

Qualitative properties of some Boltzmann like equations which do not fulfill a detailed balance condition

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July 20, 2005

Abstract

The aim of the paper, which summarized the two talks given by the authors during the summer school '04 in Santander, is to give an overview of what can be said about the qualitative properties, such that the conservation quantities, the entropies, the stationary solutions, the self-similar solutions, the long time asymptotic of generic solutions, for a family of Boltzmann like equations which do not fulfill any detailed balance condition and for which a common study strategy applies. More precisely, we will illustrate our method on the Coagulation equation (C), the Fragmentation equation (F), the Coagulation-Fragmentation equation (CF), the Growth-Fragmentation equation (GF), the Inelastic Boltzmann equation (IB) and the Pauli equation (P).

1 Introduction

The aim of this notes is to describe the collective (or statistical) dynamic, we shall say *the mesoscopic dynamic*, of a system of many particles for which we know the individual dynamic (the dynamic law that follows any given particle). The mesoscopic level of description is an intermediate level between the *microscopic level* (description of all the particles in the system) and the *macroscopic level* of the physical observables.

A common feature of all the microscopic event we will introduce below is that none of them are (a priori) reversible. That will have several strong consequences in terms of existence of equilibrium states and existence of a quantity which decreases along time (Liapunov functional or entropy) on the associated mesoscopic model as we will see.

1.1 The microscopic Level.

We are interested in some situations when the mass (or size, age) of particles (or individual) or the velocity (or wave function, energy) of particles is pertinent in order to describe the particles. We then assume that each particle is fully identified by one state variable $\xi \in \Xi$, with $\xi = y \in (0, \infty)$ in the case of the mass and with $\xi = v \in \mathbb{R}^3$ or \mathbb{T}^3 in the case of the velocity. We consider a system of many such indiscernible particles (but by its state) which undergo microscopic processes. The first step is thus to present the microscopic events that may change the state of one particle.

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1.1.1 Coagulation.

It refers to mechanisms by which two (mother) particles encounter and merge into a single (daughter) particle. This aggration mechanism may be schematically represented by

$$(1.1) \quad (y) + (y') \xrightarrow{\text{rate } a} (y + y') \quad \text{coagulation mechanism,}$$

where $a = a(y, y')$ is the rate of occurrence of the aggregation of two particles of mass y and y' . The physical mechanism which leads to aggregation of particles (and which may be very complicate) is hidden in the rate a . In any case we which assume that a is homogeneous, namely $a(sy, sy') = s^\lambda a(y, y') \forall s, y, y' > 0$ with $\lambda \in [0, 2]$. At that microscopic level, we see that the total mass of particles is conserved during a coagulation event while its number is decreasing and its mean mass is increasing.

1.1.2 Linear or spontaneous fragmentation.

It is the mechanism by which a single particle splits into two (or more) smaller pieces. Schematically, we write

$$(y) \xrightarrow{\text{rate } b} (y') + (y - y') \quad \text{fragmentation mechanism,}$$

where $b = b(y, y')$ is the rate of occurrence of the breakage of one particle of mass/size y , given rise to two particles (of mass y' and $y - y'$). Particular cases are: on the one hand the cell division for which there always holds $y' = y/2$; on the other hand the birth process (in such a case y must be understood as the age of the individual) for which there always holds $y' = 0$. Most of the time, we assume that b is homogeneous, namely $b(sy, sy') = s^{\gamma-1} b(y, y') \forall s, y, y' > 0$ with $\gamma \in \mathbb{R}$. At that microscopic level, we see that the mass is conserved during a fragmentation event while the number of particles is increasing and the mean mass is decreasing. In that rough sense, the fragmentation is a reverse mechanism with respect to the coagulation.

1.1.3 Condensation/evaporation or Growth.

It is the mechanism by which matter is transfered from the surrounding medium towards the particles or from the particles phase to the medium. That schematically can be represented by

$$(y) \xrightarrow{\text{rate } w} (y + w(y) dy) \quad \text{growth mechanism,}$$

where $w = w(y)$ is the rate of condensation/evaporation of new matter on a particle of size y . At this microscopical level, we see that the growth mechanism leaves the number of particles (or individual) constant while the size (or mass) is increasing when $w(y) > 0$ for any $y > 0$ (condensation) and it is decreasing when $w(y) < 0$ for any $y > 0$ (evaporation).

1.1.4 Inelastic Collisions.

When the particles are described by their velocity the natural event by which particles change their velocity is a collision, schematically

$$(v) + (v_*) \xrightarrow{\text{rate } B} (v') + (v'_*) \quad \text{collision mechanism.}$$

In principle, a collision conserves momentum and energy

$$v + v_* = v + v'_*, \quad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2 \quad \text{elastic collision.}$$

In many situation non elastic collisions are pertinent. It could be because kinetic energy is transfered to internal energy (heat) which is not taken into account in the (simple) model considered here (in which the particle is fully describe by its velocity). In that case, a non elastic collision schematically should write

$$(1.2) \quad v + v_* = v + v'_*, \quad |v'|^2 + |v'_*|^2 < |v|^2 + |v_*|^2 \quad \text{quadratic inelastic collision.}$$

It could also be because the collisions arise between two different kind of particles (light particles of velocity v_L and heavy particles of velocity v_H) and we only consider the dynamic of light particles, the heavy particles being at the rest for instance. Schematically, we write

$$(v_L) + (v_H) \xrightarrow{\text{rate}^\sigma} (v'_L) + (v'_H) \quad \text{linear inelastic collision.}$$

1.2 A word about physic and general references.

Growth mechanisms in the largest sense, that is coagulation/aggregation, fragmentation and evaporation/condensation, are widespread phenomenon in nature. They are met and play an important role in various fields of physics (aerosol and raindrops formation, smoke, sprays, ...), chemistry (polymer, ...), astrophysics (formation of galaxies) and biology (hematology, animal grouping, ...) and then take place at different scales. They are pertinent as soon as one consider systems of bodies (or clusters, particles, aggregates, individuals) which mass (or size, age) may change: may increase or decrease. In the simplest situations, the one we consider in that paper, one may consider that these bodies are fully identified by their mass. One of the main interest of the coagulation equation, associated to the mechanism of coagulation, and the fragmentation equation, associated to the mechanism of fragmentation that we introduce in section 1.3, is that they provide some models of change of phase (which is mathematically observed by the fact that the particles phase described by the density f loss mass), schematically

$$\text{dust } y = 0 \quad \longleftrightarrow \quad \text{particles } y \in (0, \infty) \quad \longleftrightarrow \quad \text{gel } y = \infty,$$

thought at the microscopic level, that mechanisms conserve the mass. Let emphasize that these change of phase phenomena may occur in finite time (when $\lambda > 1$, $\gamma < 0$) or asymptotically in large time (when $\lambda \leq 1$, $\gamma \geq 0$). We focus in the present paper on the second situation, and we refer the interested reader to the first (more singular) situations to [6, 8, 28, 45] for an analysis of the shattering phenomena (dust phase appear instantaneously because the fragmentation mechanism is strongest the particles are smallest, that is the case $\gamma < 0$) and to [38, 32, 21] for an analysis of the gelation phenomena (gel phase appear in finite time because the coagulation mechanism is strongest the particles are largest, that is the case $\lambda > 0$). More generally, we refer to the books and review papers of F. Leyvraz [37], P. Laurençot and S. Mischler [34], D.J. Aldous [2], J. H. Seinfeld [52], S. K. Friedlander [26] and R.L. Drake [16] for a basic physical description and motivations and an overview of available mathematical results as well as to the references therein for a more precise physical and mathematical analysis.

On the other hand, collisional mechanisms is at the root of the physic of kinetic gases. The use of Boltzmann inelastic like models to describe dilute, rapid flows of granular media started with the seminal physics paper [29], and a huge physics litterature has been developed in the last twenty years. Such a models has some application for the description of the dynamic of particles of macroscopic size moving in a fluid (grains, dust, spray) or in the void (rings of fragments in a solar system for instance). In these models, the interactions between grains or particles are dissipative. The study of granular systems in such regime is motivated by their unexpected physical behavior (with the phenomena of *collapse* –or “*cooling effect*”– at the kinetic level and *clustering* at the hydrodynamical level), their use to derive hydrodynamical equations for granular fluids, and their industrial applications. The main motivation of the paper on that model is to describe the cooling effect and to valid the Haff Law’s on the decreasing of the energy. We refer to [14] [17] [5] [10] and the references therein for a physic and mathematical introduction to the field of granular medium. Finally, the Pauli equation is a simplified model for the dynamics of a cloud of particles (electrons or ions) in a semiconductor device and we refere to [50] and the references in [4] for a mathematical introduction of such a model.

1.3 The mesoscopic level.

We now describe the system by the density $f = f(t, \xi) \geq 0$ of particles in the state ξ at times $t \geq 0$. The equation giving the collective dynamic is the ”master equation” associated to a given combination of the ”microscopic events” described before. In any case, the dynamic of the density is given by the Boltzmann like equation

$$(1.3) \quad \frac{\partial}{\partial t} f(t, \xi) = \mathcal{A}(f(t, \cdot))(\xi) \quad \text{on} \quad (0, \infty) \times \Xi,$$

where \mathcal{A} modelizes the change of state of a particles due to some microscopic events, and then it is an operator which only acts on the variable states. We complement the evolution equation (1.3) with an initial condition

$$f(0, \xi) = f_{in}(\xi) \quad \text{on} \quad \Xi.$$

In all the sequel we will use the shorthand notations $h = h(t, \xi)$ or $h = h(\xi)$ and $h^\# = h(t, \xi^\#)$ or $h^\# = h(\xi^\#)$ for any function h , any $\xi, \xi^\# \in \Xi$ and any $t \geq 0$.

Before describing with details the different models we will deal with, let present the common questioning that one can have for any of them.

- Is there any conservation law? Is there any entropy?
***** (or, mathematically speaking, any Liapunov functional)?
- Is there any particular relevant solutions? They may be stationary solutions, self-similar solutions or first eigenfunction. All of them writes

$$(1.4) \quad F(t, \xi) = \tau_1(t) G(\tau_2(t) \xi)$$

for a particular profile G and two time rescaling τ_1 and τ_2 .

- Are these solutions stables? attractives? It should mean that the dynamic of any solution is governed by these particular solutions.

1.3.1 Coagulation operator and coagulation equation.

The Smoluchowski coagulation equation, denoted by (C), is obtained considering the equation (1.3) where \mathcal{A} takes into account the sole coagulation mechanism. Namely, we take $\mathcal{A} = \mathcal{C}$ where \mathcal{C} is the coagulation operator associated to the coagulation mechanism defined by

$$\mathcal{C}(f, f)(y) := \frac{1}{2} \int_0^y a(y', y - y') f(y') f(y - y') dy' - \int_0^\infty a(y, y') f(y) f(y') dy'.$$

Under the assumption $\lambda \leq 1$ (and some additional assumptions that we do not specify here) we may show that the coagulation equation has global solutions which preserve the total mass for all time that is

$$(1.5) \quad \int_0^\infty y f(t, y) dy = \int_0^\infty y f_{in}(y) dy =: \rho \quad \text{for all } t > 0.$$

It is also known that the typical mass Y_t of a particle increases and tends to infinity, or equivalently,

$$f(t, y) \rightarrow 0 \quad \text{when } t \rightarrow \infty.$$

As a consequence $f = 0$ is the only steady state for the coagulation equation. One may look for another type of particular solutions: the self similar solutions, that is solutions which are invariant by some scaling. Their precise forms depend on the rate of coagulation. Some explicit examples are already well known, see [42, 7, 31] and the references in [34]:

- For $a(y, y') = 1$, $F(t, y) = 4t^{-2} e^{-\frac{2x}{t}}$ is the unique self-similar solution (with mass identically equal to 1).
- For $a(y, y') = y + y'$, $F(t, y) = (2\pi)^{-1/2} e^{-t} x^{-3/2} e^{-e^{-2t} x/2}$ is also the unique self-similar solution (with mass identically equal to 1).

For a general kernel a such an explicit expression for self similar solutions is not known. We then look for a particular solution F satisfying (1.4) and preserving mass (1.5). We immediately obtain that F must have the form

$$F(t, y) = \frac{1}{t^{2/(1-\lambda)}} G\left(\frac{y}{t^{1/(1-\lambda)}}\right)$$

for a given profile G satisfying the following profile equation

$$\mathcal{B}(G) := (1 - \lambda)\mathcal{C}(G, G) + \mathcal{D}_1 G = 0, \quad \text{with } \mathcal{D}_1 g := 2g + y \frac{\partial g}{\partial y}.$$

1.3.2 Fragmentation operator and fragmentation equation.

The (pure linear) fragmentation equation, denoted by (F) below, is defined taking $\mathcal{A} = \mathcal{L}$ in (1.3) where \mathcal{L} is the fragmentation operator associated to the fragmentation mechanism and it is defined by

$$\mathcal{L}(f)(y) = \int_y^\infty b(y', y) f(y') dy' - f(y) \int_0^y \frac{y'}{y} b(y, y') dy'.$$

When $\gamma \geq 0$, the conservation (1.5) holds again, but now the typical mass Y_t decreases and tends to zero, or equivalently

$$(1.6) \quad y f(t, y) \rightarrow \rho \delta_{y=0} \quad \text{when } t \rightarrow \infty.$$

The only steady states are the Dirac masses " $\rho y^{-1} \delta_{y=0}$ ", and we can ask again for a more accurate description of the above asymptotic behavior thanks to self-similar solutions. We find that a self-similar solution F must satisfies

$$F(t, y) = t^{2/\gamma} G(t^{1/\gamma} y),$$

and the profile G must satisfies

$$\mathcal{B}G := \gamma \mathcal{L}G - \mathcal{D}_1 G = 0.$$

1.3.3 Coagulation-Fragmentation equation.

The coagulation-fragmentation equation, denoted by (CF) below, is obtained as a combination of the two preceding equations. We take then $\mathcal{A} := \mathcal{C} + \mathcal{L}$. When $\gamma \geq 0$ and $\lambda \leq 1$ (also when $\gamma > 0$ and $\lambda < 1 + \gamma$) the mass is conserved (1.5). Moreover, under the detailed balance condition: there exists $M : (0, \infty) \rightarrow (0, \infty)$ such that $(1 + y) M(y) \in L^1$ and

$$(1.7) \quad a(y, y') M(y) M(y') = b(y + y', y') M(y + y') \quad \forall y, y' > 0$$

we may show the following.

1. There exists a family of equilibrium states : $M_z(y) := M(y) e^{zy}$ is a stationary solution for any $z \in \mathbb{R}$.

2. The fonctionnal

$$(1.8) \quad t \mapsto \int_0^\infty (f \ln \frac{f}{M} - f + M) dy$$

is an entropy.

3. Any solution f to the CF equation satisfies

$$(1.9) \quad f(t) \xrightarrow[t \rightarrow \infty]{} M_z \quad \text{for some } z \in \mathbb{R}.$$

4. Finally, one may identify z in (1.9) in several particular cases, see [3, 33], and one may establish a rate of convergence in (1.9), see [1, 30].

Without the detailed balance condition (1.7), the existence of stationary solutions, the existence of entropy and the long time asymptotic of generic solutions are not clear and they have to be investigated.

1.3.4 Growth-Fragmentation equation.

For the condensation (or growth) operator, we take

$$\mathcal{G}_w(f)(y) := -w(y) \partial_y f(y).$$

The Growth-Fragmentation equation, denoted by (GF) below, is then defined by (1.3) with $\mathcal{A} = \mathcal{L} + \mathcal{G}_w$.

In contrast with the case when $w(y) = y$ for which the mass of the solutions of the GF equation follows an explicit exponential growth, there is no evident law of conservation for this model when $w(y) \neq y$. Nevertheless, one can solve the dual first eigenvalue-first eigen function problem associated to the GF equation with the help of the Krein-Rutman theorem, see [49, 44]. For the sake of simplicity we consider the case

$$0 < B_0 \leq B(y) \leq B_1 < \infty, \quad w(y) \equiv 1.$$

Theorem 1.1 *Under the above assumptions there exists a couple (λ, ϕ) of first eigenvalue-first eigen function of the dual problem*

$$\mathcal{L}^* \phi + \mathcal{G}_w^* \phi = \lambda \phi, \quad \phi \geq 0, \quad \int_0^\infty \phi(y) dy = 1.$$

Considering a solution f to the GF equation, we easily see that the rescaled function $g(t, y) := f(t, y) e^{-\lambda t}$ satisfies the modified equation

$$(1.10) \quad \frac{\partial g}{\partial t} = \mathcal{B}g := \mathcal{L}g + \mathcal{G}_w g - \lambda g$$

and the following law of conservation holds:

$$\forall t \geq 0 \quad \int_0^\infty g(t, y) \phi(y) dy \equiv \int_0^\infty f_{in}(y) \phi(y) dy.$$

In order to describe the long time asymptotic of the solutions f to equation (GF) we look for the first eigensolution F , or equivalently (thanks to the change of variables linking f and g) for a stationary solution G to (1.10). It is worth mentioning that such a stationary solution is in fact given in the same time that the existence of the couple (λ, ϕ) by Theorem 1.1. We choose that presentation in order to bring the problem in a similar form than for the other models presented in the paper.

1.3.5 Inelastic Boltzmann equation.

We next define the bilinear collision operator $Q(f, f)$ which models the interaction of hard spheres by means of inelastic binary collisions preserving mass and momentum but dissipating kinetic energy (1.2). Denoting by $e \in (0, 1)$ the *normal restitution coefficient*, we define the collision operator in strong form as

$$(1.11) \quad \mathcal{Q}(g, f)(v) := \iint_{\mathbb{R}^3 \times S^2} \left(\frac{{}'f {}'g_*}{e^2} - fg_* \right) |u| d\sigma dv_*.$$

Here $u = v - v_*$ denotes the relative velocity and $'v, 'v_*$ denotes the possible pre-collisional velocities leading to post-collisional velocities v, v_* . They are defined by

$$'v = \frac{v + v_*}{2} + \frac{{}'u}{2}, \quad 'v_* = \frac{v + v_*}{2} - \frac{{}'u}{2},$$

with $'u = (1 - \beta)u + \beta|u|\sigma$ and $\beta = (e + 1)/(2e)$ ($\beta \in (1, \infty)$ since $e \in (0, 1)$).

The Inelastic Boltzmann equation, denoted below by (IB), is obtained taking $\mathcal{A} = \mathcal{Q}$ in (1.3). For any solution f of the inelastic Boltzmann equation, the mass and the momentum are conserved

$$\int_{\mathbb{R}^3} f(t, v) dv \equiv \int_{\mathbb{R}^3} f_{in}(v) dv, \quad \int_{\mathbb{R}^3} f(t, v) v dv \equiv \int_{\mathbb{R}^3} f_{in}(v) v dv,$$

but not the energy, since we have

$$\frac{d}{dt} \mathcal{E}(t) = -D_{\mathcal{E}}(t)$$

where the energy \mathcal{E} and the dissipation of energy $D_{\mathcal{E}}$ are defined by

$$\mathcal{E} = \int_{\mathbb{R}^3} f |v|^2 dv, \quad D_{\mathcal{E}} = \sigma_0 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f f_* |v - v_*|^3 dv dv_*.$$

The only steady states are $\rho \delta_{v=u}$, $\forall \rho > 0$, $u \in \mathbb{R}^3$, and for any solution f of mass $\rho > 0$ and momentum 0, the typical velocity V_t satisfies

$$V_t \rightarrow 0 \quad \text{or equivalently} \quad f(t, v) \xrightarrow[t \rightarrow \infty]{} \rho \delta_{v=0}.$$

Once again, we look for a more accurate description of the above asymptotic behavior thanks to self-similar solutions. We find that a self-similar solution F must satisfies

$$F(t, v) = t^3 G(t v)$$

and the profile G must satisfies

$$\mathcal{B}G := \mathcal{Q}(G) - \mathcal{D}_2 G = 0, \quad \text{with} \quad \mathcal{D}_2 = 3 + v \cdot \nabla_v.$$

1.3.6 Pauli equation.

Finally, the Pauli equation is obtained taking $\mathcal{A} = \mathcal{P}$ in (1.3) where the Pauli operator \mathcal{P} is defined by

$$\mathcal{P}(f)(v) = \int_{\mathbb{T}^3} (\sigma(v, v') (1 - f) f' - \sigma(v', v) (1 - f') f) dv'.$$

The nonlinearity of \mathcal{P} takes into account the quantum effect due to Pauli exclusion principle. We easily see that any solution f to the Pauli equation, preserves the total charge, that is

$$\forall t \geq 0 \quad \int_{\mathbb{T}^3} f(t, v) dv = \int_{\mathbb{T}^3} f_{in}(v) dv.$$

Moreover, under the following detailed balance condition

$$(1.12) \quad \exists K \quad \sigma(v, v') K(v) = \sigma(v', v) K(v'),$$

there exists a family of equilibrium states $(M_z(v) := (z K(v) + 1)^{-1})$ is a stationary solution for any $z \in \mathbb{R}_+$, the following functional

$$t \mapsto \int_0^\infty [f \ln(K f) + (1 - f) \ln(1 - f)] dy$$

is an entropy and the convergence (1.9) holds.

Without the detailed balance condition (1.12), the existence of stationary solutions, the existence of entropy and the long time asymptotic of generic solutions are not clear and they have to be investigated.

1.4 Good variables and good equation.

Let then summarize the different situations and results we have listed above. At this very rough level, we may differentiate three cases

- *Case 1.* For the coagulation-fragmentation equation (CF) and the Pauli equation (P), we find one conservation law and some equilibria under the (appropriate!) detailed balance hypothesis. For such an equation modeling two opposite mechanisms, the natural question is then the existence of some stationary solution:

$$(1.13) \quad \exists G \text{ with mass } \rho \text{ such that } \quad \mathcal{B}(G) = 0?$$

with $\mathcal{B} = \mathcal{A}$ an operator which does not fulfill the detailed balance hypothesis.

- *Case 2.* For the the coagulation equation (C), the fragmentation equation (F) and the Inelastic Boltzmann equation (IB) we also find one (or two) conservation law(s), but concentration (for (F) and (IB)) or dispersion (for (C)) occur (due to the one way mechanism which are involved). For these "one way mechanisms", there exists some trivial equilibria (the 0 function or the Dirac mass). The question is then whether it is possible to describe one step further these mechanisms. In that situation we look for some self-similar solutions F or equivalently for a self-similar profile G which satisfies again the stationary equation (1.13) for $\mathcal{B} = \mathcal{A} + \mathcal{D}$ where \mathcal{D} is a sum of a 0th and a 1st order derivative operator which has been specified above for each particular case.

- *Case 3.* For the fragmentation-growth equation (FG) there is no trivial conservation since that the fragmentation mechanism conserves the mass of particles and the growth mechanism conserves the number of particles. Since the model is linear we may look for a first eigenvalue and first eigenfunction F . Introducing the solution to the dual eigen problem, we see that we are leading once more to the question of the existence of a stationary solution G to the modified problem (1.13) with $\mathcal{B} := \mathcal{A} - \lambda$.

We make now the following fundamental remarks. Introducing the new evolution equation

$$(1.14) \quad \begin{aligned} \frac{\partial}{\partial t} g(t, \xi) &= \mathcal{B}(g(t, \cdot))(\xi) & \text{on } (0, \infty) \times \Xi \\ g(0, \xi) &= g_{in}(\xi) \equiv f_{in}(\xi) & \text{on } \Xi, \end{aligned}$$

with $\mathcal{B} = \mathcal{A} + \mathcal{D}$ and \mathcal{D} depending on the model into consideration, we see that a solution G to the stationary (or profile) equation (1.13) is nothing but a stationary solution to the equation (1.14). In other words, we have reduced the question of existence of particular solutions to equation (1.3) to the problem of existence of an equilibrium state for a modified equation (1.14) as in the case 1. The Boltzmann-like equation (1.14) always enjoys the following properties. There is one (or two) law(s) of conservation. It does not fulfill any detailed balance condition (because it models microscopic mechanisms without a priori micro-reversibility structure) and therefore there is not evident stationary solution nor Liapunov functional. That contrasts with the usual Boltzmann equation for which micro-reversibility implies the existence of explicit stationary solutions (the Maxwellian functions) and the existence of Liapunov functional (the Boltzmann entropy's). Nevertheless, the solutions does not undergo concentration nor dispersion (which mathematically can be expressed in term of (weak) compactness on the flow). This fact is important in

order to insure that we have obtained with equation (1.14) the relevant modification of equation (1.3).

On the other hand, we may pass from a solution f of the equation (1.3) to a solution g of the equation (1.14) (and reciprocally) by a mere change of variables. For instance, for the coagulation equation (C), the fragmentation equation (F) and the inelastic Boltzmann equation (IB), if g is a solution to (1.14) then the following function f is a solution to (1.3):

$$(1.15) \quad f(t, \xi) = (1+t)^M g(\ln(1+t)^N, (1+t)^P \xi),$$

with $M = 1/\gamma$, $N = 1/\gamma$, $P = 1/\gamma$ for (F), $M = -2/(1-\lambda)$, $N = 1/(1-\lambda)$, $P = -1/(1-\lambda)$ for (C) and $M = 3$, $N = P = 1$ for (IB). An immediate and useful consequence is the following relation between the power moments of f and g :

$$(1.16) \quad M_k(f(t, \cdot)) = (1+t)^{P(d-k)} M_k(g(\ln(1+t)^N, \cdot)),$$

where the moment M_k is defined in (2.1) below and where $d = 1$ for (F) and (C), $d = 0$ for (IB). Reciprocally, if f satisfies equation (1.3), we obtain a solution g to (1.14) defining

$$(1.17) \quad g(t, \xi) = e^{-Mt/N} f\left(e^{t/N} - 1, \xi e^{-Pt/N}\right).$$

In other words, equation (1.14) is just equivalent to equation (1.3). But, as we will see below, in order to establish properties on the long time asymptotic of solutions to equation (1.3) it is often more convenient to work on the modified equation (1.14) than directly on the equation (1.3).

Let make now more precise the question we want to answer for any of these models.

Question A. Is there any stationary solution for (1.14) and what can be said about it ?

Question A1. Existence of stationary solution?

\Rightarrow The answer is always positive. The proof is based on a rough infinity version of the Poincaré-Bendixson Theorem. This one can be applied directly, on the flow generated by the equation: it is necessary to obtain a priori bounds on the solutions to the time evolution equation (1.14). This one can also be applied on the flow generated by a "truncated version" of equation (1.14), and then it is "only" necessary to obtain a priori bounds on the solutions of the stationary (or profile) equation (1.13).

Question A2. Uniqueness of the profile?

\Rightarrow In general, we do not know how to answer. Nevertheless, for the fragmentation equation (F), the growth-fragmentation equation (GF) and the Pauli equation (P) we are able to prove that the solution of (1.13) is unique, making use the (almost) linearity of these models in order to obtain a (family of) Liapunov functional(s).

Question A3. More qualitative properties on the profile G : symmetry, smoothness, tails behavior, stability?

\Rightarrow Working on the stationary equation (1.13) and using boot-strap arguments we are able to establish some qualitative properties on the stationary solution G such as L^∞ weight estimates by above and below, smoothness (from continuity to C^∞ depending of the model). Asymptotic stability or mere stability of the profile is still an open problem.

Question B. What can be said about "generic" solutions of the equation (1.14)

Question B1. Sharp uniform (in time) bounds?

\implies We are able to prove the existence of invariant sets and attractive sets for the generic solutions g to the modified equation (1.14). These properties on g may be translated automatically thanks to (1.15) to the associated solution f to the primal equation (1.3). We establish in particular that the long time behavior of the generic solutions f is the same that the long time behavior of the self-similar solutions in the rough sense of moments decay.

Question B2. Entropy and convergence of $g(t)$ to the equilibrium G .

\implies When entropies are available (see the answer to question A2) we are able to prove the long time convergence of the generic solutions g to the profile G .

2 Main results

For any $k \in \mathbb{R}$, and any measurable function $f : \Xi \rightarrow \mathbb{R}_+$ we define the moment of f of order k by

$$(2.1) \quad M_k(f) := \int_{\Xi} f(\xi) |\xi|^k d\xi$$

and the associated Lebesgue spaces for $p \in [1, \infty)$ by

$$\dot{L}_k^p := \{f \in L_{loc}^1(\Xi), M_{kp}(|f|^p) < \infty\}, \quad L_k^p := \dot{L}_0^p \cap \dot{L}_k^p.$$

We also introduce \dot{M}_k^1 and M_k^1 the spaces of Radon measures associated to these L^1 spaces.

2.1 Fragmentation equation, the case $\gamma \geq 0$.

For the fragmentation rate we assume that

$$b(y, y') = B(y) \beta(y, y'),$$

where $B(y)$ measure the frequency with which a particle of size y breaks and $\beta(y, y')$ is the repartition probability of the daughter particles in the segment $[0, y]$ so that $\int_0^y \beta(y, y') y' dy' = y$. For the sake of simplify, we will assume that

$$B(y) = y^\gamma, \quad \gamma \geq 0, \quad \beta(y, y') = \frac{1}{y} \theta\left(\frac{y'}{y}\right),$$

where θ is a measurable function on $[0, 1]$ satisfying

$$\int_0^1 z \theta(z) dz = 1, \quad \int_0^1 z^m \theta(z) dz < \infty, \quad m < 1, \quad \theta > 0 \quad \text{a.e. on } (0, 1).$$

Theorem 2.1 1. For any $\rho > 0$, there exists a unique self-similar solution F_ρ of mass ρ to (F), that is

$$F_\rho(t, y) = t^{2/\gamma} G_\rho(t^{1/\gamma} y), \quad G \in \bigcap_{k \geq m} \dot{L}_k^1, \quad M_1(G) = \rho.$$

Moreover, $G \in W_{loc}^{1, \infty}(0, \infty)$, $y^k G \in L^\infty \forall k \geq 1 + m$ and $G > 0$ on $(0, \infty)$.

2. For any $0 \leq f_{in} \in \dot{L}_1^1$ there exists a unique weak solution $f \in C([0, \infty); \dot{L}_1^1) \cap L^1(0, \infty; L_{1+\gamma}^1)$ to the fragmentation equation (F). This one is mass conserving and satisfies

$$(2.2) \quad g(t, y) := e^{-2t} f(e^{\gamma t} - 1, e^{-t} y) \xrightarrow[t \rightarrow \infty]{} G \quad \text{strongly in } \dot{L}_1^1.$$

Theorem 2.1 gives then an accurate description of the convergence (1.6). For more details we refer to [22, 44, 45] for an analytical approach and to [6, 8] for a probabilistic approach.

2.2 Coagulation equation, the case $\lambda \in [0, 1)$.

For the coagulation rate, we assume

$$a(y, y') = y^\alpha (y')^\beta + y^\beta (y')^\alpha, \quad -1 < \alpha \leq \beta < 1, \quad \lambda := \alpha + \beta \in [0, 1).$$

Our results are still valid for linear combinations of several such rates

$$a(y, y') = (y^\nu + (y')^\nu) (y^\sigma + (y')^\sigma), \quad \nu + \sigma \in [0, 2),$$

which includes the important particular case of Smoluchowski's rate a_S introduced in [53] which is defined choosing $\nu = 1/3$ and $\sigma = -1/3$. In order to obtain our results in that case one has to take $\alpha = -1/3$, $\beta = 1/3$ in the statement of the theorem below.

Theorem 2.2 1. For any $\rho > 0$, there exists at least one self-similar solution F_ρ of mass ρ to the coagulation equation (C), that is

$$F_\rho(t, y) = t^{-\frac{2}{1-\lambda}} G_\rho(t^{-\frac{1}{1-\lambda}} y), \quad G \in \bigcap_{k \in I_a} \dot{L}_k^1, \quad M_1(G) = \rho,$$

with $I_a := \mathbb{R}$ if $\alpha < 0$, $I_a := [\lambda, \infty)$ if $\alpha = 0$, $I_a := (\lambda, \infty)$ if $\alpha > 0$. Let emphasize that F_ρ describe the dynamic of the coagulation process when starting from a pure dust initial condition, since

$$y F_\rho(t, y) \rightarrow \rho \delta_{y=0} \quad \text{when } t \rightarrow 0.$$

2. The self-similar profile G satisfies the additional properties:

2.1. $G \in C^\infty$ if $\alpha < 0$, $G \in C^1$ if $\alpha = 0$, $G \in C_{loc}^{1-\lambda, 0}$ if $\alpha > 0$.

2.2. $\forall \varepsilon \in (0, 1)$, $\exists b_\varepsilon, B_\varepsilon$ such that $e^{-b_\varepsilon y} \leq G(y) \leq e^{-B_\varepsilon y}$ for all $y \in (\varepsilon, \infty)$.

2.3. $\exists a, A$ such that $e^{-a y^\alpha} \leq G(y) \leq e^{-A y^\alpha}$ for all $y \in (0, 1)$ if $\alpha < 0$,

$$\lim_{y \rightarrow 0} y^\tau G(y) = L_0, \quad \tau := 2 - (1 - \lambda) M_\lambda(G) \quad \text{if } \alpha = 0,$$

and $G y^k \in L^\infty(0, 1)$ for any $k > 1 + \lambda$, $G y^k \notin L^\infty(0, 1)$ for any $k < 1 + \lambda$ if $\alpha > 0$.

3. For any $0 \leq f_{in} \in \dot{L}_1^1$ there exists at least one weak solution $f \in C([0, \infty]; \dot{L}_k^1)$ for any $k \in I_a \cap (-\infty, 1]$ to the coagulation equation (C) which conserves the mass. Moreover assuming that $f_{in} \in \dot{L}_1^1 \cap \dot{L}_M^1$ for some $M > 1$, that solution can be built in such a way that

$$(2.3) \quad \begin{aligned} M_k(t) &\leq C_k t^{-1} \quad \forall t \in (0, 1], \quad \forall k \in I_a \cap [\lambda, 1] \\ C_{1,k} t^{\frac{k-1}{1-\lambda}} &\leq M_k(t) \leq C_{2,k} t^{\frac{k-1}{1-\lambda}} \quad \forall t \in [1, \infty), \quad \forall k \in I_a. \end{aligned}$$

Estimate (2.3) means that $f(t, y)$ behaves in the large time asymptotic as a self-similar solution in the rough sense of moments.

4. For any $0 \leq f_{in} \in \dot{M}_1^1 \cap \dot{M}_M^1$, $M > 1$ and $0 \leq s \in L^\infty(0, \infty; \dot{M}_1^1 \cap \dot{M}_M^1)$, there exists at least a weak solution $f \in C([0, \infty); \dot{M}_1^1 - \text{weak})$ to the coagulation equation with source term

$$\frac{\partial f}{\partial t} = \mathcal{C}(f) + s, \quad f(0, \cdot) = f_{in}.$$

Moreover, this one conserves the mass

$$M_1(f(t)) = M_1(f_{in}) + \int_0^t M_1(s(\tau, \cdot)) d\tau$$

and all the matter contained in the system is in the particles phase, since that

$$M_{\frac{1+\lambda}{2}}(t) \in L_{loc}^\infty(0, \infty).$$

Theorem 2.2 is one step in the way of the description of short time asymptotic (for small masses) and the long time asymptotic (for large masses) of solution to the coagulation equation (C). For more details we refer to [22, 20, 23, 24] and the references therein.

2.3 Inelastic Boltzmann equation.

Theorem 2.3 1. For any mass $\rho > 0$, there exists one self-similar profil G with mass ρ and momentum 0 to the Inelastic Boltzmann equation:

$$0 \leq G \in L_2^1, \quad Q(G, G) = \nabla_v \cdot (v G), \quad \int_{\mathbb{R}^N} G \begin{pmatrix} 1 \\ v \end{pmatrix} dv = \begin{pmatrix} \rho \\ 0 \end{pmatrix},$$

which moreover can be built in such a way that G is radially symmetric, $G \in C^\infty$ and

$$\forall v \in \mathbb{R}^N, \quad a_1 e^{-a_2|v|} \leq G(v) \leq A_1 e^{-A_2|v|}$$

for some explicit constants $a_1, a_2, A_1, A_2 > 0$.

2. For an initial datum

$$(2.4) \quad 0 \leq f_{in} \in L_2^1 \quad \int_{\mathbb{R}^N} f_{in} \begin{pmatrix} 1 \\ v \end{pmatrix} dv = \begin{pmatrix} \rho \\ 0 \end{pmatrix},$$

the associated solution of the Inelastic Boltzmann equation (IB) satisfies the Haff's law in the sense:

$$(2.5) \quad \forall t \geq 0, \quad m(1+t)^{-2} \leq \mathcal{E}(t) \leq M(1+t)^{-2}$$

for some explicit constants $m, M > 0$.

3. More precisely, the rescaled variables function $g(t, v) := e^{-3t} f(e^t - 1, e^{-t} v)$ satisfies:

(i) There exists $\lambda > 0$ such that the solution g can be written $g = g^S + g^R$ in such a way that $g^S \geq 0$ and

$$\sup_{t \geq 0} \|g_t^S\|_{H_q^s} < +\infty, \quad \forall s \geq 0, q \geq 0, \quad \|g_t^R\|_{L_2^1} = O(e^{-\lambda t}).$$

(ii) For any $\tau > 0$ and $s \in [0, 1/2)$, there are some explicit constants $a_1, a_2, A_1, A_2 > 0$ such that

$$\forall v \in \mathbb{R}^N, \quad \liminf_{t \rightarrow \infty} g(t, v) \geq a_1 e^{-a_2 |v|}$$

and

$$\forall t \geq \tau, \quad \int_{\mathbb{R}^N} g(t, v) e^{-A_1 |v|^s} dv \leq A_2.$$

We refer to [11, 46] and the references therein for more details.

2.4 Other models and perspectives.

Coagulation-Fragmentation equation. For the equation (CF) we refer to [22] and [35] where existence of stationary solutions are proved under the assumptions we have made on a and b and the condition $\gamma \geq 0$ and $0 \leq \lambda \leq \gamma + 1$.

Growth-Fragmentation equation. For the equation (GF) we refer to [49], [44] and [43] where existence of first eigen solution-first eigen value for the primal and the dual equation is proved under quite general assumption on the fragmentation kernel by the mean of the Krein-Rutman theorem. Existence of a family of Liapunov functionals and application to the uniqueness and to the large time asymptotic is obtained in the same that for the pure fragmentation equation, see Theorem 2.1 and Theorem 5.1.

Pauli equation. For the equation (P) we refer to [4] and the references therein where existence of stationary solution is proved under the assumption $0 \leq \sigma \in L^1(\mathbb{T}^3 \times \mathbb{T}^3)$ following the same strategy that the one explained in section 3. Under some positivity assumption on σ and making use of the almost linear structure of the equation, it is shown that the flow generated by the equation is "strictly" contracting in L^1 , a property that it is used in order to prove the uniqueness of the stationary solution and the convergence of generic solutions to the stationary solution in the large time.

We then list some possible extensions of the above theorems.

Open problem 1. For any $\rho > 0$, is the self-similar profile G_ρ of mass ρ unique?

Open problem 2. For any initial datum f_{in} of mass ρ (and momentum 0), does the associated rescaled function g satisfy

$$g(t) \xrightarrow[t \rightarrow \infty]{} G_\rho \quad \text{or, equivalently,} \quad f(t) \underset{t \rightarrow \infty}{\sim} F_\rho?$$

We may weaken the conjecture 2 in the following way.

Open problem 3. Does the associated rescaled function g split

$$g(t) = g_1(t) + g_2(t) \quad \text{with} \quad g_1(t) \in \mathcal{Z}, \quad g_2(t) \rightarrow 0 \text{ in } L^1,$$

where \mathcal{Z} is smaller as possible (for instance it the space in which it has been proved that the profile G belongs)? Moreover, is the (or any) profile G locally asymptotically stable?

3 Existence of self-similar solution.

We present in this section the strategy used in order to get existence of stationary solution for the (modified) equation (1.14), giving first an abstract result and illustrating next the method on the different models in order to get theorems 2.1, 2.2 and 2.3.

3.1 An infinity dimensional Poincaré-Bendixson Theorem.

We start presenting a **first strategy** based on the following abstract theorem.

Theorem 3.1 *Let Y be a Banach space and let consider $(S_t)_{t \geq 0}$ a continuous semigroup on Y such that*

(i) S_t is weakly (sequentially) continuous as an operator acting on Y for any $t > 0$,

(ii) there exists $\mathcal{Z} \neq \emptyset$ a convex and (weakly sequentially) compact subset of Y invariant under the action of S_t (that is $S_t z \in \mathcal{Z}, \forall z \in \mathcal{Z}, t \geq 0$).

Therefore, there exists $z_0 \in \mathcal{Z}$ a stationary point under the action of S_t (that is $S_t z_0 = z_0, \forall t \geq 0$).

Proof of Theorem 3.1. Thanks to the Tykonov fixed point Theorem :

$$\forall t > 0 \quad \exists z_t \in \mathcal{Z} \quad \text{such that} \quad S_t z_t = z_t,$$

in other words, z_t is a t -periodic solution. Choosing $t = 2^{-n}$ and using the semi-group property of S_t there holds

$$(3.1) \quad S_i 2^{-m} z_{2^{-n}} = z_{2^{-n}} \quad \forall i, n, m \in \mathbb{N}, \quad m \leq n.$$

By the (weak) compactness of \mathcal{Z} : $\exists z_0 \in \mathcal{Z}, \exists$ a subsequence $(z_{2^{-n_k}})_k$ such that

$$z_{2^{-n_k}} \rightharpoonup z_0 \quad (\text{weakly}) \quad \text{in } Y.$$

By the (weak) continuity of S_t : we may pass to the limit $n_k \rightarrow \infty$ in (3.1)

$$S_t z_0 = z_0 \quad \text{for any dyadic time } t \geq 0.$$

By the (strong) continuity of the trajectories of S_t and by density of the dyadic real numbers in the real line we finally get $S_t z_0 = z_0$ for any $t \geq 0$. \square

The **first strategy** in order to get a solution G to (1.13) is therefore to exhibit a Banach space Y and a convex and (weakly sequentially) compact set $\mathcal{Z} \subset Y$ such that :

(a) for any $g_{in} \in Y$ there is a solution $g \in C([0, \infty); Y)$ to the equation (1.14) such that $g(0) = g_{in}$ and $g_{in} \in \mathcal{Z}$ implies $g(t) \in \mathcal{Z}$ for any $t \geq 0$;

(b) $\mathcal{Z} \subset \{G, \text{ mass of } G = \rho\}$ with (weakly sequentially) compact embedding;

(c) the following (weakly sequentially) stability principle holds: for any sequence (g_n) of solutions to (1.14) in $C([0, \infty); Y)$ satisfying $g_n(t) \in \mathcal{Z} \forall t \geq 0$, there exists a subsequence $(g_{n'})$ and $g \in C([0, \infty); Y)$ such that $g_{n'}(t, \cdot) \rightharpoonup g(t, \cdot)$ in Y and g is a solution to (1.14);

(d) the solution g is unique in $C([0, \infty); Y)$;

and to define the semi-group S by setting $\forall g_{in} \in Y, S(t) g_{in} := g(t)$, to which we may apply theorem 3.1.

There is some situations in which that first strategy fails, roughly speaking because we are not able to prove (d). We may circumvent that difficulty using a **second strategy** applying first the first strategy to a regularized problem (a regularized equation (1.14) associated to a regularized operator \mathcal{B} for which we may prove (d)) and then removing the regularization. Theorem 3.1 is then just one possible, but elegant, way to approximate

the problem of the existence of a solution to (1.13) by building a sequence of 2^{-n} -periodic solutions to (1.14). As a conclusion, because the point (c) is very standard since the DiPerna-Lions existence theory for Boltzmann like equation (although it may be quite technical) the fundamental point is to exhibit a convex set \mathcal{Z} such that

$$(3.2) \quad \begin{cases} g_{in} \in \mathcal{Z} \implies g(t) \in \mathcal{Z} \forall t \geq 0 \\ \mathcal{Z} \subset \{G, \text{mass of } G = \rho\} \text{ with compact embedding.} \end{cases}$$

Let finally emphasize that a third strategy in order to tackle the problem of the existence of a solution G to the equation (1.13) should be to find a Banach space X with compact unbending in $\{G, \text{mass of } G = \rho\}$ such that the following a priori estimate holds

$$(3.3) \quad \exists R = R(\rho) \quad \mathcal{B}(G) = 0, \quad \text{mass}(G) = \rho \quad \text{implies} \quad \|G\|_X \leq R,$$

in which space X we may prove a DiPerna-Lions stability principle, and to conclude again thanks to a regularization argument. Here, the equivalent to the invariant set \mathcal{Z} in the two first strategies should be $\mathcal{Z} := \{G \in X, \text{mass of } G = \rho, \|G\|_X \leq R(\rho)\}$. With regard to the problem of the existence of a solution G to the equation (1.13) it does not seem that this last strategy is more efficient than the second one. Nevertheless, with regard to the problem of the regularity of the the solution G , that third strategy is very much more efficient than the preceding ones since we know how to exhibit some space X of quite smooth functions for which (3.3) holds but for which we are not able to prove that the associated set \mathcal{Z} is invariant under the flow of the equation (1.14). We refer to section 4 for more details.

3.2 Existence of self-similar profile for the fragmentation equation

3.2.1 Existence of solutions for the evolution equation

We start with some elementary properties of the fragmentation operator that one may obtain by straightforward change of variables and which proof is leaved to the reader.

Lemma 3.2 *For any functions f and ϕ there holds*

$$\langle \mathcal{L}f, \phi \rangle = \int_0^\infty f B \ell^* \phi dy, \quad (\ell^* \phi)(y) = \int_0^1 \theta(z) (\phi(z y) - z \phi(y)) dz$$

As a consequence, for any function f and any $k \in \mathbb{R}$ there holds

$$(3.4) \quad \langle \mathcal{L}f, y^k \rangle = c_k \int_0^\infty f(y) B(y) y^k dy, \quad \langle \mathcal{L}f, y^k \text{ sign}f \rangle \leq c_k \int_0^\infty |f(y)| B(y) y^k dy,$$

with $c_k > 0$ if $k < 1$, $c_1 = 0$ and $c_k < 0$ if $k > 1$.

We next present some a priori estimates for solutions g to the fragmentation equation in self-similar variables (1.14) associated to an initial datum $g_{in} \in \dot{L}_k^1 \cap \dot{L}_K^1$, $k < 1 < K$. We assume that $B(y) = y^\gamma$ or that $0 \leq B \in L^\infty$, $B \not\equiv 0$ which corresponds to the case $\gamma = 0$ in the following computations.

- From $\langle \mathcal{L}g, y \rangle = 0$ and $\langle \mathcal{D}_1 g, y \rangle = 0$ we deduce $M_1(t) \equiv M_1(0) =: \rho$.

- For $K > 1$ we have

$$(3.5) \quad \frac{d}{dt} M_K = \langle \gamma \mathcal{L}g, y^K \rangle - \langle \mathcal{D}_1 g, y^K \rangle = -\gamma |c_K| M_{K+\gamma} + (K-1) M_K.$$

Integrating in time the equation (3.5), we get for any time $T > 0$ that there exists $C_{K,T}$ depending on $M_K(g_{in})$ such that

$$(3.6) \quad \sup_{[0,T]} M_K(t) + |c_K| \int_0^T M_{K+\gamma}(s) ds \leq C_{K,T}.$$

- For $m \leq k < 1$ we have, using the inequality $y^{\gamma+k} \leq \varepsilon^\gamma y^k + \varepsilon^{-(K-k)} y^{\gamma+K} \forall y > 0$ with $\varepsilon > 0$ small enough (for instance $\varepsilon = ((1-k)/c_k)^{1/\gamma}$),

$$(3.7) \quad \frac{d}{dt} M_k = c_k M_{k+\gamma} - (1-k) M_k \leq A_{k,K} M_{K+\gamma},$$

for some constant $A_{k,K}$. Using the previous estimate (3.6), we deduce again that for any time $T > 0$ there exists $C_{k,T}$ such that

$$(3.8) \quad \sup_{[0,T]} M_k(t) \leq C_{k,T}.$$

Lemma 3.3 *For any $g_{in} \in \dot{L}_k^1 \cap \dot{L}_K^1$ with $m \leq k < 1 < K$ there exists a unique solution $g \in C([0, T]; \dot{L}_k^1 \cap \dot{L}_K^1) \cap L^1(0, T; L_{K+\gamma}^1)$, $\forall T > 0$, to the fragmentation equation (in self-similar variables).*

Sketch of the proof of Lemma 3.3. Step 1. For any two given solutions g_i to the fragmentation equation in self-similar variables (1.14) there holds, setting $\phi := y \operatorname{sign}(g_2 - g_1)$,

$$\begin{aligned} \frac{d}{dt} \|g_2 - g_1\|_{\dot{L}_1^1} &= \langle \gamma \mathcal{L}(g_2 - g_1), \phi \rangle - \langle \mathcal{D}_1(g_2 - g_1), \phi \rangle \\ &\leq \langle \gamma \mathcal{L}|g_2 - g_1|, y \rangle - \langle \mathcal{D}_1|g_2 - g_1|, y \rangle = 0, \end{aligned}$$

from which we deduce

$$(3.9) \quad \|g_2 - g_1\|_{\dot{L}_1^1} \leq \|g_{in,1} - g_{in,2}\|_{\dot{L}_1^1} \quad \forall t \geq 0.$$

We conclude to the uniqueness of the solution.

Step 2. We assume that $B \in L^\infty$ and we verify that \mathcal{L} is a linear continuous operator on any \dot{L}_k^1 , $k \geq m$. We define the map $\Phi : E \rightarrow E$, $E := \{g \in C([0, T]; \dot{L}_k^1 \cap \dot{L}_K^1); g \geq 0, M_1(g) = \rho\}$, $g \mapsto \Phi(g) := h$, where h is the solution to the equation

$$\partial_t h + \mathcal{D}_1 h + \|B\|_\infty h = \int_y^\infty b(y', y) g' dy' + (\|B\|_\infty - B(y)) g.$$

We may apply the contraction Banach fixed point theorem to Φ and we obtain a solution to the fragmentation equation in self-similar variables (1.14) which preserve the mass.

Step 3. We next assume $B(y) = y^\gamma$ and we argue by approximation. We introduce $B_n(y) = (y \wedge n)^\gamma$ and \mathcal{L}_n the associated fragmentation operator. Thanks to the step 2, there exists a solution g_n to the fragmentation equation associated to B_n for any $n \in \mathbb{N}$. The

sequence (g_n) satisfies (3.6) with $M_{K+\gamma}$ replaced by $\|f y^k (y \wedge n)^\gamma\|_{L^1}$ and (3.8) uniformly in n . Moreover, for $n' \geq n$ and setting $\phi = y \operatorname{sign}(f_{n'} - f_n)$, we have, following the proof of the uniqueness of solutions in step 1,

$$\begin{aligned} \frac{d}{dt} \|g_{n'} - g_n\|_{L^1} &= \langle (\gamma \mathcal{L}_n - \mathcal{D}_1)(g_{n'} - g_n), \phi \rangle + \int_0^\infty (B_{n'} - B_n) g_{n'} \ell^* \phi dy \\ &\leq \int_0^\infty (n' \wedge y)^\gamma \mathbf{1}_{y \geq n} g_{n'} (\ell^* \phi) dy \leq n^{1-K} \|g_{n'} y^k (y \wedge n')^\gamma\|_{L^1}. \end{aligned}$$

We deduce that (g_n) is a Cauchy sequence and then converges in the space E . Its limit g is the unique solution to the modified fragmentation equation (1.14) associated to B which moreover preserve the mass. \square

3.2.2 Existence of self-similar profile when $\theta \in L_{comp}^\infty$

Let state an elementary maximum principle that we use many times in the sequel.

Lemma 3.4 *Let $0 \leq u \in C([0, \infty))$ satisfy $u' + k_1 u^{\theta_1} \leq k_2 u^{\theta_2} + k_3$ with $\theta_1 \geq 1$, $\theta_2 \geq 0$, $\theta_2/\theta_1 < 1$, $k_1 > 0$ and $k_2, k_3 \geq 0$. There exists $C_0 = C_0(k_i, \theta_i) \geq 0$ such that*

$$(3.10) \quad u_0 \leq C_0 \implies u(t) \leq C_0 \quad \forall t \geq 0.$$

In order to define the Banach space Y and the invariant set \mathcal{Z} we must improve the estimates already established in the preceding section.

- For $K > 1$ and $g_{in} \in \dot{L}_1^1 \cap \dot{L}_K^1$, coming back to the differential equation (3.5) and making use of the Holder inequality $M_K \leq \rho^{-\gamma/(K-1)} M_{K+\gamma}^{(K+\gamma-1)/(K-1)}$, we obtain

$$(3.11) \quad \frac{d}{dt} M_K + \gamma |c_K| \rho^{-\frac{\gamma}{K-1}} M_K^{\frac{K+\gamma-1}{K-1}} \leq (K-1) M_K.$$

We conclude thanks to Lemma 3.4 that there exists $C = C_1(\rho, K)$ such that the following estimate holds with $\|\cdot\| = M_K(\cdot)$

$$(3.12) \quad \|g_{in}\| \leq C \implies \sup_{t \geq 0} \|(g(t, \cdot))\| \leq C.$$

- For $k \in [m, 1)$ and $g_{in} \in \dot{L}_k^1 \cap \dot{L}_{1+\gamma}^1$, coming back to the differential equation (3.7) and making use of the Holder inequality in order to bound $M_{k+\gamma}$, we obtain

$$(3.13) \quad \frac{d}{dt} M_k \leq \left(\gamma c_k M_{1+\gamma}^{\gamma/(1-k+\gamma)} \right) M_k^{(1-k)/(1-k+\gamma)} - (1-k) M_k.$$

We conclude again thanks to Lemma 3.4 and the estimate on $M_{1+\gamma}$ previously established that (3.12) holds with $\|\cdot\| = M_k(\cdot)$ and some constant C_2 .

- We get now estimate on the L^2 norm for $g_{in} \in L^1 \cap L_{1+\gamma}^1 \cap L_{1/2}^2$ observing that we may take $k = 0 > m = -\infty$ because $\operatorname{supp} \theta \subset (0, 1]$. We have

$$\begin{aligned} \frac{d}{dt} \int_0^\infty \frac{g^2}{2} y dy &= \gamma \int_0^\infty g y \int_y^\infty (y')^{\gamma-1} \theta(y/y') g' dy' dt - \gamma \int_0^\infty y^{\gamma+1} g^2 dy - \langle \mathcal{D}_1 g, y g \rangle \\ &\leq \gamma \|z \theta(z)\|_{L^\infty} M_0 M_\gamma - \int_0^\infty g^2 y dy, \end{aligned}$$

and we conclude thanks to the two preceding uniform estimates on $M_{1+\gamma}$ and M_0 and thanks to lemma 3.4 that (3.12) holds with $\|\cdot\| = \|\cdot\|_{L^2_{1/2}}$ and some constant C_3 .

We present now a sketch of the proof of the existence of self-similar profile for the fragmentation using the first strategy presented in section 3.1. We define $Y = L^1 \cap L^1_{1+\gamma} \cap L^2_{1/2}$. Thanks (to a variant of) the existence result presented in section 3.2.1 we know that for any $g_{in} \in Y$ there exists a unique solution $g \in C([0, \infty); Y)$ which preserve the mass. We thus define $S(t)g_{in} = g(t)$ for any $g_{in} \in Y$. Moreover, thanks to the estimates just obtained above, we have that

$$\mathcal{Z}_1 := \{g \in Y; g \geq 0, M_1(g) = \rho, M_K(g) \leq C_1, M_k(g) \leq C_2, \|g\|_{L^2_{1/2}} \leq C_3\},$$

with $k = 0$ and $K = 1 + \gamma$, is an invariant set for S . Finally, we classically prove the following stability result, which is one of the fundamental principle of the DiPerna-Lions existence theory for Boltzmann like equations.

Lemma 3.5 *Let (g_n) be a sequence of solution associated to equation (1.14) and bounded in $L^\infty(0, T; \mathcal{Z})$. Then, up to a subsequence, (g_n) is weakly converging (in L^1) to a function g and g is a solution to the equation (1.14).*

The *first strategy* then applies without difficulty.

3.2.3 Existence of self-similar profile without restriction on θ .

Taking $Y := \dot{L}^1_m \cap \dot{L}^1_K$ with $K > 1$ arbitrary large, we observe that

$$(3.14) \quad \mathcal{Z}_0 := \{g \in Y; g \geq 0, M_1(g) = \rho, M_K(g) \leq C_1, M_m(g) \leq C_2\},$$

is an invariant set under the flow generated by the fragmentation equation in self-similar variables. We may then use the second strategy. We introduce the sequence of truncated repartition probability of daughter particles $\theta_n := \theta \mathbf{1}_{(1/n, 1)}$. Thanks to the preceding section we get the existence of a sequence of self-similar profile (G_n) associated to (θ_n) and such that $G_n \in \mathcal{Z}_0$ for any $n \geq 1$. By a compactness argument, we obtain a self-similar profile $G \in \dot{M}^1_m \cap \dot{M}^1_K$ with $M_1(G) = \rho$.

3.3 Existence of self-similar profile for the inelastic Boltzmann equation

In this section we just prove the existence of an invariant set \mathcal{Z} for the equation (1.14) associated to the inelastic Boltzmann equation for which (3.2) holds. Existence of self-similar profile can then be obtained following the *first strategy* presented in section 3.1. We refer to [46] for details. We will consider the Banach space $Y = L^1_2(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$.

We begin with some moment estimates for which we use the following weak formulation of the Inelastic Boltzmann operator

$$\langle \mathcal{Q}(g, f), \psi \rangle = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{S^2} f g_* [\psi' + \psi'_* - \psi - \psi_*] |u| d\sigma dv_*,$$

with

$$v' = \frac{v + v_*}{2} + \frac{z}{2}, \quad v'_* = \frac{v + v_*}{2} - \frac{z}{2}, \quad z = \frac{1 - e}{2} u + \frac{1 + e}{2} |u| \sigma.$$

- Mass conservation: assuming that g_{in} has mass one, we get

$$\frac{d}{dt} \int_{\mathbb{R}^3} g \, dv = 0 \implies \int_{\mathbb{R}^3} g \, dv \equiv 1.$$

- Momentum conservation: assuming that the momentum of g_{in} is zero, we get

$$\frac{d}{dt} \int_{\mathbb{R}^3} v g \, dv = 3 \int_{\mathbb{R}^3} v g \, dv \implies \int_{\mathbb{R}^3} v g \, dv \equiv 0.$$

- Energy estimate: the energy equation is

$$\frac{d}{dt} \mathcal{E} = \langle \mathcal{Q}(g, g), |v|^2 \rangle - \int_{\mathbb{R}^3} |v|^2 \operatorname{div}_v(v g) \, dv.$$

On the one hand

$$- \int_{\mathbb{R}^3} |v|^2 \operatorname{div}_v(v g) \, dv = 2 \int_{\mathbb{R}^3} |v|^2 g \, dv.$$

On the other hand

$$\langle \mathcal{Q}(g, g), |v|^2 \rangle = -\sigma_0 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} g g_* |v - v_*|^3 \, dv dv_*$$

with $\sigma_0 := \frac{1-e^2}{4} \int_{S^2} (1 - \hat{u} \cdot \sigma) \, d\sigma$ where \hat{u} stands for $u/|u|$. First, the Jensen inequality

$$j \left(\int_{\mathbb{R}^3} \varphi_* \, d\mu_* \right) \leq \int_{\mathbb{R}^3} j(\varphi_*) \, d\mu_*$$

with $j(s) = |s|^3$, $d\mu_* = g_* \, dv_*$ and $\varphi(v_*) = v - v_*$ gives

$$(3.15) \quad |v|^3 = \left| \int_{\mathbb{R}^3} g_* (v - v_*) \, dv_* \right|^3 \leq \int_{\mathbb{R}^3} g_* |v - v_*|^3 \, dv_*.$$

Next, the Hölder inequality writes

$$(3.16) \quad \int g |v|^2 \, dv \leq \left(\int g \, dv \right)^{1/3} \left(\int g |v|^3 \, dv \right)^{2/3} = \left(\int g |v|^3 \, dv \right)^{2/3}.$$

Gathering (3.15) and (3.16), we get

$$\langle \mathcal{Q}(g, g), |v|^2 \rangle \leq -\sigma_0 \int g |v|^3 \, dv \leq -\sigma_0 \left(\int g |v|^2 \, dv \right)^{3/2},$$

and then the following differential inequality on the energy

$$\frac{d}{dt} \mathcal{E} \leq 2\mathcal{E} - \sigma_0 \mathcal{E}^{3/2}.$$

We get, thanks to Lemma 3.4, that the set

$$\mathcal{Z}_0 := \left\{ g \in L^1_2, \int_{\mathbb{R}^3} g \begin{pmatrix} 1 \\ v \end{pmatrix} \, dv = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \int_{\mathbb{R}^3} g |v|^2 \, dv \leq \left(\frac{2}{\sigma_0} \right)^2 \right\}$$

is invariant under the flow generated by (1.14).

• L^p estimate, $p \in (1, \infty)$. We first recall that the collision term splits into two terms $\mathcal{Q}(g, g) = \mathcal{Q}^+(g, g) - \mathcal{Q}^-(g, g)$, namely the gain term and the loss term. The loss term may be expressed in the following way

$$\mathcal{Q}^-(g, g)(v) = g L(g), \quad L(g) = \int_{\mathbb{R}^3} g_* |v - v_*| dv_*,$$

and the gain term enjoys some regularity properties which has been investigated in [39, 54, 12, 40] for the elastic Boltzmann equation and in [46] for the inelastic Boltzmann equation.

Lemma 3.6 *There exists $\theta \in (0, 1)$ such that for any $\varepsilon > 0$*

$$(3.17) \quad \int \mathcal{Q}^+(g, g) g^{p-1} dv \leq C_\varepsilon \|g\|_{L^1_2}^{1+p\theta} \|g\|_{L^p}^{p(1-\theta)} + \varepsilon \|g\|_{L^1_2} \int g^p (1 + |v|).$$

The evolution of the L^p norm writes

$$(3.18) \quad \frac{d}{dt} \int g^p dv = \int \mathcal{Q}^+(g, g) g^{p-1} dv - \int g^p L(g) dv - \int g^{p-1} \nabla_v(v g) dv.$$

First, the Jensen inequality $L(g) \geq |v|$ gives

$$(3.19) \quad - \int g^p L(g) dv \leq - \int g^p |v| dv.$$

Next,

$$(3.20) \quad - \int g^{p-1} \nabla_v(v g) dv = \int \nabla_v(g^{p-1}) v g = c_p \int \nabla_v(g^p) v = -c_p \int g^p$$

Estimating the \mathcal{Q}^+ term thanks to (3.17) and taking ε small enough (remind that the energy is yet bounded), we obtain gathering (3.18), (3.19) and (3.20)

$$\frac{d}{dt} \int g^p dv \leq C_\varepsilon \|g\|_{L^1_2}^{1+p\theta} \|g\|_{L^p}^{p(1-\theta)} - \frac{c_p}{2} \int g^p dv.$$

Thanks to lemma 3.4, we get a new convex and weak compact invariant set $\mathcal{Z} := \{g \in \mathcal{Z}_0; \|g\|_{L^2} \leq C_3\}$. We are able to conclude because \mathcal{Z} satisfies the properties listed in (3.2).

3.4 Existence of self-similar profile for the coagulation equation

In this section we show how to get uniform estimates on a solution g to the coagulation equation (1.14) written in self-similar variables. As explained in the preceding sections we may deduce of it the existence of a self-similar profile.

Lemma 3.7 *For any $k \in (\lambda, 1)$, there exists $w_k = w_k(\lambda) \in (0, \infty)$ such that*

$$(3.21) \quad \forall t \geq 0 \quad N_k(t) \leq \max(N_k(0), w_k), \quad \text{with } N_k(t) := \int_0^\infty g(t, y) (y \wedge 1)^k dy.$$

Here and below we define $a \wedge b = \min(a, b)$ for $a, b \in \mathbb{R}$. The lemma is based on a trick introduced in [21] in order to investigate the gelation phenomenon.

Proof of Lemma 3.7. First, we define $\phi_A(y) = (y \wedge A)^m$ for $m \in (0, 1]$ and $A > 0$, and we compute

$$(3.22) \quad -\Delta_{\phi_A}(y, y') := \phi_A(y) + \phi_A(y') - \phi_A(y + y') \geq A^m \mathbf{1}_{y, y' \geq A}$$

and

$$(3.23) \quad D^* \phi_A := \phi_A - y \partial_y \phi_A \leq \phi_A.$$

Multiplying the coagulation equation in self-similar variables (1.14) by ϕ and integrating in the mass variables, we easily get

$$(3.24) \quad \frac{d}{dt} \int_0^\infty g \phi dy = \frac{1}{2} \int_0^\infty \int_0^\infty a \Delta_\phi g g' dy dy' + \int_0^\infty g D^* \phi dy.$$

We then deduce from (3.24), (3.22), (3.23) and the lower estimate $a(y, y') \geq (y y')^{\lambda/2}$

$$(3.25) \quad \frac{d}{dt} \int_0^\infty g \phi_A dy + \frac{A^m}{2} \left(\int_A^\infty g y^{\lambda/2} dy \right)^2 \leq \int_0^\infty g \phi_A dy \quad \forall A > 0.$$

Next, for a given function $\Phi : [0, \infty) \rightarrow [0, \infty)$ such that $\Phi(0) = 0$ and a given $\ell \in \mathbb{R}$, we have, using Fubini's Theorem, Cauchy-Schwarz inequality and (3.25),

$$\begin{aligned} \left(\int_0^\infty g(y) y^{\lambda/2} \Phi(y) dy \right)^2 &= \left(\int_0^\infty \Phi'(A) \int_A^\infty g(y) y^{\lambda/2} dy dA \right)^2 \\ &\leq K_0 \int_0^\infty \Phi'(A) A^\ell \left(\int_A^\infty g(y) y^{\lambda/2} dy \right)^2 dA \\ &\leq 2K_0 \int_0^\infty \Phi'(A) A^{\ell-m} \left(\int_0^\infty g \phi_A dy - \frac{d}{dt} \int_0^\infty g \phi_A dy \right) dA \\ &\leq 2K_0 \left(\int_0^\infty g \Psi dy - \frac{d}{dt} \int_0^\infty g \Psi dy \right), \end{aligned}$$

where we have set

$$(3.26) \quad K_0 := \int_0^\infty \Phi'(A) A^{-\ell} dA \quad \text{and} \quad \Psi(y) := \int_0^\infty \Phi'(A) A^{\ell-m} \phi_A(y) dA.$$

In other words, we have obtained the following differential inequality

$$(3.27) \quad \frac{d}{dt} \int_0^\infty g \Psi dy + \frac{1}{2K_0} \left(\int_0^\infty g(y) y^{\lambda/2} \Phi(y) dy \right)^2 \leq \int_0^\infty g \Psi dy.$$

Finally, we make the choices

$$(3.28) \quad \Phi(y) := \min(y^{\lambda/2+\delta}, 1), \quad \ell := \lambda/2, \quad m := \lambda + 2\delta,$$

with $\delta \in (0, (1 - \lambda)/2]$ and we easily compute

$$K_0 = \left(\frac{\lambda}{2} + \delta \right) \int_0^1 A^{\delta-1} dA < \infty, \quad \Psi(y) = \left(\frac{\lambda}{2} + \delta \right) \left(\frac{(y \wedge 1)^{\lambda+\delta}}{\lambda + \delta} + \frac{y^{\lambda+2\delta}}{\delta} (y^{-\delta} - 1) \mathbf{1}_{y \leq 1} \right).$$

As a consequence, setting $k = \lambda + \delta \in (\lambda, 1)$, there holds for some constant $C \in (0, \infty)$

$$(3.29) \quad \frac{1}{C} (y \wedge 1)^k \leq \Psi(y) \leq C (y \wedge 1)^k \quad \text{and} \quad \Phi(y) y^{\lambda/2} \geq (y \wedge 1)^k.$$

Gathering (3.27) and (3.29) we obtain, for some constant $K_1 \in (0, \infty)$, the differential inequality

$$\frac{d}{dt} \int_0^\infty g \Psi dy + K_1 \left(\int_0^\infty g \Psi dy \right)^2 \leq \int_0^\infty g \Psi dy,$$

and we conclude thanks to Lemma 3.4. \square

We define a first invariant set

$$\mathcal{Z}_0 := \{0 \leq g \in \dot{L}_1^1, M_1(g) = \rho, N_k(g) \leq w_k \quad \forall k \in (\lambda, 1)\}.$$

Lemma 3.8 *For any $k > 1$, there exists a constant A_k such that $g_{in} \in \mathcal{Z}_0$ implies*

$$(3.30) \quad \sup_{[0, \infty)} M_k(t) \leq \max(A_k, M_k(0)).$$

Proof of Lemma 3.8. For a given $k > 1$, let us define

$$(3.31) \quad \Lambda_k(y, y') := (y^\alpha (y')^\beta + y^\beta (y')^\alpha) ((y + y')^k - y^k - (y')^k) \geq 0.$$

For $y > y'$, denoting $z = y'/y \in (0, 1]$, we have for any $\mu \leq 1$

$$\begin{aligned} \Lambda_k(y, y') &= y^{\lambda+k} (z^\alpha + z^\beta) ((1+z)^k - z^k - 1) \\ &\leq y^{\lambda+k} (2z^\alpha) (C_k z) \leq 2C_k y^{\lambda+k} z^{\mu+\alpha} \\ &\leq 2C_k [y^{\beta-\mu+k} (y')^{\mu+\alpha} + (y')^{\beta-\mu+k} y^{\mu+\alpha}] =: \bar{\Lambda}_k(y, y'). \end{aligned}$$

for a constant $C_k > 0$. Therefore, since Λ_k and $\bar{\Lambda}_k$ are symmetric functions, the inequality $\Lambda_k(y, y') \leq \bar{\Lambda}_k(y, y')$ holds for any $y, y' \geq 0$. We then deduce from (3.24) with $\phi = y^k$, the following differential inequality

$$\frac{d}{dt} M_k \leq C_k M_{k+\beta-\mu} M_{\alpha+\mu} - (k-1) M_k \quad \text{for any } \mu \leq 1.$$

Making the choice $\mu := \beta + \min(\frac{k-1}{2}, \frac{1-\lambda}{2}, 1-\beta) \in (0, 1]$, we obtain

$$\frac{d}{dt} M_k \leq C_k M_{k_1} M_{k_2} - (k-1) M_k,$$

with $k_1 := k + \beta - \mu = k - \min(\frac{k-1}{2}, \frac{1-\lambda}{2}, 1-\beta) \in (1, k)$, $k_2 := \alpha + \mu = \lambda + \min(\frac{k-1}{2}, \frac{1-\lambda}{2}, 1-\beta) \in (\lambda, 1]$. Finally, using the Hölder inequality $M_{k_1} \leq M_1^{1-\theta} M_k^\theta$, with $\theta \in (0, 1)$, we deduce

$$(3.32) \quad \frac{d}{dt} M_k \leq C_1 M_k^\theta M_{k_2} - C_2 M_k.$$

Since $g_{in} \in \mathcal{Z}_0$, we deduce that M_{k_2} is bounded and we may apply Lemma 3.4 in order to conclude. \square

We define a second weak compact invariant set $\mathcal{Z}_1 := \{g \in \mathcal{Z}_0, M_k(g) \leq A_k \quad \forall k > 1\}$ which satisfies the properties listed in (3.2) and with the help of which we are able to use the second strategy in order to conclude.

4 Qualitative properties of G .

4.1 The fragmentation equation.

We yet know that $G \in \dot{M}_k^1$ for any $k \geq m$ and then $\mathcal{L}G \in \dot{M}_k^1$. Writing for $k \geq 1+m$

$$(4.1) \quad \frac{\partial}{\partial y}(y^k G) = \frac{\partial}{\partial y}(y^{k-2} y^2 G) = (k-2)y^{k-1}G + y^{k-1}\mathcal{L}G$$

we deduce that $y^k G \in L^\infty$ for any $k \geq 1+m$. Furthermore, gathering (4.1) with $BG \in L_{loc}^\infty$ and

$$\begin{aligned} (\mathcal{L}^+G)(x) &:= \int_x^\infty (y)^{\gamma-1} \theta(x/y) G(y) dy \leq \|G x^{2+\gamma}\|_{L^\infty} \int_x^\infty (y)^{-3} \theta(x/y) dy \\ &\leq \|G x^{2+\gamma}\|_{L^\infty} \int_0^1 \frac{z^3}{x^3} \theta(z) x \frac{dz}{z^2} = \|G x^{2+\gamma}\|_{L^\infty} x^{-2} \in L_{loc}^\infty, \end{aligned}$$

we obtain that $y^2 G \in W_{loc}^{1,\infty}$. That concludes the proof of the regularity estimate. Finally, there holds

$$(4.2) \quad \frac{\partial}{\partial y} \left(y^2 G(y) e^{y^\gamma/\gamma} \right) = y (\mathcal{L}^+G)(y) e^{y^\gamma/\gamma}.$$

Since $G \not\equiv 0$ there exists $x_0, x_1 \in (0, \infty)$, $x_0 < x_1$ such that $G > 0$ on $[x_0, x_1]$. On the one hand, integrating (4.2) between 0 and x , for any $x \in (0, x_0)$, we have

$$x^2 G(x) e^{x^\gamma/\gamma} \geq \int_{x_0}^{x_1} G(y) y^\gamma e^{y^\gamma/\gamma} \int_0^{x/x_1} \theta(z) z e^{z^\gamma/\gamma} dz dy > 0.$$

On the other hand, for any $x > x_0$, integrating (4.2) between x_0 and x and using the fact that $\mathcal{L}^+G \geq 0$, we find

$$G(x) \geq C(x_0) x^{-2} e^{-x^\gamma/\gamma} > 0 \quad \text{on } (x_0, \infty),$$

and that conclude the proof of the positivity of G . \square

4.2 L^∞ exponential weight bound for the inelastic Boltzmann equation

First, because the inelastic Boltzmann equation is invariant under rotations, we may prove that G is radially symmetric. Next, using a trick introduced in Bobylev paper's [9] for the Boltzmann equation, Bobylev, Gamba and Panferov have proved in [11] the following exponential moment bound for self-similar profiles to the inelastic Boltzmann equation:

$$C_0 := \int_0^\infty G(r) e^{Ar} r^2 dr = \int_{\mathbb{R}^3} G(v) e^{A|v|} dv < \infty.$$

The stationary equation (1.13) in radial variable $r = |v|$ reads

$$Q(G, G) - 3G - rG'(r) = 0.$$

For any $R \geq 1$, we have

$$G(R) = - \int_R^\infty G'(r) dr \leq \int_R^\infty 3 \frac{G(r)}{r} + \frac{Q^-(G, G)}{r} dr \leq C_1 \int_R^\infty G(r) r^2 dr,$$

where we have used $L(G) \leq C r$, $r \geq 1$. As a consequence, we get

$$\forall R \geq 1 \quad G(R) e^{AR} \leq C_1 \int_R^\infty G(r) e^{Ar} r^2 dr \leq C_1 C_0,$$

which conclude the L^∞ exponential weight estimate.

On the other hand, for the L^∞ lower bound in theorem 2.3 we use the spreading effect of positivity of the gain term and of the drift operator as introduced in [13] and next developed in [51, 27, 46].

5 Long time behavior for generic solutions

We present in this section some results which are available for generic solutions. On the one hand, for the inelastic Boltzmann equation and the coagulation equation rough moment estimates can be proved, and we just present the proof of the first one, referring to [20] for the second one. On the other hand, quite complete description of the long time asymptotic can be obtained in the case of the fragmentation equation.

5.1 Haff's law for the inelastic Boltzmann equation

On the one hand, we have yet proved

$$\forall t \geq 0 \quad \int_{\mathbb{R}^3} g(t, v) |v|^2 dv \leq C_1.$$

On the other hand

$$\begin{aligned} \forall t \geq 0 \quad \int_{\mathbb{R}^3} g(t, v) |v|^2 dv &\geq r_0^2 \int_{|v| \geq r_0} g dv \geq r_0^2 \left(1 - \int_{|v| \leq r_0} g dv \right) \\ &\geq r_0^2 \left(1 - r_0^{1/2} \|g\|_{L^2} \right) \geq \frac{r_0^2}{2} \end{aligned}$$

for $r_0 > 0$ small enough. We conclude that the Haff's law (2.5) holds using the change of functions (1.15).

5.2 More about the fragmentation equation.

5.2.1 A family of Liapunov functional.

We consider in this section the modified fragmentation equation (1.14) with $\mathcal{B} = \mathcal{L} - \gamma^{-1} \mathcal{D}_1$ for the pure fragmentation equation and $\mathcal{B} = \mathcal{L} - w \partial_y - \lambda$ for the growth-fragmentation equation and (where w is defined in section 1.3.4) and λ is the first eigen value defined in theorem 1.1. Let consider a solution ψ to the dual equation of the modified equation (1.14) associated to the fragmentation equation or to the growth-fragmentation equation

$$(5.1) \quad \mathcal{D}^* \psi = \mathcal{L}^* \psi,$$

with

$$(5.2) \quad \mathcal{D}^* \psi = -w \frac{\partial \psi}{\partial y} + \sigma \psi, \quad (\mathcal{L}^* \psi)(y) = \int_0^y b(y, y') \psi(y') dy' - B(y) \psi(y).$$

where $w = \gamma^{-1}y$, $\sigma = \gamma^{-1}$ for the fragmentation equation and $\sigma = \partial_y w + \lambda$ for the growth-fragmentation equation.

We start establishing a general relative entropy principle which is based on the following elementary computation that we lead to the reader.

Theorem 5.1 *For any solution g and any solution $G > 0$ to equation (1.14) associated to the fragmentation equation or to the growth-fragmentation equation, any solution $\psi \geq 0$ to the dual equation (5.1) and any function $j : \mathbb{R} \rightarrow \mathbb{R}$ there holds*

$$\begin{aligned} & \frac{\partial}{\partial t} \left[\psi(x) G(t, x) j\left(\frac{g(t, x)}{G(t, x)}\right) \right] + \frac{\partial}{\partial x} \left[w(x) \psi(x) G(t, x) j\left(\frac{g(t, x)}{G(t, x)}\right) \right] \\ & + \int_0^\infty \left[b(x, y) \psi(y) G(t, x) j\left(\frac{g(t, x)}{G(t, x)}\right) - b(y, x) \psi(x) G(t, y) j\left(\frac{g(t, y)}{G(t, y)}\right) \right] dy \\ & = \int_0^\infty b(y, x) \psi(x) G(t, y) \left[j\left(\frac{g(t, x)}{G(t, x)}\right) - j\left(\frac{g(t, y)}{G(t, y)}\right) + j'\left(\frac{g(t, x)}{G(t, x)}\right) \left[\frac{g(t, y)}{G(t, y)} - \frac{g(t, x)}{G(t, x)} \right] \right] dy. \end{aligned}$$

Considering now the case when j is convex, we integrate in the x -variable the preceding identity. Since the second and third terms vanish, we get

$$(5.3) \quad \frac{d}{dt} J(g(t, \cdot) | G(t, \cdot)) = -\mathcal{J}(g(t, \cdot) | G(t, \cdot)) \leq 0,$$

with

$$J(g|G) := \int_0^\infty \psi(x) G(x) j\left(\frac{g(x)}{G(x)}\right) dx$$

and

$$\mathcal{J}(g|G) := \int_0^\infty \int_0^\infty b(y, x) \psi(x) G(y) \left[j\left(\frac{g(x)}{G(x)}\right) - j\left(\frac{g(y)}{G(y)}\right) + j'\left(\frac{g(x)}{G(x)}\right) \left(\frac{g(y)}{G(y)} - \frac{g(x)}{G(x)} \right) \right] dx dy.$$

5.2.2 Uniqueness of the self-similar profile for the fragmentation equation.

Let consider two solutions G_i of (1.13) associated to the fragmentation equation. From the above entropy principle with $\psi = y$ and $j(s) = |s - 1|$ or equivalently (3.4) there holds, since $G := G_2 - G_1$ is a solution to equation (1.14) with mass 0:

$$\begin{aligned} 0 = \mathcal{J}_1(G) & := \mathcal{J}(G_2 | G_1) = \langle \mathcal{L}G, y \text{ sign}(G) \rangle \\ & = \int_0^\infty \int_0^y b(y, y') y' (G \text{ sign}(G') - |G|) dy' dy. \end{aligned}$$

By positivity of b we get

$$G \text{ sign}(G') = |G| \quad \text{for any } y, y' \in (0, \infty),$$

and then $\text{sign} G$ is constant on $(0, \infty)$. Gathering this last result with the fact that $M_1(G) = 0$ we get $G \equiv 0$ and we conclude of the uniqueness of the self-similar profile for the fragmentation equation.

5.2.3 Long time behavior.

We prove (2.2). We split the proof in three steps.

Step 1. Let us first assume that $y \mapsto y g_{in}^2(y) G^{-1}(y) \in L^1$, where G stands for the self-similar profile with same mass than g_{in} , or equivalently, defining by J_2 and \mathcal{J}_2 the entropy term and the entropy dissipation term associated to $j(s) = (s - 1)^2$, that $J_2(g_{in}|G) < \infty$. Thanks to (5.3) and using a variant of the existence result established in section 3.2.1, there exists a unique solution g associated to the initial data g_{in} such that

$$(5.4) \quad \forall t \geq 0 \quad J_2(g(t, \cdot)|G) = \int_0^\infty g(t, y)^2 G(y)^{-1} y dy \leq J_2(g_{in}|G),$$

and, using the fact that for any $\xi, \xi' \geq 0$ there holds $j(\xi) - j(\xi') + j'(\xi')(\xi' - \xi) = (\xi - \xi')^2$,

$$(5.5) \quad \mathcal{J}_2(g(t, \cdot)|G) := \int_0^\infty \int_0^\infty b(x, y) G(x) y \left(\frac{g(t, x)}{G(x)} - \frac{g(t, y)}{G(y)} \right)^2 dx dy \in L_t^1(0, \infty).$$

Consider now a sequence (t_n) such that $t_n \rightarrow \infty$, a time $T > 0$ and define $g_n(t, y) := g(t + t_n, y)$. From $0 < G \in W_{loc}^{1, \infty}$ and (5.4), we know that the sequence (g_n) is bounded in $L_{loc}^2([0, T] \times (0, \infty))$ and we may extract a subsequence still denoted by (t_n) such that $g_n \rightharpoonup \bar{g}$ weakly in $L_{loc}^2([0, T] \times (0, \infty))$. On the one hand, for any function $\varphi \in C_c^1([0, \infty[)$, using the equation (1.14) and the estimates induced by the fact that the set \mathcal{Z} defined in (3.14) is invariant we have

$$\frac{d}{dt} \int_0^\infty g_n \varphi dx \quad \text{is bounded in} \quad L^1(0, T).$$

We deduce that

$$(5.6) \quad \int_0^\infty g_n \varphi dx \xrightarrow{n \rightarrow \infty} \int_0^\infty \bar{g} \varphi dx \quad \text{in} \quad L^1(0, T) \quad \forall \varphi \in C_c^1([0, \infty[).$$

On the other hand, we introduce for any $\varepsilon \in (0, 1)$ the truncated dissipation entropy

$$(5.7) \quad \mathcal{J}_{2, \varepsilon}(g|G) := \int_\varepsilon^{1/\varepsilon} \int_\varepsilon^{1/\varepsilon} b(x, y) G(x) y \left(\frac{g(x)}{G(x)} - \frac{g(y)}{G(y)} \right)^2 dx dy.$$

Thanks to standard convexity arguments (see [15]), we see that $g \mapsto \mathcal{J}_{2, \varepsilon}(g|G)$ is l.s.c. for the above sense of convergence (5.6) for (g_n) and therefore using (5.5) for any $\varepsilon > 0$:

$$(5.8) \quad \int_0^T \mathcal{J}_{2, \varepsilon}(\bar{g}|G) dt \leq \liminf_{n \rightarrow \infty} \int_0^T \mathcal{J}_{2, \varepsilon}(g_n|G) dt \leq \liminf_{n \rightarrow \infty} \int_{t_n}^\infty \mathcal{J}_{2, \varepsilon}(g|G) ds = 0.$$

Gathering (5.7) and (5.8) and letting $\varepsilon \rightarrow 0$, we get

$$(5.9) \quad \bar{g}(t, x)/G(x) = \bar{g}(t, y)/G(y) \quad \text{for a.e. } t, x, y.$$

We have then prove that for some $\alpha = \alpha(t) \in [0, \infty)$ there holds $g(t, y) = \alpha(t) G(y)$ for a.e. $(t, y) \in (0, T) \times (0, \infty)$, and the mass condition implies $\alpha(t) \equiv 1$. In other words, we have yet proved

$$(5.10) \quad g_n(t, \cdot) \rightharpoonup G \quad \text{weakly in} \quad \dot{L}_1^1 \cap L_{loc}^2,$$

Step 2 We prove that the convergence holds in fact in the strong sense. Let fix $\varepsilon_0 \in (0, 1)$ such that

$$\int_{\varepsilon_0}^{1/\varepsilon_0} z \theta(z) dz \geq 1/2.$$

For any $\varepsilon \in (0, \varepsilon_0)$ because of the positivity of G , there exists $\eta_\varepsilon > 0$ such that

$$\begin{aligned} \eta_\varepsilon \int_0^T \int_\varepsilon^{1/\varepsilon} (g_n(t, x) - G(x))^2 dx dt &\leq \\ &\leq \int_0^T \int_\varepsilon^{1/\varepsilon} \int_\varepsilon^{1/\varepsilon} b(x, y) G(x) y \left[\left(\frac{g(t, x)}{G(x)} - 1 \right)^2 + \left(1 - \frac{g(t, y)}{G(y)} \right)^2 \right] dx dy dt \\ &= \int_0^T \mathcal{J}_{2, \varepsilon}(g|G) dt \\ &\quad + 2 \int_0^T \int_\varepsilon^{1/\varepsilon} \int_\varepsilon^{1/\varepsilon} b(x, y) G(x) y \left[\frac{g(t, x)}{G(x)} \frac{g(t, y)}{G(y)} + 1 - \frac{g(t, x)}{G(x)} - \frac{g(t, y)}{G(y)} \right] dx dy dt. \end{aligned}$$

Thanks to (5.10), (5.6) and (5.8) we easily deduce that

$$\eta_\varepsilon \int_0^T \int_\varepsilon^{1/\varepsilon} (g_n(t, x) - G(x))^2 dx dt \rightarrow 0 \quad \forall \varepsilon > 0.$$

We conclude that (2.2) holds using the contraction principle (3.9) applied to $g_1 = G$ and $g_2 = g(t_n + \tau, \cdot)$ for some $\tau \in (0, T)$.

Step 3. For $g_{in} \in \dot{L}_k^1 \cap \dot{L}_K^1$ with $k < 1 < K$, we consider a sequence $(g_{in, n})$ such that $J_2(g_{in, n}|G) < \infty$, the mass associated to $g_{in, n}$ is ρ and $g_{in, n} \rightarrow g_{in}$ in $\dot{L}_m^1 \cap \dot{L}_M^1$. On the one hand, thanks to the preceding analysis, the solution g_n associated to $g_{in, n}$ satisfies $\|g_n - G\|_{\dot{L}_1^1} \rightarrow 0$. On the other hand, the contraction principle (3.9) implies that $\|g(t, \cdot) - g_n(t, \cdot)\|_{\dot{L}_1^1} \leq \|g_{in} - g_{in, n}\|_{\dot{L}_1^1}$. As a conclusion g satisfies the asymptotic property (2.2). For a general initial datum $g_{in} \in \dot{L}_1^1$ we argue as above using "generalized" moments rather than "power" moments. \square

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