the L^1 theory. Nevertheless, it is more convenient in other aspects since it deals directly with the physical collision kernel.

Here we assume that the collision kernel writes $B(v - v_*, \cos\theta) = |v - v_*|\, b(\cos\theta)$ with

(2.5) $$\|b\|^2_{L^2(\mathbb{S}^{N-1})} = \int_0^\pi b(\cos\theta)^2\, \sin^{N-1}\theta\, d\theta = c_2(b) < +\infty$$

(this assumption is obviously satisfied when b satisfies (1.4)). Then we have the

Theorem 2.4 *Let B be a collision kernel satisfying (2.5). Then the gain part Q^+ satisfies, for all $s \in \mathbb{R}_+$ and $\eta \in \mathbb{R}_+$,*

$$\|Q^+(g,f)\|_{H^{s+(N-1)/2}_\eta} \le C(s,B) \left[\|g\|_{H^s_{\eta+2}} \|f\|_{H^s_{\eta+2}} + \|g\|_{L^1_{\eta+2}} \|f\|_{L^1_{\eta+2}} \right]$$

for some explicit constant $C(s,B) > 0$ depending only on s and B.

Proof of Theorem 2.4. We follow closely the method in [10]. We write it for $\eta = 0$ but the general case is strictly similar.

Let us denote $F(v,v_*) = f(v)\, g(v_*)\, |v - v_*|$. The same arguments as in [10] easily lead to

$$\mathcal{F}Q^+(\xi) = \int_{\mathbb{S}^{N-1}} \widehat{F}(\xi^+, \xi^-)\, b(\hat{\xi}\cdot\sigma)\, d\sigma$$

where $\mathcal{F}Q^+$ denotes the Fourier transform of Q^+ according to v, $\widehat{F}$ denotes the Fourier transform of F according to v, v_*, and

$$\xi^+ = \frac{3-e}{4}\,\xi - \frac{1+e}{4}\,|\xi|\sigma, \qquad \xi^- = \frac{1+e}{4}\,\xi - \frac{1+e}{4}\,|\xi|\sigma.$$

Thus

$$|\mathcal{F}Q^+(\xi)|^2 \le \|b\|^2_{L^2(\mathbb{S}^{N-1})} \left(\int_{\mathbb{S}^{N-1}} |\widehat{F}(\xi^+,\xi^-)|^2\, d\sigma \right)$$

Let us consider frequencies ξ such that $|\xi| \ge 1$. As

$$\begin{aligned}
&\int_{\mathbb{S}^{N-1}} |\widehat{F}(\xi^+,\xi^-)|^2\, d\sigma \\
= &\int_{\mathbb{S}^{N-1}} \int_{|\xi|}^{+\infty} -\frac{\partial}{\partial r} \left| \widehat{F}\left(\frac{3-e}{4}\,\xi - \frac{1+e}{4}\, r\sigma, \frac{1+e}{4}\,\xi - \frac{1+e}{4}\, r\sigma \right) \right|^2 d\sigma\, dr \\
\le &\int_{\mathbb{S}^{N-1}} \int_{|\xi|}^{+\infty} \left| \widehat{F}\left(\frac{3-e}{4}\,\xi - \frac{1+e}{4}\, r\sigma, \frac{1+e}{4}\,\xi - \frac{1+e}{4}\, r\sigma \right) \right| \times \\
&\qquad \left| (\nabla_2 - \nabla_1)\widehat{F}\left(\frac{3-e}{4}\,\xi - \frac{1+e}{4}\, r\sigma, \frac{1+e}{4}\,\xi - \frac{1+e}{4}\, r\sigma \right) \right| d\sigma\, dr \\
\le &\int_{|\zeta| \ge |\xi|} \left| \widehat{F}\left(\frac{3-e}{4}\,\xi - \frac{1+e}{4}\, \zeta, \frac{1+e}{4}\,\xi - \frac{1+e}{4}\, \zeta \right) \right| \times \\
&\qquad \left| (\nabla_2 - \nabla_1)\widehat{F}\left(\frac{3-e}{4}\,\xi - \frac{1+e}{4}\, \zeta, \frac{1+e}{4}\,\xi - \frac{1+e}{4}\, \zeta \right) \right| \frac{d\zeta}{|\zeta|^{N-1}},
\end{aligned}$$

where we have made the spherical change of variable $\zeta = r\sigma$, we deduce

$$\begin{aligned}
&\int_{|\xi|\ge 1} |\mathcal{F}Q^+(\xi)|^2\, |\xi|^{2s+(N-1)}\, d\xi \\
&\le \|b\|^2_{L^2(\mathbb{S}^{N-1})} \int_{1 \le |\xi| \le |\zeta|} \left| \widehat{F}\left(\frac{3-e}{4}\,\xi - \frac{1+e}{4}\, \zeta, \frac{1+e}{4}\,\xi - \frac{1+e}{4}\, \zeta \right) \right| \times \\
&\left| (\nabla_2 - \nabla_1)\widehat{F}\left(\frac{3-e}{4}\,\xi - \frac{1+e}{4}\, \zeta, \frac{1+e}{4}\,\xi - \frac{1+e}{4}\, \zeta \right) \right| \frac{|\xi|^{2s+(N-1)}}{|\zeta|^{N-1}}\, d\xi\, d\zeta.
\end{aligned}$$

Finally we make the change of variable

$$X = \frac{3-e}{4}\,\xi - \frac{1+e}{4}\,\zeta, \qquad Y = \frac{1+e}{4}\,\xi - \frac{1+e}{4}\,\zeta,$$

to obtain

$$\begin{aligned}
\int_{|\xi|\geq 1} |\mathcal{F}Q^+(\xi)|^2 |\xi|^{2s+(N-1)}\,d\xi \;\; &\leq \;\; C\,\|b\|^2_{L^2(\mathbb{S}^{N-1})} \int_{\mathbb{R}^N\times\mathbb{R}^N} \left|\widehat{F}(X,Y)\right| \times \\
&\qquad \left|(\nabla_2 - \nabla_1)\widehat{F}(X,Y)\right| \, \langle X\rangle^{2s} \, \langle Y\rangle^{2s} \, dX\,dY \\
&\leq \;\; C\,\|b\|^2_{L^2(\mathbb{S}^{N-1})} \, \|F\|_{H^s} \, \|(v-v_*)F\|_{H^s} \\
&\leq \;\; C\,\|b\|^2_{L^2(\mathbb{S}^{N-1})} \, \|g\|^2_{H^s_2} \|f\|^2_{H^s_2}.
\end{aligned}$$

Then small frequencies are controlled thanks to the L^1 norms of f and g, which concludes the proof. $\square$

2.4 Estimates on the global collision operator in Lebesgue spaces

We consider a collision kernel $B = \Phi\, b$ with $\Phi(u) = |u|$ and b integrable. We shall make a splitting of Q^+ as in [33, Section 3.1]. We denote by $\mathbf{1}_E$ the usual indicator function of the set E.

Let $\Theta : \mathbb{R} \to \mathbb{R}_+$ be an even C^∞ function such that $\operatorname{support}\Theta \subset (-1,1)$, and $\int_{\mathbb{R}} \Theta\,dx = 1$. Let $\widetilde{\Theta} : \mathbb{R}^N \to \mathbb{R}_+$ be a radial C^∞ function such that $\operatorname{support}\widetilde{\Theta} \subset B(0,1)$ and $\int_{\mathbb{R}^N} \widetilde{\Theta}\,dx = 1$. Introduce the regularizing sequences

$$\begin{cases} \Theta_m(x) = m\,\Theta(mx), & x \in \mathbb{R}, \\ \widetilde{\Theta}_n(x) = n^N \widetilde{\Theta}(nx), & x \in \mathbb{R}^N. \end{cases}$$

We use these mollifiers to split the collision kernel into a smooth and a non-smooth part. As a convention, we shall use subscripts S for "smooth" and R for "remainder". First, we set

$$\Phi_{S,n} = \widetilde{\Theta}_n * (\Phi\ \mathbf{1}_{\mathcal{A}_n})\,, \qquad \Phi_{R,n} = \Phi - \Phi_{S,n},$$

where $\mathcal{A}_n$ stands for the annulus $\mathcal{A}_n = \left\{x \in \mathbb{R}^N \;;\; \frac{2}{n} \leq |x| \leq n\right\}$. Similarly, we set

$$b_{S,m}(z) = \Theta_m * (b\ \mathbf{1}_{\mathcal{I}_m})\,(z), \qquad b_{R,m} = b - b_{S,m},$$

where $\mathcal{I}_m$ stands for the interval $\mathcal{I}_m = \left\{x \in \mathbb{R} \;;\; -1 + \frac{2}{m} \leq |x| \leq 1 - \frac{2}{m}\right\}$ (b is understood as a function defined on $\mathbb{R}$ with compact support in $[-1,1]$). Finally, we set

$$Q^+ = Q^+_S + Q^+_R,$$

where

$$Q^+_S(g,f) = e^{-2} \int_{\mathbb{R}^N\times\mathbb{S}^{N-1}} \Phi_{S,n}(|v-v_*|)\, b_{S,m}(\cos\theta)\, {}'g_*\, {}'f \, d\sigma\, dv_*$$

and

$$Q^+_R = Q^+_{RS} + Q^+_{SR} + Q^+_{RR}$$

with the obvious notation

$$\begin{cases} Q^+_{RS}(g,f) = e^{-2} \displaystyle\int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \Phi_{R,n}\, b_{S,m}\, 'g_* \, 'f \, dv_* \, d\sigma \\ Q^+_{SR}(g,f) = e^{-2} \displaystyle\int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \Phi_{S,n}\, b_{R,m}\, 'g_* \, 'f \, dv_* \, d\sigma \\ Q^+_{RR}(g,f) = e^{-2} \displaystyle\int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \Phi_{R,n}\, b_{R,m}\, 'g_* \, 'f \, dv_* \, d\sigma. \end{cases}$$

Now we follow the proof as in [33, Section 4.1] since we have the same functional inequalities in Sobolev and Lebesgue spaces:

Proposition 2.5 *For any $\varepsilon > 0$ there is $C_\varepsilon > 0$ depending on ε and the collision kernel (and blowing up as $\varepsilon \to 0$) such that*

$$\int_{\mathbb{R}^N} Q^+(f,f)\, f^{p-1}\, dv \le C_\varepsilon \, \|f\|_{L^1}^{1+p\theta} \, \|f\|_{L^p}^{p(1-\theta)} + \varepsilon \, \|f\|_{L^1_2} \, \|f\|^p_{L^p_{1/p}}.$$

Remark 2.6 *Note that all the estimates in this section are valid only for $e > 0$ (and the constants blow up as $e \to 0$).*

Proof of Proposition 2.5. Let us fix $\varepsilon > 0$. We split Q^+ as $Q^+_S + Q^+_{RS} + Q^+_{SR} + Q^+_{RR}$ and we estimate each term separately. From the beginning we assume, without loss of generality, that the angular part $b(\cos\theta)$ of the collision kernel has its support included in $[0, \pi/2]$ (see the discussion on the symmetrization of b in Subsection 2.1). Remember that the truncation parameters n (for the kinetic part) and m (for the angular part) are implicit in the decomposition of Q^+.

By Corollary 2.3, there exists a constant $C(m,n) > 0$, blowing up as m or n goes to infinity, such that

$$\left\| Q^+_S(f,f) \right\|_{L^p} \le C(m,n) \|f\|_{L^q} \|f\|_{L^1},$$

for some $q < p$, namely

$$q = \begin{cases} \dfrac{(2N-1)p}{N + (N-1)p} & \text{if } p \in (1; 2N] \\ \dfrac{p}{N} & \text{if } p \in [2N; +\infty) \end{cases} \tag{2.6}$$

(the roles of p and q are exchanged here with respect to Corollary 2.3).

Next we fix a weight $\eta \ge -1$ and we estimate the L^p_η norm of $Q^+_R(f,f)$. We use that $\|b_{R,m}\|_{L^1(\mathbb{S}^{N-1})}$ goes to 0 as m goes to infinity (since b is integrable on the sphere), and we write, using Theorem 2.1 with $k = 1$,

$$\|Q^+_{RR}(f,f)\|_{L^p_\eta} \le \epsilon(m) \, \|f\|_{L^1_{|1+\eta|+|\eta|}} \|f\|_{L^p_{1+\eta}},$$

for some $\epsilon(m)$ going to 0 as m goes to infinity. A similar estimate holds true for $\|Q^+_{SR}\|_{L^p_\eta}$. Since $1+\eta \geq 0$, we can write $|1+\eta|+|\eta| = 1+2\eta_+$, where $\eta_+ = \max(\eta, 0)$.

It remains to estimate Q^+_{RS}. Let us consider separately large and small velocities: we write $f = f_r + f_{r^c}$, where

$$\begin{cases} f_r = f\, \mathbf{1}_{\{|v|\leq r\}}, \\ f_{r^c} = f\, \mathbf{1}_{\{|v|>r\}}. \end{cases}$$

On the one hand, we pick a $1 < k \leq 2$ and use Theorem 2.1. By direct computation, one can easily prove

$$\|\Phi_{R,n}\|_{L^\infty_{-k}} \leq C\, n^{-\min\{1,k-1\}} = C\, n^{-(k-1)}.$$

It follows

$$\begin{aligned} \big\|Q^+_{RS}(f, f_r)\big\|_{L^p_\eta} &\leq C\, \|f\|_{L^1_{|k+\eta|+|\eta|}}\, \|f_r\|^p_{L^p_{k+\eta}}\, \|\Phi_{R,n}\|_{L^\infty_{-k}} \\ &\leq C\, \|f\|_{L^1_{|k+\eta|+|\eta|}}\, r^{k-1} \|f\|_{L^p_{1+\eta}} n^{1-k} \\ &\leq C\left(\frac{r}{n}\right)^{k-1} \|f\|_{L^1_{k+2\eta_+}} \|f\|_{L^p_{1+\eta}}. \end{aligned}$$

(here $\theta_b = \pi/2$ thanks to the symmetrization). On the other hand, the support of $b_{S,m}$ lies a positive distance $(O(1/m))$ away from 0, so (2.2) holds true with $\theta_b = C\, m^{-1}$. Thus we can apply Theorem 2.1 with f and g exchanged, to find

$$\big\|Q^+_{RS}(f, f_{r^c})\big\|_{L^p_\eta} \leq C\, m^\alpha\, \|f_{r^c}\|_{L^1_{|1+\eta|+|\eta|}} \|f\|_{L^p_{1+\eta}}.$$

where $\alpha = \max(-\eta, 0) + 2/p' > 0$. Since we assume $1 + \eta \geq 0$, this can be bounded by

$$C\, m^\alpha\, r^{1-k}\, \|f\|_{L^1_{k+\eta+|\eta|}} \|f\|_{L^p_{1+\eta}} = C\, m^\alpha\, r^{1-k}\, \|f\|_{L^1_{k+2\eta_+}} \|f\|_{L^p_{1+\eta}}.$$

To sum up, we have obtained

$$\|Q^+_R(f, f)\|_{L^p_\eta} \leq C\Big[\epsilon(m) + \left(\frac{r}{n}\right)^{k-1} + \frac{m^\alpha}{r^{k-1}}\Big] \|f\|_{L^1_{k+2\eta_+}} \|f\|_{L^p_{1+\eta}}.$$

Then by choosing first m large enough, then r large enough, then n large enough, one gets

$$\|Q^+_R(f, f)\|_{L^p_\eta} \leq \varepsilon\, \|f\|_{L^1_{k+2\eta_+}}\, \|f\|_{L^p_{1+\eta}}.$$

Now by Hölder's inequality,

$$\begin{aligned} \int_{\mathbb{R}^N} f^{p-1} Q^+_S(f, f)\, dv &\leq \left[\int_{\mathbb{R}^N} f^p\, dv\right]^{\frac{p-1}{p}} \left[\int_{\mathbb{R}^N} (Q^+_S)^p\, dv\right]^{\frac{1}{p}} \\ &= \|f\|^{p-1}_{L^p} \|Q^+_S(f, f)\|_{L^p}, \end{aligned}$$

and

$$\int_{\mathbb{R}^N} f^{p-1} Q_R^+(f,f)\,dv = \int_{\mathbb{R}^N} (f\langle v\rangle^{1/p})^{p-1} \frac{Q_R^+}{\langle v\rangle^{\frac{1}{p'}}}\,dv$$

$$\leq \left[\int_{\mathbb{R}^N} (f\langle v\rangle^{1/p})^p\,dv\right]^{\frac{p-1}{p}} \left[\int_{\mathbb{R}^N} (Q_R^+\langle v\rangle^{-1/p'})^p\,dv\right]^{\frac{1}{p}} = \|f\|^{p-1}_{L^p_{1/p}} \|Q_R^+(f,f)\|_{L^p_{-1/p'}}.$$

By using the estimates above on Q_S^+ and for Q_R^+ with $\eta = -1/p'$ and $k = 2$, one can find $C_\varepsilon > 0$ such that

$$\int_{\mathbb{R}^N} f^{p-1} Q^+(f,f)\,dv \leq C_\varepsilon \|f\|_{L^q} \|f\|_{L^1} \|f\|^{p-1}_{L^p} + \varepsilon\, \|f\|_{L^1_2}\, \|f\|^p_{L^p_{1/p}}$$

where q is defined by (2.6). Combining this with elementary interpolation and the uniform bounds on the mass and kinetic energy, we deduce that there exists $\theta \in (0,1)$, only depending on N and p, and a constant $C_\varepsilon > 0$, only depending on N, p, B and ε, such that

$$\begin{aligned}\int_{\mathbb{R}^N} f^{p-1} Q^+(f,f)\,dv \quad &\leq C_\varepsilon\, \|f\|^{1+p\theta}_{L^1}\, \|f\|^{1-p\theta}_{L^p}\, \|f\|^{p-1}_{L^p} + \varepsilon\, \|f\|^p_{L^p_{1/p}} \\ &\leq C_\varepsilon\, \|f\|^{1+p\theta}_{L^1}\, \|f\|^{p(1-\theta)}_{L^p} + \varepsilon\, \|f\|_{L^1_2}\, \|f\|^p_{L^p_{1/p}}.\end{aligned}$$

This concludes the proof. □

2.5 Abrahamsson-Mischler-Wennberg estimate on the iterated gain term

In this subsection we shall extend results of [31, 1] to the case of granular gases. These results say essentially that the regularizing effect of the gain part of the collision operator can be translated into some quantified non-concentration estimates on the iterated gain term, namely an increase of Lebesgue integrability.

Lemma 2.7 *For any $1 \leq p < 3$, there exits an explicit constant $C(p) > 0$ depending on p, e and b (and blowing up as $e \to 0$) such that for any $f, g, h \in L^1_2(\mathbb{R}^N)$ there holds*

(2.7) $$\|Q^+(f, Q^+(g,h))\|_{L^p(\mathbb{R}^N)} \leq C(p)\, \|f\|_{L^1_2}\, \|g\|_{L^1_2}\, \|h\|_{L^1_2}.$$

Proof of Lemma 2.7. We follow [31, lemma 2.1] and [1, lemma 2.1] and we make use of the Carleman representation introduced in Proposition 1.4. Let us consider $f, g, h \in L^1_2(\mathbb{R}^N)$ and $\phi \in L^\infty(\mathbb{R}^N)$. We start applying twice the weak formulation of the gain term

$$\begin{aligned}&\int_{\mathbb{R}^N} Q^+(f, Q^+(g,h))(v)\, \phi(v)\, dv \\ &= \int_{\mathbb{R}^N} Q^+(g,h)(v)\, \left[\int_{\mathbb{R}^N} f(v_2)\, |v - v_2| \int_{\mathbb{S}^{N-1}} \phi(w_2')\, d\sigma_2\, dv_2\right] dv \\ &= \int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N} g(v)\, h(v_1)\, f(v_2)\, \left[|v - v_1| \int_{\mathbb{S}^{N-1}} |v_1' - v_2| \int_{\mathbb{S}^{N-1}} \phi(v_2'')\, d\sigma_2\, d\sigma_1\right] dv\, dv_1\, dv_2\end{aligned}$$

with $w_2' = V'(v_2, v, \sigma_2)$, $v_1' = V'(v, v_1, \sigma_1)$ and therefore $v_2'' = V'(v_2, v_1', \sigma_2)$. In these expressions, we have defined, for any given $v, v_*, \sigma \in \mathbb{R}^3$,

$$w = \frac{v + v_*}{2}, \quad u = v - v_*, \quad \gamma = \frac{1+e}{2}, \quad u' = (1-\gamma)\, u + \gamma\, |u|\, \sigma$$

and then

$$\begin{cases} V' = V'(v, v_*, \sigma) = \dfrac{w}{2} + \dfrac{u'}{2} = v + \dfrac{\gamma}{2}\,(|u|\,\sigma - u) \\ V_*' = V_*'(v, v_*, \sigma) = \dfrac{w}{2} - \dfrac{u'}{2} = v_* - \dfrac{\gamma}{2}\,(|u|\,\sigma - u). \end{cases}$$

We denote by $\Phi = \Phi(v, v_1, v_2)$ the term between brackets in the last integral. Introducing the point w_1 and the set $S_{v,v_1,\varepsilon}$ defined by

$$w_1 := (1 - \gamma/2)\, v + (\gamma/2)\, v_1, \qquad S_{v,v_1,\varepsilon} := \Big\{ z \in \mathbb{R}^N; \ \Big| |z - w_1| - |v - w_1| \Big| \le \varepsilon/2 \Big\},$$

we get

$$\Phi = (2/\gamma)^{N-1}\, |v - v_1|^{2-N} \lim_{\varepsilon \to 0} \frac{\Psi_\varepsilon}{\varepsilon} \tag{2.8}$$

with

$$\Psi_\varepsilon = \int_{\mathbb{R}^N \times \mathbb{S}^{N-1}} \mathbf{1}_{S_{v,v_1,\varepsilon}}(v_1')\, |v_1' - v_2|\, \phi(v_2'')\, d\sigma_2\, dv_1'.$$

Remarking that $v_2'' = v_2 + (\gamma/2)\,(|u_2|\,\sigma_2 - u_2)$ with $u_2 = v_1' - v_2$, we observe that the integral term Ψ_ε is very similar to the collision term Q^+: here v_2 (resp. v_1, σ_2, γ, v_2'') plays the role of v (resp. v_1, σ, β, $'v$) in the gain term). Therefore we may give a Carleman representation of Ψ_ε. The same computations as performed in Proposition 1.4 yield

$$\Psi_\varepsilon = (2/\gamma)^{N-1} \int_{\mathbb{R}^N} \int_{E^1_{v_2, v_2''}} \mathbf{1}_{S_{v,v_1,\varepsilon}}(v_1')\, |v_2'' - v_2|^{-1}\, \phi(v_2'')\, dE(v_3'')\, dv_2''$$

with $E^1_{v_2,v_2''}$ is the hyperplan orthogonal to the vector $v_2 - v_2''$ and passing through the point $\Omega_{v_2,v_2''} = v_2 + (1 - \gamma^{-1})\,(v_2 - v_2'')$. Here v_3'' stands for the postcollisional velocity issued from v_1', that is $v_3'' = V_*'(v_2, v_1', \sigma_2)$. Hence, thanks to the momentum conservation, $v_1' := v_2'' + v_3'' - v_2$, we finally define $E^2_{v_2,v_2''}$ as the hyperplan orthogonal to the vector $v_2 - v_2''$ and passing through the point $\Omega'_{v_2,v_2''} = v_2'' + (1 - \gamma^{-1})\,(v_2 - v_2'')$ and we get by a translation change of variable

$$\Psi_\varepsilon = (2/\gamma)^{N-1} \int_{\mathbb{R}^N} \int_{E^2_{v_2, v_2''}} \mathbf{1}_{S_{v,v_1,\varepsilon}}(v_1')\, |v_2'' - v_2|^{-1}\, \phi(v_2'')\, dE(v_1')\, dv_2''. \tag{2.9}$$

Now, arguing as in [1, lemma 2.1], we see that $\mathrm{mes}(\Gamma_\varepsilon)$, the $(N-1)$-dimensional measure of the intersection Γ_ε between the plane $\Pi_{v_2,v_2''}$ and the thickened sphere $S_{v,v_1,\varepsilon}$ is bounded by $\mathrm{cst}\,\varepsilon\,\gamma\,|v-v_1|^{N-2}$ and that $v_1' \in \Gamma_\varepsilon$ implies that $v_2'' \in B^\varepsilon$ with

$$B^\varepsilon := \left\{ z \in \mathbb{R}^N\,;\ |z|^2 \le |v|^2 + |v_1|^2 + 2\varepsilon(|v|+|v_1|) + \varepsilon^2 \right\}.$$

Gathering these estimates with (2.8) and (2.9) we get

$$\begin{aligned}
\Phi &= (2/\gamma)^{2N-2}\,|v-v_1|^{2-N}\,\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{\mathbb{R}^N}\frac{\phi(v_2'')}{|v_2''-v_2|}\,\mathrm{mes}(\Gamma_\varepsilon)\,dv_2'' \\
&\le \mathrm{cst}\,2^{2N-2}\,\gamma^{3-2N}\,\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\frac{\phi(v_2'')}{|v_2''-v_2|}\,\mathbf{1}_{B^\varepsilon}(v_2'')\,dv_2'' \\
&= \mathrm{cst}\,2^{2N-2}\,\gamma^{3-2N}\,\int_{\mathbb{R}^N}\frac{\phi(v_2'')}{|v_2''-v_2|}\,\mathbf{1}_{B^0}(v_2'')\,dv_2''
\end{aligned}$$

where we have defined $B^0 := \{z \in \mathbb{R}^N\,;\ |z|^2 \le |v|^2 + |v_1|^2\}$. Using [1, lemma 2.2] we may conclude as in the end of [1, lemma 2.1] and therefore (2.7) follows. $\square$

3 Regularity study in the rescaled variables

In this section we show the uniform propagation of Lebesgue and Sobolev norms and the exponential decay of singularities for the solutions of (1.10).

3.1 Uniform propagation of moments - Povzner Lemma

Let us prove that the kinetic energy of g remains uniformly bounded from above as t goes to infinity. Using (1.10) and (1.8), we get

$$\frac{d}{dt}\int_{\mathbb{R}^N} g\,|v|^2\,dv \le -\tau\int_{\mathbb{R}^N\times\mathbb{R}^N} g\,g_*\,|u|^3\,dv_*dv + 2\int_{\mathbb{R}^N} g\,|v|^2\,dv.$$

On the one hand, from Jensen's inequality (see for instance [29, Lemma 2.2]), there holds

$$\int_{\mathbb{R}^N} g_*\,|u|^3\,dv_* \ge \rho\,|v|^3.$$

On the other hand, Hölder's inequality yields

$$\int_{\mathbb{R}^N} g\,|v|^2\,dv \le \left(\int_{\mathbb{R}^N} g\,dv\right)^{1/3}\left(\int_{\mathbb{R}^N} g|v|^3\,dv\right)^{2/3},$$

which implies that

$$\int_{\mathbb{R}^N} g\,|v|^3\,dv \ge \rho^{1/2}\left(\int_{\mathbb{R}^N} g|v|^2\,dv\right)^{3/2}.$$

Thus

$$\frac{d}{dt}\int_{\mathbb{R}^N} g\,|v|^2\,dv \le -\tau\,\rho^{3/2}\left(\int_{\mathbb{R}^N} g\,|v|^2\,dv\right)^{3/2} + 2\left(\int_{\mathbb{R}^N} g\,|v|^2\,dv\right)$$
$$\le \tau\,\rho^{3/2}\left(\int_{\mathbb{R}^N} g\,|v|^2\,dv\right)\left[\frac{2}{\tau\rho^{3/2}} - \left(\int_{\mathbb{R}^N} g\,|v|^2\,dv\right)^{1/2}\right],$$

and by maximum principle we deduce

$$\sup_{t\ge 0}\int_{\mathbb{R}^N} g\,|v|^2\,dv \le C_E = \max\left\{\left(\frac{4}{\rho^3\,\tau^2}\right), \int_{\mathbb{R}^N} g_{\rm in}\,|v|^2\,dv\right\}. \tag{3.1}$$

The same argument, together with sharp versions of Povzner inequalities (see [29, Proof of Proposition 3.2]) shall yield uniform bounds on every moments of the solution. Indeed we prove the

Proposition 3.1 *Let g be a solution in $C(\mathbb{R}_+; L^1_2) \cap L^1(\mathbb{R}_+; L^1_3)$ to the rescaled Boltzman equation (1.10), with initial datum $g_{\rm in}$. Then it satisfies the following additional moment properties:*

(i) For any $s \ge 2$, there is an explicit constant $C_s > 0$, depending only on B, e, and $g_{\rm in}$, such that

$$\sup_{t\in[0,\infty)} \|g(t,\cdot)\|_{L^1_s} \le \max\{\|g_{\rm in}\|_{L^1_s}, C_s\}.$$

(ii) If $g_{\rm in}\, e^{r\,|v|^\eta} \in L^1(\mathbb{R}^N)$ for $r > 0$ and $\eta \in (0,1]$, there exists $C_1, r' > 0$, depending only on B, e, and $g_{\rm in}$, such that

$$\sup_{t\in[0,\infty)} \int_{\mathbb{R}^N} g(t,v)\, e^{r'\,|v|^\eta}\,dv \le C_1.$$

(iii) For any $\eta \in (0,1/2)$ and $\tau > 0$, there exists $a_\eta, C_\eta \in (0,\infty)$, depending only on B, e, τ and $g_{\rm in}$, such that

$$\sup_{t\in[\tau,\infty)} \int_{\mathbb{R}^N} g(t,v)\, e^{a_\eta\,|v|^\eta}\,dv \le C_\eta.$$

Let us emphasize that the constant C_s, a_η, C_η may depend on $g_{\rm in}$ only through its mass ρ and its kinetic energy $\mathcal{E}_{\rm in}$.

Proof of Proposition 3.1. The proof is just a copy with minor modifications of classical proofs. For the proof of (i) we refer for intance to [31, 36, 19] and the references therein. The proofs of (ii) and (iii) are variants of the proof of [29, Proposition 3.2], which itself follows closely the proof of [5, Theorem 3] and use arguments introduced in [19, 9]. The starting point is the following differential equation on the moments

$$\frac{d}{dt} m_p = \int_{\mathbb{R}^N} Q(g,g)\, |v|^{2p}\, dv + p\, m_p \quad \text{with} \quad m_p := \int_{\mathbb{R}^N} g\, |v|^{2p}\, dv.$$

Proceeding along the lines of [29, Proof of Proposition 3.2], we introduce the new rescaled moment function

$$z_p := \frac{m_p}{\Gamma(a\, p + 1/2)}, \qquad Z_p := \max_{k=1,\dots,k_p} \left\{ z_{k+1/2}\, z_{p-k},\ z_k\, z_{p-k+1/2} \right\},$$

for some fixed $a \geq 2$, and we obtain the differential inequality

$$\frac{dz_p}{dt} \leq A'\, p^{a/2-1/2}\, Z_p - A''\, p^{a/2}\, z_p^{1+1/2p} + p\, z_p \tag{3.2}$$

for any $p = 3/2, 2, \dots$ and for some constants $A', A'' > 0$. Note that (3.2) is nothing but [29, equation (3.18)], with an additional term $p\, z_p$ due to the additional term $-\nabla_v \cdot (v\, g)$ in equation (1.10).

On the one hand, we remark, by an induction argument, that taking $p_0 = p_0(a, A', A'')$ and $x_0 = x_0(a, A', A'')$ large enough, the sequence of functions $z_p := x^p$ is a sequence of supersolution of (3.2) for any $x \geq x_0$ and $p \geq p_0$. Let us emphasize here that we have to take $a \geq 2$ (i.e. $\eta \leq 1$ in [29, Proof of Proposition 3.2]) because of the additional term $p\, z_p$. On the other hand, choosing x_1 large enough, which may depend on p_0, we have from (i) that the sequence of functions $z_p := x^p$ is a sequence of supersolution of (3.2) for any $x \geq x_1$ and for $p \in \{0, 1/2, ..., p_0\}$. As a consequence, we have proved that there exists $x_2 := \max(x_0, x_1)$ such that the set

$$\mathcal{C}_x := \left\{ z = (z_p); \quad z_p \leq x^p \ \ \forall\, p \in \frac{1}{2}\, \mathbb{N} \right\} \tag{3.3}$$

is invariant under the flow generated by the Boltzmann equation for any $x \geq x_2$: if $g(t_1) \in \mathcal{C}_x$ then $g(t_2) \in \mathcal{C}_x$ for any $t_2 \geq t_1$. The end of the proof is exactly similar to that of [29, Proof of Proposition 3.2]. □

The integral upper bound in point (ii) of Theorem 1.2 follows from point (iii) of Proposition 3.1.

3.2 Stability in L^1

The stability result [29, Proposition 3.4] translates for (1.10) into:

$$\|g - h\|_{L^1} + e^{-2T}\, \|(g-h)|v|^2\|_{L^1} \leq e^{C(e^{2T}-1)} \left[\|g_{\rm in} - h_{\rm in}\|_{L^1} + \|(g_{\rm in} - h_{\rm in})|v|^2\|_{L^1} \right]$$

for any solutions g and h in $C(\mathbb{R}_+, L^1_2) \cap L^\infty(\mathbb{R}_+, L^1_3)$ with initial datum $0 \leq g_{\rm in}, h_{\rm in} \in L^1_3$. This shows (together with the propagation of the L^1_3 norm) that, in the Banach space L^1_2, the evolution semi-group S_t of (1.10) satisfies: for any $t \geq 0$, S_t is (strongly) continuous in any L^1_3 bounded subset of L^1_2. However we shall prove a more precise stability result, working directly on the rescaled equation (1.10).

Proposition 3.2 *Let $0 \le g_{\rm in}, h_{\rm in} \in L^1_3$ and let g and h be the two solutions of (1.10) (in $C(\mathbb{R}_+, L^1_2) \cap L^\infty(\mathbb{R}_+, L^1_3)$). Then there is $C_{\rm stab} > 0$ depending only on B and $\sup_{t \ge 0} \|g+h\|_{L^1_3}$ such that*

$$\forall\, t \ge 0, \quad \|g_t - h_t\|_{L^1_2} \le \|g_{\rm in} - h_{\rm in}\|_{L^1_2}\, e^{C_{\rm stab} t}.$$

Proof of Proposition 3.2. We multiply the equation satisfied by $g-h$ by $\phi(t,v) = \mathrm{sgn}(g(t,v) - h(t,v))\,(1+|v|^2)$. We use on the one hand the same arguments as in [29, Proposition 3.4] to treat

$$I = \int_{\mathbb{R}^N} [Q(g,g) - Q(h,h)]\; \phi(t,v)\, dv,$$

which gives

$$I \le C \left(\int_{\mathbb{R}^N} (g+h)\,(1+|v|^3)\, dv \right) \left(\int_{\mathbb{R}^N} |g-h|\,(1+|v|^2)\, dv \right).$$

On the other hand we use that

$$\begin{aligned} - \int_{\mathbb{R}^N} \nabla_v \cdot (v\,(g-h))\, \phi(t,v)\, dv \;&=\; -N \int_{\mathbb{R}^N} |g-h|\,(1+|v|^2)\, dv \\ &\quad + \int_{\mathbb{R}^N} |g-h|\, \nabla_v \cdot (v + v|v|^2)\, dv \\ &=\; 2 \int_{\mathbb{R}^N} |g-h|\, |v|^2\, dv. \end{aligned}$$

This concludes the proof with

$$C_{\rm stab} = C \sup_{t \ge 0} \|g+h\|_{L^1_3} + 2.$$

$\square$

3.3 Uniform propagation of Lebesgue norms

Let us take $1 < p < +\infty$. We compute the time derivative of the L^p norm of g using equation (1.10):

$$\frac{1}{p}\,\frac{d}{dt} \int_{\mathbb{R}^N} g^p\, dv = \int_{\mathbb{R}^N} Q^+(g,g)\, g^{p-1}\, dv - \int_{\mathbb{R}^N} g^p\, L(g)\, dv - \int_{\mathbb{R}^N} g^{p-1}\, \nabla_v (v\, g)\, dv.$$

We use the control (1.16), and

$$\int_{\mathbb{R}^N} \nabla_v \cdot (vg) g^{p-1}\, dv = N \left(1 - \frac{1}{p}\right) \|g\|^p_{L^p}.$$

Gathering all these estimates, we deduce

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbb{R}^N} g^p\,dv \leq \int_{\mathbb{R}^N} Q^+(g,g)\,g^{p-1}\,dv - \min\left\{1, N\left(1-\frac{1}{p}\right)\right\}\int_{\mathbb{R}^N} g^p(1+|v|)\,dv.$$

Concerning the gain term, Theorem 2.2 yields, for any $\varepsilon > 0$,

$$\int_{\mathbb{R}^N} Q^+(g,g) f^{p-1}\,dv \leq C_\varepsilon\, \|g\|_{L^1_2}^{1+p\theta}\, \|g\|_{L^p}^{p(1-\theta)} + \varepsilon\, \|g\|_{L^1_2}\, \|g(1+|v|)\|_{L^p}^p.$$

Hence, using the bound C_E on the kinetic energy, if we fix ε such that

$$C_E^{p(1-\theta)}\varepsilon < 1/2\,\min\left\{1, N\left(1-\frac{1}{p}\right)\right\},$$

we obtain

$$\frac{d}{dt}\|g\|_{L^p}^p \leq C_+\,\|g\|_{L^p}^{p(1-\theta)} - K_-\,\|g\|_{L^p_{1/p}}^p$$

for some explicit constants C_+, $K_- > 0$. By maximum principle, it shows that the L^p norm of g is uniformly bounded by

$$\sup_{t\geq 0}\|g_t\|_{L^p} \leq \max\left\{\left(\frac{C_+}{K_-}\right)^{\frac{1}{p\theta}}, \|g_{\text{in}}\|_{L^p}\right\}.$$

3.4 Non-concentration in the rescaled variables and Haff's law

In this subsection we give a short proof of Haff's law, even if a stronger pontwise result on the tail will be proved in the next section. Let $f_{\text{in}} = g_{\text{in}}$ be an initial datum in $L^1_3 \cap L^p$ (with $1 < p < +\infty$). Hence according to the previous subsection, the rescaled solution g of (1.10) with initial datum g_{in} satisfies

$$\sup_{t\geq 0}\|g_t\|_{L^p} \leq C_p$$

for some explicit constant $C_p > 0$ depending on the collision kernel and the mass, kinetic energy and L^p norm of f_{in}. By using Cauchy-Schwarz inequality, this non-concentration estimate implies that for any $r > 0$

$$\forall\, t \geq 0, \quad \int_{|v|\leq r} g(t,v)\,dv \leq C\, r^{\frac{p-1}{p}N}.$$

Thus there is $r_0 > 0$ such that

$$\forall\, t \geq 0, \quad \int_{|v|\leq r_0} g(t,v)\,dv \leq 1/2$$

and thus

$$
(3.4) \qquad \forall\, t \ge 0, \quad \int_{\mathbb{R}^N} g(t,v)\,|v|^2\,dv \ \ge \ \int_{|v|\ge r_0} g(t,v)\,|v|^2\,dv
$$
$$
\ge \ r_0^2 \int_{|v|\ge r_0} g(t,v)\,dv
$$
$$
\ge \ r_0^2 \left(1 - \int_{|v|\le r_0} g(t,v)\,dv\right) \ge \frac{r_0^2}{2}.
$$

As a conclusion, gathering (3.1) and (3.4), we have proved that for some constants $C_0, C_1 \in (0,\infty)$ there holds

$$
C_0 \le \mathcal{E}(g(t,\cdot)) \le C_1,
$$

and Haff's law (1.14) follows thanks to (1.13), which proves Theorem 1.3.

Remark 3.3 *The inequality $\mathcal{E}(f(t,\cdot)) \le M\, t^{-2}$ (or equivalently $\mathcal{E}(g(t,\cdot)) \le C_1$) was already known: see for instance [4, equations (2.5)-(2.6)] where it is proved for a quasi-elastic one-dimensional model with the same evolution equation (1.8) on the kinetic energy, by comparison to a differential equation. Indeed the harder part in Haff's law is the first inequality, which means that the solution does not cool down faster than the self-similar profil. As emphasized by the proof above, this is related to the impossiblity of asymptotic concentration in the rescaled equation (1.10).*

3.5 Uniform propagation of Sobolev norms

The study of propagation of regularity and exponential decay of singularities is based on a Duhamel representation of the solution we shall introduce. Let us denote

$$
L(t,v) = \left(\int_{\mathbb{R}^N} g(v_*)\,|v-v_*|\,dv_*\right),
$$

and

$$
S_t g = g(e^{-t}v)\,\exp\left[-Nt - \int_0^t L(s, e^{-(t-s)}v)\,ds\right]
$$

the evolution semi-group associated to

$$
Tg = -\left(\int_{\mathbb{R}^N} g(v_*)\,|v-v_*|\,dv_*\right) g(v) - \nabla_v \cdot (v\,g).
$$

Then the solution of (1.10) represents as

$$
g_t = S_t g_{\rm in} + \int_0^t S_{t-s} Q^+(g_s, g_s)\,ds.
$$

We give a proposition similar to [33, Proposition 5.2]

Proposition 3.4 *There are some constants $\alpha > 0$, $\delta > 0$, $K > 0$ and $k > 0$ such that for any $s, \eta \geq 0$, we have*

$$\|S_t g_{\rm in}\|_{H^{s+\alpha}_\eta} \leq C_{\rm Duh}\, e^{-Kt}\, \|g_{\rm in}\|_{H^{s+\alpha}_{\eta+\delta}} \sup_{0\leq \bar t\leq t} \|g(\bar t, \cdot)\|^{s+k}_{H^s_{\eta+\delta}}$$

$$\left\|\int_0^t S_{t-s} Q^+(g_s, g_s)\, ds\right\|_{H^{s+\alpha}_\eta} \leq C_{\rm Duh} \sup_{0\leq \bar t\leq t} \|g(\bar t, \cdot)\|^{s+k}_{H^s_{\eta+\delta}}.$$

Proof of Proposition 3.4. The proof is exactly similar to [33, Proof of Proposition 5.2]. Indeed the semi-group in [33, Proof of Proposition 5.2] is

$$\bar S_t g = g(v)\, \exp\left[-\int_0^t L(s,v)\, ds\right]$$

and thus the estimates on the Sobolev norm in v can only improve for S_t according to $\bar S_t$. The main tool of [33, Proof of Proposition 5.2], i.e. the Bouchut-Desvillettes-Lu regularity result on Q^+, has been proved in our case in Theorem 2.4. $\square$

Now results follows as in [33]:

Theorem 3.5 *Let $0 \leq g_{\rm in} \in L^1_2$ be an initial datum and let g be the unique solution of (1.10) in $C(\mathbb{R}_+, L^1_2) \cap L^1(\mathbb{R}_+, L^1_3)$ associated with $g_{\rm in}$. Then for all $s > 0$ and $\eta \geq 1$, there exists $w(s) > 0$ (explicitly $w(s) = \delta \lceil s/\alpha \rceil$, where α is defined in Proposition 3.4) such that*

$$g_{\rm in} \in H^s_{\eta+w} \Longrightarrow \sup_{t\geq 0} \|g(t,\cdot)\|_{H^s_\eta} < +\infty$$

with uniform bounds.

Proof of Theorem 3.5. Let $n \in \mathbb{N}$ be such that $n\alpha \geq s$ ($n = \lceil s/\alpha \rceil$). Let $w(s) = \delta \lceil s/\alpha \rceil$. The proof is made by an induction comprising n steps, proving successively that g is uniformly bounded in $H^{i\alpha}_{\eta + \frac{n-i}{n} w}$ for $i = 0, 1, \ldots, n$.

Let us write the induction. The initialisation for $i = 0$, i.e g uniformly bounded in $L^2_{\eta+w}$ is proved by the previous study of uniform propagation of weighted L^p norms in Subsection 3.3. Now let $0 < i \leq n$ and suppose the induction assumption to be satisfied for all $0 \leq j < i$. Then proposition 3.4 implies

$$\|S_t g_{\rm in}\|_{H^{i\alpha}_{\eta+\frac{n-i}{n}w}} \leq C_{\rm Duh}\, e^{-Kt}\, \|g_{\rm in}(\cdot)\|_{H^{i\alpha}_{\eta+\frac{n-i}{n}w+\delta}} \sup_{0\leq t_0\leq t} \|g(t_0,\cdot)\|^{i\alpha+k}_{H^{(i-1)\alpha}_{\eta+\frac{n-i}{n}w+\delta}},$$

and

$$\left\|\int_0^t S_{t-s} Q^+(g_s, g_s)\, ds\right\|_{H^{i\alpha}_{\eta+\frac{n-i}{n}w}} \leq C_{\rm Duh} \sup_{0\leq t_0\leq t} \|g(t_0,\cdot)\|^{i\alpha+k}_{H^{(i-1)\alpha}_{\eta+\frac{n-i}{n}w+\delta}}.$$

Moreover as $i \geq 1$,

$$\eta + \frac{n-i}{n} w + \delta \leq \eta + \frac{n-(i-1)}{n} w.$$

Thus, using the induction assumption for $i-1$, g is uniformly bounded in $H^{i\alpha}_{\eta+\frac{n-i}{n}w}$, which concludes the proof. $\square$

3.6 Exponential decay of singularities

Theorem 3.6 *Let* $0 \leq g_{\rm in} \in L^1_2$ *and let* g *be the unique solution of (1.10) in* $C(\mathbb{R}_+, L^1_2) \cap L^1(\mathbb{R}_+, L^1_3)$ *associated with* $g_{\rm in}$. *Let* $s \geq 0$, $q \geq 0$ *be arbitrarily large. Then* g *can be written* $g^S + g^R$ *in such a way that*

$$\begin{cases} \sup_{t \geq 0} \left\| g_t^S \right\|_{H^s_q \cap L^1_2} < +\infty, \quad g^S \geq 0 \\ \exists\, \lambda > 0; \quad \left\| g_t^R \right\|_{L^1_2} = O\left(e^{-\lambda t}\right). \end{cases}$$

All the constants in this theorem can be computed in terms of the collision kernel, the mass and kinetic energy of $g_{\rm in}$.

Proof of Theorem 3.6. The proof of Theorem 3.6 is exactly similar to [33, Proof of Theorem 7.2], since the only tools of the proof are the stability result, the estimate on the Duhamel representation, the uniform propagation of Sobolev norms, and the non-concentration estimate on the iterated gain term, which have been proved respectively in Proposition 3.2, Proposition 3.4, Proposition 3.5 and Proposition 2.7. The propagation and appearance of moments in L^1 (used in this proof) were proved in Proposition 3.1. $\square$

Point (i) of Theorem 1.2 is deduced from this theorem.

Remark 3.7 *A suggested by this study, the self-similar variables are not only useful for proving the existence of self-similar profils, but it seems that they also provide the good framework for studying precisely the regularity of the solution. For instance, coming back to the original variables, Theorem 3.6 shows the algebraic decay of singularities for the solutions of (1.1).*

4 Self-similar solutions and tail behavior

In this section we achieve the proofs of Theorem 1.1 and Theorem 1.2 by showing the existence of self-similar solutions, and obtaining estimates on their tail and the tail of generic solutions.

4.1 Existence of self-similar solutions

The starting point is the following result, see for instance [19, Theorem 5.2] or [3, 17].

Theorem 4.1 *Let* $\mathcal{Y}$ *be a Banach space and* $(S_t)_{t \geq 0}$ *be a continuous semi-group on* $\mathcal{Y}$. *Assume that there exists* $\mathcal{K}$ *a nonempty convex and weakly (sequentially) compact subset of* $\mathcal{Y}$ *which is invariant under the action of* S_t *(that is* $S_t y \in \mathcal{K}$ *for any* $y \in \mathcal{K}$ *and* $t \geq 0$*), and such that* S_t *is weakly (sequentially) continuous on* $\mathcal{K}$ *for any* $t > 0$. *Then there exists* $y_0 \in \mathcal{K}$ *which is stationary under the action of* S_t *(that is* $S_t y_0 = y_0$ *for any* $t \geq 0$*).*

Proof of Theorem 1.1 (existence part). The existence of self-similar solutions follows from the application of this result to the evolution semi-group of (1.10). The continuity properties of the semi-group are proved by the study of the Cauchy problem, recalled in Section 3. On the Banach space $\mathcal{Y} = L^1_2$, thanks to the uniform bounds on the L^1_3 and L^p norms, the nonempty convex subset of $\mathcal{Y}$

$$\mathcal{K} = \left\{ f \in \mathcal{Y}, \quad \|f\|_{L^1_3} + \|f\|_{L^p} \leq M \right\}$$

is stable by the semi-group provided M is big enough. This set is weakly compact in $\mathcal{Y}$ by Dunford-Pettis Theorem, and the continuity of S_t for all $t \geq 0$ on $\mathcal{K}$ follows from Proposition 3.2. This shows that there exists a stationary solution to (1.10) in $L^1_3 \cap L^p$ for any given mass, that is a self-similar solution for the original problem (1.1).

Then one can apply Theorem 3.6, which proves that the stationary solution of (1.10) obtained above belongs to C^∞ (in fact it proves that it belongs to the Schwartz space of C^∞ functions decreasing faster than any polynomial at infinity). Moreover, since the property of being radially symmetric is stable along the flow of (1.10), this sationary solution can be shown to exist within the set of radially symmetric functions by the same arguments. $\square$

4.2 Tail of the self-similar profils

In this subsection we prove pointwise bounds on the tail behavior of the self-similar solutions. The starting point is the following result extracted from [9, Theorem 1]; notice that it is also a consequence of the construction of invariant sets $\mathcal{C}_x$ for z_p with $a = 2$, as defined in (3.3).

Theorem 4.2 (Bobylev-Gamba-Panferov) *Let G be a steady state of (1.10) with finite moments of all orders. Then G has exponential tail of order 1, that is*

$$r^* = \sup \left\{ r \geq 0, \quad \int_{\mathbb{R}^N} G(v) \, \exp(r|v|) \, dv < +\infty \right\}$$

belongs to $(0, +\infty)$.

Note that if we define more generally (for $s > 0$)

$$r^*_s = \sup \left\{ r \geq 0, \quad \int_{\mathbb{R}^N} G(v) \, \exp(r|v|^s) \, dv < +\infty \right\},$$

a simple consequence of this result is that $r^*_s = +\infty$ for any $s < 1$, and $r^*_s = 0$ for any $s > 1$.

First let us prove the pointwise bound from above on the steady state. Since the evolution equation (1.10) makes all the moments appear (see Proposition 3.1), we assume that G has finite moments of all orders. Moreover, as discussed above, we can also assume that G is smooth and radially symmetric. We denote $r = |v|$. We thus have the

Proposition 4.3 *Let $G \in C^1$ be a radially symmetric steady state of (1.10) with finite moments of all order. Then there exists $A_1, A_2 > 0$ such that*

$$\forall\, v \in \mathbb{R}^N, \quad G(v) \leq A_1\, e^{-A_2\,|v|}.$$

Proof of Proposition 4.3. The differential equation satisfied by $G = G(r)$ writes

$$Q(G,G) - N\,G - r\,G' = 0.$$

Since G is smooth and integrable, it goes to 0 at infinity. By integrating this equation between $r = R$ and $r = +\infty$, we obtain

$$G(R) = N \int_R^{+\infty} \frac{G(r)}{r}\, dr - \int_R^{+\infty} \frac{Q(G,G)}{r}\, dr.$$

One deduces the following upper bound

$$G(R) \leq N \int_R^{+\infty} \frac{G(r)}{r}\, dr + \int_R^{+\infty} \frac{Q^-(G,G)}{r}\, dr.$$

Since $Q^-(G,G) = G\,(G * \Phi)$, we have

$$Q^-(G,G)(v) \leq C\,(1 + |v|)\, G.$$

Hence, taking $R \geq 1$ leads to

$$G(R) \leq C \int_R^{+\infty} G(r)\, r^{N-1}\, dr.$$

Finally, since we have by Theorem 4.2

$$\int_0^{+\infty} G(r)\, \exp(A_2\, r)\, r^{N-1}\, dr \leq A_0 < +\infty$$

for some constants $A_0, A_2 > 0$, we deduce that

$$G(R) \leq C \int_R^{+\infty} G(r)\, r^{N-1}\, dr \leq C\, A_0\, \exp(-A_2\, R) = A_1\, \exp(-A_2\, R).$$

This concludes the proof. $\square$

For the pointwise lower bound, we give here a proof based on a maximum principle argument, inspired from the works [19, 20]. We shall in the next subsection give a more general result for generic solutions of (1.10), based on the spreading effect of the gain term and the dispersion (or transport) effect of the evolution semi-group of (1.10) (due to the anti-drift term).

Proposition 4.4 *Let $G \in C^1$ be a steady state of (1.10) with finite moments of orders 0 and 2. Then there exists $a_1, a_2 > 0$ such that*

$$\forall\, v \in \mathbb{R}^N, \quad G(v) \geq a_1\, e^{-a_2\,|v|}.$$

We first start with a lemma.

Lemma 4.5 *For any $r_0, a_1, \rho_0, \rho_1 > 0$, there exists $a_2 > 0$ such that the function $h(v) := a_1 \exp(-a_2\,|v|)$ satisfies*

$$(4.1) \qquad \forall\, v,\ |v| \geq r_0, \quad Q^-(g,h) + \nabla_v(v\,h) \leq 0$$

for any function g such that

$$\int_{\mathbb{R}^N} g(v)\,dv = \rho_0, \qquad \int_{\mathbb{R}^N} g(v)\,|v|\,dv = \rho_1.$$

Proof of Lemma 4.5. On the one hand, it is straightforward that

$$Q^-(g,h) := (g * \Phi)\,h \leq (\rho_1 + \rho_0\,|v|)\,h.$$

On the other hand, simple computations show that

$$\nabla_v(v\,h) = (N - a_2\,|v|)\,h.$$

Gathering these two inequalities there holds

$$\forall\, v,\ |v| \geq r_0, \quad Q^-(g,h) + \nabla_v(v\,h) \leq (\rho_1 + N + \rho_0\,|v| - a_2\,|v|)\,h \leq 0$$

for a_2 large enough. $\square$

Proof of Proposition 4.4. Since $G \in C^1$ and radially symmetric, there holds $G'(0) = 0$. As a consequence, the equation satisfied by G reads in $v = 0$

$$Q(G,G)(0) - N\,G(0) = 0$$

and then

$$G(0) = \frac{Q^+(G,G)(0)}{\rho_1 + N} > 0$$

since G is not zero everywhere. By continuity, $G(v) > 2\,a_1$ on $B(0,r_0)$ for some $a_1, r_0 > 0$.

Let us define

$$\rho_0 := \int_{\mathbb{R}^N} G(v)\,dv = \rho_0, \quad \rho_1 := \int_{\mathbb{R}^N} G(v)\,|v|\,dv.$$

and $a_2 > 0$ by Lemma 4.5. On the one hand $h(v) := a_1 \exp(-a_2\,|v|)$ satisfies (4.1) for $g = G$ and, on the other hand, G satisfies

$$(4.2) \qquad \forall\, v \in \mathbb{R}^N, \quad Q^-(G,G) + \nabla_v(v\,G) = Q^+(G,G) \geq 0.$$

Introducing the auxiliary function $W := G - h$, we deduce from (4.1) and (4.2)

$$\forall\, v,\ |v| \geq r_0, \quad (G * \Phi)\,W + \nabla_v(v\,W) \geq 0$$

and $W(r_0) = G(r_0) - h(r_0) \geq G(r_0)/2 > 0$. By the Gronwall Lemma (using that all the functions involved in this inequality are radially symmetric), we get $W(v) \geq 0$ for any v, $|v| \geq r_0$, which concludes the proof. $\square$

4.3 Positivity of the rescaled solution

We start with three technical lemmas.

Lemma 4.6 *Let g_0 satisfies*

$$\int_{\mathbb{R}^N} g_0\,dv = 1, \quad \int_{\mathbb{R}^N} g_0\,|v|^2\,dv \le C_1, \quad \int_{\mathbb{R}^N} g_0^2\,dv \le C_2. \tag{4.3}$$

There exist $R > r > 0$ and $\eta > 0$ depending only on C_1, C_2, and $(v_i)_{i=1,\dots,4}$ such that $|v_i| \le R$, $i = 1,\dots,4$, and $|v_i - v_j| \ge 3r$ for $1 \le i \ne j \le 3$, and

$$\int_{B(v_i,r)} g_0(v)\,dv \ge \eta \quad \textit{for} \quad i = 1,\, 2,\, 3, \tag{4.4}$$

$$\forall\, w_i \in B(v_i,r), \quad E^e_{w_3,w_4} \cap S^e_{w_1,w_2} \textit{ is a sphere of radius larger than } r, \tag{4.5}$$

where $E^e_{v,'v}$ stands for the plane defined in Proposition 1.4 and S^e_{v,v_*} stands for the sphere of all possibles post-collisional velocity v' defined by (1.19).

Proof of Theorem 4.6. Let C_R denotes the hypercube $[-R,R]^N$ centered at $v = 0$ with length $2R > 0$. Thanks to the mass condition and the energy bound in (4.3), for R large enough, there holds

$$\int_{C_R} g_0\,dv \ge \frac{1}{2}. \tag{4.6}$$

Then we define $(K_i)_{i=1,\dots,I}$ the familly of $I = (2\,R/r)^N$ hypercubes of length $r > 0$ (with $R/r \in \mathbb{N}$), included in C_R and such that the union of K_i is almost equal to C_R. For any given $\lambda > 0$ to be later fixed, we may find $r > 0$ such that

$$\int_{K_i+B(0,\lambda r)} g_0\;dv \le |K_i + B(0,\lambda r)|^{1/2} \left(\int_{K_i+B(0,\lambda r)} g_0^2\;dv\right)^{1/2} \le C\,[(\lambda+1)r]^{N/2} \le 1/4 \tag{4.7}$$

for any $i = 1,\dots,I$. Hence we can choose K_{i_0} such that the mass of g_0 in K_i is maximal for $i = i_0$. Because of (4.6) there holds

$$\int_{K_{i_0}} g_0\;dv \ge 1/4\,(2\,R/r)^{-N}. \tag{4.8}$$

Gathering (4.6) and (4.7) we may find $K_{j_0} \subset C_R$ such that $\mathrm{dist}(K_{i_0}, K_{j_0}) > \lambda r$ and (4.8) also holds for $i = j_0$.

Next, we fix $\lambda := 200\,\beta$. We define v_1 (respectively v_2) as the center of the hypercube K_{i_0} (respectively K_{j_0}), and $v_3 = (v_1 + v_2)/2$ and $v_4 = v_2$. Then we have

$$\Omega(v_3, v_4) = v_1 + \frac{\beta^{-1}}{2}\,(v_2 - v_1) \in [v_1, v_2],$$

which implies

$$|\Omega - v_1| = \frac{\beta^{-1}}{2}\,|v_2 - v_1| \geq \frac{\beta^{-1}}{2}\,(\lambda\, r) \geq 100\, r.$$

Thus $E^e_{v_3,v_4} \cap S^e(v_1,v_2)$ is a $(N-2)$-dimensional sphere of radius larger than $100\, r$ (because $B(\Omega, 100r)$ is included in the convex hull of $S^e(v_1,v_2)$), and (4.5) follows straightforwardly. $\square$

Lemma 4.7 *Let us fix $R > r > 0$ and $\eta > 0$. Then there exists $\delta_0 > 0$, $\eta_0 > 0$, $\xi_0 \in (0,1)$ (depending on $R > r > 0$, $\eta > 0$ and B) such that, for any functions f, h, ℓ satisfying (4.4)-(4.5) for some velocities $(v_i)_{i=1,\dots,4}$ such that $|v_i| \leq R$, $i = 1,\dots,4$ and $|v_i - v_j| \geq 3r$, $1 \leq i \neq j \leq 3$, and for any $\xi \in (\xi_0, 1)$, there holds*

$$Q^+(f, Q^+_\xi(h,\ell)) \geq \eta_0\, \mathbf{1}_{B(v_3,\delta_0)},$$

where we define here and below $Q^+_\xi(\cdot,\cdot)(v) = Q^+(\cdot,\cdot)(\xi\, v)$.

Proof of Theorem 4.7. We first establish a convenient formula to handle representations of the iterated gain term. For any f, h and ℓ and any $v \in \mathbb{R}^N$ there holds (setting $'v = w$ and $'v_* = w_*$)

$$Q^+(f, Q^+_\xi(h,\ell))(v) = C'_b \int_{\mathbb{R}^N} \frac{f(w)}{|v - w|} \left\{ \int_{E^e_{v,w}} Q^+_\xi(h,\ell)(w_*)\, dw_* \right\} dw.$$

From the following identity

$$Q^+_\xi(h,\ell)(w_*) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} h(w_1)\,\ell(w_2)\, Q^+_\xi(\delta_1,\delta_2)(w_*)\, dw_1 dw_2$$

where δ_i stands for the Dirac measure at w_j, the term between brackets, that we denote by A, write

$$A(v,w) = \int_{\mathbb{R}^N\times\mathbb{R}^N} h(w_1)\,\ell(w_2) \left\{ \lim_{\varepsilon\to 0} \frac{1}{2\,\varepsilon} \int_{\mathbb{R}^N} Q^+(\delta_1,\delta_2)(\xi\, w_*)\, \Xi_\varepsilon(w_*)\, dw_* \right\} dw_1\, dw_2$$

where Ξ_ε denotes the indicator function of the set $\{w_*\,;\mathrm{dist}(w_*, E_{v,w}) < \varepsilon\}$. Denoting now by D_ε the integral just after the limit sign in the term between brackets, and using the weak formulation (1.18), there holds

$$\begin{aligned} D_\varepsilon &= \frac{\xi^{-1}}{2\,\varepsilon} \int_{\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{S}^{N-1}} \delta_1(z)\,\delta_1(z_*)\,|z - z_*|\, b(\sigma\cdot\hat{z})\, \Xi_\varepsilon(\xi^{-1}\, z')\, d\sigma\, dz\, dz_* \\ &= |w_1 - w_2|\,\xi^{-1}\, C_b \int_{\mathbb{S}^{N-1}} \frac{\Xi_\varepsilon(z'\,\xi^{-1})}{2\varepsilon}\, d\sigma, \end{aligned}$$

where in these integrales z' is defined from (z, z_*, σ) and next from (w_1, w_2, σ) thanks to formula (1.19). We define $\xi_0 = (1 + r/R)^{-1}$ in such a way that $|\xi^{-1}\, z' - z'| \leq r$ for any $z' \in B(0,R)$ and $\xi \in (\xi_0, 1)$. Taking $v \in B(v_3, r)$, $w \in B(v_4, r)$, $w_1 \in B(v_1, r)$, $w_2 \in B(v_2, r)$, we have thanks to (4.5) and $|w_2 - w_1| \geq r$:

$$D_0(v, w, w_1, w_2) := \lim_{\varepsilon \to 0} D_\varepsilon \geq r\, \xi_0^{-1}\, C_b\, C\, r^{N-2}.$$

As a consequence, for any $v \in B(v_3, r)$,

$$\begin{aligned} Q^+(f, Q^+_\xi(h,\ell))(v) &\geq Q^+(f\, \mathbf{1}_{B(v_4,r)}, Q^+(h\, \mathbf{1}_{B(v_1,r)}, \ell\, \mathbf{1}_{B(v_2,r)}))(v) \\ &\geq C_b' \int_{B(v_1,r)} \int_{B(v_2,r)} \int_{B(v_4,r)} \frac{f(w)}{|v-w|}\, h(w_1)\, \ell(w_2)\, D_0\, dw_1\, dw_2\, dw \\ &\geq C_b'\, \eta^3\, \frac{1}{2\,R}\, r\, \xi_0^{-1}\, C_b\, C\, r^{N-2} =: \eta_0. \end{aligned}$$

This concludes the proof. □

Lemma 4.8 *For any $\bar v \in \mathbb{R}^N$ and $\delta > 0$, there exists $\kappa = \kappa(\delta) > 0$ such that*

$$\mathcal{Q}^+(v) := Q^+(\mathbf{1}_{B(\bar v, \delta)}, \mathbf{1}_{B(\bar v, \delta)}) \geq \kappa\, \mathbf{1}_{B\left(\bar v, \frac{\sqrt5}{2}\delta\right)}. \tag{4.9}$$

Proof of Lemma 4.8. The homogeneity property (1.9) of Q^+ and the invariance by translation allow to reduce the proof of (4.9) to the case $\bar v = 0$ and $\delta = 1$. The invariance by rotations implies that $\mathcal{Q}^+$ is radially symmetric and the homogeneity property again allows to conclude that the support of $\mathcal{Q}^+$ is a ball B'. More precisely, taking a C^∞ radially symmetric function ϕ such that $\phi > 0$ on $B = B(0,1)$ and $\phi \leq \mathbf{1}_B$ on $\mathbb{R}^N$, we have $Q^+(\phi,\phi)$ is continuous, $\mathcal{Q}^+ \geq Q^+(\phi,\phi)$ on $\mathbb{R}^N$ and $Q^+(\phi,\phi) > 0$ on the ball B'. As a consequence, for any ball B'' strictly included in B', there exists $\kappa > 0$ such that $\mathcal{Q}^+ \geq \kappa\, \mathbf{1}_{B''}$. In order to conclude, we just need to estimate the support of $\mathcal{Q}^+$.

Let us fix $R \in (0,1)$ and choose $'v, 'v_* \in B(0,1)$ such that $'v \perp\ 'v_*$, $|'v| = |'v_*| = R$. Then for any $\sigma \in \mathbb{S}^{N-1}$, $\sigma \perp\ 'v -' v_*$, the function $\mathcal{Q}^+$ is positive at the post-collisional associated velocity v defined by

$$v = \frac{'v +' v_*}{2} + \frac{1-e}{4}\,('v -' v_*) + \frac{1+e}{4}\,|'v -' v_*|\,\sigma.$$

Remarking that $|'v +' v_*|^2 = |'v -' v_*|^2 = 2\,R^2$, $('v -' v_*) \cdot ('v +' v_*) = 0$ and $('v +' v_*) \cdot \sigma = \sqrt{2}\,R$, we easily compute

$$|v|^2 = R^2 \left[1 + \left(\frac{1+e}{2}\right)^2\right] > \frac{5}{4}\,R^2,$$

and the radius of B' is strictly larger than $\sqrt5/2$. □

Theorem 4.9 *Let g_{in} satisfy the hypothesis of Theorem 1.2 and let g be the solution to the rescaled equation (1.10) associated to the initial datum g_{in}. Then for any $t_* > 0$, $g(t,\cdot) > 0$ a.e. on $\mathbb{R}^N$ for any $t \geq t_*$, and there exists $a_1, a_2, c > 0$ such that*

$$\forall\, t \geq t_*, \quad g(t,v) \geq a_1\, e^{-a_2\,|v|}\, \mathbf{1}_{|v| \leq c\, e^{t-t_*}} \quad \textit{for a.e. } v \in \mathbb{R}^N.$$

Proof of Theorem 4.9. We split the proof into four steps.
Step 1. The starting point is the evolution equation satisfied by g written in the form

$$\partial_t g + v \cdot \nabla_v g + (N + |v|)\, g = Q^+(g,g) + (|v| - L(g))\, g.$$

Let us introduce the semigroup S_t associated to the operator $v \cdot \nabla_v + \lambda(v)$, where $\lambda(v) := N + |v|$. Thanks to the Duhamel formula and (1.16), we have

$$g(t,\cdot) \geq S_t\, g(0,\cdot) + \int_0^t S_{t-s} Q^+(g(s,\cdot), g(s,\cdot))\, ds, \tag{4.10}$$

where the semigroup S_t is defined by

$$(S_t\, h)(v) = h(v\, e^{-t})\, \exp\left(- \int_0^t \lambda(v\, e^{-s})\, ds \right).$$

Notice that

$$\left(- \int_0^t \lambda(v\, e^{-s})\, ds \right) \geq -(|v| + N\, t).$$

Step 2. Let us fix $t_0 > 0$ and define $\tilde{g}_0(t,\cdot) := g(t_0 + t, \cdot)$. Using twice the Duhamel formula (4.10), we find

$$\begin{aligned} \tilde{g}_0(t,\cdot) \ &\geq\ \int_0^t S_{t-s} Q^+ \left(\tilde{g}_0(s,\cdot), \int_0^s S_{s-s'} Q^+(\tilde{g}_0(s',\cdot), \tilde{g}_0(s',\cdot))\, ds' \right) ds \\ &\geq\ \int_0^t \int_0^s S_{t-s} Q^+ \left(S_s \tilde{g}_0, S_{s-s'} Q^+ (S_{s'} \tilde{g}_0, S_{s'} \tilde{g}_0) \right) ds'\, ds. \end{aligned} \tag{4.11}$$

We apply now Lemma 4.6 to $\tilde{g}_0$ and set $R_0 := 2R$. Since S_t is continuous in L^1, there exists $T_1 > 0$, such that for any $s \in [0, T_1]$, there holds

$$\int_{B(v_i,r)} S_s(\tilde{g}_0)(v)\, dv \geq \eta/2 \quad \text{for} \quad i = 1,\, 2,\, 3,$$

and $e^{-T_1} > \xi_0$. For $v \in B(0, R_0)$ and $t \in [0, T_1]$ we may estimate $S_t h$ from below in the following way

$$(S_t\, h)(v) \geq \gamma\, h_{e^{-t}}(v)$$

for some constant $\gamma = \gamma_{R_0, T_1}$. The bound from below (4.11) then yields (using Lemma 4.7)

$$\begin{aligned} \tilde{g}_0(t,\cdot) \ &\geq\ \gamma^2 \int_0^t \int_0^s Q^+_{e^{s-t}} \left(S_s \tilde{g}_0, Q^+_{e^{s'-s}} (S_{s'} \tilde{g}_0, S_{s'} \tilde{g}_0) \right) ds'\, ds \\ &\geq\ \gamma^2 \int_0^t \int_0^s \eta_0\, \mathbf{1}_{v\, e^{s-t} \in B(v_3, r)}\, ds'\, ds. \end{aligned}$$

We have then proved that there exists $T_1 > 0$ and for any $t_1 \in (0, T_1/2]$ there exists $\eta_1 > 0$ such that (for some $\bar v \in B(0,R)$)

$$\forall\, t \in [0, T_1/2], \quad \tilde g_1(t,\cdot) := \tilde g_0(t+t_1,\cdot) \ge \eta_1\, \mathbf{1}_{B(\bar v,\delta_1)}.$$

Step 3. Using again the Duhamel formula (4.10) and the preceding step we have

$$\tilde g_1(t,\cdot) \ge \int_0^t S_{t-s} Q^+(\tilde g_1(s,\cdot), \tilde g_1(s,\cdot))\, ds.$$

Thanks to Lemma 4.7, on the ball $B(0,R_0)$, there holds

$$\begin{aligned} \tilde g_1(t,\cdot) &\ge \eta_1^2 \int_0^t S_{t-s} Q^+(\mathbf{1}_{B(\bar v,\delta_1)}, \mathbf{1}_{B(\bar v,\delta_1)})\, ds \\ &\ge \eta_1^2\, \kappa(\delta_1)\, e^{-(R_0+N\,t)} \int_0^t \mathbf{1}_{e^{-t}\, v \in B(\bar v, \sqrt5\,\delta_1/2)}\, ds \\ &\ge \eta_1^2\, \kappa(\delta_1)\, e^{-(R_0+N\,T_1)}\, t\, \mathbf{1}_{B(\bar v,\sqrt{19}\,\delta_1/4)} \end{aligned}$$

on $[0,T_2]$ with $T_2 \in (0, T_1/2]$ small enough, and then

$$\tilde g_2(t,\cdot) := \tilde g_1(t+t_2,\cdot) \ge \eta_2\, \mathbf{1}_{B(\bar v,\delta_2)} \quad \text{on} \quad [0, T_2/2]$$

with $\delta_2 := \sqrt{19}\,\delta_1/4$ and $t_2 \in (0, T_2/2]$ arbitrarily small, $\eta_2 > 0$. Repeating the argument we obtain

$$\tilde g_k(t,\cdot) := g\left(t + \sum_{i=0}^k t_i, \cdot\right) \ge \eta_k\, \mathbf{1}_{B(\bar v,\delta_k)} \text{ on } [0, T_k/2], \quad \text{with } \delta_k := (\sqrt{19}/4)^k\, \delta_1$$

with $k \ge 1$ and some $t_i \in [0, T_i/2]$ arbitrarily small, $\eta_k > 0$. As a consequence, taking k large enough in such a way that $\delta_k R_0$, we get for some explicit constant $\eta_* > 0$ and some (arbitrarily small) time $t_* > 0$

$$\forall\, t_0 \ge 0, \quad g(t_* + t_0, \cdot) \ge \eta_*\, \mathbf{1}_{B(0,R)}. \tag{4.12}$$

Step 4. Coming back to the Duhamel formula (4.10) where we only keep the first term, we have, for any $t_0 \ge 0$,

$$\forall\, t \ge t_*, \quad g(t_0+t, v) \ge \eta_*\, \mathbf{1}_{|v| \le R\, e^{t-t_*}}\, \exp(-|v| - N\,(t-t_*)).$$

As a consequence, for any $t > t_*$,

$$\begin{aligned} g(t,v) &\ge \mathbf{1}_{|v| \le R\, e^{t-t_*}} \left(\sup_{s \in [0,t-t_*]} \mathbf{1}_{|v| = R\, e^s}\, \exp(-|v| - N\, s) \right) \\ &\ge \mathbf{1}_{|v| \le R\, e^{t-t_*}} \left(\sup_{s \in [0,t-t_*]} \mathbf{1}_{|v| = R\, e^s} \right) \exp(-|v| - N\, \ln^+(|v|/R)), \end{aligned} \tag{4.13}$$

and we conclude gathering (4.12) and (4.13). $\square$

It is straightforward that Theorem 4.9 implies the lower bound in point (ii) of Theorem 1.2.

5 Perspectives

As a conclusion, we discuss some possible perspectives arising from our study.

Let us denote

$$\mathcal{P} \;=\; \Big\{G \in C^\infty,\ G \text{ radially symmetric,} \\ \exists\, a_1, a_2, A_1, A_2 > 0 \ \mid\ a_1 e^{-a_2|v|} \le G(v) \le A_1 e^{-A_2|v|}\Big\}.$$

Conjecture 1. For any mass $\rho > 0$, the self-similar profil G_ρ with mass ρ and momentum 0 is unique.

Conjecture 2. (Strong version) For any initial datum with mass ρ and momentum 0 (maybe with some regularity or moment assumptions), the associated solution satisfies (in rescaled variables)

$$g_t \to_{t\to\infty} G_\rho,$$

for the steady state G_ρ with mass ρ and momentum 0 of (1.10).

Conjecture 2. (Weak version) For any initial datum with mass ρ and momentum 0 (maybe with some regularity or moment assumptions), the associated solution satisfies (in rescaled variables)

$$g_t = g_t^S + g_t^R$$

with $g_t^S \in \mathcal{P}$ and $g_t^R \to_{t\to\infty} 0$ in L^1.

Note that the weak version of conjecture 2 still makes sense when the self-similar profil with mass ρ and momentum 0 is not unique and even if there is no convergence towards some self-similar profil (which could be the case for instance if the solution in rescaled variables "oscillates" asymptotically between several self-similar profils).

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