General relative entropy inequality: an illustration on growth models

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March 3, 2005

Abstract

We introduce the notion of General Relative Entropy Inequality for several linear PDEs. This concept extends to equations that are not concervation laws, the notion of relative entropy for conservative parabolic, hyperbolic or integral equations. These are particularly natural in the context of biological applications where birth and death can be described by zeroth order terms. But the concept also has applications to more general growth models as the fragmentation equations. We give several types of applications of the General Relative Entropy Inequality: a priori estimates and existence of solution, long time asymptotic to a steady state, attraction to periodic solutions for periodic forcing.

Key-words: Relative entropy, fragmentation equations, cell division, self-similar solutions, periodic solutions, long time asymptotic.

AMS class. No: 35B40, 35B10, 82C21, 92B05, 92D25

1 Introduction: Hyperbolic, Parabolic and scattering equations

Many linear Partial Differential Equations or Integral equations with non constant coefficients satisfy some entropy dissipation property. The purpose of this paper is to give on several exemples the entropy functional, the difficulty being that it depends upon the coefficients in a very specific form which does not seems to be known. As we show it below, the most general case of interest is when the equation is not a conservative law, otherwise the principle is known and can be related the Markov process underlying the equation, see for instance [30]. These are particularly natural in the context of biological applications where birth and death can be described by zeroth order terms. To the best of our knowledge this General Relative Entropy

(GRE in short) inequality has been introduced, in a less general framework, in [27], and some of the results of the present paper have been announced in [26].

We first exemplify the notion of GRE on the standard hyperbolic-parabolic equation on the unknown $n = n(t,x)$

$$\frac{\partial n}{\partial t} - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\big(a_{ij}\frac{\partial n}{\partial x_j}\big) + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\big(b_i n\big) + dn = 0 \quad \text{on} \quad (0,\infty)\times\mathbb{R}^d, \tag{1.1}$$

where the coefficients depend on $t \geq 0$ and $x \in \mathbb{R}^d$, $d \equiv d(t,x)$ (no sign assumed), $b_i \equiv b_i(t,x)$, and the symmetric matrix $A(t,x) = \big(a_{ij}(t,x)\big)_{1\leq i,j\leq d}$ satisfies $A(t,x) \geq 0$. We could also set the equation on a domain and assume Dirichlet, Neuman, Robin or periodic boundary conditions without substantial changes in the above calculation. In full generality, it is not obvious to derive a priori bounds on the solution $n(t,x)$, by opposition to the case $A \geq \nu Id > 0$, $\mathrm{div} b + d(x) \geq 0$ where the maximum principle holds.

Consider the associated dual problem (it should be understood as a final time problem)

$$-\frac{\partial \psi}{\partial t} - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\big(a_{ij}\frac{\partial \psi}{\partial x_j}\big) - \sum_{i=1}^{d} b_i \frac{\partial}{\partial x_i}\psi + d\psi = 0 \quad \text{on} \quad (0,\infty)\times\mathbb{R}^d, \tag{1.2}$$

with solution $\psi = \psi(t,x)$.

A straightforward computation leads to the following result.

Lemma 1.1 *(General Relative Entropy, parabolic-hyperbolic equation) For any solutions $p(t,x) > 0$ and $n(t,x)$ to the primal equation (1.1), any solution $\psi(t,x)$ to the dual equation (1.2) and any function $H : \mathbb{R} \to \mathbb{R}$ there holds*

$$\frac{\partial}{\partial t}\left[\psi\, p\, H\left(\frac{n}{p}\right)\right] - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left\{\psi^2 a_{ij} \frac{\partial}{\partial x_j}\left[\frac{p}{\psi} H\left(\frac{n}{p}\right)\right]\right\} + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left[b_i\,\psi\, p\, H\left(\frac{n}{p}\right)\right] =$$
$$= -\psi\, p\, H''\left(\frac{n}{p}\right) \sum_{i,j=1}^{d} a_{ij} \frac{\partial}{\partial x_i}\left(\frac{n}{p}\right) \frac{\partial}{\partial x_j}\left(\frac{n}{p}\right).$$

The interest of such a formula appears clearly for H convex and $\psi > 0$ because it provides a Liapunov functional for the primal equation (1.1). More precisely, if the different quantities have enough decay at infinity (this are the cases below), we can integrate over x the above identity. Then using that the two terms in divergence form (at the left hand side) vanish and that the right hand side is nonpositive, we obtain

$$t \mapsto \mathcal{H}_\psi(n|p) := \int_{\mathbb{R}^d} \psi\, p\, H\left(\frac{n}{p}\right) dx \quad \text{is decreasing.} \tag{1.3}$$

Up to our knowledge the above entropy principle is only known and used in conservative cases.

Example 1. We assume $d(t,x) \equiv 0$, $A = Id$ and $b(x) = -\nabla V(x)$ for a given potential V. In that case, the steady state solutions of (1.1) and (1.2) are

$$p = N(x) := e^{-V(x)} \qquad \psi(x) \equiv 1.$$

When moreover $V(x) \to \infty$ as $|x| \to \infty$ fast enough in order to fulfill appropriate integrability conditions, one arrive at the Relative Entropy Inequality

$$\frac{d}{dt}\int_{\mathbb{R}^d} N(x)\, H\left(\frac{n(t,x)}{N(x)}\right)\, dx = -\int_{\mathbb{R}^d} N(x)\, H''(\frac{n}{N}) \left|\nabla\left(\frac{n(t,x)}{N(x)}\right)\right|^2 dx \leq 0.$$

See Carillo *et al* [9], [3] for similar issues in relation with Monge-Kantorovich mass transportation. It is also related, as far as the control of the entropy by the entropy dissipation is concerned, to logarithmic Sobolev inequalities [5, 29, 2, 9] and the references therein. The fact that N can be taken more generally as a time dependent solution to this equation was also noticed independently by [21].

Another class of classical equations satisfies the same kind of General Relative Entropy, namely the scattering (linear Boltzmann) equation

$$\frac{\partial}{\partial t} n(t,x) + k_T(t,x)\, n(t,x) = \int_{\mathbb{R}^d} K(t,y,x)\; n(t,y)\; dy. \tag{1.4}$$

Here $0 \leq k_T(\cdot) \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^d)$ and $0 \leq K(t,x,y) \in L^\infty\big(\mathbb{R}^+; L^1 \cap L^\infty(\mathbb{R}^d)\big)$ and especially we consider the non-conservative and non-symmetric case as motivated by [12, 22, 8]. The associated dual problem reads now

$$-\frac{\partial}{\partial t}\psi(t,x) + k_T(t,x)\, \psi(t,x) = \int_{\mathbb{R}^d} K(t,x,y)\; \psi(t,y)\; dy. \tag{1.5}$$

Again a straightforward computation leads to the following result.

Lemma 1.2 *(General Relative Entropy, scattering equation) For any solutions $p(t,x) > 0$ and $n(t,x)$ to the primal equation (1.4), any solution $\psi(t,x)$ to the dual equation (1.5) and any function $H : \mathbb{R} \to \mathbb{R}$ there holds*

$$\frac{\partial}{\partial t}\big[\psi(t,x)\; p(t,x)\; H\big(\frac{n(t,x)}{p(t,x)}\big)\big]$$

$$+\int_{\mathbb{R}^d}\left[K(t,x,y)\,\psi(t,y)\,p(t,x)\,H\big(\frac{n(t,x)}{p(t,x)}\big) - K(t,y,x)\,\psi(t,x)\,p(t,y)\,H\big(\frac{n(t,y)}{p(t,y)}\big)\right] dy$$

$$= \int_{\mathbb{R}^d} K(t,y,x)\,\psi(t,x)\,p(t,y)\big[H\big(\frac{n(t,x)}{p(t,x)}\big) - H\big(\frac{n(t,y)}{p(t,y)}\big) + H'\big(\frac{n(t,x)}{p(t,x)}\big)\big[\frac{n(t,y)}{p(t,y)} - \frac{n(t,x)}{p(t,x)}\big]\big]\, dy.$$

When H is convex and $\psi \geq 0$ the above identity provides again a Liapunov functional for the primal equation (1.4): integrating in the x variable we see that the second term vanishes and the right hand side is nonpositive so that (1.3) holds again. A classical case for which the entropy principle (1.3) is known is the following.

Example 2. We assume that the kernels $k_T = k_T(x)$ and $K = K(x,y)$ do not dependent of time, that they are linked by the relation

$$k_T(x) = \int_{\mathbb{R}^d} K(x,y)dy,$$

and that the following detailed balance condition holds

$$\exists\, N; \quad N(x) > 0, \quad K(x,y)N(x) = K(y,x)N(y).$$

We easily check that $\psi \equiv 1$ is a solution of the dual equation (1.5) (that means that the primal equation is conservative) and that $p = N(x)$ is a solution of the primal equation (1.4). As a consequence, we obtain again the usual relative entropy inequality: for all convex function H there holds

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d} N(x) H\big(\frac{n(t,x)}{N(x)}\big)\, dx = \\
&\quad = -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} K(x,y)\, N(x) \left[H'\big(\frac{n(t,x)}{N(x)}\big) - H'\big(\frac{n(t,y)}{N(y)}\big)\right] \left(\frac{n(t,x)}{N(x))} - \frac{n(t,y)}{N(y))}\right) dxdy \leq 0.
\end{aligned}$$

The aim of this paper is to present and to use this general relative entropy principle on a family of fragmentation-growth type equations issued from physical, biological and ecological situations and which take form as a particular case of the combination of the two above equations.

In section 2, we present the general framework and give the three examples we want to deal with, namely the pure fragmentation equation, the cell division equation and the renewal equation with periodic coefficients. We also present the general problematic: first, the problem of existence of particular relevant solutions p and ψ to the primal and dual equations; next, the use of the GRE inequality in order to get some insight on the long time dynamic of the models under consideration. Two kinds of long time behaviors are treated in the following sections: attraction to a steady state or to a periodic solution.

The sections 3, 4 and 5 are then dedicated to study of the three mentioned models and to illustrate in these specific cases the use of the GRE inequality.

2 Growth models and first consequences of GRE inequality

From now on, we are interested in growth models which take the form of a *mass preserving* fragmentation equation complemented with a drift term. More precisely, we denote by $n =$

$n(t,x) \geq 0$ the density of particles/cells of size $x > 0$ at time $t \geq 0$ or the density of individuals of age $x \geq 0$ at time $t \geq 0$ and we consider that the time dynamic of the population of particles/cells/individuals is given by the following equation

$$\begin{cases} \frac{\partial n}{\partial t} + \mathcal{D}_0\, n = \mathcal{F} n \quad \text{on} \quad (0,\infty) \times (0,\infty), \\ \text{boundary condition in } x = 0, \end{cases} \tag{2.1}$$

where $\mathcal{F}$ is a *mass conservative* fragmentation operator

$$(\mathcal{F}n)(t,x) = \int_0^\infty b(t,y,x)\, n(t,y)\, dy - n(t,x)\, B(t,x)$$

and $\mathcal{D}_0$ is a drift term with velocity $v(x) \geq 0$,

$$(\mathcal{D}_0 n)(t,x) = \frac{\partial}{\partial x}\left(v(x)\, n(t,x)\right) + w(t,x)\, n(t,x).$$

We also complement the equation by an initial condition, namely

$$n(t=0,x) = n_0(x). \tag{2.2}$$

Notice that when $\int_0^{\cdot} \frac{1}{v(x)} dx = \infty$ the boundary condition at $x = 0$ in (2.1) is not needed. This is the case of hematopoiesis ([1]) and also of example 3 below. Anyway the boundary condition will be made precise for any example treated below. Also, we would like to make clear that all the equations are to be understood in distributional sense.

The fragmentation operator $\mathcal{F}$ models the division of a single particle of size x into two or more pieces of size $x_k \geq 0$, or in other words, the event

$$\{x\} \xrightarrow{b} \{x_1\} + ... + \{x_k\} + ... , \tag{2.3}$$

in such a way that the mass is conserved

$$x = \sum_k x_k, \quad 0 \leq x_k \leq x.$$

Then $b(x,y)$ is the production rate of particles of size y as the result of the fragmentation event (2.3). For consistency with the modelling we assume

$$b(t,x,y) \geq 0, \qquad b(t,x,y) = 0 \quad \text{for} \;\; y > x, \tag{2.4}$$

$$B(t,x) = \int_0^x \frac{y}{x}\, b(t,x,y)\, dy. \tag{2.5}$$

It the fragmentation creates in the average, k_0 new particles, with $1 < k_0 < \infty$, then we have

$$\int_0^y b(t,x,y)\, dy = k_0 B(t,x), \tag{2.6}$$

For individuals or cells, in the examples 4 and 5 below, this is the case with $k_0 = 2$. At odds with this case, we do not need the condition (2.6) in example 3, where $k_0 = \infty$ is allowed, which means that a fragmentation event may produce an infinite number of particles (with finite total mass!).

The drift term $\mathcal{D}_0$ models the growth (for particles and cells) or the ageing (for individuals) which can be schematically represented by

$$\{x\} \to \{x + v\,dx\}.$$

For the equation (2.1), the associated dual equation reads

$$-\frac{\partial}{\partial t}\psi(t,x) + \mathcal{D}_0^*\,\psi = \mathcal{F}^*\psi, \tag{2.7}$$

with

$$\mathcal{D}_0^*\,\psi = -v\,\frac{\partial \psi}{\partial x} + w\,\psi, \qquad (\mathcal{F}^*\psi)(t,x) = \int_0^x b(t,x,y)\psi(t,y)dy - B(t,x)\psi(t,x). \tag{2.8}$$

We start establishing the GRE principle in the present context.

Theorem 2.1 *(General Relative Entropy, fragmentation drift equation) For any solutions $n(t,x)$ and $p(t,x) > 0$ to (2.1) and any solution $\psi(t,x) \geq 0$ to the dual equation (2.7) and any function $H : \mathbb{R} \to \mathbb{R}$ there holds*

$$\frac{\partial}{\partial t}\big[\psi(t,x)\;p(t,x)\;H\big(\frac{n(t,x)}{p(t,x)}\big)\big] + \frac{\partial}{\partial x}\left[v(t,x)\,\psi(t,x)\,p(t,x)\,H\left(\frac{n(t,x)}{p(t,x)}\right)\right]$$

$$+\int_0^\infty \left[b(t,x,y)\,\psi(t,y)\,p(t,x)\,H\big(\frac{n(t,x)}{p(t,x)}\big) - b(t,y,x)\,\psi(t,x)\,p(t,y)\,H\big(\frac{n(t,y)}{p(t,y)}\big)\right]\,dy$$

$$= \int_0^\infty b(t,y,x)\,\psi(t,x)\,p(t,y)\big[H\big(\frac{n(t,x)}{p(t,x)}\big) - H\big(\frac{n(t,y)}{p(t,y)}\big) + H'\big(\frac{n(t,x)}{p(t,x)}\big)\big[\frac{n(t,y)}{p(t,y)} - \frac{n(t,x)}{p(t,x)}\big]\big]\,dy.$$

Following the argument given in the introduction, we consider now the case when H is convex and there is enough decay for x large. Again, we can integrate in the x-variable. Since the second and third terms vanish, (1.3) holds and we can quantify it as

$$\frac{d}{dt}\mathcal{H}_\psi(n|p) = -\mathcal{D}_\psi(n|p) \leq 0 \tag{2.9}$$

with

$$\mathcal{D}_\psi(n|p) := \int_0^\infty \int_0^\infty b(t,y,x)\,\psi(t,x)\,p(t,y)$$
$$\Big[H\big(\tfrac{n(t,x)}{p(t,x)}\big) - H\big(\tfrac{n(t,y)}{p(t,y)}\big) + H'\big(\tfrac{n(t,x)}{p(t,x)}\big)\Big(\tfrac{n(t,y)}{p(t,y)} - \tfrac{n(t,x)}{p(t,x)}\Big)\Big]\;dxdy. \tag{2.10}$$

This theorem is nothing but a combination of the similar results in the parabolic and scattering cases (Lemmas 1.1 and 1.2) and relies on an easy calculation that we leave to the reader. We list now the three examples we have in mind.

Example 3. Pure fragmentation with scaling invariant fragmentation rate. We assume that $B(t,x) = B(x) = x^\gamma$, $\gamma > 0$ and $b(t,x,y) = B(x)\,\beta(y/x)/x$ where β is a measure on $[0,1]$ such that

$$\beta \geq 0, \quad \int_0^1 z\,\beta(dz) = 1, \quad \int_0^1 z^m\,\beta(dz) < \infty \;\text{ for some }\; m < 1, \tag{2.11}$$

and β satisfies the following positivity condition

$$\exists\,\beta_0 > 0,\; 0 < \delta_1 < \delta_2 < 1 \qquad \beta(z) \geq \beta_0 \quad \forall\, z \in [\delta_1, \delta_2]. \tag{2.12}$$

The pure fragmentation model is then obtained for $\mathcal{D}_0 \equiv 0$ in (2.1). This equation arises in physics to describe fragmentation processes, [23, 7, 4, 6, 16, 4]. For this equation the only steady states are the Dirac masses, namely $x\,n(t,x) = \rho\,\delta_{x=0}$, and then the GRE principle is not pertinent. On the other hand, if n is a solution to the pure fragmentation equation, we may introduce the rescaled density g defined by

$$g(t,x) = e^{-2t} n\left(e^{\gamma\,t} - 1, x e^{-t}\right), \tag{2.13}$$

which is a solution to the fragmentation equation in self-similar variables (see for instance [16])

$$\frac{\partial}{\partial t} g + \frac{\partial}{\partial x}(x\,g) + g = \gamma \mathcal{F}\,g. \tag{2.14}$$

This is a mass preserving equation with no detailed balance condition and then the GRE principle may be used in order to understand in an accurate way the dynamic of the fragmentation mechanism. We refer to the section 3 below which deals with this model.

Example 4. The cell division equation. We consider a population of cells which grow at constant rate and divide through a binary fragmentation mechanism. We denote by $n = n(t,x)$ the density of cells/organisms with mass or volume $x > 0$ at time $t \geq 0$. The general cell division equation (see [24]) reads then

$$\frac{\partial}{\partial t} n(t,x) + \frac{\partial}{\partial x} n(t,x) + B(x) n(t,x) = \int_0^\infty b(y,x) n(t,y) dy \tag{2.15}$$

which we complement with a flux condition at the $x = 0$, namely

$$n(t, x = 0) = 0, \quad t \geq 0. \tag{2.16}$$

In order to take into account that the cell division is a binary and symmetric fragmentation process we assume

$$\int_0^x b(x, y)\, dy = 2\, B(x) \quad \text{and} \quad b(x, y) = b(x, x - y). \tag{2.17}$$

We can recover the equal mitosis equation as some particular example of this equation, with the following appropriate choices for b:

$$\text{(equal mitosis)} \qquad b(x, y) = 2\, B(x)\, \delta(y = x/2) \tag{2.18}$$

which yields the equation

$$\frac{\partial}{\partial t} n(t, x) + \frac{\partial}{\partial x} n(t, x) + B(x) n(t, x) = 4B(2x) n(t, 2x).$$

This equation is studied in [28] for $B(x)$ close to a constant and especially long time convergence to a steady state is proved with an exponential rate. We refer to section 4 where we consider this model.

Example 5. Renewal equation with periodic coefficients. In order to illustrate the case of periodic coefficients, we finally consider a population of individuals with age $x \geq 0$ and which is described by the renewal equation

$$\frac{\partial}{\partial t} n(t, x) + \frac{\partial}{\partial x} n(t, x) + d(t, x)\, n(t, x) = 0, \qquad n(t, x = 0) = \int_0^\infty B(t, y) n(t, y) dy. \tag{2.19}$$

Here we assume that there is $T > 0$ such that d and B are T-periodic.

Although our method also applies to the general cell-division equation, (2.19) allows us a much simpler proof and also, sometimes, to access explicit formulas that can serve as guidelines for our assumptions. Notice that it can also be handled via Volterra integral equations and thus via Laplace transform ([17, 24]) but these methods have not been extended to general cell division equations. Notice that the renewal equation can also be seen as a particular example of the cell division equation (2.15) making the following choice for b:

$$b(t, x, y) = B(t, x)\, [\delta(y = x) + \delta(y = 0)], \qquad \text{(renewal equation)}. \tag{2.20}$$

This choice satisfies the assumptions (2.4)–(2.6) with $k_0 = 2$. Because it rises a Dirac mass at $x = 0$ in the right hand side of the cell division equation (2.15), it can be interpreted, in distribution sense, as a boundary data at $x = 0$ which is the renewal equation. We refer to section 5 where we study this model.

We give now three types of possible applications of the GRE principle: we show a priori bounds on any solution n by comparison to p, we also state a contraction principle in the space L^1 with weight ψ and finally state a result on the long time behavior. For each of the three examples, we prove these results under specific assumptions. They imply the non-degeneracy of the drift and fragmentation terms, and that,

$$p(t,x) > 0 \text{ for } x > 0, \qquad \psi(t,x) > 0, \qquad \int_0^\infty p(t,x)\psi(t,x)dx \equiv 1.$$

Theorem 2.2 *(Existence and a priori bounds) Let $\psi > 0$ be a solution to the dual equation (2.7) with initial condition $\psi(0,.) = \psi_0$. For any initial datum n_0 such that $n_0\psi_0 \in L^1(0,\infty)$, there exists a (unique) solution to equation (2.1) such that*

$$\int_0^\infty n(t,x)\psi(t,x)\,dx = \int_0^\infty n_0\psi_0\,dx \qquad \forall\, t \geq 0. \tag{2.21}$$

Moreover, let $p > 0$ be a solution to (2.1) with initial condition $p(0,.) = p_0$, for any initial datum n_0 such that $n_0\, p_0^{1/q-1}\psi_0^{1/q} \in L^q(0,\infty)$, $q \in (1,\infty)$, (resp. $\exists C_0$, $|n_0| \leq C_0 p_0$), the solution n satisfies

$$\int_0^\infty \frac{|n(t,x)|^q}{p(t,x)^{q-1}}\,\psi(t,x)\,dx \leq \int_0^\infty \frac{|n_0(x)|^q}{p_0(x)^{q-1}}\,\psi_0(x)\,dx \quad \Big(\textit{resp. } |n(t,x)| \leq C_0 p(t,x)\Big) \quad \forall\, t \geq 0. \tag{2.22}$$

Theorem 2.3 *(L^1 contraction) Let $\psi > 0$ be a solution to the dual equation (2.7) with initial condition $\psi(0,.) = \psi_0$. For any initial datum $n_0, m_0 \in L^1(0,\infty;\psi_0\,dy)$ the associated solutions n and m to (2.1) satisfy*

$$\int_0^\infty |n(t,x) - m(t,x)|\,\psi(t,x)\,dx \leq \int_0^\infty |n_0(x) - m_0(x)|\,\psi(0,x)\,dx.$$

The next question, usual when entropy inequalities are available ([13, 31]), is to derive the long time asymptotic of solutions. This is possible under the assumptions of Theorem 2.2 and appropriate additional assumptions of positivity of the fragmentation operator $\mathcal{F}$. Introducing the "total mass" $\rho \geq 0$ associated to the conserved quantity (see (2.21))

$$\int_0^\infty n(0,y)\,\psi_0(y)\,dy = \rho \int_0^\infty p(0,y)\,\psi_0(y)\,dy,$$

there holds

$$\int_0^\infty |n(t,x) - \rho\, p(t,x)|\,\psi(t,x)\,dx \underset{t\to\infty}{\longrightarrow} 0. \tag{2.23}$$

This result is based on the mixing property of the equation (2.1). It acts in such a way that the initial condition is asymptotically forgotten and the solution only keeps memory of the single information contained in the conservation law (2.21). The property (2.23) will be proved in any example under appropriate assumptions of positivity of the fragmentation operator which guarantees the mixing property of the flow. The asymptotic behavior (2.23) is particularly relevant when (for instance) p is a stationary solution for coefficients independent of time or when p is a periodic solution for time periodic coefficients. The former phenomena is known as 'desynchronization' ([10]), the later is resynchronization (on a circadian or seasonal rhythm for instance) [20].

In the theory we develop here, the first question one has to answer in order to obtain pertinent general relative entropy is precisely to find the pertinent particular solution p. In the case of example 3 the model is mass conservative and it is possible to prove existence of a stationary solution with the help of the Schauder Theorem (see for instance [18, 16] for details), in other words 0 is the first eigenvalue. On the other hand, in the case of the models described in examples 4 and 5, the equations are not conservative and do not have stationary solutions. One has to solve simultaneously the eigenvalue problem associated to the primal and the dual equations. More precisely, we look for (λ_0, p, ψ) such that

$$\begin{cases} \frac{\partial p}{\partial t} + \mathcal{D}_0\, p + \lambda_0\, p = \mathcal{F} p \quad \text{on} \quad (0,\infty)\times(0,\infty), \\ \\ -\frac{\partial \psi}{\partial t} + \mathcal{D}_0^*\, \psi + \lambda_0\, \psi = \mathcal{F}^* \psi \quad \text{on} \quad (0,\infty)\times(0,\infty), \end{cases} \tag{2.24}$$

with appropriate boundary conditions, initial conditions and stationary or periodicity conditions. Here in very particular cases an explicit computation may be performed (see [28]) but in general existence of (λ_0, p, ψ) is obtained by the mean of the Krein-Rutman Theorem.

The second question is to understand how the GRE inequality, based on these particular solutions may be used in order to get some information on generic solutions. While Theorems 2.2 and 2.3 are standard, the question of long time behavior is more subtle and require more attention (and additional assumptions) and will be treated for each example separately.

We conclude this section stating some problems of interest which are closely related to the present work.

1. Rate of convergence to the steady state, or to periodic solution, in (2.23). See however [27, 28, 19].

2. Dependance of λ_0 with respect to the coefficients involved in the model? As a biological interpretation, one can expect to observe in nature only those species that maximize λ_0 in a given environment.

3. Use of the entropy method for nonlinear problems (see [14] for finite dimensional models)

3 The pure fragmentation equation

In this section we consider the pure fragmentation equation in self-similar variables (2.14) as motivated in example 3. We assume that b fulfills the assumptions (2.11)–(2.12) as stated in the presentation of example 3 above. Let first consider the dual problem

$$-\frac{\partial}{\partial t}\,\psi + \mathcal{D}_0^*\psi = \gamma\,\mathcal{F}^*\,\psi.$$

It has a simple solution $\psi(x) = x$ since $\mathcal{D}_0^* h = x\frac{\partial h}{\partial x} - h$ and $\mathcal{F}^* x = 0$ by assumption (2.5). Therefore, using (2.21), we deduce that (2.14) is a mass conservative equation, that is

$$\int_0^\infty x\,g(t,x)\,dx \equiv \text{cst} \qquad \forall\, t \geq 0.$$

In order to apply the GRE inequality we need next to find particular relevant solutions to the equation (2.14) which are here stationary solutions. More precisely, we are looking for a steady solution N to the self-similar profile fragmentation equation

$$\frac{\partial}{\partial x}\,(x\,N) + N = \mathcal{F}\,N, \quad N \geq 0, \quad \int_0^\infty x\,N(x)\,dx = 1. \tag{3.1}$$

The self-similar profile is given by the following. Here and below we denote

$$\dot{L}^1_k = \{g \in L^1_{loc}(0,\infty);\ x^k\,g(x) \in L^1\}.$$

Theorem 3.1 *With assumptions (2.11)–(2.12), there exists a unique solution N in $\dot{L}^1_1$ to equation (3.1). Moreover $N \in W^{1,\infty}_{loc}(0,\infty)$, $y^k\,N \in L^\infty$ $\forall\, k \geq 1+m$ and $N > 0$ on $(0,\infty)$.*

We may now give a consequence of the GRE inequality on the long time behavior.

Theorem 3.2 *For any $g_0 \in \dot{L}^1_m \cap \dot{L}^1_M$ with $M > 1$ and $\rho := \int_0^\infty x\,g_0(x)\,dx$, there exists a unique solution $g \in C([0,T);\dot{L}^1_1) \cap L^1(0,T;\dot{L}^1_{\gamma+M})$ ($\forall\, T > 0$) to the fragmentation equation (2.14), and*

$$\int_0^\infty x\,g(t,x)\,dx = \rho \quad \textit{for all}\ \ t \geq 0.$$

Moreover, g satisfies

$$(g(t))_{t\geq 1} \quad \textit{is uniformly bounded in}\ \ \dot{L}^1_k \quad \forall\, k \geq m, \tag{3.2}$$

$$\lim_{t\to+\infty} \int_0^\infty x\,|g(t,x) - \rho N(x)|\,dx = 0. \tag{3.3}$$

Back to the pure fragmentation equation (2.1), its solution

$$n(t,x) = (1+t)^{\frac{2}{\gamma}} g\left(\frac{1}{\gamma}\ln(1+t), (1+t)^{\frac{1}{\gamma}}\, x\right) \tag{3.4}$$

converges as $t \to \infty$ to a Dirac mass. Then, our Theorem gives the precise convergence speed and the profile. Those are determined as

$$n(t,x) \approx (1+t)^{\frac{2}{\gamma}} N\left((1+t)^{\frac{1}{\gamma}}\, x\right) \quad \text{when} \quad t \to \infty.$$

Proof of Theorem 3.1. We refer to [16] for the existence of solution $N \in \dot{L}^1_1$ to the equation (3.1) such that $N \in \dot{L}^1_k$, $\mathcal{F}\, N \in \dot{L}^1_k$ for any $k \geq m$. Writing for $k \geq 1+m$

$$\frac{\partial}{\partial y}(y^k\, N) = \frac{\partial}{\partial y}(y^{k-2}\, y^2\, N) = (k-2)\, y^{k-1} N + y^{k-1}\, \mathcal{F}N \tag{3.5}$$

we deduce that $y^k\, N \in L^\infty$ for any $k \geq 1+m$. Furthermore, gathering (3.5) with $B\, N \in L^\infty_{loc}$ and

$$\begin{aligned}
(\mathcal{F}^+ N)(x) \quad &:= \quad \int_x^\infty (y)^{\gamma-1} \beta(x/y)\, N(y)\, dy \leq \|N\, x^{2+\gamma}\|_{L^\infty} \int_x^\infty (y)^{-3}\, \beta(x/y)\, dy \\
&\leq \quad \|N\, x^{2+\gamma}\|_{L^\infty} \int_0^1 \frac{z^3}{x^3}\, \beta(z)\, x\, \frac{dz}{z^2} = \|N\, x^{2+\gamma}\|_{L^\infty}\, x^{-2} \in L^\infty_{loc},
\end{aligned}$$

we obtain that $y^2\, N \in W^{1,\infty}_{loc}$. That concludes the proof of the regularity estimate.

Finally, there holds

$$\frac{\partial}{\partial y}\left(y^2\, N(y)\, e^{y^\gamma/\gamma}\right) = y\, (\mathcal{F}^+ N)(y)\, e^{y^\gamma/\gamma}. \tag{3.6}$$

Since $N \not\equiv 0$ there exists $x_0 \in (0,\infty)$ such that $N(x_0) > 0$. On the one hand, integrating (3.6) between 0 and x, for any $x \in (\delta_2\, x_0, x_0)$, we have

$$\begin{aligned}
x^2\, N(x)\, e^{x^\gamma/\gamma} \quad &\geq \quad \int_0^\infty N(y)\, y^\gamma\, e^{y^\gamma/\gamma} \int_0^1 B(z)\, z\, e^{z^\gamma/\gamma}\, \mathbf{1}_{z \leq x/y}\, dz\, dy \\
&\geq \quad C \int_{x_0}^{(\delta_1+\delta_2)/(2\delta_1)} N(y) \int_0^1 B(z)\, z\, \mathbf{1}_{\delta_1 \leq 2\delta_1\delta_2/(\delta_1+\delta_2)}\, dz\, dy > 0.
\end{aligned}$$

By an iterative argument, we find $N > 0$ on $(0, x_0)$. On the other hand, for any $x > x_0$, integrating (3.6) between x_0 and x and using the fact that $\mathcal{F}^+ N \geq 0$, we find

$$N(x) \geq cst\, x^{-2}\, e^{-x^\gamma/\gamma} > 0 \quad \text{on} \quad (x_0, \infty),$$

and that conclude the proof of positivity property on N. $\square$

Proof of Theorem 3.2. From [16] we already know that, with the assumptions made above, there exists a unique solution g satisfying the estimate (3.2) and we just have to prove (3.3). This will be achieved in several steps.

Step 1. Let us first assume that $y \mapsto y\, g_0^2(y)\, N^{-1}(y) \in L^1$. We use Theorem 2.1 with $H(s) = (s-1)^2$ and denote simply by $\mathcal{H}$ and $\mathcal{D}$ the corresponding entropy and entropy dissipation. Then, thanks to Theorem 2.2, there exists a unique solution g associated to the initial data g_0 such that

$$\mathcal{H}(g|N) := \int_0^\infty g^2\, N^{-1}\, y\, dy \le \mathcal{H}(g_0|N) < \infty \tag{3.7}$$

and, using the fact that for any $\xi, \xi' \ge 0$ there holds $H(\xi) - H(\xi') + H'(\xi')\,(\xi' - \xi) = (\xi - \xi')^2$,

$$\mathcal{D}(g|N) := \int_0^\infty \int_0^\infty b(x,y)\, N(x)\, y \left(\frac{g(x)}{N(x)} - \frac{g(y)}{N(y)} \right)^2 dxdy \in L^1_t(0,\infty). \tag{3.8}$$

Consider now a sequence (t_n) such that $t_n \to \infty$, a time $T > 0$ and define $g_n(t,y) := g(t+t_n, y)$. From $0 < N \in W^{1,\infty}_{loc}$ and (3.7), we know that the sequence (g_n) is bounded in $L^2_{loc}([0,T] \times (0,\infty))$ and we may extract a subsequence still denoted by (t_n) such that $g_n \rightharpoonup \bar{g}$ weakly in $L^2_{loc}([0,T] \times (0,\infty))$. On the one hand, for any function $\varphi \in C^1_c(]0,\infty[)$, using the equation (2.14) and the estimate (3.2) we have

$$\frac{d}{dt} \int_0^\infty g_n\, \varphi\, dx \quad \text{is bounded in} \quad L^1(0,T),$$

from which we deduce that

$$\int_0^\infty g_n\, \varphi\, dx \underset{n\to\infty}{\longrightarrow} \int_0^\infty \bar{g}\, \varphi\, dx \quad \text{in} \quad L^1(0,T) \quad \forall\, \varphi \in C^1_c(]0,\infty[). \tag{3.9}$$

On the other hand, we introduce for any $\varepsilon \in (0,1)$ the truncated dissipation entropy

$$\mathcal{D}_\varepsilon(g|N) := \int_\varepsilon^{1/\varepsilon} \int_\varepsilon^{1/\varepsilon} b(x,y)\, N(x)\, y \left(\frac{g(x)}{N(x)} - \frac{g(y)}{N(y)} \right)^2 dxdy. \tag{3.10}$$

Thanks to (3.9) and standard convexity arguments (see [15]), we see that $g \mapsto \mathcal{D}(g|N)$ is l.s.c. for the above sense of convergence for (g_n) and therefore using (3.8)

$$\int_0^T \mathcal{D}_\varepsilon(\bar{g}|N)\, dt \le \liminf_{n\to\infty} \int_0^T \mathcal{D}_\varepsilon(g_n|N)\, dt \le \liminf_{n\to\infty} \int_{t_n}^\infty \mathcal{D}(g|N)\, ds = 0 \qquad \forall\, \varepsilon > 0. \tag{3.11}$$

We set $\xi(x) := \bar{g}(t,x)/N(x)$ and combine (3.10) and (3.11), then let $\varepsilon \to 0$. We get

$$\xi(y) = \xi(x) \quad \text{for a.e. } t,\ x,\ y \quad \text{s.t. } y/x \in [\delta_1, \delta_2]. \tag{3.12}$$

Step 2. We prove that (3.12) implies $\bar{g} = \rho\, N$. On the one hand, for any $y, z > 0$ there exists $n, m \in \mathbb{N}^*$ s.t.

$$]\delta_1^n\, y, \delta_2^n\, y[\, \cap\,]\delta_1^m\, z, \delta_2^m\, z[\, \neq \emptyset. \tag{3.13}$$

Indeed, assuming for instance $y < z$, we may first find $k \in \mathbb{N}$ such that $\delta_2^{k+1}\, z \le y < \delta_2^k\, z$. We next define $n \in \mathbb{N}$ such that

$$\delta_2^{r+1+k}\, z < \delta_1^r\, \delta_2^k\, z \ \text{ for all } \ r = 0, ..., n-1, \quad \text{ and } \quad \delta_2^{n+1+k}\, z \ge \delta_1^n\, \delta_2^k\, z.$$

As a consequence,

$$\delta_1^{n+1+k}\, z < \delta_1^n\, \delta_2^{1+k}\, z \le \delta_1^n\, y < \delta_1^n\, \delta_2^k\, z \le \delta_2^{n+1+k}\, z$$

and (3.13) holds with $m := n+1+k$.

On the other hand, we define $K = \{x \in (0,\infty);\ \xi(y) = \xi(x) \ \ for \ \ a.e. \ \ y \in [\delta_1 x, \delta_2 x]\}$ and fixing $x \in K$ we define $A_+ = \{y \in (0,\infty);\ \xi(y) = \xi(x)\}$. From the definition of K and x there holds $|A_+| > 0$. Define $A_- := (0,\infty)\backslash A_+$ and assume by contradiction that $|A_-| > 0$. That means that, there exists $y \in A_+$, $z \in A_-$ such that $\forall\, \varepsilon > 0$ $|B(y,\varepsilon) \cap A_+| > 0$, $|B(z,\varepsilon) \cap A_-| > 0$. Thanks to (3.13) we may find $\varepsilon > 0$ such that for any $y' \in B(y,\varepsilon)$, $z' \in B(z,\varepsilon)$ there holds

$$]\delta_1^n\, y', \delta_2^n\, y'[\cap]\delta_1^m\, z', \delta_2^m\, z'[\ne \emptyset.$$

As a consequence, for a.e. $y' \in A_+ \cap B(y,\varepsilon)$, for a.e. $z' \in A_- \cap B(z,\varepsilon)$ there holds $\xi(y') = \xi(z')$ and that is absurd. Therefore we have $|A_-| = 0$ so that $\xi \equiv \xi(x)$ a.e. Then, we have proved that for some $x = x(t) \in (0,\infty)$,

$$g(t,y) = \xi(t,x)\, N(y) \quad \text{for a.e.} \quad (t,y) \in (0,T) \times (0,\infty)$$

and the mass condition implies $\xi(t,x) = \rho$ for any $t \in (0,T)$.

Step 3. Combining (3.2) with the results obtained in steps 1 and 2, we have yet proved that

$$g_n(t,.) \rightharpoonup \rho\, N \quad \text{weakly in} \quad \dot L^1_1 \cap L^2_{loc}, \tag{3.14}$$

and we have to prove that this convergence holds in fact in the strong sense. Let fix $\varepsilon_0 \in (0,1)$ such that

$$\int_{\varepsilon_0}^{1/\varepsilon_0} z\, \beta(z)\, dz \ge 1/2.$$

For any $\varepsilon \in (0,\varepsilon_0)$, there exists $\eta_\varepsilon > 0$ such that there holds

$$\begin{aligned}
\eta_\varepsilon \int_0^T \int_\varepsilon^{1/\varepsilon} & (g_n(t,x) - \rho\, N(x))^2\, dxdt \le \\
& \le \int_0^T \int_\varepsilon^{1/\varepsilon} \int_\varepsilon^{1/\varepsilon} b(x,y)\, N(x)\, y \left[\left(\frac{g(t,x)}{N(x)} - \rho \right)^2 + \left(\rho - \frac{g(t,y)}{N(y)} \right)^2 \right] dxdydt \\
& = \int_0^T D_{2,\varepsilon}(g|N)\, dt \\
& \quad + 2 \int_0^T \int_\varepsilon^{1/\varepsilon} \int_\varepsilon^{1/\varepsilon} b(x,y)\, N(x)\, y \left[\frac{g(t,x)}{N(x)} \frac{g(t,y)}{N(y)} + \rho^2 - \rho \frac{g(t,x)}{N(x)} - \rho \frac{g(t,y)}{N(y)} \right] dxdydt.
\end{aligned}$$

Thanks to (3.14)–(3.9) and (3.11) we easily deduce that

$$\eta_\varepsilon \int_0^T \int_\varepsilon^{1/\varepsilon} (g_n(t,x) - \rho\, N(x))^2 \, dx dt \to 0 \quad \forall\, \varepsilon > 0,$$

and we conclude that (3.3) holds using (3.14) and the contraction principle stated in Theorem 2.3 applied to $n_0 = \rho\, N$ and $m_0 = g(t_n + \tau, .)$ for some $\tau \in (0,T)$.

Step 4. For $g_0 \in \dot{L}^1_m \cap \dot{L}^1_M$ we consider a sequence $(g_{0,n})$ such that $\mathcal{H}(g_{0,n}|N) < \infty$, the mass associated to $g_{0,n}$ is ρ and $g_{0,n} \to g_0$ in $\dot{L}^1_m \cap \dot{L}^1_M$. On the one hand, the solution g_n associated to $g_{0,n}$ satisfies $\|g_n - \rho\, N\|_{\dot{L}^1_1} \to 0$. On the other hand, the contraction principle stated in Theorem 2.3 implies that $\|(g - g_n)(t)\|_{\dot{L}^1_1} \le \|g_0 - g_{0,n})\|_{\dot{L}^1_1}$. As a conclusion g satisfies the asymptotic property (3.3). $\square$

4 Cell division, existence and steady states

In this Section we consider the cell division equation of example 4 of Section 2. We restrict our attention to the case of coefficients independent of time

$$b(t,x,y) \equiv b(x,y), \qquad B(t,x) = B(x), \tag{4.1}$$

A classical question is the existence of a global attractive steady state, the so-called Stable Size Distribution ([24]), i.e., that is observed in practice. Steady states do not always exist because an exponential growth is expected. Therefore we have to settle this in an eigenvalue problem and we use the notation $N(x) = p(t,x)e^{-\lambda_0 t}$ and $\phi(x) = \psi(t,x)e^{-\lambda_0 t}$. Then, the problem is first to find $(\lambda_0, N(x), \phi(x))$ such that

$$\begin{cases} \frac{\partial}{\partial x} N(x) + (\lambda_0 + B(x))N(x) = \int_x^\infty b(y,x)N(y)dy, & x \ge 0, \\ \\ N(x=0) = 0, \qquad N(x) > 0 \ \text{ for } x > 0, \qquad \int N = 1, \end{cases} \tag{4.2}$$

$$\begin{cases} \frac{\partial}{\partial x} \phi(x) - (\lambda_0 + B(x))\phi(x) = -\int_0^x b(x,y)\phi(y)dy, & x \ge 0, \\ \\ \phi(x) > 0, \qquad \int \phi N = 1. \end{cases} \tag{4.3}$$

Also the precise dynamic of the system is better described after renormalizing n taking into account the exponential growth. Therefore, we set $g(t,x) = n(t,x)e^{-\lambda_0 t}$ and obtain

$$\begin{cases} \frac{\partial}{\partial t} g + \frac{\partial}{\partial x} g + (\lambda_0 + B(x))g = \int_x^\infty b(y,x)g(t,y)dy, & x \ge 0, \\ \\ g(x=0) = 0. \end{cases} \tag{4.4}$$

The existence of eigenelements (λ_0, N, ϕ) relies on the balance between transport (to larger values of x) and division (that reduces x and increases n). Such an eigenvalue problem does not always have a solution since we have

Lemma 4.1 *With the assumptions (2.4), (2.5), (2.6) with $k_0 = 2$, (2.17) and (4.1), if a solution to (4.2) exists, then*

$$\int_0^\infty B(x)dx \geq 1/2. \tag{4.5}$$

Proof. First, we integrate the equation (4.2) in the size variable all over $\mathbb{R}_+$, then using (2.6), we get

$$\lambda_0 = \int B(x)N(x)dx > 0.$$

Next, integrating again the equation (4.2) in the size variable, but between 0 and x, we find

$$N(x) \leq \int_{z=0}^{x} \int_{y=0}^{\infty} b(z,y)N(y)dydz \leq \int_0^\infty \int_0^\infty b(z,y)N(y)dydz = 2\int_0^\infty B(y)N(y)dy, \quad \forall x \geq 0,$$

and thus

$$\|N\|_{L^\infty} \leq 2\lambda_0.$$

Finally, we come back to the first identity and we obtain

$$\lambda_0 = \int_0^\infty B(x)N(x)dx \leq \int_0^\infty B(x)dx\|N\|_{L^\infty} \leq 2\lambda_0 \int_0^\infty B(x)dx.$$

Hence, if there is a solution, then we should have (4.5). $\square$

In view of Lemma 4.1, we consider only a simple case for existence, better conditions can be found in [25]. But optimal conditions are known only in the case of the renewal equation (2.19), a special case (see example 5 equation (2.20)) where we find as a necessary and sufficient condition $\int B > 1$.

Theorem 4.2 *(First eigenvectors) Assume (2.4), (2.5), (2.6) with $k_0 = 2$, (4.1) and*

$$0 < B_m = \min_{x \geq 0} B(x), \qquad \max_{x \geq 0} B(x) = B_M < \infty. \tag{4.6}$$

There exists a unique lipschitz continuous solution (λ_0, N, ϕ) to (4.2), (4.3) and

$$B_m \leq \lambda_0 \leq B_M, \tag{4.7}$$

$$\int_0^\infty N(x)e^{\mu x}dx \leq \frac{\lambda_0}{\lambda_0 - \mu}, \qquad \sup_{x \in (0,\infty)} N(x)e^{\mu x} \leq \lambda_0 + \frac{\lambda_0 B_M}{\lambda_0 - \mu}, \qquad \forall \mu \in [0, \lambda_0),$$

$$\exists C > 0, \quad s.t. \qquad 0 \leq \phi(x) \leq C(1 + x).$$

The exponential decay for N is (close to be) sharp with our assumptions since for the renewal equation (2.19), we have exactly $N(x) = \lambda_0 e^{-\lambda_0 x}$. See [25] for more precise estimates in this direction.

Theorem 4.3 *We make the assumptions of Theorem 4.2 and*

$$\exists C_0, \;\; s.t. \;\; \forall x \qquad |g(0,x)| \leq C_0 N(x). \tag{4.8}$$

There is a unique solution to (4.4) and for all $t > 0$,

$$|g(t,.)| \leq C_0 N, \qquad \int g(t,y)\phi(y)dy = \int g(0,y)\phi(y)dy := \rho, \tag{4.9}$$

$$\int |g(t,y)|\phi(y)dy \leq \int |g(0,y)|\phi(y)dy \qquad \textit{(contraction principle)}. \tag{4.10}$$

With the following non-degeneracy condition on the support of b: there exists a C^1 function $\Gamma : (0,\infty) \to (0,\infty)$ such that

$$\{(x,\Gamma(x)), \quad x \geq 0\} \subseteq \Delta = Supp_{[0,\infty[\times[0,\infty[}\; b(x,y) \;\; and \;\; \frac{\partial}{\partial x}\Gamma(x) \neq 1 \;\; \forall x \neq 0, \tag{4.11}$$

we have

$$\lim_{t\to\infty} \| \, g(t,.) - \rho \, N \, \|_{L^q(N^{1-q}\phi dx)} = 0 \qquad \forall q \in [1,\infty). \tag{4.12}$$

Remark 4.4 *The condition (4.11) is much more general than the non-degeneracy condition (2.12) for scaling invariant fragmentation kernels b. In this case, the condition (4.11) holds with $\Gamma(x) = \frac{\delta_1+\delta_2}{2}x$ but is not enough to prove (3.3) in Theorem 3.2. The condition (4.11) is also fullfiled for equal mitosis $b(x,y) = 2B(x)\delta_{y=x/2}$ while condition (2.12) is of course not fullfiled for such a kernel.*

The exponential rate of convergence here is known in special cases. For the renewal equation (2.19), an abstract argument due to [17] proves the exponential rate (but the rate is not explicitely known) for B with compact support. In [27], an explicit rate is given when supp$B$ is an interval that contains $x = 0$ and an a recent improvment is due to [19]. For equal mitosis (2.18), an explicit rate is also given in [28] when $B(x)$ is close to a constant. We now turn to the proof of these two Theorems.

Proof of Theorem 4.2. We refer to [28, 25] for the method and ideas developed here, and we only sketch the main estimates. The rigorous proof goes through an approximation process which is written in details in the above references. Then, we only need to prove a priori estimates that imply compactness of (λ_0, N, ϕ).

Step 1. Bounds on λ_0. After multiplying equation (4.2) by 1 and x and integrating, we obtain

$$\lambda_0 = \int_0^\infty B(y)N(y)dy \qquad \text{and} \qquad \lambda_0 \int_0^\infty yN(y)dy = 1. \tag{4.13}$$

The upper and lower bounds on λ_0 follows from the first idendity, the assumption (4.6) and the normalization of N in (4.2).

Step 2. Bounds on N. We firstly prove that

$$\int_0^\infty b(x,y)e^{\mu y}dy \leq (1 + e^{\mu x})B(x). \tag{4.14}$$

To do this, we notice that, because of $y < x$ in the integrals below (thanks to (2.17)), and using (2.5),

$$\int_0^\infty y^p b(t,x,y)dy = x^p \int_0^x b(t,x,y)(\frac{y}{x})^p dy \leq x^p B(t,x), \qquad \forall p = 2,3,4,\dots$$

Secondly, we deduce

$$\int_0^\infty b(x,y)\frac{(\mu y)^p}{p!}dy \leq \frac{(\mu x)^p}{p!}B(x), \qquad p \geq 1,$$

and thus, using (2.17), the inequality (4.14) holds.
Therefore, multiplying equation (4.2) by $e^{\mu x}$ with $\mu < \lambda_0$ and integrating, we obtain

$$\begin{aligned} \forall x \qquad N(x)e^{\mu x} + \textstyle\int_0^x [\lambda_0 - \mu + B(z)]N(z)e^{\mu z}dz &\leq \textstyle\int_0^\infty \int_0^\infty b(y,z)e^{\mu z}N(y)dzdy \\ &\leq \textstyle\int_0^\infty [B(y) + B(y)e^{\mu y}]N(y)dy. \end{aligned}$$

Letting $x \to \infty$ and using (4.14), we deduce that

$$\int_0^\infty (\lambda_0 - \mu)N(z)e^{\mu z}dz \leq \int_0^\infty B(y)N(y)dy = \lambda_0.$$

This is the first bound on N, the second one follows from the same inequality, using the information

$$N(x)e^{\mu x} \leq \int_0^\infty [B(y) + B(y)e^{\mu y}]N(y)dy \leq \lambda_0 + B_M \int e^{\mu y}N(y)dy.$$

Step 3. Estimate on ϕ. We refer to [28] to prove the existence of a constant C such that $\phi(y) \leq C(1 + y^k)$ for some $k > 0$ in the case of equal mitosis. Here we improve the proof in order to get the linear growth and treat more general kernels b.

We follow the proof in [28], using a solution $(N_L,\ \lambda_L,\ \phi_L)$ of the eigenproblem on a bounded interval $(0, L)$ with $\phi_L(L) = 0$. Then firstly, one can derive, as above, an priori bounds on N_L . Secondly one derives local bounds on ϕ_L. We write, integrating equation (4.3) on $(0, x_L)$,

$$\sup_{(0,x_L)} \phi_L(y) \leq \phi_L(x_L) + \sup_{(0,x_L)} \phi_L(y) \int_0^{x_L} \int_0^y b(y, y')dy'dy,$$

and choose $x_L = a$ such that $\int_0^a \int_0^y b(y, y')dy'dy = 1/2$. Then $\sup_{(0,a)} \phi_L(y) \leq 2\phi_L(a)$. It remains to bound $\phi_L(a)$ which we do using that

$$\phi_L(a) \leq \frac{\int_0^a N_L(x)\phi_L(x)e^{\int_x^a (\lambda+B(s))ds}dx}{\int_0^a N_L(x)dx} \leq \frac{\sup_{0\leq x\leq a} e^{\int_x^a (\lambda+B(s))ds}dx}{\int_0^a N_L(x)dx},$$

that we deduce because $\phi_L(x)e^{-\int_0^x (\lambda+B(s))ds}$ is decreasing and finite by the choice $a > 0$ (therefore $\int_0^a N_L$ is uniformly positive). Thirdly, and this is the new point here, we find a supersolution (independent of L) for the equation on ϕ_L. We notice that $\overline{v}(y) = C(L - y)$ is a supersolution of the equation on $\overline{\phi}_L(y) = \phi_L(L - y)$. Indeed $\overline{\phi}_L(y)$ satisfies

$$\frac{\partial}{\partial y}\overline{\phi}_L(y) + (\lambda_L + B(L - y))\overline{\phi}_L(y) = \int_0^{L-y} b(L - y, y')\overline{\phi}_L(L - y')dy',$$

and using $\int_0^x yb(x, y)dy = B(x)x$, we find that $\overline{v}(y)$ is a supersolution if $L - y$ is large enough, indeed

$$-C + C\lambda_L(L - y) + C[B(L - y)(L - y) - \int_0^{L-y} b(L - y, y')y'dy'] \geq 0,$$

if $L - y \geq 1/\lambda_L$. Therefore we have indeed $\phi(y) \leq C(1 + y)$ and Theorem 4.2 is proved. $\square$

Proof of Theorem 4.3. We first notice that the first inequality in (4.9) follows directly from the GRE inequality (2.1) with for instance $H(h) = (h - C_0)_+^2$. This is a nonnegative convex function, therefore it gives $\int_0^\infty N\phi H\big(g(t)/N\big)dx \leq 0$ for all $t > 0$, and thus $H\big(g(t)/N\big) = 0$, i.e., $g(t)/N \leq C_0$. A similar argument proves the inequality $g(t)/N \geq -C_0$. The equality in (4.9) follows also directly from the GRE inequality with $H(h) = h$. Finally, the contraction principle (4.10) follows from the GRE inequality with $H(h) = |h|$. It remains to prove (4.12) which we do in several steps.

Step 1. We proceed along the lines of the proof of Theorem 3.2. Arguing as in *Step 4* of Theorem 3.2, we see that we can restrict ourselves to consider a smooth initial data g_0 such that $h^0 = g_0/N \in C_0^1$.

Step 2. We then introduce the sequence of function $g_n(t, y) = g(t + t_n, y)$. As in the *Step*

1 of Theorem 3.2, we have $g_n \rightharpoonup \overline{g}$ and $\overline{g}/N(t,x) = \overline{g}/N(t,y)$ $\forall t \forall (x,y) \in \Delta$. Therefore the function $u := \overline{g}/N$ satisfies

$$u(t,\Gamma(x)) = u(t,x), \qquad \forall t > 0,\ x \geq 0.. \tag{4.15}$$

Step 3. In the limit, the entropy dissipation (2.10) vanishes in (2.9), and thus this function u satisfies

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}u = 0. \tag{4.16}$$

Step 4. Thanks to Lemma 4.5 below we have $u(t,x) = cst$ and the mass condition allows us to conclude $\overline{g} = \rho N$.
Step 5. We conclude the proof as in Theorem 3.2 using the contraction property. □

Lemma 4.5 *Any function u satisfying (4.16), (4.15) is constant.*

Proof. On one hand we have

$$(\partial_t u)(t,x) = (\partial_t u(t,\Gamma(x))) = (\partial_t u)(t,\Gamma(x)). \tag{4.17}$$

On the other hand we have

$$(\partial_x u)(t,x) = (\partial_x u(t,x)) = (\partial_x u(t,\Gamma(x))) = \Gamma'(x)(\partial_x u)(t,\Gamma(x)). \tag{4.18}$$

We deduce gathering (4.17), (4.18) and using (4.16) that

$$(\partial_t u)(t,\Gamma(x)) + \Gamma'(x)(\partial_x)u(t,\Gamma(x)) = 0, \qquad \forall t > 0,\ x \geq 0, \tag{4.19}$$

and from (4.16) we also have

$$(\partial_t u)(t,\Gamma(x)) + (\partial_x)u(t,\Gamma(x)) = 0, \qquad \forall t > 0, x \geq 0. \tag{4.20}$$

Combining (4.19), (4.20) we get

$$(\Gamma'(x) - 1)(\partial_x)u(t,\Gamma(x)) = 0,$$

from which we dedue, since $\Gamma'(x) \neq 1$,

$$(\partial_x)u(t,x) = \Gamma'(x)(\partial_x)u(t,\Gamma(x)) = 0.$$

Finally usning again the transport equation (4.16) we obtain indeed that u is constant. □

5 Renewal equation and periodic solutions

We now consider the renewal equation with T-periodic death and birth rates d and B,

$$\begin{cases} \frac{\partial}{\partial t} n + \frac{\partial}{\partial x} n + d(t,x)\, n = 0, \\ \\ n(t,0) = \int_0^\infty B(t,y)\, n(t,y))\, dy, \\ \\ n(t=0,x) = n_0(x). \end{cases} \tag{5.1}$$

and we make the following assumptions on the nonnegative functions d, B,

$$\sup_{t\in(0,T)} \int_0^\infty B(\cdot,y) e^{-\int_0^y d(\cdot+y'-y,y')dy'} dy < \infty, \qquad \inf_{t\in(0,T)} \int_0^\infty B(\cdot,y) e^{-\int_0^y d(\cdot+y'-y,y')dy'} dy > 1, \tag{5.2}$$

$$B(t,x) > 0, \tag{5.3}$$

$$d,\ B \in W^{1,\infty}. \tag{5.4}$$

These conditions could be relaxed, to the expense of more steps in the proof. Especially the positivity of B on the half-line can be reduced to the positivity on an interval using a compactness argument. We also refer to [27] for the variant in the proof when B can vanish. Finally, similar results as below hold for the general cell division equation, but the proof goes through discrete approximation that is longer to develop.

As in Section 4 for steady states, the theory uses an eigenvalue problem to find the periodic solution. Therefore we consider the problem

$$\begin{cases} \frac{\partial}{\partial t} N(t,x) + \frac{\partial}{\partial x} N(t,x) + (\lambda_0 + d(t,x)) N(t,x) = 0, \qquad t \geq 0,\ x \geq 0, \\ \\ N(t,x=0) = \int_0^\infty B(t,y) N(t,y) dy, \qquad t \geq 0, \\ \\ N(t,x) > 0, \qquad \int_0^T \int_0^\infty N(t,x) dx\, dt = 1, \quad N \text{ is } T\text{-periodic}, \end{cases} \tag{5.5}$$

$$\begin{cases} \frac{\partial}{\partial t} \phi(t,x) + \frac{\partial}{\partial x} \phi(t,x) - (\lambda_0 + d(t,x)) \phi(t,x) = -B(t,x)\phi(t,0), \qquad t \geq 0,\ x \geq 0, \\ \\ \phi(t,x) > 0, \qquad \int N(t,x)\phi(t,x) dx = 1, \qquad \phi \text{ is } T\text{-periodic}. \end{cases} \tag{5.6}$$

Following the previous sections, we prove

Theorem 5.1 *With the assumptions (5.2)–(5.4), there exists a unique solution* (λ_0, N, ϕ) *to the eigenvalue problem (5.5)–(5.6) and* $N,\ \phi \in C([0,T]; W^{1,\infty})$.

Theorem 5.2 *(Attraction to periodic solutions) With the assumptions (5.2)– (5.4), and* $n_0 \in L^1(\mathbb{R}^+, \phi(0,x)dx)$*, then the solution to (5.1) satisfies*

$$\int \left| n(t,x)e^{-\lambda_0 t} - \rho\, N(t,x) \right| \phi(t,x)dx \underset{t\to\infty}{\longrightarrow} 0,$$

with $\rho = \int n(0,x)\phi(0,x)dx$.

The existence of periodic solutions (Theorem 5.1) is not surprising and in spirit combines compactness arguments with Floquet's theory for a positive matrix (see [11], - chap.3 sec.5, for instance) although our proof is more direct. The attraction to the periodic solution requires a dissipative mechanism which, in our approach, is expressed by the dissipation of entropy.

Proof of Theorem 5.1. By opposition to the case when the coefficients are independent of time, here we do not have explicit solutions at hand. Nevertheless, it can be solved as an eigenvalue problem thanks to Krein-Rutman Theorem considering, as it is classical, an operator on the boundary $x = 0$. To do that, we use the explicit solutions to (5.5)–(5.6)

$$N(t,x) = \mathcal{N}(t-x)e^{-\int_0^x (\lambda_0+d)(t+y'-x,y')dy'}, \tag{5.7}$$

$$\phi(t,x) = \int_x^\infty B(t+y-x,y)\mathcal{U}(t+y-x)e^{-\int_x^y (\lambda_0+d(t+y'-x,y'))dy'}dy, \tag{5.8}$$

and we reduce the problems (5.5) and (5.6) to the integral equations

$$\mathcal{N}(t) = \int_0^\infty B(t,y)e^{-\int_0^y (\lambda_0+d)(t+y'-y,y')dy'}\mathcal{N}(t-y)dy, \tag{5.9}$$

$$\mathcal{U}(t) = \int_0^\infty B(t+y,y)e^{-\int_0^y (\lambda_0+d(t+y',y'))dy'}\mathcal{U}(t+y)dy. \tag{5.10}$$

Finally, we directly obtain the solutions to (5.5), (5.6) (their properties follow without any difficulty) from the

Lemma 5.3 *With the assumptions (5.2)–(5.4), there is a unique solution* $(\lambda_0, \mathcal{N}, \mathcal{U})$ *to (5.9), (5.10) with* $\mathcal{N}$ *and* $\mathcal{U}$ *two* T*-periodic functions, and* $\mathcal{N}(t) > 0$, $\mathcal{U}(t) > 0$.

Proof of Lemma 5.3 We consider a parameter $\lambda > 0$, the Banach space $X = \mathcal{C}_{\text{per}}(0,T)$ and the operator which, to $\mathcal{M} \in X$ associates $\mathcal{N} \in X$ given by

$$\mathcal{N}(t) = \int_0^\infty B(t,y)e^{-\int_0^y (\lambda+d)(t+y'-y,y')dy'}\mathcal{M}(t-y)dy,$$

and its dual, which, to $\mathcal{V} \in X$ associates $\mathcal{U} \in X$ given by

$$\mathcal{U}(t) = \int_0^\infty B(t+y,y)e^{-\int_0^y (\lambda+d(t+y',y'))dy'}\mathcal{V}(t+y)dy.$$

This linear operator is continuous (using the first inequality in (5.2)), strictly positive (thanks to assumption (5.3)) and compact (it is convolution like). Therefore, using Krein-Rutman Theorem it admits a simple first eigenvalue $\nu(\lambda) > 0$, the corresponding eigenvector (positive and normalized with unit mass) is denoted $\mathcal{N}_\lambda(t)$ and a dual eigenvector is denoted by $\mathcal{U}_\lambda(t) > 0$. One readily checks that, using the second inequality in (5.2),

$$\nu(0) \geq \min \int_0^\infty B(\cdot, y) e^{-\int_0^y d(\cdot + y' - y, y') dy'} dy > 1.$$

It is also clear that $\nu(\infty) = 0$ and that $\nu'(\lambda) < 0$ (just by the maximum principle). Therefore, there is a unique λ_0 such that $\nu(\lambda_0) = 1$, and thus a solution to (5.9), (5.10). $\square$

Proof of Theorem 5.2. We do not repeat the details of the proof which were already given in sections 3 and 4. From Theorem 2.1, we have the entropy inequality for $g = ne^{-\lambda_0 t}$,

$$\frac{d}{dt} \mathcal{H}_\phi(g|N) = -\mathcal{D}_\phi(g|N) \leq 0, \tag{5.11}$$

where $\mathcal{H}_\psi(g|N)$ is defined in (1.3) and

$$\mathcal{D}_\phi(g|N) := \phi(t,0) \int_0^\infty B(t,x)N(t,x)dx \int_0^\infty \left[H\Big(\int_{y=0}^\infty \frac{g(t,y)}{N(t,y)} d\mu_t(y) \Big) - H\Big(\frac{g(t,x)}{N(t,x)} \Big) \right] d\mu_t(x) \,,$$

$$d\mu_t(x) := B(t,x)N(t,x)dx \Big/ \int_0^\infty B(t,x)N(t,x)dx.$$

We are in the same situation as in the proof of Theorem 3.2 and Theorem 4.3. For the convex function $H(\cdot) = |\cdot|$, and applying the GRE inequality to $g - \rho N$, we find that

$$\int \big| n(t,x)e^{-\lambda_0 t} - \rho\, N(t,x) \big| \phi(t,x) dx \downarrow L \qquad \text{as } t \to \infty.$$

It remains to prove that $L = 0$. By weak compactness there is a subsequence (but we keep again the notation of the full sequence) $g_k(t,x) = g(t+k,x)$ which converges. From the entropy dissipation term, we deduce that the limit $\bar{g}$ satisfies,

$$\frac{\bar{g}(t,x)}{N(t,x)} = C(t),$$

Thanks to the mass conservation, this implies that $\bar{g}(t,x) = \rho N(t,x)$ and the strong convergence holds as proved in Section 3. From this it follows that the limit $L = 0$. $\square$

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