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Singular solutions for the Uehling–Uhlenbeck equation

M. Escobedo
Departamento de Matemáticas, Universidad del País Vasco, Apartado 644, 48080 Bilbao, Spain (mtpesmam@lg.ehu.es)

S. Mischler
Ceremade, Université Paris IX-Dauphine, place du Maréchal de Lattre de Tasigny, 75016 Paris, France (mischler@ceremade.dauphine.fr)

J. J. L. Velázquez
Departamento de Matemática Aplicada, Facultad de Matemáticas, Universidad Complutense, 28040 Madrid, Spain (jj_velazquez@mat.ucm.es)

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In this paper we prove the existence of solutions of the Uehling–Uhlenbeck equation that behave like $k^{-7/6}$ as $k \to 0$. From the physical point of view, such solutions can be thought as particle distributions in the space of momentum having a sink (or a source) of particles with zero momentum. Our construction is based on the precise estimates of the semigroup for the linearized equation around the singular function $k^{-7/6}$ that we obtained in an earlier paper.

1. Introduction

We consider the initial-value problem associated with the Uehling–Uhlenbeck (UU) equation [14]:

$$\frac{\partial f}{\partial t}(t,k) = Q(f)(t,k), \tag{1.1}$$

$$f(0,k) = f_0(k), \tag{1.2}$$

where

$$Q(f)(k_1) = \int_{D(k_1)} W(k_1,k_2,k_3,k_4) q(f)\,\mathrm{d}k_3\,\mathrm{d}k_4, \tag{1.3}$$

$$q(f) = f_3 f_4 (1+f_1)(1+f_2) - f_1 f_2 (1+f_3)(1+f_4), \tag{1.4}$$

$$D(k_1) \equiv \{(k_3,k_4) : k_3 + k_4 \geqslant k_1\}, \tag{1.5}$$

$$W(k_1,k_2,k_3,k_4) = \frac{\min(\sqrt{k_1},\sqrt{k_2},\sqrt{k_3},\sqrt{k_4})}{\sqrt{k_1}}, \tag{1.6}$$

$$k_2 = k_3 + k_4 - k_1. \tag{1.7}$$

We are interested in solutions which are singular at the origin and, more particularly, behave like $k^{-7/6}$ as $k \to 0$. The choice of this specific asymptotic behaviour is due to the fact that, as proved in [3], $Ak^{-7/6}$ is a stationary solution of the equation

$$\tilde{Q}(f)(k_1) = 0 \tag{1.8}$$

for all $A > 0$, where

$$\tilde{Q}(f)(k_1) = \int_{D(k_1)} W(k_1, k_2, k_3, k_4)\tilde{q}(f)\,\mathrm{d}k_3\,\mathrm{d}k_4, \tag{1.9}$$

and

$$\tilde{q}(f) = f_3 f_4(f_1 + f_2) - f_1 f_2(f_3 + f_4). \tag{1.10}$$

Note that $\tilde{q}(f)$ contains the largest terms of $q(f)$ for large values of f. We therefore consider the initial data f_0, which also behave in this way at the origin.

1.1. Physical motivation

Let us define

$$\rho_0 := \int_0^\infty \frac{\sqrt{k}\,\mathrm{d}k}{\mathrm{e}^k - 1} = \frac{\sqrt{\pi}}{2}\zeta\left(\frac{3}{2}\right), \tag{1.11}$$

where ζ is the classical Riemann zeta function.

The UU equation describes a dilute gas of Bose particles. It has a one-parameter family of steady states $\mathcal{B}_\rho$ characterized by their total density $\rho > 0$ as follows:

(i) if $0 < \rho \leqslant \rho_0$, then

$$\mathcal{B}_\rho(k) \equiv F_\mu(k) := \frac{1}{\mathrm{e}^{k+\mu} - 1}, \quad \text{where } \rho = \int_0^\infty \frac{\sqrt{k}\,\mathrm{d}k}{\mathrm{e}^{\mu+k} - 1},\ \mu \geqslant 0; \tag{1.12}$$

(ii) if $\rho > \rho_0$, then

$$\mathcal{B}_\rho(k) \equiv \frac{1}{\mathrm{e}^k - 1} + (\rho - \rho_0)\frac{\delta_0}{\sqrt{k}}. \tag{1.13}$$

Note that in both cases $\int_0^\infty \mathcal{B}_\rho(k)\sqrt{k}\,\mathrm{d}k = \rho$. The solutions $\mathcal{B}_\rho(k)$ in (1.12) are the classical Bose–Einstein equilibrium distributions if $\mu > 0$ and the Planck distribution if $\mu = 0$. On the other hand, the solutions (1.13) are the classical distributions that describe the thermal equilibrium of a family of bosons with the Bose–Einstein condensate of particles having zero momentum.

In this paper we construct solutions of (1.1)–(1.7) that behave like $k^{-7/6}$ near the origin. The physical meaning of such asymptotics is that these particle distributions have a non-zero flux of particles towards the origin (cf. [3, 7, 8]). More precisely, the asymptotics

$$f(t, k) \sim a(t)k^{-7/6} \quad \text{as } k \to 0 \tag{1.14}$$

means that the rate gain of particles towards the particles with zero momentum is

$$\lim_{K \to 0} \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{|k_1| \leqslant K} \sqrt{k_1} f(k_1, t)\,\mathrm{d}k_1\right) = -\frac{(a(t))^3}{3}U'\left(\frac{7}{6}\right), \tag{1.15}$$

where

$$U(\nu) := \int_{D(1)} a(\xi_2, \xi_3, \xi_4)\, \mathrm{d}\xi_3\, \mathrm{d}\xi_4$$

and

$$a(\xi_2, \xi_3, \xi_4) := [W(\xi_1, \xi_2, \xi_3, \xi_4) q(\xi^{-\nu})]|_{\xi_1 = 1}.$$

There are several different ways of deriving (1.15). One possibility is to make a careful count of the number of particles leaving the region $\{k : |k| \leqslant \delta\}$ towards $\{k : |k| > \delta\}$, as well as the particles entering $\{k : |k| \leqslant \delta\}$ from $\{k : |k| > \delta\}$, under assumption (1.14). An alternative way, analogous to the method used in [2], is to approximate the singular behaviour $k^{-7/6}$ by the less singular behaviour $k^{-7/6+\delta}$, $\delta > 0$, and compute the rate of change in the number of particles. After deriving some asymptotics for the arising integrals we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\biggl(\int_{|k_1| \leqslant k} \sqrt{k_1} f(k_1, t)\, \mathrm{d}k_1\biggr) = -\frac{(a(t))^3}{3} U'\biggl(\frac{7}{6}\biggr) + \mathcal{O}(k^{1/10}) \quad \text{as } k \to 0,$$

where the last term is uniform on δ for $0 < \delta = \frac{7}{6} - \nu$ sufficiently small. Taking the limit $\delta \to 0$, the result follows [2].

The presence of a non-zero flux of particles towards the particles of zero momentum makes it tempting to think that the solutions constructed in this paper could provide some information about the dynamic growth of Bose–Einstein condensates. However, this does not seem to be the case, since the zero-momentum particles would not interact at all with the particles outside the condensate. Actually, a more careful analysis yields more complicated models (cf. [1, 8, 12, 13]) in which the condensate interacts with the particles that are not in the condensate. Some of the models proposed in these papers will be studied more carefully elsewhere.

There exist other kinetic equations describing fluxes of some physical quantity in some mathematical space (momentum, energy, etc.). One of the most typical examples is the case of gelation in coagulation processes described by means of the Smoluchovski equation [9]. The solutions obtained in the current paper have several analogies with the explicit examples that describe gelation in such processes. Other physically relevant cases arise in the theory of weak turbulence, which can be applied to describe the distribution of energy in fields of gravity waves, capillary waves, Langmuir waves in plasmas, acoustic waves, etc. A detailed description of these examples can be found in [15]. A particularly simple example of solutions behaving like those found here has been constructed for the Kompaneets equation, which describes the energy of photons in plasma physics [5].

In all these cases, there exists a stationary solution to the corresponding kinetic equation of the form $f(k) = k^{-\beta}$, which plays a role analogous to the distribution $k^{-7/6}$ in our case. Physically, such solutions describe a flux of some physical quantity (particles, energy, etc.) from high to low values of the quantity or vice versa, as in the classical Kolmogorov theory of turbulence.

We are not aware of any situation where the solutions constructed in this paper could have any clear physical meaning. However, we think that the mathematical methods employed in their construction can be used to treat some of the physical examples mentioned above.

1.2. Mathematical motivation

From the mathematical point of view, this paper is the continuation of the previous work, [6]. In that paper we studied the linear problem that results linearizing the leading term in the collision integral $\tilde{Q}$ defined in (1.9) and (1.10). The paper [6] contains a detailed description of the fundamental solution associated with such a linear problem. Here we construct singular solutions which behave like $k^{-7/6}$ near the origin, estimating carefully the nonlinear parts in equation (2.3) in suitable functional spaces.

The solutions constructed in this paper are, as far as we are aware, the first example of singular solutions of a nonlinear kinetic equation with precise singular behaviour for general initial data that has been rigorously obtained. Indeed, the solutions that we obtain have the precise asymptotic behaviour $f \sim a(t)k^{-7/6}$ as $k \to 0$. There is of course a large literature devoted to the study of bounded solutions of Boltzmann-type kinetic equations. On the other hand, Lu has recently proved the global existence of weak solutions for the Uehling–Uhlenbeck equation [10,11]. Moreover, these papers also describe the long time asymptotics towards the stationary solutions as $t \to \infty$.

One of the mathematical consequences of our analysis that seems noteworthy is the presence of some kind of regularizing effects for the problem (1.1), (1.2). At first glance this could seem surprising, because the structure of this equation suggests a 'hyperbolic' non-regularizing behaviour for its solutions. These regularizing effects are, however, restricted to the values of f at the particular point $k = 0$. Some typical examples of the kind of 'smoothing effects' associated with this equation are theorem 3.2 and lemma 3.21 in § 3.5, below. The estimates for $\partial a(t)/\partial t$ when (1.14) holds bear more resemblance to a typical estimate for parabolic than for hyperbolic equations. Actually, a large number of the methods used in the proofs of our results are very similar to the standard semigroup arguments for parabolic equations. On the other hand, (3.27) indicates that such regularizing effects do not take place away from the origin. Indeed, the presence of the Dirac mass term shows that the smoothness of the initial data does not increase if $k \neq 0$.

Finally, let us note that it is most likely that the solutions obtained in this paper cannot be extended globally in time. Indeed, the numerical calculations in [7,8,12,13] suggest that the regular solutions of the UU equation might blow up in finite time and it would not be surprising to find the same type of behaviour for the singular solutions derived in this paper.

2. Outline of the paper

Our goal is to obtain an existence and uniqueness theory for singular solutions of the equation

$$\frac{\partial f}{\partial t}(t,k) = Q(f)(t,k), \tag{2.1}$$

$$f(0,k) = f_0(k), \tag{2.2}$$

where $Q(f)$ is defined as in (1.1)–(1.7). The initial data $f_0 \geqslant 0$ are assumed to satisfy the following conditions:

$$|f_0(k) - Ak^{-7/6}| \leqslant \frac{B}{k^{7/6-\delta}}, \quad 0 \leqslant k \leqslant 1, \tag{2.3}$$

$$|f_0'(k) + \tfrac{7}{6}Ak^{-13/6}| \leqslant \frac{B}{k^{13/6-\delta}}, \quad 0 \leqslant k \leqslant 1, \tag{2.4}$$

$$f_0(k) \leqslant B\frac{\mathrm{e}^{-Dk}}{k^{7/6}}, \quad k \geqslant 1, \tag{2.5}$$

for some positive constants A, B, D and δ. The key assumption on $f_0(k)$ is that it behaves like the stationary solution $k^{-7/6}$ near the origin.

The main result that we prove in this paper is the following.

THEOREM 2.1. *For any f_0 satisfying (2.3)–(2.5), there exists a unique solution $f \in \boldsymbol{C}^{1,0}((0,T) \times (0,\infty))$ of (2.1), (2.2) as well as a function $a(t)$, satisfying*

$$0 \leqslant f(t,k) \leqslant L\frac{\mathrm{e}^{-Dk}}{k^{7/6}}, \quad k > 0, \quad t \in (0,T), \tag{2.6}$$

$$|f(t,k) - a(t)k^{-7/6}| \leqslant Lk^{-7/6+\delta/2}, \qquad k \leqslant 1, \quad t \in (0,T), \tag{2.7}$$

$$|a(t)| \leqslant L, \qquad t \in (0,T), \tag{2.8}$$

for some positive constant L and for some $T = T(A,B,\delta) > 0$.

REMARK 2.2. The space of functions $\boldsymbol{C}^{1,0}((0,T) \times (0,\infty))$ is the set of functions that are continuously differentiable with respect to the first variable in $(0,\infty)$ and continuous with respect to the second variable on $(0,\infty)$.

In order to construct the desired solution, we will argue as follows. It is convenient to consider first the problem (2.1), (2.2), replacing the kernel $W(k_1,k_2,k_3,k_4)$ by the truncated kernel

$$W_{M,M'}(k_1,k_2,k_3,k_4) = W(k_1,k_2,k_3,k_4)\chi\biggl(\frac{|k_3-k_4|}{M}\biggr)\chi\biggl(\frac{|k_1|}{M'}\biggr), \tag{2.9}$$

where M and M' are large positive constants, $\chi(z) = 1$ if $0 \leqslant z \leqslant 1$, $\chi(z) = 0$ if $z > 1$. Similar cut-offs are often used in the study of other kinetic equations (see [4]). The reason for this cut-off in our case is to control the 'Boltzmann-like' quadratic terms in f in (1.4), that otherwise would yield divergences in some of the terms arising later. Using this truncation, the problem (2.1), (2.2) becomes the truncated problem:

$$\frac{\partial f}{\partial t}(t,k) = Q_{M,M'}(f)(t,k), \tag{2.10}$$

$$f(0,k) = f_0(k), \tag{2.11}$$

where

$$Q_{MM'}(f)(k_1) = \int_{D(k_1)} W_{M,M'}(k_1,k_2,k_3,k_4)q(f)\,\mathrm{d}k_3\,\mathrm{d}k_4. \tag{2.12}$$

Note that f also depends on M and M' but, for the sake of simplicity, we will not write this dependence explicitly.

As the next step, we will obtain solutions of (2.10)–(2.12) in the form

$$f(k,t) = \lambda(t) f_0(k) + g(k,t), \tag{2.13}$$

where $\lambda(t)$ will be chosen uniquely by means of the condition

$$\lim_{k\to 0} k^{7/6} g(t,k) = 0 \quad \text{for all } t>0, \tag{2.14}$$

which means that g is less singular near the origin than $k^{-7/6}$. Moreover, we will assume that $\lambda(0)=1$, whence (cf. (2.3)) we obtain

$$g(0,k) = 0, \quad k \geqslant 0. \tag{2.15}$$

We introduce the notation

$$q(f_0+g) = q(f_0) + \ell(f_0,g) + n(f_0,g), \tag{2.16}$$

where $\ell(f_0,g)$ is a linear function on g and $n(f_0,g)$ contains the quadratic and higher-order terms on g. The equation (2.10) might then be written as

$$\frac{\partial g}{\partial t}(t,k_1) = \mathcal{L}_k(\lambda(t)f_0,g)(k_1,t) + \mathcal{R}_1(t,k_1) + \mathcal{R}_2(t,k_1,g) - \lambda'(t) f_0, \tag{2.17}$$

where, for $t>0$, $k_1>0$,

$$\mathcal{L}_k(\lambda(t)f_0,g)(k_1,t) = \int_{D(k_1)} W_{M,M'}(k_1,k_2,k_3,k_4)\ell(\lambda(t)f_0,g)\,\mathrm{d}k_3\,\mathrm{d}k_4, \tag{2.18}$$

$$\mathcal{R}_1(t,k_1) = \int_{D(k_1)} W_{M,M'}(k_1,k_2,k_3,k_4) q(\lambda(t)f_0)\,\mathrm{d}k_3\,\mathrm{d}k_4, \tag{2.19}$$

$$\mathcal{R}_2(t,k_1,g) = \int_{D(k_1)} W_{M,M'}(k_1,k_2,k_3,k_4) n(\lambda(t)f_0,g)\,\mathrm{d}k_3\,\mathrm{d}k_4. \tag{2.20}$$

It may be convenient to reformulate the problem (2.10)–(2.12) using the new time variable

$$\tau = \int_0^t \lambda^2(s)\,\mathrm{d}s. \tag{2.21}$$

Then, the problem (2.10)–(2.12) becomes

$$\begin{aligned}\frac{\partial g}{\partial \tau}(\tau,k_1) &= \mathcal{L}_{k,2}(f_0,g)(k_1,\tau) + \frac{1}{\lambda(\tau)}\mathcal{L}_{k,1}(f_0,g)(k_1,\tau)\\ &\quad + \frac{1}{\lambda^2(\tau)}(\mathcal{R}_1(\tau,k_1) + \mathcal{R}_2(\tau,k_1,g)) - \lambda_\tau f_0(k_1), \qquad (2.22)\\ g(0,k_1) &= 0. \qquad (2.23)\end{aligned}$$

where, with some abuse of notation, we still set $g(\tau,k_1)\equiv g(t,k_1)$, $\lambda(\tau)=\lambda(t)$, $\lambda_\tau = \lambda'(t)/\lambda^2(t)$, and $\mathcal{L}_{k,2}(f_0,\tilde{g}_1)$ is quadratic with respect to f_0 and $\mathcal{L}_{k,1}(f_0,\tilde{g}_1)(k_1,t)$ is linear with respect to f_0. Note that, as long as $0<c_1\leqslant\lambda(\tau)\leqslant c_2$, the two equations (2.22) and (2.17) are equivalent or, more precisely, a solution of (2.17)

with the regularity given in theorem 2.1 exists if and only if there exists a solution of (2.22) with the same regularity.

Our strategy in order to solve the problem (2.14), (2.15), (2.17) is the following. It turns out that the most relevant terms to describe the asymptotics of $g(k,t)$ as $k \to 0$ are $\partial g/\partial\tau$ and $\mathcal{L}_k(\lambda(\tau)f_0, g)$. If only these terms are kept in the equation, we obtain a linear problem that can be analysed using the results of [6]. This is made in § 3. The reason that the term $\mathcal{R}_1$ is less relevant than the linear terms in (2.16) is that f_0 behaves like the stationary solution $k^{-7/6}$ near the origin and this yields a cancellation in the integral term in (2.19), and as a consequence this term is smaller than $\mathcal{L}_k(\lambda(t)f_0, g)$ as $k \to 0$. On the other hand, the term $\mathcal{R}_2$ contains only quadratic terms in g and, due to (2.14), its contribution is also smaller than that due to the linear terms.

The solution of (2.15), (2.17) can be written using the results for the linear semigroups in § 3 by means of the variation-of-constants formula. In particular, such formula can be used to compute the limit $\lim_{k\to 0} k^{7/6} g(t,k)$. Then, the condition (2.14) becomes an integrodifferential equation for λ that is solved under suitable regularity assumptions on the initial data f_0 (cf. § 4).

Moreover, we obtain uniform estimates on λ and g for M and M' sufficiently large (cf. § 5). Using these estimates, it is not difficult to take the limit as M and M' tend to infinity in order to obtain a solution to (2.1), (2.2). Similar arguments also provide the uniqueness in the class of functions under consideration.

3. On the linearized equation

3.1. Functional framework and main results

In this section we study the solutions of the Cauchy problem

$$\frac{\partial h}{\partial \tau} = \mathcal{L}_{k,2}(f_0, h)(k_1, \tau) + \frac{1}{\lambda(\tau)}\mathcal{L}_{k,1}(f_0, h)(k_1, \tau) + \nu(k_1, \tau), \tag{3.1}$$

$$h(0,k) = h_0(k), \tag{3.2}$$

for some given function ν. To this end we rewrite (3.1) in a more convenient manner. We define the functions

$$\tilde{q}(f) = f_3 f_4(f_1 + f_2) - f_1 f_2(f_3 + f_4), \tag{3.3}$$

$$r(f) = f_3 f_4 - f_1 f_2, \tag{3.4}$$

as well as

$$\tilde{q}(f_0 + g) = \tilde{q}(f_0) + \tilde{\ell}(f_0, g) + \tilde{n}(f_0, g), \tag{3.5}$$

$$r(f_0 + g) = r(f_0) + s(f_0, g) + r(g), \tag{3.6}$$

where $\tilde{\ell}$ and s contain only linear terms on g. Note that, since $q(f) = \tilde{q}(f) + r(f)$, we have

$$\ell(f_0, g) = \tilde{\ell}(f_0, g) + s(f_0, g).$$

For further reference, it is convenient to define the operator,

$$\tilde{\mathcal{L}}_k(k^{-7/6}, g)(k_1, t) = \int_{D(k_1)} W(k_1, k_2, k_3, k_4)\tilde{\ell}(k^{-7/6}, g)\,\mathrm{d}k_3\,\mathrm{d}k_4. \tag{3.7}$$

A detailed (and complicated) expression of $\tilde{\ell}(k^{-7/6}, g)$ can be found in [6, equation (2.2) for $q_l(F)$], but we do not use that expression here.

We now introduce some suitable functional spaces,

$$\boldsymbol{X}_{p,q,r}(T) = \{\varphi \in \mathbb{C}([0,T]),\ L^{\infty}_{\mathrm{loc}}(\mathbb{R}^+) \cap \mathbb{C}(\mathbb{R}^+);\ t^{1-r}\|\varphi\|_{p,q} < \infty\}, \tag{3.8}$$

endowed with the norm

$$|||\varphi|||_{p,q,r} = \sup_{0\leqslant t\leqslant T} t^{1-r}\|\varphi\|_{p,q}, \tag{3.9}$$

$$\|\varphi\|_{p,q} = \sup_{0\leqslant k\leqslant 1}\{k^p|\varphi(k)|\} + \sup_{k\geqslant 1}\{k^q|\varphi(k)|\}. \tag{3.10}$$

where p, q and r are three arbitrary real numbers. Since we will use these spaces repeatedly with $r = 1$, we write them, for convenience, using the particular notation

$$\boldsymbol{Y}_{p,q}(T) := \boldsymbol{X}_{p,q,1}(T) = \{\varphi \in \mathbb{C}([0,T]),\ L^{\infty}_{\mathrm{loc}}(\mathbb{R}^+) \cap \mathbb{C}(\mathbb{R}^+);\ |||\varphi|||_{p,q} < \infty\}, \tag{3.11}$$

where

$$|||\varphi|||_{p,q} := |||\varphi|||_{p,q,1} = \sup_{0\leqslant \tau\leqslant T} \|\varphi(\tau,\cdot)\|_{p,q}.$$

Using the homogeneity of $\tilde{\ell}$ we can rewrite (3.1) as

$$h_\tau = \tilde{\mathcal{L}}_k(k^{-7/6}, h)(k_1,\tau) + \mathcal{U}(k;\lambda,h) + \nu(k,\tau), \tag{3.12}$$

where

$$\mathcal{U}(k_1;\lambda,h) = \mathcal{U}_1(k_1;\lambda,h) + \mathcal{U}_2(k_1;\lambda,h) + \mathcal{U}_3(k_1;\lambda,h),$$

$$\mathcal{U}_1(k_1;\lambda,h) = \int_{D(k_1)} W_{M,M'}(\tilde{\ell}(f_0,h) - \tilde{\ell}(k^{-7/6},h))\,\mathrm{d}k_3\,\mathrm{d}k_4, \tag{3.13}$$

$$\mathcal{U}_2(k_1;\lambda,h) = \lambda(\tau)^{-1}\int_{D(k_1)} W_{M,M'}s(f_0,h)\,\mathrm{d}k_3\,\mathrm{d}k_4, \tag{3.14}$$

$$\mathcal{U}_3(k;\lambda,h) = \int_{D(k_1)} (W_{M,M'} - W)\tilde{\ell}(k^{-7/6},h)\,\mathrm{d}k_3\,\mathrm{d}k_4. \tag{3.15}$$

We will say that a function h solves equation (3.12) with initial data $h(0,k) = h_0(k)$ in the integral sense if the integral equality

$$h(\tau,k) = \int_0^\infty G(\tau,k,k_0)h_0(k_0)\,\mathrm{d}k_0 + \int_0^\tau \mathrm{d}s\int_0^\infty \mathrm{d}k_0\, G(\tau-s,k,k_0)[\mathcal{U}(k,\lambda(s),h(s)) + \nu(k,s)] \tag{3.16}$$

holds, where $G(\tau,k,k_0)$ is the Green function associated with the Cauchy problem,

$$\frac{\partial h}{\partial \tau} = \tilde{\mathcal{L}}_k(k^{-7/6},h), \tag{3.17}$$

$$h(0,k) = \delta(k-k_0), \tag{3.18}$$

which was obtained in [6]; detailed properties of this function are recalled in theorem 3.5, below. The main results proved in this section are the following.

THEOREM 3.1. *Suppose that the function $\lambda(\tau)$ satisfies*

$$\lambda(0) = 1 \quad \text{and} \quad \tfrac{1}{2} \leqslant \lambda(\tau) \leqslant 2 \text{ for all } \tau \in [0,1]. \tag{3.19}$$

Let us also assume that $\|h_0\|_{7/6,\beta} < \infty$ and that $\nu \in \boldsymbol{Y}_{\alpha,\beta}(T')$ for some $T' > 0$, where $\alpha = \frac{3}{2} - \delta$ and $\beta = \frac{11}{6} - \delta$ with $\delta > 0$ sufficiently small.

Then, for any $M > 1$ and $M' > 1$ there exists $T > 0$ and a unique solution h of (3.1), (3.2) in the integral sense in the space $\boldsymbol{Y}_{7/6,\beta}(T)$. Moreover,

$$|||h|||_{7/6,\beta} \leqslant C_{M,M'}(\|h_0\|_{7/6,\beta} + T^{3\delta}|||\nu|||_{\alpha,\beta}). \tag{3.20}$$

On the other hand, there exists a function $a \in L^\infty([0,T])$ such that

$$\|h - a(\tau)k_1^{-7/6}\chi_{\{0\leqslant k_1\leqslant 1\}}\|_{7/6-\delta/2,\beta} \leqslant C_{M,M'}(\tau^{-3\delta/2}\|h_0\|_{7/6,\beta} + \tau^{3\delta/2}|||\nu|||_{\alpha,\beta}), \tag{3.21}$$

$$|a(\tau)| \leqslant C_{M,M'}(\|h_0\|_{7/6,\beta} + \tau^{3\delta}|||\nu|||_{\alpha,\beta}). \tag{3.22}$$

THEOREM 3.2. *Suppose that (3.19) holds. Suppose that $\|h_0\|_{\alpha,\beta} < \infty$ and that $|||\nu|||_{\alpha,\beta,\gamma} < \infty$, where $\alpha = \frac{3}{2} - \delta$ and $\beta = \frac{11}{6} - \delta$ with $\delta > 0$, $\gamma > 0$ sufficiently small.*

Then, for any $M > 1$ and $M' > 1$, there exists $T > 0$ sufficiently small and a unique solution h of (3.1), (3.2) in the integral sense for $0 < \tau < T$ such that

$$\|h(\tau,\cdot)\|_{7/6,\beta} \leqslant \frac{C}{\tau^{1-3\delta}}\|h_0\|_{\alpha,\beta} + C_{M,M'}T^{\gamma}\frac{|||\nu|||_{\alpha,\beta,\gamma}}{\tau^{1-3\delta}}.$$

On the other hand, there exists a function $a(\tau)$ such that

$$\begin{aligned}\|h - a(\tau)k_1^{-7/6}&\chi_{\{0\leqslant k_1\leqslant 1\}}\|_{7/6-\delta/2,\beta}\\ &\leqslant C_{M,M'}(\tau^{-1+9\delta/2}\|h_0\|_{\alpha,\beta} + |||\nu|||_{\alpha,\beta,\gamma}\tau^{-1+\gamma+3\delta/2}),\end{aligned} \tag{3.23}$$

$$|a(\tau)| \leqslant C_{M,M'}(\tau^{-1+6\delta}\|h_0\|_{\alpha,\beta} + |||\nu|||_{\alpha,\beta,\gamma}\tau^{-1+\gamma+3\delta}). \tag{3.24}$$

REMARK 3.3. The main difference between both theorems is that theorem 3.1 requires stronger boundedness assumptions on the initial data h_0 as $k \to 0$.

REMARK 3.4. The existence time T in the theorems above could depend, in principle, on M and M'. It will be shown in § 5 that it is possible to derive uniform lower estimates for T if M and M' are large enough.

The key ingredient in the proof of theorem 3.1 is the description of the solution of the linear problem (3.17), (3.18) that we recall here for the reader's convenience.

THEOREM 3.5 (Escobedo *et al.* [6]). *For each $k_0 > 0$ there exists a unique solution of (3.17), (3.18) in the class of measures of the form*

$$G(\tau,k,k_0) = \alpha(\tau)\delta(k-k_0) + H(\tau,k,k_0),$$

where

$$\left.\begin{aligned} &H(\tau,\,\cdot,k_0)\in L^\infty_{\mathrm{loc}}(\mathbb{R}^+), && \\ &|H(\tau,k,k_0)|\leqslant \frac{C}{k^{7/6}}, && k\leqslant \tfrac12 k_0, \\ &|H(\tau,k,k_0)|\leqslant \frac{C}{k^{11/6}}, && k\geqslant 2k_0, \\ &|H(\tau,k,k_0)|\leqslant \frac{C}{|k-k_0|^{5/6}}, && |k-k_0|\leqslant \tfrac12 k_0. \end{aligned}\right\} \tag{3.25}$$

Moreover, $G(\tau,k,k_0)$ has the self-similar form

$$G(\tau,k,k_0)=\frac{1}{k_0}G\biggl(\frac{\tau}{k_0^{1/3}},\frac{k}{k_0},1\biggr) \tag{3.26}$$

and the function $G(\tau,k,1)$ satisfies the following estimates. For $k\in(0,2)$ we have

$$G(\tau,k,1)=\mathrm{e}^{-a\tau}\delta(k-1)+\sigma(\tau)k^{-7/6}+\mathcal{R}_1(\tau,k)+\mathcal{R}_2(\tau,k), \tag{3.27}$$

where $\sigma\in \boldsymbol{C}[0,\infty)$ satisfies

$$\sigma(\tau)=\begin{cases} A\tau^4+\mathcal{O}(\tau^{4+\varepsilon}) & \text{as } \tau\to 0^+, \\ \mathcal{O}(\tau^{-(3v_0-5/2)}) & \text{as } \tau\to\infty, \end{cases} \tag{3.28}$$

where $\mathcal{R}_1$, $\mathcal{R}_2$ can be estimated as

$$\begin{aligned} &\mathcal{R}_1(\tau,k)\equiv 0 \quad \text{for } |k-1|\geqslant \tfrac12, \\ &|\mathcal{R}_1(\tau,k)|\leqslant C\frac{\mathrm{e}^{-(a-\varepsilon)\tau}}{|k-1|^{5/6}} \quad \text{for } |k-1|\leqslant \tfrac12, \end{aligned} \tag{3.29}$$

$$\mathcal{R}_2(\tau,k)\leqslant C\psi_1(\tau)\biggl(\frac{\tau^3}{k}\biggr)^{\tilde b} \tag{3.30}$$

with

$$\psi_1(\tau)=\begin{cases} \dfrac{1}{\tau^{(5/2)+\varepsilon}} & \text{for } 0\leqslant\tau\leqslant 1, \\ \dfrac{1}{\tau^{3v_0-\varepsilon}} & \text{for } \tau>1. \end{cases} \tag{3.31}$$

On the other hand, for $k>2$ we have

$$G(\tau,k,1)\leqslant C\psi_2(\tau)\biggl(\frac{\tau^3}{k}\biggr)^{11/6}, \tag{3.32}$$

$$\psi_2(\tau)=\begin{cases} \dfrac{1}{\tau^{(9/2)+\varepsilon}} & \text{for } 0\leqslant\tau\leqslant 1, \\ \dfrac{1}{\tau^{1+3v_0-\varepsilon}} & \text{for } \tau>1. \end{cases} \tag{3.33}$$

In these formulae, $A\in\mathbb{R}$, $\varepsilon>0$ is an arbitrarily small number, $\tilde b$ is an arbitrary number in the interval $(1,\frac76)$ and $v_0=1.84020\cdots$. The constant C depends on ε and $\tilde b$ but is independent of k_0 and τ.

REMARK 3.6. The constants $\tilde{b}$, v_0 and ε will have the same meaning throughout the rest of the paper.

REMARK 3.7. Note that, since the right-hand sides of (3.31) and (3.33) are monotonically decreasing, we can assume without loss of generality that the functions ψ_1 and ψ_2 are globally decreasing in τ; this assumption will be made from now on.

REMARK 3.8. Although not explicitly stated among the results in [6], the function $G(t,k,k_0)$ is differentiable with respect to t, for $k>0$, $k_0>0$ and $t>0$, as can be seen using the explicit representation formula of G obtained in [6, (4.17), (4.19) and (4.25)]. Moreover, the function $\tau|\partial G/\partial\tau|$ satisfies the estimates that are obtained by differentiating formally and multiplying the resulting formulae by τ.

3.2. Some estimates for the semigroup generated by $\tilde{\mathcal{L}}_k$

The two lemmas in this subsection provide some estimates for the semigroup generated by $\tilde{\mathcal{L}}_k$ with initial data bounded near the origin or at infinity by power laws.

LEMMA 3.9. *Suppose that φ is the solution to*

$$\frac{\partial\varphi}{\partial\tau} = \tilde{\mathcal{L}}_k(k^{-7/6},\varphi),$$
$$\varphi(0,k) = \varphi_0(k),$$

where

$$|\varphi_0(k)| \leqslant k^{-\alpha}\chi_{\{k\leqslant 1\}}, \tag{3.34}$$

with $\alpha \in [\frac{7}{6},\frac{3}{2})$. Then, there exists a function $a \in L^\infty([0,1])$ such that, for any $\tau \in [0,1]$,

$$|\varphi(\tau,k) - a(\tau)k^{-7/6}| \leqslant C\tau^{-3\alpha}\Phi(y) \quad \text{for } 0 \leqslant k \leqslant 2, \tag{3.35}$$

$$|a(\tau)| \leqslant C\tau^{7/2-3\alpha}, \tag{3.36}$$

where $y = k\tau^{-3}$ and

$$\Phi(y) = \min\{y^{-\tilde{b}}, y^{-7/6}\}. \tag{3.37}$$

On the other hand

$$|\varphi(\tau,k)| \leqslant Cy^{-11/6}\tau^{-9/2-\varepsilon} \quad \text{for } k > 2, \tag{3.38}$$

for any $\tau \in [0,1]$ and where ε is as in theorem 3.5.

Proof. We assume in the rest of the proof that $0 \leqslant \tau \leqslant 1$. Using the fundamental solution G described in theorem 3.5 as well as remark 3.8, we can write

$$\begin{aligned}\varphi(\tau,k) &= \int_0^1 \frac{1}{k_0} G\left(\frac{\tau}{k_0^{1/3}}, \frac{k}{k_0}, 1\right)\varphi_0(k_0)\,\mathrm{d}k_0,\\ &= \int_0^{\min(k/2,1)} \cdots\,\mathrm{d}k_0 + \int_{\min(k/2,1)}^1 \cdots\,\mathrm{d}k_0 \equiv I_1 + I_2.\end{aligned}$$

We first estimate I_1. Using (3.32) we have

$$|I_1| \leqslant C\left(\frac{\tau^3}{k}\right)^{11/6} \int_0^{\min(k/2,1)} \psi_2\left(\frac{\tau}{k_0^{1/3}}\right) k_0^{-(\alpha+1)}\,\mathrm{d}k_0$$
$$= C\left(\frac{\tau^3}{k}\right)^{11/6} \tau^{-3\alpha} \int_0^{\min(k/2,1)\tau^{-3}} \psi_2\left(\frac{1}{\xi^{1/3}}\right)\xi^{-(\alpha+1)}\,\mathrm{d}\xi.$$

Using the fact that ψ_2 is monotonically decreasing, we deduce that

$$|I_1| \leqslant C\left(\frac{\tau^3}{k}\right)^{11/6} \psi_2\left(\frac{\tau}{\min(k/2,1)^{1/3}}\right)\min(\tfrac{1}{2}k,1)^{-\alpha}. \tag{3.39}$$

Combining (3.39) and (3.33) we obtain

$$\left.\begin{aligned} &|I_1| \leqslant C\tau^{-3\alpha}\min\{y^{v_0-3/2-\alpha-\epsilon/3},\ y^{-\alpha-1/3+\varepsilon/3}\}, && 0<k\leqslant 2,\\ &|I_1| \leqslant C\tau^{-9/2-\epsilon}y^{-11/6}, && k\geqslant 2.\end{aligned}\right\} \tag{3.40}$$

We now estimate the term I_2. By definition, $I_2=0$ for $k>2$. On the other hand, using (3.27) we can rewrite I_2 for $0\leqslant k\leqslant 2$ as

$$\begin{aligned} I_2 &= a(\tau)k^{-7/6} + \varphi_0(k)\exp\left(-\frac{a\tau}{k^{1/3}}\right)\chi_{\{k\leqslant 1\}} + \int_0^{k/2}\sigma\left(\frac{\tau}{k_0^{1/3}}\right)\left(\frac{k_0}{k}\right)^{7/6}\varphi_0(k_0)\frac{\mathrm{d}k_0}{k_0}\\ &\quad + \int_{k/2}^1 \mathcal{R}_1\left(\frac{\tau}{k_0^{1/3}},\frac{k}{k_0}\right)\varphi_0(k_0)\frac{\mathrm{d}k_0}{k_0} + \int_{k/2}^1 \mathcal{R}_2\left(\frac{\tau}{k_0^{1/3}},\frac{k}{k_0}\right)\varphi_0(k_0)\frac{\mathrm{d}k_0}{k_0}\\ &\equiv a(\tau)k^{-7/6} + I_{2,1}+I_{2,2}+I_{2,3}+I_{2,4}, \end{aligned} \tag{3.41}$$

where

$$a(\tau) = \int_0^1 \sigma\left(\frac{\tau}{k_0^{1/3}}\right)k_0^{7/6}\varphi_0(k_0)\frac{\mathrm{d}k_0}{k_0}.$$

Therefore, using (3.34) and (3.28),

$$|a(\tau)| \leqslant \tau^{7/2-3\alpha}\int_0^{1/\tau^3}\sigma(\xi^{-1/3})\xi^{1/6-\alpha}\,\mathrm{d}\xi \leqslant C\tau^{7/2-3\alpha} \quad \text{for } 0\leqslant\tau\leqslant 1. \tag{3.42}$$

Again using (3.34), we can estimate the second term on the right-hand side of (3.41) as

$$|I_{2,1}| \leqslant \tau^{-3\alpha}y^{-\alpha}\exp(-ay^{-1/3}). \tag{3.43}$$

A similar argument yields

$$|I_{2,2}| \leqslant \tau^{-3\alpha}y^{-7/6}\int_0^{y/2}\sigma(\xi^{-1/3})\xi^{1/6-\alpha}\,\mathrm{d}\xi, \tag{3.44}$$

$$|I_{2,3}| \leqslant C\tau^{-3\alpha}\int_{y/2}^{3y/2}\frac{\exp(-(a-\varepsilon)/\xi^{1/3})}{|y-\xi|^{5/6}}\xi^{-1/6-\alpha}\,\mathrm{d}\xi, \tag{3.45}$$

$$|I_{2,4}| \leqslant C\tau^{-3\alpha}y^{-\tilde{b}}\int_{y/2}^{\infty}\psi_1\left(\frac{1}{\xi^{1/3}}\right)\frac{\mathrm{d}\xi}{\xi^{1+\alpha}}. \tag{3.46}$$

The right-hand sides of the formulae (3.43)–(3.46) have a self-similar structure of the form $\tau^{-3\alpha}\Theta(y)$, $y \equiv k/\tau^3$. Therefore, it only remains to estimate the different functions Θ for $y \to 0$ and $y \to \infty$. The corresponding functions Θ in (3.43) and (3.45) have an exponential decay as $y \to 0$. Using (3.28) and (3.31), it follows that the contributions of the functions Θ in (3.44) and (3.46) behave like $y^{v_0-5/6-\alpha}$ and $y^{-\tilde{b}}$, respectively, as $y \to 0$. Since $v_0 - \frac{5}{6} - \alpha > -\tilde{b}$, and $\tilde{b} > 1$ it follows that all the terms in (3.43)–(3.46) might be bounded as $C\tau^{-3\alpha}y^{-\tilde{b}}$ when $y \to 0$. On the other hand, the functions Θ might be estimated in an analogous manner for $y \to \infty$. In particular, the functions Θ in (3.43) and (3.45) are bounded like $Cy^{-\alpha}$ as $y \to \infty$. The corresponding function Θ in (3.44) and (3.46) are bounded by $Cy^{-7/6}$ and $y^{5/6-\tilde{b}+\varepsilon/3}y^{-\alpha}$, respectively, as $y \to \infty$. Since $\alpha \geqslant \frac{7}{6}$ and $\frac{5}{6} - \tilde{b} + \frac{1}{3}\varepsilon < 0$, all the terms in (3.43)–(3.46) are bounded as $Cy^{-7/6}$ as $y \to \infty$. Combining the estimates obtained for the different functions Θ for large and small values of y we obtain (3.35). Finally, (3.36) follows from (3.42) and (3.38) is a consequence of (3.40). □

LEMMA 3.10. *Suppose that φ solves*

$$\varphi_\tau = \tilde{\mathcal{L}}_k(k^{-7/6}, \varphi),$$
$$\varphi(0, k) = \varphi_0(k),$$

where

$$|\varphi_0(k)| \leqslant k^{-\beta}\chi_{\{k\geqslant 1\}}, \tag{3.47}$$

with $\beta = \frac{11}{6} - \delta$ and $\delta > 0$ small enough. Then, for $\tau \in [0, 1]$, the following inequalities hold

$$|\varphi(\tau, k) - \beta(\tau)k^{-7/6}| \leqslant Ck^{-\beta}\chi_{\{k\geqslant 1\}} + C\tau^{-5/2-\epsilon}y^{-\tilde{b}}, \quad 0 \leqslant k \leqslant 2, \tag{3.48}$$

where $y = k\tau^{-3}$ and

$$|\beta(\tau)| \leqslant C\tau^4. \tag{3.49}$$

Moreover,

$$|\varphi(\tau, k)| \leqslant Ck^{-\beta}, \quad k \geqslant 2. \tag{3.50}$$

Proof. Using the fundamental solution G described in theorem 3.5 as well as in remark 3.8, we can write

$$\begin{aligned}\varphi(\tau, k) &= \int_1^\infty \frac{1}{k_0} G\left(\frac{\tau}{k_0^{1/3}}, \frac{k}{k_0}, 1\right)\varphi_0(k_0)\,\mathrm{d}k_0 \\ &= \int_1^{\max(k/2,1)} \cdots\,\mathrm{d}k_0 + \int_{\max(k/2,1)}^\infty \cdots\,\mathrm{d}k_0 \equiv J_1 + J_2.\end{aligned}$$

We first estimate J_1. Using (3.26), (3.32) and (3.47), we obtain

$$\begin{aligned}|J_1| &\leqslant C\chi_{\{k\geqslant 2\}}y^{-11/6}\int_1^{k/2} \psi_2\left(\frac{\tau}{k_0^{1/3}}\right)\frac{\mathrm{d}k_0}{k_0^{1+\beta}} \\ &\leqslant C\tau^{-3\beta}\chi_{\{k\geqslant 2\}}y^{-11/6}\int_0^{y/2} \psi_2(\xi^{-1/3})\frac{\mathrm{d}\xi}{\xi^{1+\beta}}.\end{aligned} \tag{3.51}$$

On the other hand, $J_1 = 0$ for $k < 2$.

We now estimate J_2. Using (3.27), we can rewrite J_2 for $0 \leqslant k \leqslant 2$ as

$$\begin{aligned} J_2 - \beta(\tau) k^{-7/6} &= \varphi_0(k) \mathrm{e}^{-a\tau/k^{1/3}} \chi_{\{k \geqslant 1\}} \\ &\quad + \int_1^\infty \mathcal{R}_1\left(\frac{\tau}{k_0^{1/3}}, \frac{k}{k_0}\right) \varphi_0(k_0) \frac{\mathrm{d}k_0}{k_0} \\ &\quad + \int_1^\infty \mathcal{R}_2\left(\frac{\tau}{k_0^{1/3}}, \frac{k}{k_0}\right) \varphi_0(k_0) \frac{\mathrm{d}k_0}{k_0}, \\ &\equiv J_{2,1} + J_{2,2} + J_{2,3}, \end{aligned}$$

where

$$\beta(\tau) = \int_1^\infty \sigma\left(\frac{\tau}{k_0^{1/3}}\right) k_0^{7/6} \varphi_0(k_0) \frac{\mathrm{d}k_0}{k_0}.$$

Taking into account (3.28) and (3.47), we obtain

$$|\beta(\tau)| \leqslant C\tau^4. \tag{3.52}$$

On the other hand, again using (3.47) as well as (3.29) and (3.30), we obtain

$$\begin{aligned} |J_{2,1}| &\leqslant C\tau^{-3\beta} y^{-\beta} \exp(-ay^{-1/3}) \chi_{\{k \geqslant 1\}}, \\ |J_{2,2}| &\leqslant C \int_{2k/3}^{2k} \chi_{\{k_0 \geqslant 1\}} \frac{\exp(-(a-\varepsilon)\tau/k_0^{1/3})}{|k/k_0 - 1|^{5/6}} \frac{\mathrm{d}k_0}{k_0^{1+\beta}}, \end{aligned}$$

which vanish for k small enough. For $k \leqslant 2$, the term with $\mathcal{R}_2$ gives

$$|J_{2,3}| \leqslant C y^{-\tilde{b}} \int_1^\infty \psi_1\left(\frac{\tau}{k_0^{1/3}}\right) \frac{\mathrm{d}k_0}{k_0^{1+\beta}} \leqslant C\tau^{-5/2-\varepsilon} y^{-\tilde{b}}. \tag{3.53}$$

Combining (3.52) and (3.53) yields (3.48).

We now estimate J_2 for $k \geqslant 2$. To this end we rewrite J_2 as

$$\begin{aligned} J_2 = \varphi_0(k) \exp\left(-\frac{a\tau}{k^{1/3}}\right) &+ \int_{k/2}^\infty \sigma\left(\frac{\tau}{k_0^{1/3}}\right) \left(\frac{k_0}{k}\right)^{7/6} \varphi_0(k_0) \frac{\mathrm{d}k_0}{k_0} \\ &+ \int_{k/2}^\infty \left(\mathcal{R}_1\left(\frac{\tau}{k_0^{1/3}}, \frac{k}{k_0}\right) + \mathcal{R}_2\left(\frac{\tau}{k_0^{1/3}}, \frac{k}{k_0}\right)\right) \varphi_0(k_0) \frac{\mathrm{d}k_0}{k_0}. \end{aligned}$$

Using (3.28), (3.29) and (3.30) we deduce that

$$\begin{aligned} |J_2| \leqslant Ck^{-\beta} + Ck^{-7/6} &\int_{k/2}^\infty \sigma\left(\frac{\tau}{k_0^{1/3}}\right) k_0^{7/6-\beta-1} \,\mathrm{d}k_0 \\ &+ \left(\frac{\tau^3}{k}\right)^{\tilde{b}} \int_{k/2}^\infty \psi_1\left(\frac{\tau}{k_0^{1/3}}\right) k_0^{-\beta-1} \,\mathrm{d}k_0 + C \int_{2k/3}^{2k} \frac{k_0^{-\beta-1/6}}{|k-k_0|^{5/6}} \,\mathrm{d}k_0. \end{aligned}$$

Using a rescaling argument, the last integral term can be estimated as $k^{-\beta}$. Therefore,

$$|J_2| \leqslant C\tau^{-3\beta} \Theta(y), \quad y = k\tau^{-3}, \tag{3.54}$$

where

$$\Theta(y) := y^{-\beta} + y^{-7/6} \int_{y/2}^{\infty} \sigma(\xi^{-1/3}) \xi^{7/6-\beta-1} \, \mathrm{d}\xi + y^{-\tilde{b}} \int_{y/2}^{3y/2} \psi_1\left(\frac{1}{\xi^{1/3}}\right) \frac{\xi^{-\beta-1/6}}{|y-\xi|^{5/6}} \, \mathrm{d}\xi. \tag{3.55}$$

Using (3.28), (3.31) and (3.33), it follows that, for large values of y, the second term on the right-hand side of (3.55) can be bounded as $Cy^{-4/3-\beta}$, and the third one can be bounded as $Cy^{-\tilde{b}+5/6+\varepsilon/3} y^{-\beta}$. Therefore, $\Theta(y) \leqslant Cy^{-\beta}$ for $y > 1$. On the other hand, combining (3.33) and (3.51), it follows that $|J_1| \leqslant \tau^{-3\beta} y^{-11/6}$ for $y > 1$. Then using (3.54) as well as the fact that we have $y \geqslant 2$ for $k \geqslant 2$ and $0 \leqslant \tau \leqslant 1$, the estimate (3.50) follows. □

We now derive similar results for the non-homogeneous equation.

PROPOSITION 3.11. *Let us define*

$$\theta \equiv \sup_{0 \leqslant \tau \leqslant T} \Big(\sup_{0 \leqslant k \leqslant 1} \{k^{\alpha} |\mu(\tau, k)|\} + \sup_{k \geqslant 1} \{k^{\beta} |\mu(\tau, k)|\} \Big), \tag{3.56}$$

where $\alpha = \frac{3}{2} - \delta$, $\beta = \frac{11}{6} - \delta$ *with* $\delta > 0$ *arbitrarily small. Suppose that* $0 \leqslant T \leqslant 1$. *Then, there exists a function* $y \in L^{\infty}([0, T])$ *and a constant* $C > 0$ *independent of* θ *and* T *such that the solution in the integral sense of*

$$\frac{\partial h}{\partial \tau} = \tilde{\mathcal{L}}_k(k^{-7/6}, h) + \mu(\tau, k_1),$$
$$h(0, k_1) = 0.$$

satisfies

$$|h(\tau, k_1) - y(\tau) k_1^{-7/6}| \leqslant C\theta \tau^{3\delta/2} k_1^{-7/6+\delta/2} \quad \textit{for } 0 \leqslant k \leqslant 1, \tag{3.57}$$

$$|h(\tau, k_1)| \leqslant C\theta \tau k_1^{-\beta} \quad \textit{for } k > 1, \tag{3.58}$$

where

$$|y(\tau)| \leqslant C\theta \tau^{3\delta}, \tag{3.59}$$

for $0 \leqslant \tau \leqslant T$.

Proof. The idea is to use the estimates derived in lemma 3.9 with $\alpha = \frac{3}{2} - \delta$ and lemma 3.10. Combining (3.36) and (3.49) and the variation-of-constants formula, we obtain (3.59). On the other hand,

$$\begin{aligned} |h(\tau, k_1) - y(\tau) k_1^{-7/6}| &\leqslant C_M \int_0^{\tau} (\tau - s)^{-3\alpha} \Phi\left(\frac{k}{(\tau - s)^3}\right) \mathrm{d}s \\ &\quad + \frac{C}{k_1^{\tilde{b}}} \int_0^{\tau} (\tau - s)^{3\tilde{b}-5/2-\varepsilon} \, \mathrm{d}s + C\theta \tau k^{-\beta} \chi\{k \geqslant 1\} \\ &= Ck_1^{1/3-\alpha} \int_0^{\tau/k_1^{1/3}} u^{-3\alpha} \Phi(u^{-3}) \, \mathrm{d}u + Ck_1^{-\tilde{b}} \tau^{3\tilde{b}-3/2-\varepsilon} \\ &\quad + C\theta \tau k^{-\beta} \chi\{k \geqslant 1\}. \end{aligned} \tag{3.60}$$

Then, using (3.37), we deduce that

$$\int_0^{\tau/k_1^{1/3}} u^{-3\alpha}\Phi(u^{-3})\,\mathrm{d}u$$

is convergent as $k_1 \to 0$ and it behaves like $(k_1/\tau^3)^{-\delta}$ as $\tau^3/k_1 \to 0$. Therefore, (3.58) follows.

To obtain (3.57), we use (3.56) as well as the estimates (3.35), (3.50) in lemmas 3.9 and 3.10. $\square$

3.3. Estimates for the higher-order terms

For convenience, let us rewrite equation (3.12) in the form

$$h_\tau = \tilde{\mathcal{L}}_k(k^{-7/6}, h)(k_1, \tau) + \mathcal{U}(k; \lambda, h) + \nu(k_1, \tau). \tag{3.61}$$

In this subsection we obtain some technical estimates for the terms $\mathcal{U}$ that are linear on h but less singular near the origin than $\tilde{\mathcal{L}}_k(k^{-7/6}, h)(k_1, \tau)$. These estimates are written in terms of suitable functional norms of the function h itself. The results in this subsection will allow us to prove theorem 3.1 by means of a standard fixed-point argument.

We rewrite $\tilde{q}$ and r in (3.3), (3.4) as

$$\left.\begin{aligned} \tilde{q}(f) &= f_1\tilde{q}_1(f) + \tilde{q}_2(f), \\ r(f) &= r_1(f) - f_1 f_2, \end{aligned}\right\} \tag{3.62}$$

where

$$\tilde{q}_1(f) = f_3 f_4 - f_2 f_3 - f_2 f_4, \tag{3.63}$$
$$\tilde{q}_2(f) = f_2 f_3 f_4, \tag{3.64}$$
$$r_1(f) = f_3 f_4.$$

Note that the functions $\tilde{q}_1(f)$, $\tilde{q}_2(f)$, $r(f)$ do not depend on f_1. On the other hand, we introduce the linearizations of these functions by means of

$$\tilde{q}_i(f_0 + g) = \tilde{q}_i(f_0) + \tilde{\ell}_i(f_0, g) + \tilde{n}_i(f_0, g), \quad i = 1, 2, \tag{3.65}$$
$$r_1(f_0 + g) = r_1(f_0) + s_1(f_0, g) + r_1(g), \tag{3.66}$$

where $\tilde{\ell}_i$ and s_1 contain only linear terms on g. Combining (3.5), (3.6) and (3.65), (3.66), we obtain

$$\tilde{\ell}(f_0, g) = \tilde{q}_1(f_0)g_1 + [f_{0,1}\tilde{\ell}_1(f_0, g) + \tilde{\ell}_2(f_0, g)], \tag{3.67}$$
$$s(f_0, g) = -g_1 f_{0,2} + s_1(f_0, g) - f_{0,1} g_2 \tag{3.68}$$

($f_{0,i} \equiv f_0(k_i)$). Using (3.67) and (3.68), we can rewrite $\mathcal{U}_1$, $\mathcal{U}_2$ and $\mathcal{U}_3$ in (3.13), (3.14) as

$$\begin{aligned} \mathcal{U}_1 &= h_1\mathcal{U}_{1,1} + \mathcal{U}_{1,2}, \\ \mathcal{U}_2 &= h_1\mathcal{U}_{2,1} + \mathcal{U}_{2,2}, \\ \mathcal{U}_3 &= h_1\mathcal{U}_{3,1} + \mathcal{U}_{3,2}, \end{aligned}$$

where

$$\left.\begin{aligned}
\mathcal{U}_{1,1} &= \int_{D(k_1)} W_{M,M'}(k_1,k_2,k_3,k_4)(\tilde{q}_1(f_0)-\tilde{q}_1(k^{-7/6}))\,\mathrm{d}k_3\,\mathrm{d}k_4,\\
\mathcal{U}_{1,2} &= \int_{D(k_1)} W_{M,M'}(k_1,k_2,k_3,k_4)\\
&\qquad\times(f_{0,1}\tilde{\ell}_1(f_0,h)-k_1^{-7/6}\tilde{\ell}_1(k^{-7/6},h)+\tilde{\ell}_2(f_0,h)-\tilde{\ell}_2(k^{-7/6},h))\,\mathrm{d}k_3\,\mathrm{d}k_4,\\
\mathcal{U}_{2,1} &= -\lambda(\tau)^{-1}\int_{D(k_1)} W_{M,M'}(k_1,k_2,k_3,k_4)f_{0,2}\,\mathrm{d}k_3\,\mathrm{d}k_4,\\
\mathcal{U}_{2,2} &= \lambda(\tau)^{-1}\int_{D(k_1)} W_{M,M'}(k_1,k_2,k_3,k_4)(s_1(f_0,h)-f_{0,1}h_2)\,\mathrm{d}k_3\,\mathrm{d}k_4,\\
\mathcal{U}_{3,1} &= \int_{D(k_1)} (W_{M,M'}-W)(k_1,k_2,k_3,k_4)\tilde{q}_1(k^{-7/6})\,\mathrm{d}k_3\,\mathrm{d}k_4\\
\mathcal{U}_{3,2} &= \int_{D(k_1)} (W_{M,M'}-W)(k_1^{-7/6}\tilde{\ell}_1(k^{-7/6},h)+\tilde{\ell}_2(k^{-7/6},h))\,\mathrm{d}k_3\,\mathrm{d}k_4.
\end{aligned}\right\} \tag{3.69}$$

The dependence of the functions $\mathcal{U}_{i,j}$ on their arguments will not be written explicitly unless necessary. As a general rule, we will only note the dependent variables that are relevant to the argument.

LEMMA 3.12. *There exists a positive constant C, depending only on A, B, D, δ in (2.3)–(2.5) such that, for all (k_1,k_2,k_3,k_4) satisfying $k_2=k_3+k_4-k_1$, the following formula holds:*

$$|\tilde{q}_1(f_0)-\tilde{q}_1(Ak^{-7/6})| \leqslant C\left(\frac{k_3^\delta+k_4^\delta}{k_3^{7/6}k_4^{7/6}}+\frac{k_2^\delta+k_4^\delta}{k_2^{7/6}k_4^{7/6}}+\frac{k_2^\delta+k_3^\delta}{k_2^{7/6}k_3^{7/6}}\right).$$

Proof. Note that

$$|\tilde{q}_1(f_0)-\tilde{q}_1(Ak^{-7/6})| \leqslant \sum_{i,j=2;i<j}^{4} |f_{0,i}f_{0,j}-k_i^{-7/6}k_j^{-7/6}|$$

and, since

$$|f_{0,j}-Ak_j^{-7/6}||f_{0,i}| \leqslant Ck_j^{-7/6+\delta}k_i^{-7/6}, \qquad |f_{0,i}-Ak_i^{-7/6}||k_j^{-7/6}| \leqslant Ck_i^{-7/6+\delta}k_j^{-7/6},$$

the result follows. □

LEMMA 3.13. *There exists a positive constant C as in lemma 3.12 such that*

$$|\mathcal{U}_{1,1}| \leqslant \frac{C}{k_1^{1/3-\delta}}.$$

Proof. The result follows using lemma 3.12, rescaling the variables of integration as $k_3=k_1\xi_3$, $k_4=k_1\xi_4$ and using the expression for W. □

LEMMA 3.14. *There exists a positive constant C as in lemma 3.12 such that*

$$|f_{0,1}\tilde{\ell}_1(f_0,h) - Ak_1^{-7/6}\tilde{\ell}_1(Ak^{-7/6,},h)| \\ \leqslant C\bigg[k_1^{-7/6+\delta}\sum_{i,j=2;i\neq j}^{4} k_i^{-7/6}\zeta(k_j) + k_1^{-7/6}\sum_{i,j=2;i\neq j}^{4} k_i^{-7/6+\delta}\zeta(k_j)\bigg]|||h|||_{7/6,\beta}, \tag{3.70}$$

where $\zeta(k) = k^{-7/6}$ if $0 \leqslant k \leqslant 1$ and $\zeta(k) = k^{-11/6+\delta}$ if $k \geqslant 1$.

Proof. We write

$$f_{0,1}\tilde{\ell}_1(f_0,h) - Ak_1^{-7/6}\tilde{\ell}_1(Ak^{-7/6},h) \\ = (f_{0,1} - Ak_1^{-7/6})\tilde{\ell}_1(f_0,h) + k_1^{-7/6}(\tilde{\ell}_1(f_0,h) - \tilde{\ell}_1(Ak^{-7/6},h)).$$

The first term is estimated using

$$|(f_{0,1} - Ak_1^{-7/6})\tilde{\ell}_1(f_0,h)| \leqslant Ck_1^{-7/6+\delta}|||h|||_{7/6,\beta}\sum_{i,j=2;i\neq j}^{4} k_i^{-7/6}\zeta(k_j).$$

The second term is estimated as in lemma 3.12. □

LEMMA 3.15. *There exists a positive constant $C = C(A,B,D,\delta)$ such that*

$$|\tilde{\ell}_2(f_0,h) - \tilde{\ell}_2(Ak^{-7/6},h)| \leqslant C|||h|||_{7/6,\beta}\sum_{i,j,\ell=2;i\neq j,i\neq \ell,j\neq \ell}^{4} k_i^{-7/6}k_j^{-7/6}(k_i^{\delta}+k_j^{\delta})\zeta(k_\ell). \tag{3.71}$$

Proof. Formula (3.71) is a consequence of the definition of $\tilde{\ell}_2$ as well as of (2.3)–(2.5). □

LEMMA 3.16. *There exists a positive constant $C_M \equiv C(A,B,D,\delta,M)$, independent of M', such that the following estimates hold:*

$$|\mathcal{U}_{1,2}(h) - \mathcal{U}_{1,2}(\tilde{h})| \leqslant \frac{C_M}{k_1^{3/2-\delta}}|||h-\tilde{h}|||_{7/6,\beta} \quad \textit{for } 0 \leqslant k_1 \leqslant 1, \tag{3.72}$$

$$|\mathcal{U}_{1,2}(h) - \mathcal{U}_{1,2}(\tilde{h})| \leqslant \frac{C_M}{k_1^{17/6-\delta}}|||h-\tilde{h}|||_{7/6,\beta} \quad \textit{for } k_1 > 1. \tag{3.73}$$

Proof. Let us suppose by simplicity that $\tilde{h} \equiv 0$, since the argument in the general case is similar. Using (3.70), (3.71) in (3.69) we deduce

$$|\mathcal{U}_{1,2}| \leqslant C\int_{D(k_1)} W_{M,M'}\bigg\{\sum_{i,j=1;i\neq j}^{4}\sum_{\ell=2;\ell\neq i,\ell\neq j}^{4} k_i^{-7/6}k_j^{-7/6}(k_i^{\delta}+k_j^{\delta})\zeta(k_\ell)\bigg\}\,\mathrm{d}k_3\,\mathrm{d}k_4. \tag{3.74}$$

In order to obtain (3.72), we bound $\zeta(k_\ell)$ by $k_\ell^{-7/6}$ in (3.70) and (3.71). Using the rescaling $k_j = k_1\xi_j$ for $j = 2,3,4$ yields the result. For $k_1 > 1$, the largest contribution to the integral in (3.74) is due to the terms where $\ell = 2$. On the other hand, due to the cut-off in $W_{M,M'}$, k_3 and k_4 are of order k_1 for k_1 large. Because of this, the corresponding integral can be estimated by $k_1^{-1/2}k_1^{-7/3+\delta}$ and this yields (3.73). □

LEMMA 3.17. *For all $\varepsilon > 0$ arbitrarily small, there exists a positive constant C_M with the same dependencies as in lemma 3.16 and depending also on ε such that*

$$|\mathcal{U}_{2,1}| \leqslant \frac{C_M}{k_1^{1/2}} \quad \textit{for all } k_1 > 1.$$

Proof. Using the fact that $W \leqslant \sqrt{k_2}/\sqrt{k_1}$ and that the kernel $W_{M,M'}$ is not zero only if $|k_3 - k_4| \leqslant M$, the result follows. □

LEMMA 3.18. *There exists a positive constant C_M as in lemma 3.17 such that*

$$|\mathcal{U}_{2,2}(h)| \leqslant \frac{C_M}{k_1^{5/3}}|||h|||_{7/6,\beta} \qquad \textit{for } 0 \leqslant k_1 \leqslant 1 \tag{3.75}$$

and

$$|\mathcal{U}_{2,2}(h)| \leqslant \frac{C_M}{k_1^{\beta+1/2}}|||h|||_{7/6,\beta} \quad \textit{for } k_1 > 1. \tag{3.76}$$

Proof. When $0 \leqslant k_1 \leqslant 1$, the term due to $s_1(f_0, h)$ may be estimated as

$$|s_1(f_0, h)| \leqslant \sum_{i,j=2,i\neq j}^{4} k_i^{-7/6}k_j^{-7/6}.$$

The corresponding estimate follows using the rescaling $k_j = k_1\xi_j$, $j = 2, 3, 4$. Alternatively, the term in $\mathcal{U}_{2,2}$ containing $f_{0,1}h_2$ can be estimated, after integrating in k_3, k_4, as $C_M k_1^{-7/6}k_1^{-1/2}$ for $k_1 \leqslant 1$. In order to make this integration, it is convenient to change the integral variables from k_3, k_4 to $k_2, k_3 - k_4$. Then the function $W_{M,M'}$ can be bounded by 1 for $k_2 \geqslant k_1$ and by $\sqrt{k_2}/\sqrt{k_1}$ for $k_2 \leqslant k_1$, whence estimate (3.75) follows. On the other hand, in order to derive the estimate for $k_1 > 1$, we use the fact that, due to the cut-off, k_3 and k_4 are of order k_1. The contribution to $\mathcal{U}_{2,2}$ due to the term $f_{0,1}h_2$ may be estimated by $C_M\mathrm{e}^{-Dk_1}$ after integration in k_3, k_4. To estimate the remaining terms in $\mathcal{U}_{2,2}$ we use the fact that

$$|s_1(f_0, h)| \leqslant \sum_{i,j=3;i\neq j}^{4} f_{0,i}h_j.$$

For $k_1 > 1$, the largest contribution to $\mathcal{U}_{2,2}$ is due to the terms with $i = 2$. The resulting contribution can be bounded as $k_1^{-\beta-1/2}$, whence (3.76) follows. □

LEMMA 3.19. *There exists a positive constant C_M as in lemma 3.17 such that*

$$|\mathcal{U}_{3,1}(h)| \leqslant C_M \qquad \textit{for } 0 \leqslant k_1 \leqslant 1 \tag{3.77}$$

and

$$|\mathcal{U}_{3,1}(h)| \leqslant \frac{C_M}{k_1^{1/3}} \quad \textit{for } k_1 \geqslant 1.$$

Proof. The estimate (3.77) for $0 \leqslant k_1 \leqslant 1$ follows using the fact that, due to the cut-off, the domain of integration is contained in a fixed domain independent of k_1. For $k_1 \geqslant 1$, we estimate $|W_{M,M'} - W|$ by $2W$ and use in the resulting integral the rescaling $k_j = k_1\xi_j$, $j = 2, 3, 4$. □

LEMMA 3.20. *There exists a positive constant C_M as in lemma 3.17 such that*

$$|\mathcal{U}_{3,2}(h)| \leqslant \frac{C_M}{k_1^{7/6}} |||h|||_{7/6,\beta} \qquad \text{for } 0 \leqslant k_1 \leqslant 1$$

and

$$|\mathcal{U}_{3,2}(h)| \leqslant \frac{C_M}{k_1^{1/3+\beta}} |||h - \tilde{h}|||_{7/6,\beta} \quad \text{for } k_1 \geqslant 1.$$

Proof. The proof is essentially similar to that of the previous lemma. □

3.4. Proof of theorems 3.1 and 3.2

We can reformulate the original problem (3.1), (3.2) as a fixed-point problem. To this end we use the variation-of-constants formula in (3.2), (3.61) to obtain

$$\begin{aligned} h(\tau, k_1) = \int_0^\infty G(\tau, k_1, \xi) h_0(\xi) + \int_0^\tau \mathrm{d}s \int_0^\infty \mathrm{d}\xi G(\tau - s, k_1, \xi) \mathcal{U}(\xi; \lambda(s), h(s, \xi)) \\ + \int_0^\tau \mathrm{d}s \int_0^{\mathrm{d}} \xi G(\tau - s, k_1, \xi) \nu(s, \xi) \\ \equiv \mathcal{T}(h)(\tau, k_1), \end{aligned} \tag{3.78}$$

where $G(\tau, k_1, \xi)$ is the fundamental solution of the problem (3.17), (3.18) described in theorem 3.5.

Proof of theorem 3.1. The theorem will follow by proving that the operator $\mathcal{T}$ defined in (3.78) is contractive in the space $\boldsymbol{Y}_{7/6,\beta}(T)$ for $T > 0$ small enough.

To this end, note that, using lemma 3.13 as well as lemmas 3.16–3.20, we obtain

$$\begin{aligned} \left| \sum_{j=1}^{3} (h_1 \mathcal{U}_{j,1} + \mathcal{U}_{j,2})(\xi; \lambda(s), h(s, \xi)) + \nu(s, \xi) \right| \\ \leqslant C_M \xi^{-3/2+\delta} (|||h|||_{7/6,\beta} + |||\nu|||_{\alpha,\beta}) \quad \text{for } 0 \leqslant \xi \leqslant 1, \end{aligned} \tag{3.79}$$

$$\begin{aligned} \left| \sum_{j=1}^{3} (h_1 \mathcal{U}_{j,1} + \mathcal{U}_{j,2})(\xi; \lambda(s), h(s, \xi)) + \nu(s, \xi) \right| \\ \leqslant C_M \xi^{-\beta-1/3+\delta} (|||h|||_{7/6,\beta} + |||\nu|||_{\alpha,\beta}) \quad \text{for } \xi > 1. \end{aligned} \tag{3.80}$$

Combining these estimates with proposition 3.11, we obtain

$$|||\mathcal{T}(h - \tilde{h})|||_{7/6,\beta} \leqslant C_M T^{3\delta/2} |||h - \tilde{h}|||_{7/6,\beta},$$

where C_M is a positive constant as in lemma 3.16. The existence and uniqueness parts for small T in theorem 3.1 follow by means of a standard fixed-point argument.

On the other hand, combining (3.79) and (3.80) with proposition 3.11, we obtain

$$|||\mathcal{T}(h)|||_{7/6,\beta} \leqslant C_M(\|h_0\|_{7/6,\beta} + T^{3\delta/2}|||h|||_{7/6,\beta}) + T^{3\delta/2}|||\nu|||_{7/6,\beta}, \tag{3.81}$$

which yields the estimate (3.20).

The proof of (3.21), (3.22) follows from proposition 3.11, which yields an estimate for the contribution due to the term ν, as well as from lemma 3.9 with $\alpha = \frac{7}{6}$, which provides bounds for the contribution due to h_0. □

Proof of theorem 3.2. This is very similar to that of theorem 3.1, although we must use the functional space $\boldsymbol{X}_{7/6,\beta,3\delta}(T)$. We first rewrite the equation as

$$h_\tau = \mathcal{L}_k(\lambda(\tau)Ak^{-7/6}, h) + \mu(k_1,\tau) + \nu, \tag{3.82}$$

where

$$\mu(\tau,k_1) = \mathcal{L}_k(\lambda(\tau)f_0, h) - \mathcal{L}_k(\lambda(\tau)Ak^{-7/6}, h). \tag{3.83}$$

Then, arguing as in the proofs of formulae (3.79) and (3.80), we obtain

$$\|\mu(\tau,\,\cdot)\|_{3/2-\delta,\beta} \leqslant \frac{C}{\tau^{1-3\delta}}|||h|||_{7/6,\beta,3\delta}, \quad 0 \leqslant \tau \leqslant T.$$

We use now the usual fixed-point argument. Given h in $\boldsymbol{X}_{7/6,\beta,3\delta}(T)$, we define μ as in (3.83) and then solve (3.82) with $h(0,k_1) = h_0(k_1)$. This defines an operator $\mathcal{T}(h)$. Using the variation-of-constants formula as well as lemmas 3.9 and 3.10, we obtain

$$\begin{aligned}\|\mathcal{T}(h)(\tau,\,\cdot)\|_{7/6,\beta} &\leqslant C\|h_0\|_{7/6,\beta} + C\int_0^\tau \frac{\mathrm{d}s}{(\tau-s)^{1-3\delta}}\left\{\frac{|||h|||_{7/6,\beta,3\delta}}{s^{1-3\delta}} + \frac{|||\nu|||_{\alpha,\beta,\gamma}}{s^{1-\gamma}}\right\}\\ &\leqslant C\|h_0\|_{7/6,\beta} + CT^{3\delta}\frac{|||h|||_{7/6,\beta,3\delta}}{\tau^{1-3\delta}} + CT^{\gamma}\frac{|||\nu|||_{\alpha,\beta,\gamma}}{\tau^{1-3\delta}}\end{aligned}$$

and, similarly,

$$|||\mathcal{T}(h_1-h_2)|||_{7/6,\beta,3\delta} \leqslant CT^{3\delta}|||h_1-h_2|||_{7/6,\beta,3\delta}.$$

The existence and uniqueness of solution of (3.1), (3.2) in the space $\boldsymbol{X}_{7/6,\beta,3\delta}(T)$ follow for $T > 0$ sufficiently small by the usual contraction argument. Finally, (3.23) and (3.24) follow by a small modification of the proof of proposition 3.11. More precisely, if $\tilde{h}$ is a solution of (3.82) with initial data $\tilde{h}_0(k) = 0$, then, arguing as in the derivation of (3.60), we have

$$\begin{aligned}|\tilde{h}(\tau,k_1) - y(\tau)k_1^{-7/6}| &\leqslant C_M\int_0^\tau (\tau-s)^{-3\alpha}\varPhi\left(\frac{k}{(\tau-s)^3}\right)j(s)\,\mathrm{d}s\\ &\quad + \frac{C}{k_1^{\tilde{b}}}\int_0^\tau (\tau-s)^{3\tilde{b}-5/2-\varepsilon}j(s)\,\mathrm{d}s,\end{aligned} \tag{3.84}$$

where

$$j(s) \equiv \left\{\frac{\|h\|_{\alpha,\beta,\delta}}{s^{1-3\delta}} + \frac{|||\nu|||_{\alpha,\beta,\gamma}}{s^{1-\gamma}}\right\} \quad \text{and} \quad \tilde{b} = \tfrac{7}{6} - \tfrac{1}{2}\delta.$$

We can now estimate the first term on the right-hand side of this inequality by splitting the integral into the intervals $(0, \frac{1}{2}\tau)$ and $(\frac{1}{2}\tau, \tau)$. In the second term, we can bound $s^{-1+3\delta}$ and $s^{-1+\gamma}$ by $C\tau^{-1+3\delta}$ and $C\tau^{-1+\gamma}$, respectively, and estimate the remaining integral as in (3.81). Eventually, for $0 \leqslant \tau \leqslant T$ and $0 \leqslant k_1 \leqslant 1$, this gives

$$\int_{\tau/2}^{\tau} (\tau - s)^{-3\alpha} \Phi\biggl(\frac{k}{(\tau - s)^3}\biggr) j(s)\,\mathrm{d}s \\ \leqslant C(\|h\|_{\alpha,\beta,\delta}\tau^{-1+9\delta/2} + |||\nu|||_{\alpha,\beta,\gamma}\tau^{-1+\gamma+3\delta/2})k_1^{-7/6+\delta/2}.$$

On the other hand, the contribution due to the integral for $0 \leqslant s \leqslant \frac{1}{2}\tau$ is estimated using the monotonicity of the function Φ defined in (3.37). Then

$$\begin{aligned} \int_0^{\tau/2} (\tau - s)^{-3\alpha} \Phi\biggl(\frac{k}{(\tau - s)^3}\biggr) j(s)\,\mathrm{d}s &\leqslant C\frac{1}{\tau^{3\alpha}}\Phi\biggl(\frac{k}{\tau^3}\biggr)\int_0^{\tau/2} j(s)\,\mathrm{d}s \\ &\leqslant C(\|h_0\|\tau^{-1+9\delta/2} + |||\nu|||_{\alpha,\beta,\gamma}\tau^{-1+\gamma+3\delta/2})k_1^{-7/6+\delta/2}. \end{aligned}$$

The second integral on the right-hand side of (3.84) is estimated using similar arguments. Finally, the bound (3.22) for $a(\tau)$ follows as in proposition 3.11, using (3.36) and (3.49). □

3.5. Some regularity results for the time derivatives

We now prove some regularity properties with respect to the initial time for the function $a(\tau)$ (whose existence is asserted in (3.21)), which will be needed later.

Lemma 3.21. *Let us suppose that f_0 satisfies (2.3), (2.4), (2.5) and $\frac{1}{2} \leqslant \lambda(\tau) \leqslant 1$ for $\bar\tau \leqslant \tau \leqslant T$. Let us denote by H the unique solution of the problem*

$$\frac{\partial H}{\partial \tau}(\tau, \bar\tau, k) = \mathcal{L}_{k,2}(f_0, H(\tau, \bar\tau)) + \frac{1}{\lambda(\tau)}\mathcal{L}_{k,1}(f_0, H(\tau, \bar\tau)) \quad \text{for } \bar\tau \leqslant \tau \leqslant T, \tag{3.85}$$

$$H(\bar\tau, \bar\tau) = f_0 \tag{3.86}$$

in $\boldsymbol{Y}_{7/6,\beta}(T)$. Suppose also that

$$|\lambda'(\tau)| \leqslant C, \quad 0 \leqslant \tau \leqslant T. \tag{3.87}$$

Then, the function $a(\tau, \bar\tau)$, defined as

$$a(\tau, \bar\tau) = \lim_{k_1 \to 0} k_1^{7/6} H(\tau, \bar\tau, k_1), \tag{3.88}$$

satisfies

$$a(\bar\tau^+, \bar\tau) = A, \tag{3.89}$$

$$\left|\frac{\partial}{\partial \bar\tau} a(\tau, \bar\tau)\right| \leqslant C(\tau - \bar\tau)^{-1+3\delta}, \quad \bar\tau \leqslant t \leqslant T, \tag{3.90}$$

$$\left\| \frac{\partial H}{\partial \bar\tau} \right\|_{7/6,\beta} \leqslant \frac{C \sup_{0\leqslant\tau\leqslant T} |\lambda(\tau)|}{(\tau-\bar\tau)^{1-3\delta}}, \tag{3.91}$$

$$\left\| \frac{\partial H}{\partial \tau} \right\|_{7/6,\beta} \leqslant \frac{C}{(\tau-\bar\tau)^{1-3\delta}} \Big\{ \sup_{0\leqslant\tau\leqslant T} |\lambda(\tau)| + \sup_{0\leqslant\tau\leqslant T} |\lambda'(\tau)| \Big\}, \tag{3.92}$$

and

$$\left| \frac{\partial a}{\partial \tau}(\tau,\bar\tau) \right| \leqslant \frac{C}{(\tau-\bar\tau)^{1-3\delta}} \Big\{ \sup_{0\leqslant\tau\leqslant T} |\lambda(\tau)| + \sup_{0\leqslant\tau\leqslant T} |\lambda'(\tau)| \Big\} \tag{3.93}$$

for $\bar\tau \leqslant \tau \leqslant T$.

Finally, under the same assumptions,

$$\|H(\tau,\bar\tau) - a(\tau,\bar\tau)k^{-7/6}\|_{7/6-\delta/2,\beta} \leqslant C, \tag{3.94}$$

$$\left\| \frac{\partial H}{\partial \tau}(\tau,\bar\tau) - \frac{\partial a}{\partial \tau}(\tau,\bar\tau)k^{-7/6} \right\|_{7/6-\delta/2,\beta} \leqslant \frac{C}{(\tau-\bar\tau)^{1-3\delta/2}} \tag{3.95}$$

for $\bar\tau \leqslant \tau \leqslant T$.

Proof of lemma 3.21. The existence and uniqueness of the solution H follows from theorem 3.1 with $\nu = 0$. Now using (3.78) we obtain

$$H(\tau,\bar\tau,k_1) = \int_0^\infty G(\tau-\bar\tau,k_1,\xi) f_0(\xi)\,\mathrm{d}\xi + \int_{\bar\tau}^{\tau} \mathrm{d}s \int_0^\infty \mathrm{d}\xi G(\tau-s,k_1,\xi)\mathcal{U}(\xi;\lambda(s),H(s,\bar\tau,\xi)). \tag{3.96}$$

Multiplying by $k_1^{7/6}$ and taking the limit as $k_1 \to 0$, we obtain

$$a(\tau,\bar\tau) = \int_0^\infty \xi^{1/6}\sigma\left(\frac{\tau-\bar\tau}{\xi^{1/3}}\right) f_0(\xi)\,\mathrm{d}\xi + \int_{\bar\tau}^{\tau} \mathrm{d}s \int_0^\infty \mathrm{d}\xi\, \xi^{1/6}\sigma\left(\frac{\tau-s}{\xi^{1/3}}\right)\mathcal{U}(\xi;\lambda(s),H(s,\bar\tau,\xi)) \tag{3.97}$$

for all $\tau < \bar\tau$, where the convergence of the different integrals is ensured by the estimates (3.79) and (3.80). We now take the limit of (3.97) as $\tau \to \bar\tau$. To this end, we use in the first integral of the right-hand side the change of variables $\xi = \zeta\tau^3$ and (2.3), whence

$$\lim_{\tau\to\bar\tau} \int_0^\infty \xi^{1/6}\sigma\left(\frac{\tau-\bar\tau}{\xi^{1/3}}\right) f_0(\xi)\,\mathrm{d}\xi = A\int_0^\infty \sigma(\zeta^{-1/3})\zeta^{-1}\,\mathrm{d}\zeta. \tag{3.98}$$

Differentiating the identity $\tilde{Q}(Ak^{-7/6}) = 0$ (c.f. (1.8)) with respect to A, we obtain $\mathcal{L}_{k,2}(Ak^{-7/6},H_s) = 0$. Therefore, if $f_0(\xi) = \xi^{-7/6}$ and $\mathcal{U} = 0$ in (3.98) it would follow that $a(\bar\tau,\tau) = A$, whence

$$\int_0^\infty \sigma(\zeta^{-1/3})\zeta^{-1}\,\mathrm{d}\zeta = 1. \tag{3.99}$$

On the other hand, using (3.79) and (3.80) and lemmas 3.9 and 3.10, we deduce that

$$\left| \int_0^\infty \mathrm{d}\xi\, \xi^{1/6} \sigma\Big(\frac{\tau - s}{\xi^{1/3}}\Big) \mathcal{U}(\xi; \lambda(s), H(s, \bar\tau, \xi)) \right| \leqslant C(\tau - s)^{-1+3\delta}.$$

Integrating this formula in the interval $(\bar\tau, \tau)$, we derive an estimate for the second term on the right-hand side of (3.97) in the form $C(\tau - \bar\tau)^{-3\delta}$. Taking the limit $\tau \to \bar\tau$ and using (3.98), (3.99), we obtain (3.89).

The function $H(\tau, \bar\tau, k)$ satisfies (3.85) in the classical sense. To check this we could differentiate formally in (3.96), after rewriting the second integral on the right-hand side as

$$\int_0^{\tau - \bar\tau} \mathrm{d}s \int_0^\infty \mathrm{d}\xi\, G(s, k_1, \xi) \mathcal{U}(\xi; \lambda(\tau - s), H(\tau - s, \bar\tau, \xi))$$

to obtain

$$\begin{aligned}\frac{\partial H}{\partial \tau}(\tau, \bar\tau, k_1) = \int_0^\infty \frac{\partial G}{\partial \tau}(\tau - \bar\tau, k_1, \xi) f_0(\xi)\, \mathrm{d}\xi + \int_0^\infty G(\tau - \bar\tau, k_1, \xi) \mathcal{U}(\xi; \lambda(\bar\tau), f_0(\xi))\, \mathrm{d}\xi \\ + \int_0^{\tau - \bar\tau} \mathrm{d}s \int_0^\infty \mathrm{d}\xi\, G(s, k_1, \xi) \Big\{ \frac{\partial \mathcal{U}}{\partial \lambda} \lambda'(\tau - s) + \frac{\partial \mathcal{U}}{\partial H} \frac{\partial H}{\partial \tau}(\tau - s, \bar\tau, \xi) \Big\}.\end{aligned}$$

Use of Gronwall's lemma would then yield that H is a classical solution of (3.85). To make this argument rigorously, we merely replace $\partial/\partial\tau$ by the incremental quotients and pass to the limit.

Let us first indicate the formal arguments that we will use to prove (3.90) and (3.92)–(3.95). In order to prove (3.90) we differentiate (3.85) and (3.86) with respect to $\bar\tau$ to obtain

$$\frac{\partial}{\partial \tau}\Big(\frac{\partial H}{\partial \bar\tau}\Big)(\tau, \bar\tau, k) = \mathcal{L}_k\Big(\lambda(\tau) f_0, \Big(\frac{\partial H}{\partial \bar\tau}\Big)(\tau, \bar\tau)\Big), \tag{3.100}$$

$$\frac{\partial H}{\partial \bar\tau}(\bar\tau, \bar\tau) = -\frac{\partial H}{\partial \tau}(\bar\tau, \bar\tau) = -\mathcal{L}_k(\lambda(\bar\tau) f_0, f_0). \tag{3.101}$$

Using (2.3), we obtain

$$\|\mathcal{L}_k(\lambda(\bar\tau) f_0, f_0)\|_{\alpha,\beta} \leqslant C \tag{3.102}$$

with $\alpha = \frac{3}{2} - \delta$ and $\beta = \frac{11}{6} - \delta$. The estimate (3.90) is then a consequence of theorem 3.2.

The analogous argument to prove (3.92) and (3.93) would be as follows. We note first that, due to (3.87), estimating the derivative of a function with respect to t is equivalent to estimating its derivative with respect to τ. Differentiating (3.85) with respect to τ, and using (3.100) and (3.101), we see that $\partial H/\partial\tau$ solves

$$\begin{aligned}\frac{\partial}{\partial \tau}\Big(\frac{\partial H}{\partial \tau}\Big) &= \mathcal{L}_{k,2}\Big(f_0, \Big(\frac{\partial H}{\partial \tau}\Big)\Big) \\ &\quad + \frac{1}{\lambda(\tau)} \mathcal{L}_{k,1}\Big(f_0, \Big(\frac{\partial H}{\partial \tau}\Big)\Big) + \frac{\partial}{\partial \tau}\Big(\frac{1}{\lambda(\tau)}\Big) \mathcal{L}_{k,1}(f_0, H),\end{aligned} \tag{3.103}$$

$$\frac{\partial H}{\partial \tau}(\bar\tau, \bar\tau) = \frac{1}{\lambda(\bar\tau)} \mathcal{L}_k(\lambda(\bar\tau) f_0, f_0). \tag{3.104}$$

Combining (2.3), (2.5) and (3.87), as well as the fact that $H \in \boldsymbol{Y}_{7/6,\beta}(T)$, it follows that

$$\left\| \frac{\partial}{\partial \tau}\left(\frac{1}{\lambda(\tau)}\right)\mathcal{L}_{k,1}(f_0, H)\right\|_{\alpha,\beta} \leqslant C.$$

Applying theorem 3.2 to (3.102), we deduce (3.92). Formula (3.93) follows from (3.22).

Analogously, in order to derive (3.94), we use the fact that the equation satisfied by $W = H - f_0$, which may be derived using (3.85), (3.86), is linear with zero initial data and source terms bounded by $Ck_1^{-3/2+\delta}$ for $k_1 \leqslant 1$. Therefore, (3.94) follows, using variation of the constants as above and theorem 3.1. The proof of (3.95) is similar, but uses (3.103), (3.104) instead of (3.85), (3.86), and theorem 3.2 instead of theorem 3.1.

The above computations can be made rigorous by replacing the derivatives $\partial/\partial\bar{\tau}$ and $\partial/\partial\tau$ by the corresponding incremental quotients. □

4. Solving the nonlinear truncated equation

In this section we prove the following result.

THEOREM 4.1. *Suppose that f_0 satisfies (2.3), (2.5). Then, for any $M > 0$ and $M' > 0$, there exist a $T = T(M, M') > 0$ and a unique solution of (2.10)–(2.12) of the form $f(t) = \lambda(t)f_0 + g(t)$, where $g \in \mathcal{C}[[0,T] \times (0,\infty)]$, $g \in \boldsymbol{Y}_{7/6-\delta/2,\beta}(T)$, $\beta = \frac{11}{6} - \delta$, for $\delta > 0$ sufficiently small, and $\lambda \in \mathbb{C}[0,T] \cap \mathbb{C}^1(0,T)$. Moreover,*

$$|||g|||_{7/6-\delta,\beta}(T) \leqslant C_{M,M'}T^{\delta/2}. \tag{4.1}$$

REMARK 4.2. Note that the condition $g \in \boldsymbol{Y}_{7/6-\delta,\beta}(T)$ implies that (2.14) holds.

The idea of the proof of theorem 4.1 is to use a fixed-point argument for (2.17) under the constraint (2.14). First we will obtain a proof of the result in the τ variable instead of t because, by (2.21), both formulations are equivalent as long as $\frac{1}{2} \leqslant \lambda \leqslant 2$. The statement in the t variable immediately follows for the same reason. As a first step, we derive suitable estimates for the terms $\mathcal{R}_1$ and $\mathcal{R}_2$ defined in (2.19), (2.20).

LEMMA 4.3. *Suppose that f_0 satisfies (2.3), (2.5) and $\frac{1}{2} \leqslant \lambda(\tau) \leqslant 2$ for $0 \leqslant \tau \leqslant T$ and some $T > 0$. Then the function $\mathcal{R}_1(\tau, k_1)$ defined in (2.19) satisfies*

$$\sup_{0\leqslant\tau\leqslant T} |\mathcal{R}_1(\tau,k_1)| \leqslant \begin{cases} \dfrac{C_M}{k_1^{3/2-\delta}}, & k_1 \leqslant 1, \\[2mm] \dfrac{C_M}{k_1^{7/3-2\delta}}, & k_1 \geqslant 1, \end{cases}$$

where $C_M = C(A, B, D, \delta, M)$ is a positive constant independent of M'.

Proof of lemma 4.3. Using the fact that $q(f) = \tilde{q}(f) + r(f)$ as well as (3.5) with $g = \lambda(\tau)(f_0 - Ak^{-7/6})$, we can rewrite $\mathcal{R}_1$ as

$$\begin{aligned}\mathcal{R}_1(\tau, k_1) = &\int_{D(k_1)} W_{M,M'}\tilde{q}(\lambda(\tau)Ak^{-7/6})\,\mathrm{d}k_3\,\mathrm{d}k_4\\ &+\int_{D(k_1)} W_{M,M'}\tilde{\ell}(\lambda(\tau)Ak^{-7/6}, \lambda(\tau)(f_0 - Ak^{-7/6}))\,\mathrm{d}k_3\,\mathrm{d}k_4\\ &+\int_{D(k_1)} W_{M,M'}\tilde{n}(\lambda(\tau)Ak^{-7/6}, \lambda(\tau)(f_0 - Ak^{-7/6}))\,\mathrm{d}k_3\,\mathrm{d}k_4\\ &+\int_{D(k_1)} W_{M,M'}r(\lambda(\tau)f_0)\,\mathrm{d}k_3\,\mathrm{d}k_4\\ \equiv\ &\mathcal{R}_{1,1} + \mathcal{R}_{1,2} + \mathcal{R}_{1,3} + \mathcal{R}_{1,4}.\end{aligned} \tag{4.2}$$

The $\tilde{q}(f)$ term may be bounded by $C_M k_1^{-7/3}\min(1, k_2^{-7/6})$ since $W_{M,M'}$ is supported in the region $|k_3 - k_4| \leqslant M$ (due to (3.3)). Using the fact that

$$W(k_1, k_2, k_3, k_4) \leqslant \min\left(1, \frac{\sqrt{k_2}}{\sqrt{k_1}}\right),$$

we then deduce that

$$|\mathcal{R}_{1,1}| \leqslant C_M k_1^{-7/3}\int_0^\infty \min\left(1, \frac{\sqrt{k_2}}{\sqrt{k_1}}\right)\min(1, k_2^{-7/6})\,\mathrm{d}k_2.$$

Splitting the integral into the three regions $0 < k_2 < 1$, $1 < k_2 < k_1$ and $k_1 < k_2 < \infty$, we obtain

$$|\mathcal{R}_{1,1}| \leqslant \frac{C_M}{k_1^{5/2}}, \quad k_1 \geqslant 1. \tag{4.3}$$

On the other hand, since

$$\int_{D(k_1)} W\tilde{q}(Ak^{-7/6})\,\mathrm{d}k_3\,\mathrm{d}k_4 = 0, \tag{4.4}$$

we can rewrite $\mathcal{R}_{1,1}$ as

$$\mathcal{R}_{1,1} = \int_{D(k_1)} (W_{M,M'} - W)\tilde{q}(\lambda(\tau)Ak^{-7/6})\,\mathrm{d}k_3\,\mathrm{d}k_4. \tag{4.5}$$

Using the fact that $W_{M,M'} - W$ vanishes for $|k_3 - k_4| < M$, we obtain

$$|\mathcal{R}_{1,1}| \leqslant \frac{C_M}{k_1^{7/6}}, \quad k_1 \leqslant 1. \tag{4.6}$$

We now consider $\mathcal{R}_{1,2}$. By (2.3) the estimate $|\lambda(\tau)(f_0 - Ak^{-7/6})| \leqslant Ck_1^{-7/6+\delta}$ holds for all $k_1 > 0$. Making the change of variables $k_3 = k_1\xi_3$, $k_4 = k_1\xi_4$, it follows that

$$|\mathcal{R}_{1,2}| \leqslant \frac{C}{k_1^{3/2-\delta}} \quad \text{for } k_1 \leqslant 1. \tag{4.7}$$

Arguing as in the derivation of (4.3) we obtain

$$|\mathcal{R}_{1,2}| \leqslant C_M \mathrm{e}^{-Bk_1} \quad \text{for } k_1 \geqslant 1. \tag{4.8}$$

Similar arguments yield

$$|\mathcal{R}_{1,3}| \leqslant \begin{cases} \dfrac{C_M}{k_1^{5/2}} & \text{for } k_1 \geqslant 1, \\[2ex] \dfrac{C_M}{k_1^{3/2-2\delta}} & \text{for } k_1 \leqslant 1, \end{cases} \tag{4.9}$$

as well as

$$|\mathcal{R}_{1,4}| \leqslant \begin{cases} \dfrac{C_M}{k_1^{4/3}} & \text{for } k_1 \leqslant 1, \\[2ex] C_M \mathrm{e}^{-Bk_1} & \text{for } k_1 \geqslant 1. \end{cases} \tag{4.10}$$

Putting together (4.3) and (4.6)–(4.10), lemma 4.3 follows. □

LEMMA 4.4. *Suppose that $g \in \boldsymbol{Y}_{7/6-\delta/2,\beta}(T)$, for some $T > 0$, with β as in theorem 4.1. Suppose that λ also satisfies the assumptions in theorem 4.1 and $\frac{1}{2} \leqslant \lambda(\tau) \leqslant 2$. Then the function $\mathcal{R}_2(\tau, k_1, g)$ defined by (2.20) satisfies*

$$\sup_{0\leqslant\tau\leqslant T} |\mathcal{R}_2(\tau, k_1, g)| \leqslant \frac{C_M}{k_1^{3/2-\delta}}, \quad k_1 \leqslant 1, \tag{4.11}$$

$$\sup_{0\leqslant\tau\leqslant T} |\mathcal{R}_2(\tau, k_1, g)| \leqslant \frac{C_M}{k_1^{\beta}}, \qquad k_1 \geqslant 1, \tag{4.12}$$

where $C_M = C(A, B, D, \delta, M, |||g|||_{7/6-\delta/2,\beta})$ is uniformly bounded if $|||g|||_{7/6-\delta/2,\beta}$ is bounded and is independent of M'. Moreover, suppose that $g, \bar{g}$ are such that

$$|||g|||_{7/6-\delta/2,\beta} + |||\bar{g}|||_{7/6-\delta/2,\beta} \leqslant \rho \tag{4.13}$$

for some positive constant ρ. Then,

$$|||\mathcal{R}_2(\cdot,\cdot,g) - \mathcal{R}_2(\cdot,\cdot,\bar{g})|||_{3/2-\delta,\beta} \leqslant C_M |||g - \bar{g}|||_{7/6-\delta/2,\beta}, \tag{4.14}$$

where $C_M = C(A, B, D, \delta, M, \rho)$.

REMARK 4.5. Lemma 4.4 will play a crucial role in the forthcoming argument. The reason is that it states that the function $\mathcal{R}_2(\tau, k_1, g)$ is smaller near the origin than the leading linear term $\mathcal{L}_{k,2}(f_0, g)(\tau, k_1)$ in (2.22). Indeed, given $g \in \boldsymbol{Y}_{7/6-\delta/2,\beta}(T)$, it follows that $\mathcal{L}_{k,2}(f_0, g)(\tau, k_1)$ is pointwise bounded by $Ck_1^{-3/2+\delta/2}$ for $0 < k_1 \leqslant 1$. On the other hand, the term $\mathcal{R}_2(\tau, k_1, g)$ can be estimated by the smaller quantity $Ck_1^{-3/2+\delta}$ for $0 < k_1 \leqslant 1$. This additional smallness, which is due to the fact that $\mathcal{R}_2(\tau, k_1, g)$ is quadratic with respect to g, allows us to handle the final term in a perturbative manner.

Proof of lemma 4.4. The function $n(\lambda(\tau) f_0, g)$ contains two types of term, depending on their homogeneity. Some of the terms are those in $\tilde{n}$ which are quadratic in g and linear in f_0. These terms can be estimated for $0 \leqslant k_1 \leqslant 1$ using (2.3) and

$g \in Y_{7_6-\delta/2,\beta}(T)$. Using the change of variables $k_3 = k_1\xi_3$, $k_4 = k_1\xi_4$, we deduce an estimate of the form (4.11) for the contribution due to these terms. The remaining terms in $n(\lambda(\tau)f_0, g)$ are the ones in $r(\lambda(\tau)f_0, g)$. Their contribution can be estimated as $C_M k_1^{-7/6-\delta}$ when $k_1 \leqslant 1$, which is smaller than the right-hand side of (4.11). Finally, (4.12) follows by using the same arguments as in the proof of lemma 4.3. Estimates for the differences (4.14) are obtained in the same way. □

Proof of theorem 4.1. Recall that we are looking for a solution of the problem (2.22), (2.23) of the form

$$f(\tau, k) = \lambda(\tau) f_0(k) + g(\tau, k),$$

where $\lambda(\tau)$ will be prescribed, imposing $g \in Y_{7/6-\delta/2,\beta}(T)$ for some $T > 0$. Moreover, we also have $g(0, k) = 0$ for $k \geqslant 0$ (cf. (2.15)).

Let us introduce a suitable functional framework. We define the space

$$\Lambda(T) \equiv \{\lambda \in \mathbb{C}([0,T]) \cap \mathbb{C}^1(0,T) : |\lambda(\tau) - \lambda(0)| \leqslant \tfrac{1}{4},\ |\lambda'(\tau)| \leqslant C,\ 0 \leqslant \tau \leqslant T\} \tag{4.15}$$

endowed with the norm

$$\|\lambda\|_{1,\infty} = \sup_{0\leqslant\tau\leqslant T} \{|\lambda(\tau)| + |\lambda'(\tau)|\}. \tag{4.16}$$

Let us introduce the functional spaces

$$\mathcal{W}(T) = \left\{ g \in \mathcal{Y}_{7/6-\delta/2,\beta}(T),\ \frac{\partial g}{\partial \tau} \in \mathcal{Y}_{7/6-\delta/2,\beta}(T) \right\} \tag{4.17}$$

with the norm

$$\|g\|_{\mathcal{W}} = |||g|||_{7/6-\delta/2,\beta} + \left|\left|\left| \frac{\partial g}{\partial \tau} \right|\right|\right|_{7/6-\delta/2,\beta} \tag{4.18}$$

and $\mathcal{Z}(T) = \mathcal{W} \times \Lambda(T)$. We define an operator $\mathcal{T}$ from $\mathcal{Z}$ into itself as follows. Given $(g, \lambda) \in \mathcal{Z}$, let $\tilde{g}_1$ be the solution of

$$\begin{aligned} \frac{\partial \tilde{g}_1}{\partial \tau}(\tau, k_1) &= \mathcal{L}_{k,2}(f_0, \tilde{g}_1)(k_1, \tau) + \frac{1}{\lambda(\tau)} \mathcal{L}_{k,1}(f_0, \tilde{g}_1)(k_1, \tau) \\ &\quad + \frac{1}{\lambda^2(\tau)} (\mathcal{R}_1(\tau, k_1) + \mathcal{R}_2(\tau, k_1, g)), \end{aligned} \tag{4.19}$$

$$\tilde{g}_1(0) = 0. \tag{4.20}$$

The function $\tilde{g}_1$ is uniquely defined due to theorem 3.1. Moreover, the limit

$$b(\tau) \equiv b_{g,\lambda}(\tau) \equiv \lim_{k\to 0} k^{7/6} \tilde{g}_1(\tau, k) \tag{4.21}$$

exists. We define the function $\tilde{\lambda}(t)$ as the solution of the integral equation

$$\tilde{\lambda}(\tau) \equiv a(\tau, 0) + \frac{1}{A} \int_0^\tau \frac{\partial a}{\partial \bar{\tau}}(\tau, \bar{\tau}) \tilde{\lambda}(\bar{\tau})\, \mathrm{d}\bar{\tau} - b(\tau) \equiv \mathcal{S}(\tilde{\lambda}), \tag{4.22}$$

where a is defined by (3.88) in lemma 3.21. Let us suppose for the moment that the function $\tilde{\lambda}(\tau)$, the solution of (4.22), is well defined. We then define a function

$\tilde{g}_2$ by means of

$$\tilde{g}_2(\tau,k)=\frac{1}{A}\left\{H(\tau,\tau,k)\tilde{\lambda}(\tau)-H(\tau,0,k)\tilde{\lambda}(0)-\int_0^\tau\frac{\partial H}{\partial\bar{\tau}}(\tau,\bar{\tau},k)\tilde{\lambda}(\bar{\tau})\,\mathrm{d}\bar{\tau}\right\},\quad(4.23)$$

where H is the solution of the problem (3.85), (3.86) whose existence and uniqueness is asserted in lemma 3.21.

After all these preliminaries we define

$$\left.\begin{aligned}\mathcal{T}:\mathcal{Z}\to\mathcal{Z},\\ \mathcal{T}(g,\lambda)=(\tilde{g},\tilde{\lambda}),\\ \tilde{g}=\tilde{g}_1+\tilde{g}_2.\end{aligned}\right\}\quad(4.24)$$

Note that a fixed point of the operator $\mathcal{T}$ is a solution of the integral equation associated with the problem (2.22), (2.23). Moreover, we remark that the solution of such an integral equation solves the differential equation (2.22), (2.23). Indeed, this follows from the differentiability of the function $\tilde{g}_2$ defined in (4.23) with respect to τ for $k>0$. Such a regularity can be seen by differentiating formally the right-hand side of (3.100) with respect to τ and using the regularity properties of the function H proved in lemma 3.21 (see (3.91) and (3.92)).

We then proceed to check that the operator $\mathcal{T}$ is well defined. As a first step we derive a local well-posedness result for (4.22). To this end we first prove an auxiliary result. Let us denote by $\boldsymbol{T}(g;\lambda)=\tilde{g}_1$ the solution of (4.19), (4.20) and let $\boldsymbol{S}(g;\lambda)=\boldsymbol{T}(g)-b_{g,\lambda}k^{-7/6}$. We then have the following lemma.

LEMMA 4.6. *Suppose that* $\lambda\in\Lambda(T)$ *satisfies* $\|\lambda\|_{1,\infty}<\infty$ *with* $\|\lambda\|_{1,\infty}$ *defined in (4.16) and* $g,\partial g/\partial\tau\in\boldsymbol{Y}_{7/6-\delta/2,\beta}(T)$. *Then the function* $b(\tau)$ *defined in (4.21) satisfies*

$$|b(\tau)|+|b'(\tau)|\leqslant C\tau^{3\delta},\quad 0\leqslant\tau\leqslant T.\quad(4.25)$$

Moreover,

$$|b_{g,\lambda}(\tau)-b_{h,\mu}(\tau)|+|b'_{g,\lambda}(\tau)-b'_{h,\mu}(\tau)|\leqslant C\tau^{3\delta}(\|\lambda-\mu\|_{1,\infty}+\|g-h\|_{\mathcal{W}})\quad(4.26)$$

and

$$\|\boldsymbol{S}(g;\lambda)-\boldsymbol{S}(h;\mu)\|_{\mathcal{W}}\leqslant CT^{3\delta/2}(\|g-h\|_{\mathcal{W}}+\|\lambda-\mu\|_{L^\infty(0,T)})\quad(4.27)$$

for $0\leqslant\tau\leqslant T$, *where* $C=C(A,B,D,\delta,M,M',d)$ *and* $d=|||g|||_{\mathcal{W}}+|||h|||_{\mathcal{W}}+\|\lambda\|_{1,\infty}+\|\mu\|_{1,\infty}+\|g-h\|_{\mathcal{W}}$.

Proof of lemma 4.6. The existence of the functions $\tilde{g}_1$ and $b(\tau)$ and the part of the estimate (4.25) for b is just a consequence of theorem 3.1.

In order to estimate $b'(\tau)$, we differentiate (4.19) with respect to τ. The resulting equation has the form

$$\begin{aligned}\frac{\partial}{\partial\tau}\left(\frac{\partial\tilde{g}_1}{\partial\tau}\right)&=\mathcal{L}_{k,2}\left(f_0,\frac{\partial\tilde{g}_1}{\partial\tau}\right)(k_1,\tau)\\&\quad+\frac{1}{\lambda(\tau)}\mathcal{L}_{k,1}\left(f_0,\frac{\partial\tilde{g}_1}{\partial\tau}\right)(k_1,\tau)+\mathcal{F}\left(k_1,g,\tilde{g}_1,\frac{\partial g}{\partial\tau},\tau\right).\end{aligned}\quad(4.28)$$

Arguing as in the proof of lemmas 4.3 and 4.4, we deduce

$$\left|\left|\left|\mathcal{F}\left(k_1, g, \tilde{g}_1, \frac{\partial g}{\partial \tau}, \tau\right)\right|\right|\right|_{3/2-\delta,\beta} \leqslant C\|g\|_{\mathcal{W}}. \tag{4.29}$$

The estimate for $b'(\tau)$ in (4.25) then follows from theorem 4.1. Combining (4.14) and theorem 3.1 we obtain

$$|b_{g,\lambda} - b_{h,\mu}| \leqslant CT^{3\delta}(|||g-h|||_{7/6-\delta/2,\beta} + \|\lambda-\mu\|_{L^\infty(0,T)}), \tag{4.30}$$

$$|||\boldsymbol{S}(g;\lambda) - \boldsymbol{S}(h;\mu)|||_{7/6-\delta/2,\beta} \leqslant CT^{3\delta/2}(|||g-h|||_{7/6-\delta/2,\beta} + \|\lambda-\mu\|_{L^\infty(0,T)}). \tag{4.31}$$

Arguing as in the proof of (4.29) we obtain

$$\left|\left|\left|\mathcal{F}\left(k_1, g, \boldsymbol{T}(g;\lambda), \frac{\partial g}{\partial \tau}, \tau\right) - \mathcal{F}\left(k_1, h, \boldsymbol{T}(h;\mu), \frac{\partial h}{\partial \tau}, \tau\right)\right|\right|\right|_{3/2-\delta,\beta} \leqslant C(\|g-h\|_{\mathcal{W}} + \|\lambda-\mu\|_{L^\infty(0,T)}).$$

Using theorem 3.1 again, we deduce that

$$|b'_{g,\lambda} - b'_{h,\mu}| \leqslant CT^{3\delta}(|||g-h|||_{\mathcal{W}} + \|\lambda-\mu\|_{L^\infty(0,T)}), \tag{4.32}$$

$$\left|\left|\left|\frac{\partial}{\partial t}\boldsymbol{S}(g;\lambda) - \frac{\partial}{\partial t}\boldsymbol{S}(h;\mu)\right|\right|\right|_{7/6-\delta/2,\beta} \leqslant CT^{3\delta/2}(\|g-h\|_{\mathcal{W}} + \|\lambda-\mu\|_{L^\infty(0,T)}). \tag{4.33}$$

This concludes the proof of lemma 4.6. □

We can now prove a local well-posedness result for (4.22).

Lemma 4.7. *For any $M > 0$ and $M' > 0$ there exist a T such that*

$$T = T(A, B, D, \delta, M, M')$$

and a unique $\lambda \in \mathbb{C}([0,T])$ solving (4.22) for $0 \leqslant \tau \leqslant T$. Moreover,

$$|\lambda(\tau) - A| \leqslant C(A, B, D, \delta, M, M')T^{3\delta}, \quad 0 \leqslant \tau \leqslant T. \tag{4.34}$$

Proof of lemma 4.7. We note that the operator $\mathcal{S}$ defined in (4.22) maps $\mathcal{C}[0,T]$ onto $\mathcal{C}[0,T]$ and is contractive for T sufficiently small. Indeed, by (4.26) and (3.90) we have

$$|\mathcal{S}(\lambda_1)(\tau) - \mathcal{S}(\lambda_2)(\tau)| \leqslant Cq(T)T^{3\delta}\|\lambda_1 - \lambda_2\|_{1,\infty}, \tag{4.35}$$

where $C = C(\delta)$ and

$$q(T) = \sup_{0\leqslant\tau\leqslant\bar{\tau}\leqslant T}\left|(\tau-\bar{\tau})^{1-3\delta}\frac{\partial a}{\partial\bar{\tau}}(\tau,\bar{\tau})\right|.$$

Moreover,

$$\|\mathcal{S}(\lambda) - b(\cdot) - a(\cdot,0)\|_\infty \leqslant q(T)T^{3\delta}(\|b\|_\infty + \|a(\cdot,0)\|_\infty + \|\lambda - b(\cdot) - a(\cdot,0)\|_\infty).$$

By theorem 3.1 and lemma 3.21, we have, for some $\tilde{T} = \tilde{T}(A, B, D, \delta, M, M') > 0$

$$\|b(\cdot)\|_\infty + \|a(\cdot, 0)\|_\infty + q(T) \leqslant C(A, B, D, \delta, M, M'), \quad 0 \leqslant T \leqslant \tilde{T}.$$

Therefore, a standard fixed-point argument concludes the proof of the lemma. □

LEMMA 4.8. *The function $\tilde{\lambda}$ solution of the integral equation (4.22) satisfies*

$$|\tilde{\lambda}(\tau)| \leqslant C(\|a(\cdot, 0)\|_\infty + \|b\|_\infty), \quad 0 \leqslant \tau \leqslant T, \tag{4.36}$$

$$|\tilde{\lambda}_\tau(\tau)| \leqslant C\|b'\|_\infty, \quad 0 \leqslant \tau \leqslant T, \tag{4.37}$$

for $T > 0$ sufficiently small.

Proof of lemma 4.8. The inequality (4.36) is a consequence of (3.90) and (4.22). On the other hand, in order to derive (4.37), note that integration by parts in (4.22) yields

$$\frac{1}{A}\int_0^\tau a(\tau, \bar{\tau})\tilde{\lambda}'(\bar{\tau})\,\mathrm{d}\bar{\tau} + b(\tau) = 0. \tag{4.38}$$

Differentiating this equation, we obtain

$$\tilde{\lambda}'(\tau) + \frac{1}{A}\int_0^\tau \frac{\partial a}{\partial \tau}(\tau, \bar{\tau})\tilde{\lambda}'(\bar{\tau})\,\mathrm{d}\bar{\tau} + b'(\tau) = 0, \tag{4.39}$$

which, combined with (3.93), gives (4.37). □

We now complete the proof of theorem 4.1. This reduces to show that the operator $\mathcal{T}$ defined in (4.24) is a contraction for T small enough. Note that

$$\mathcal{T}(g, \lambda) = (\boldsymbol{T}(g) + \tilde{g}_2, \tilde{\lambda}). \tag{4.40}$$

Let us first show that $\boldsymbol{T}(g) + \tilde{g}_2 \in \mathcal{W}(T)$. Indeed, using (4.23) and (3.88), we obtain

$$\lim_{k\to 0} k^{7/6}\tilde{g}_2(\tau, k) = \lambda(\tau) - a(\tau, 0)\lambda(0) - \frac{1}{A}\int_0^\tau \frac{\partial a}{\partial \tau}(\bar{\tau}, \bar{\tau})\,\mathrm{d}\bar{\tau}. \tag{4.41}$$

Combining (4.21), (4.22) and (4.41), it then follows that

$$\lim_{k\to 0}(k^{7/6}(\boldsymbol{T}(g) + \tilde{g}_2)) = 0. \tag{4.42}$$

Then, the fact that $\boldsymbol{T}(g) + \tilde{g}_2 \in \mathcal{W}(T)$ follows from (3.94), (3.95), (4.23) and (4.27). Moreover, we also obtain

$$\|(\boldsymbol{T}(g) + \tilde{g}_2) - (\boldsymbol{T}(h) + \tilde{h}_2)\|_{\mathcal{W}} \leqslant \tfrac{1}{4}(\|g - h\|_{\mathcal{W}} + \|\lambda - \mu\|_{1,\infty}) \tag{4.43}$$

for $T > 0$ sufficiently small.

On the other hand, in order to keep track of the dependence of $a(\cdot, 0)$ with respect to λ, we denote by $H_\lambda(t, 0)$ the solution of (3.85) and by a_λ the function defined by (3.88) in lemma 3.21. Lemma 4.8 then yields

$$\|\tilde{\lambda} - \tilde{\mu}\|_{1,\infty} \leqslant C(\|b_{g,\lambda} - b_{h,\mu}\|_{1,\infty} + \|a_\lambda(\cdot, 0) - a_\mu(\cdot, 0)\|_\infty). \tag{4.44}$$

The first term on the right-hand side of (4.44) was estimated in (4.26). Additionally, the second term may be estimated as follows. Consider

$$\frac{\partial}{\partial\tau}(H_\lambda - H_\mu) = \mathcal{L}_{k,1}(f_0, H_\lambda - H_\mu) + \frac{1}{\lambda(\tau)}\mathcal{L}_{k,2}(f_0, H_\lambda - H_\mu) + \left(\frac{1}{\lambda(\tau)} - \frac{1}{\mu(\tau)}\right)\mathcal{L}_k(f_0, H_\mu). \quad (4.45)$$

Using both the fact that

$$\left\|\left(\frac{1}{\lambda(\tau)} - \frac{1}{\mu(\tau)}\right)\mathcal{L}_{k,2}(f_0, H_\mu)(t)\right\|_{3/2-\delta,\beta} \leqslant C\|\lambda - \mu\|_\infty$$

and theorem 3.1 we deduce that

$$\|a_\lambda(\cdot, 0) - a_\mu(\cdot, 0)\|_\infty \leqslant C\|\lambda - \mu\|_{\mathcal{W}}. \quad (4.46)$$

Combining (4.26) and (4.46) we obtain

$$\|\tilde{\lambda} - \tilde{\mu}\|_{1,\infty} \leqslant \tfrac{1}{4}(\|g - h\|_{\mathcal{W}} + \|\lambda - \mu\|_{1,\infty}) \quad (4.47)$$

for $T > 0$ sufficiently small. Formulae (4.41), (4.43) and (4.47) imply that $\mathcal{T}$ is a contractive operator, from whence we see that the operator $\mathcal{T}$ defined in (4.24) has a unique fixed point. Finally, changing to the time variable t using (2.21) yields theorem 4.1. □

REMARK 4.9. We note that the dependence on M, M' of the different constants C used in the proof of theorem 4.1 is due to the dependence on M, M' of the terms $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{U}_k$, $k = 1, 2, 3$ in (2.19), (2.20) and (3.13)–(3.15). This fact is relevant because, in the next section we will derive refined estimates for the solution f of (2.10), (2.11) which, in particular, will provide estimates on the terms $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{U}_k$, $k = 1, 2, 3$, independent of M, M'. This will make it possible to show that the solution f constructed in theorem 4.1 can be extended on a time interval independent of M and M'.

5. The limit $M, M' \to \infty$

5.1. Uniform bounds

The aim of this subsection is to obtain uniform bounds on the solutions of the truncated nonlinear problem (2.10)–(2.12) with respect to the truncation parameters M and M'. The main result that we prove is an estimate of the form

$$0 \leqslant f(t, k) \leqslant L\frac{\mathrm{e}^{-Dk}}{k^{7/6}} \quad \text{if } k > 0,\ t \in (0, T), \quad (5.1)$$

with L and T independent of M and M' and with D as in (2.5). We recall that, although the functions f depend on M and M', we will not state this dependence explicitly unless it is necessary.

Note that, by (2.5) and (2.5)–(2.11), for all $M > 0$ and $M' > 0$, we have

$$f(t, k) = f(0, k) \leqslant L\frac{\mathrm{e}^{-Dk}}{k^{7/6}} \quad \text{for all } k > M',\ t > 0, \quad (5.2)$$

whence we see that (5.1) holds immediately for all $k > M'$. Our goal now is to extend the range of validity of this inequality to the values $k < M'$.

Owing to the interaction between the regions of small and large values of k, it is not possible to obtain the estimate (5.1) without also estimating the function $f(t,k)$ for k of order 1. More precisely, in the derivation of (5.1), we will also obtain

$$|f(t,k) - a(t)k^{-7/6}| \leqslant Lk^{-7/6+\delta/2}, \quad k \leqslant 1, \quad t \in (0,T), \tag{5.3}$$

$$|a(t)| \leqslant L \qquad\qquad t \in (0,T), \tag{5.4}$$

with L and T as in (5.1). The key idea for proving (5.1), (5.3), (5.4) is to use a standard continuity argument. More precisely, it turns out that the functions $f(t,k)$, solutions of problems (2.10)–(2.12), satisfy (5.1), (5.3), (5.4) in an interval of time $t \in [0, T(M,M')]$. This is proved in the next lemma. In the rest of this subsection we extend the range of validity of these inequalities to a time interval independent of M and M'. Since we are interested in the limit as M and M' approach to ∞, we will assume from now on that M and M' are larger than a positive fixed number.

LEMMA 5.1. *For any $M > 0$ and $M' > 0$, there exists $T(M,M')$ such that the solution f of (2.10)–(2.12), with f_0 as in (2.3)–(2.5), obtained in theorem 4.1, satisfies (5.1), (5.3), (5.4) with $L = 4B$, where B is as in (2.3)–(2.5), for $t \in [0, T(M,M')]$.*

Proof of lemma 5.1. For $k > M'$ this is a consequence of the fact that $W_{M,M'}$ vanishes. For $k \leqslant M'$ the result is a consequence of (4.1) in theorem 4.1. □

Our purpose is now to extend these estimates to a finite time T independent of M'. From now on let us denote by $T_{\max}(M,M',L)$ the size of the largest interval of the form $[0,T]$ where (5.1), (5.3), (5.4) hold.

LEMMA 5.2. *Let f be the solution of (2.10)–(2.12). There exists a $T > 0$, $T = T(L)$ independent of M and M' such that*

$$f(t,k) \geqslant \tfrac{1}{2} f_0(k), \quad 1 \leqslant k \leqslant 2, \quad t \in [0, \min\{T, T_{\max}(M,M',L)\}). \tag{5.5}$$

Proof of lemma 5.2. Note that

$$\frac{\partial f_1}{\partial t} \geqslant -f_1 \int_{D(k_1)} f_2(1+f_3+f_4) W_{M,M'} \, \mathrm{d}k_3 \, \mathrm{d}k_4 \quad \text{for } 0 \leqslant t \leqslant T_{\max}(M,M',L). \tag{5.6}$$

In order to derive a lower estimate for $\partial f/\partial t$ we need an upper estimate for the integral term on the right-hand side of (5.6). To this end we first use

$$\int_{D(k_1)} f_2 W_{M,M'} \, \mathrm{d}k_3 \, \mathrm{d}k_4 \leqslant \frac{1}{\sqrt{k_1}} \int_0^\infty \int_{-k_2-k_1}^{k_2+k_1} \sqrt{k_2} f_2 \, \mathrm{d}\xi \, \mathrm{d}k_2, \tag{5.7}$$

where $\xi = k_4 - k_3$. Therefore,

$$\begin{aligned} \int_{D(k_1)} f_2 W_{M,M'} \, \mathrm{d}k_3 \, \mathrm{d}k_4 &\leqslant \frac{2L}{\sqrt{k_1}} \int_0^\infty \sqrt{k_2} \frac{\mathrm{e}^{-Dk_2}}{k_2^{7/6}} (k_2+k_1) \, \mathrm{d}k_2 \\ &= CL(k_1^{-1/2} + k_1^{1/2}), \end{aligned} \tag{5.8}$$

where C is a positive constant independent of M, M' and L. On the other hand, a straightforward calculation, using (5.1), gives

$$\int_{D(k_1)} f_2(f_3+f_4)W_{M,M'}\,\mathrm{d}k_3\,\mathrm{d}k_4 \leqslant CL^2k_1^{-2/3}, \tag{5.9}$$

where C is a positive constant independent of M, M' and L. Combining (5.8) and (5.9) we obtain

$$\int_{D(k_1)} f_2(1+f_3+f_4)W_{M,M'}\,\mathrm{d}k_3\,\mathrm{d}k_4 \leqslant CL(k_1^{-1/2}+k_1^{1/2})+CL^2k^{-2/3} \tag{5.10}$$

for $0\leqslant t\leqslant T(M,M')$. Therefore, by (5.6),

$$\frac{\partial f}{\partial t} \geqslant -CL(k_1^{-1/2}+k_1^{1/2})-CL^2k^{-2/3} \quad \text{for } 0\leqslant t\leqslant T_{\max}(M,M',L). \tag{5.11}$$

Integrating this equation for $k\in(1,2)$, lemma 5.2 follows. □

We now prove the following lemma.

Lemma 5.3. *Suppose that f is a solution to (2.10)–(2.12) with initial data f_0. Then, there exist two positive constants, $\rho=\rho(L)$ and $\kappa=\kappa(L)$, independent of M and M' such that*

$$\int_{D(k_1)} f_2(1+f_3+f_4)W_{M,M'}\,\mathrm{d}k_3\,\mathrm{d}k_4 \geqslant \frac{\kappa}{\sqrt{k_1}}\min\{M,k_1\}\chi\left(\frac{k_1}{M'}\right) \tag{5.12}$$

for $0\leqslant t\leqslant T_{\max}(M,M',L)$.

Proof of lemma 5.3. We have

$$\begin{aligned}
&\int_{D(k_1)} f_2(1+f_3+f_4)W_{M,M'}\,\mathrm{d}k_3\,\mathrm{d}k_4 \\
&\qquad \geqslant \chi\left(\frac{k_1}{M'}\right)\int_{D(k_1)} f_2W_{M,M'}\,\mathrm{d}k_3\,\mathrm{d}k_4 \\
&\qquad \geqslant \chi\left(\frac{k_1}{M'}\right)\int_0^{k_1/2}\int_{-k_1-k_2}^{k_1+k_2} W_{M,M'}\,\mathrm{d}\xi\, f_2\,\mathrm{d}k_2 \\
&\qquad \geqslant \frac{2}{\sqrt{k_1}}\chi\left(\frac{k_1}{M'}\right)\int_0^{k_1/2}\sqrt{k_2}f_2\int_0^{k_1-k_2}\chi\left(\frac{\xi}{M}\right)\mathrm{d}\xi\,\mathrm{d}k_2 \\
&\qquad = \frac{1}{\sqrt{k_1}}\chi\left(\frac{k_1}{M'}\right)\min\{k_1,M\}\int_0^{\infty}\sqrt{k_2}f_2\,\mathrm{d}k_2.
\end{aligned} \tag{5.13}$$

Using lemma 5.2, we derive a uniform lower estimate for the last integral and lemma 5.3 follows. □

Lemma 5.4. *Suppose that f is a solution to (2.10)–(2.12) satisfying (5.1) with initial data f_0. Then, there exists a positive constant $\rho=\rho(L)$, independent of M and M', such that*

$$\int_{D(k_1)} f_3f_4(1+f_1+f_2)W_{M,M'}\,\mathrm{d}k_3\,\mathrm{d}k_4 \leqslant C\chi\left(\frac{k_1}{M'}\right)\frac{\mathrm{e}^{-Dk_1}}{k_1^{7/3}}\min\{k_1,M\} \tag{5.14}$$

for $k_1\geqslant\rho$ and $t\leqslant T_{\max}(M,M')$.

Proof of lemma 5.4. Estimate (5.1) implies that there exists $\rho = \rho(L) > 0$ such that $f_1 \leqslant 1$ for $k_1 \geqslant \rho$. Then, for $k_1 \geqslant \rho$,

$$\int_{D(k_1)} f_3 f_4 (1 + f_1 + f_2) W_{M,M'} \, \mathrm{d}k_3 \, \mathrm{d}k_4 \leqslant 2 \int_{D(k_1)} f_3 f_4 (1 + f_2) W_{M,M'} \, \mathrm{d}k_3 \, \mathrm{d}k_4. \tag{5.15}$$

Again using (5.1), we may write

$$\int_{D(k_1)} f_3 f_4 (1 + f_2) W_{M,M'} \, \mathrm{d}k_3 \, \mathrm{d}k_4 = C \mathrm{e}^{-Dk_1} \int_0^\infty \mathrm{d}k_2 (1 + f_2) \mathrm{e}^{-Dk_2} J(k_1, k_2), \tag{5.16}$$

$$J(k_1, k_2) = (k_1 + k_2)^{-7/6} \int_0^{k_1 + k_2} \mathrm{d}\xi \frac{W_{M,M'}}{(k_1 + k_2 - \xi)^{7/6}}, \tag{5.17}$$

where we have used the change of variables $k_2 = k_3 + k_4 - k_1$, $\xi = k_4 - k_3$ and the fact that $k_1 + k_2 + \xi \geqslant k_1 + k_2$ for $\xi \geqslant 0$.

Consider first the case when $\rho < k_1 \leqslant 2M$ and $k_2 \geqslant k_1$. Using the estimate $W_{M,M'} \leqslant k_3^{1/2} k_1^{-1/2} = (k_1 + k_2 - \xi)^{1/2} k_1^{-1/2}$, we deduce that

$$J(k_1, k_2) \leqslant C k_1^{-1/2} (k_1 + k_2)^{-5/6} \leqslant C k_1^{-4/3}. \tag{5.18}$$

On the other hand, we use the fact that

$$W_{M,M'} \leqslant \min\{k_2^{1/2} k_1^{-1/2}, (k_1 + k_2 - \xi)^{1/2} k_1^{-1/2}\}$$

holds if $\rho < k_1 \leqslant 2M$ and $k_2 \leqslant k_1$. Therefore, an explicit computation yields

$$J(k_1, k_2) \leqslant C (k_1 + k_2)^{-7/6} k_2^{1/3} k_1^{-1/2} \leqslant \frac{1}{k_1^{4/3}} \left(\frac{k_2}{k_1} \right)^{1/3}, \tag{5.19}$$

where, in the derivation of this formula, we split the domain of integration in (5.17) into the intervals $(0, k_1 - k_2)$ and $(k_1 - k_2, k_1 + k_3)$. In the original variables these regions are equivalent to $k_4 \leqslant k_1$ and $k_4 \geqslant k_1$, respectively.

Suppose now that $k_1 \geqslant 2M$. In this case, a geometrical argument shows that, for the values of k_3 and k_4 where $W_{M,M'} \neq 0$, they can be estimated from below by means of k_1. More precisely, there exists a positive constant κ, independent of k_1, k_3, k_4, M and M' such that, for $(k_3, k_4) \in D(k_1)$ and $W_{M,M'} \neq 0$, $k_3 \geqslant \kappa k_1$ and $k_4 \geqslant \kappa k_1$ hold. Using $W_{M,M'} \leqslant \min(1, k_2^{1/2} k_1^{-1/2})$, it then follows that

$$J(k_1, k_2) \leqslant C k_1^{-7/3} M \min\{1, k_2^{1/2} k_1^{-1/2}\} \leqslant C k_1^{-7/3} M \min\{1, k_2^{1/3} k_1^{-1/3}\}. \tag{5.20}$$

By (5.18)–(5.20) we obtain

$$J(k_1, k_2) \leqslant \frac{C}{k_1^{7/3}} \min\{k_1, M\} \min\left\{1, \left(\frac{k_2}{k_1}\right)^{1/3}\right\}. \tag{5.21}$$

Plugging this into (5.16), and using (5.1), we conclude the proof of lemma 5.4. □

Combining now the two previous lemmas, we can obtain the following upper estimate for the solutions.

Lemma 5.5. *Suppose that f is a solution to (2.10)–(2.12) satisfying (5.1) in $0 \leqslant t \leqslant T_{\max}(M, M')$ with initial data f_0 satisfying (2.3)–(2.5). Then, there exists $\rho = \rho(L)$ independent of M and M' such that*

$$f(t, k_1) \leqslant \tfrac{1}{2} L k_1^{-7/6} \mathrm{e}^{-Dk_1}, \quad k_1 \geqslant \rho,\ 0 \leqslant t \leqslant T_{\max}(M, M'). \tag{5.22}$$

Proof of lemma 5.5. Using the estimates (5.12) and (5.14) in (2.10), we obtain

$$\frac{\partial f}{\partial t} \leqslant \biggl(C k_1^{-7/3} \mathrm{e}^{-Dk_1} - \frac{\kappa}{\sqrt{k_1}} f \biggr) \min\{M, k_1\} \chi\biggl(\frac{k_1}{M'}\biggr). \tag{5.23}$$

By the maximum principle, we obtain

$$f(k_1, t) \leqslant \max\biggl\{ f_0(k_1), \frac{C}{\kappa} \frac{\mathrm{e}^{-Dk_1}}{k_1^{11/6}} \biggr\}. \tag{5.24}$$

Combining (5.1) and (5.24) yields lemma 5.5. □

As a final step, we prove that (5.22) also holds for $0 < k_1 \leqslant \rho$ as well as the improved estimates (5.3), (5.4). To this end we use the regularity estimates derived for the solutions of (2.10)–(2.12) in § 3.

Proposition 5.6. *Suppose that f is a solution to (2.10)–(2.12) satisfying (5.1), (5.3) and (5.4) in $0 \leqslant t \leqslant T_{\max}(M, M')$ with initial data f_0 satisfying (2.3)–(2.5). There exists $T^* = T^*(A, B, \delta)$ such that, if $T_{\max}(M, M') \leqslant T^*$,*

$$f(t, k_1) \leqslant \tfrac{1}{2} L k_1^{-7/6} \mathrm{e}^{-Dk_1}, \quad 0 < k_1 \leqslant \rho, \tag{5.25}$$

$$|f(t, k) - a(t) k^{-7/6}| \leqslant \tfrac{1}{2} L k^{-7/6+\delta/2}, \qquad k \leqslant 1, \tag{5.26}$$

$$|a(t)| \leqslant \tfrac{1}{2} L \tag{5.27}$$

for $0 \leqslant t \leqslant T_{\max}(M, M')$.

Remark 5.7. The key point in proposition 5.6 is that T^* is independent of M, M'.

Proof of proposition 5.6. Let us pick $M_0 > 0$ sufficiently large but fixed ($M_0 = 4$ for example). We assume from now on that $M \geqslant M_0$, $M' \geqslant M_0$. The equation satisfied by f may be written as

$$\begin{aligned} \frac{\partial f}{\partial t} = \int_{D(k_1)} & W_{M_0, M_0} \tilde{q}(f) \,\mathrm{d}k_3 \,\mathrm{d}k_4 \\ & + \int_{D(k_1)} (W_{M, M'} - W_{M_0, M_0}) \tilde{q}(f) \,\mathrm{d}k_3 \,\mathrm{d}k_4 \\ & \qquad + \int_{D(k_1)} W_{M, M'} (f_3 f_4 - f_1 f_2) \,\mathrm{d}k_3 \,\mathrm{d}k_4. \end{aligned} \tag{5.28}$$

Using (3.62)–(3.65), we can rewrite (5.28) as follows:

$$\frac{\partial f}{\partial t} = \lambda^2(t) \int_{D(k_1)} W_{M_0, M_0} \tilde{\ell}(f_0, g) \,\mathrm{d}k_3 \,\mathrm{d}k_4 + \mathcal{S}, \tag{5.29}$$

$$\begin{aligned}\mathcal{S} = \lambda(t) &\int_{D(k_1)} W_{M_0,M_0}\tilde{n}(f_0,g)\,\mathrm{d}k_3\,\mathrm{d}k_4 + \int_{D(k_1)} (W_{M,M'} - W_{M_0,M_0})\tilde{q}(f)\,\mathrm{d}k_3\,\mathrm{d}k_4 \\ &+ \int_{D(k_1)} W_{M,M'} r(f)\,\mathrm{d}k_3\,\mathrm{d}k_4 + \lambda^3(t)\int_{D(k_1)} W_{M_0,M_0}\tilde{q}(f_0)\,\mathrm{d}k_3\,\mathrm{d}k_4 \\ = &\ \mathcal{S}_1 + \mathcal{S}_2 + \mathcal{S}_3 + \mathcal{S}_4. \end{aligned} \tag{5.30}$$

In order to apply theorem 3.1 we need to bound the source term $\mathcal{S}$ in (5.30). This is done in the following lemma.

LEMMA 5.8. *Suppose that f satisfies the assumptions in proposition 5.6. Then,*

$$\|\mathcal{S}(t)\|_{3/2-\delta,\beta} \leqslant C(L), \quad 0 < t < T_{\max}(M, M') \tag{5.31}$$

Proof of lemma 5.8. The term $\mathcal{S}_1$ in (5.30) is estimated using the same method as for the term $\mathcal{R}_2$ in lemma 4.4. In the third term $\mathcal{S}_3$, in order to obtain an estimate uniform with respect to M we use the exponential decay of f in (5.1) to bound the integral in the region where $k_3 \geqslant 1$ or $k_4 \geqslant 1$. To estimate the contribution in the region where $k_3 \leqslant 1$, $k_4 \leqslant 1$ we use the fact that $r(f)$ is quadratic with respect to f and therefore its contribution is of lower order. Actually, the argument is exactly the same as that which was used in lemma 4.4 to estimate the quadratic terms of $\mathcal{R}_2$. The main novelty arises in the estimate of $\mathcal{S}_2$. Note that the support of $W_{M,M'} - W_{M_0,M_0}$ is contained in the region where $|k_3 - k_4| \geqslant M_0$. On the other hand, we write

$$|\mathcal{S}_2| \leqslant a_1(k_1) f_1 + a_2(k_1), \tag{5.32}$$

where

$$a_1(k_1) = \int_{D(k_1)} |W_{M,M'} - W_{M_0,M_0}|(f_3 f_4 + f_2(f_3 + f_4))\,\mathrm{d}k_3\,\mathrm{d}k_4, \tag{5.33}$$

$$a_2(k_1) = \int_{D(k_1)} |W_{M,M'} - W_{M_0,M_0}| f_2 f_3 f_4\,\mathrm{d}k_3\,\mathrm{d}k_4. \tag{5.34}$$

Since the integration in these two formulae takes place in the region where $|k_3 - k_4| > M_0$, the functions a_1 and a_2 in (5.33), (5.34) can be bounded by a constant independent of M and M' due to the exponential decay of f. Moreover, functions a_1 and a_2 both decay exponentially fast as $k_1 \to \infty$, due to the exponential decay of the function f. □

We now complete the proof of proposition 5.6. The basic idea is once more to apply theorem 3.1. Note that theorem 3.1 is written using the time variable τ, instead of t. However, (2.21) and the fact that $\frac{1}{2} \leqslant \lambda(t) \leqslant 2$ imply that the result of theorem 3.1 can also be applied in the t variable, as it has in §§ 3 and 4. Therefore, theorem 3.1 combined with lemma 5.8 yields

$$|f(k,t)| \leqslant \frac{B}{k^{7/6}}\mathrm{e}^{-Dk} + C\frac{T^{3\delta}}{k^{7/6}}, \quad 0 < k < \rho,\ 0 \leqslant t \leqslant T_{\max}(M, M'), \tag{5.35}$$

where C depends on M_0 but is independent of M and M'. Formula (5.25) follows by choosing T small enough but independent of M and M'. Similarly,

$$\left.\begin{aligned} |a(t)| &\leqslant B + CT^{3\delta/2}, \qquad\qquad 0 \leqslant t \leqslant 1, \\ |f(t,k) - a(t)k^{-7/6}| &\leqslant (B + CT^{-3\delta/2}k^{-7/6+\delta/2}). \end{aligned}\right\} \tag{5.36}$$

Hence, (5.26), (5.27) also follow by choosing T small enough but independent of M, M'. This concludes the proof of proposition 5.6. □

Lemma 5.9. *Suppose that f is a solution to (2.10)–(2.12) satisfying (5.1), (5.3) and (5.4) in $0 \leqslant t \leqslant T_{\max}(M, M')$, with initial data f_0 satisfying (2.3)–(2.5). Then, the solution f constructed in theorem 4.1 can be extended to a time interval $[0, T)$, where $T = (A, B, D, \delta)$ is independent of M and M'.*

Proof of lemma 5.9. Let us denote by $T_{\text{exist}}(M, M')$ the maximal existence time of the solutions constructed in theorem 4.1. If for some $M > M_0$, $M' > M_0$ we have $T_{\text{exist}}(M, M') \geqslant T^*$, the lemma will follow. Let us suppose that, on the contrary, for some $M > M_0$ and $M' > M_0$, we have $T_{\text{exist}}(M, M') < T^*$. By definition, $T_{\max}(M, M') \leqslant T_{\text{exist}}(M, M')$. Moreover, we claim that $T_{\max}(M, M') = T_{\text{exist}}(M, M')$. Indeed, if $T_{\max}(M, M') < T_{\text{exist}}(M, M')$, lemma 5.5 and proposition 5.6 yield a contradiction since estimates (5.1), (5.3) and (5.4) could be extended beyond $T_{\max}(M, M')$. Therefore, as long as the solution of (2.10)–(2.12) exists, these estimates hold. The constants arising in the contractivity argument that gives theorem 4.1 are then independent of M and M' (see remark 4.9). We deduce that there exists a lower bound for $T_{\text{exist}}(M, M')$ independent of M and M' and the result follows. □

5.2. Taking the limit $M \to \infty$, $M' \to \infty$

Proposition 5.10. *Suppose that $f = f_{M,M'}$ are the solutions of (2.10)–(2.12) constructed in theorem 4.1, and defined in the interval of time T independent of M and M'. Then $\lim_{M,M' \to \infty} f_{M,M'}(t, k) = \bar{f}(t, k)$ uniformly on compact sets of $\mathbb{R}^+ \times [0, \bar{T})$. The function $\bar{f}$ is such that $\bar{f} \in \boldsymbol{Y}_{7/6,\beta}$, $\partial_t \bar{f} \in \boldsymbol{Y}_{3/2,\beta}$, it solves (2.1), (2.2) for $0 \leqslant t \leqslant T$ and, moreover, satisfies (5.1), (5.3), (5.4).*

Proof of proposition 5.10. The idea is to prove that the family $\{f_M\}_{M>M_0}$ satisfies the Cauchy condition with the norm $|||f|||_{7/6-\delta/2,\beta}$. Let us write $f = f_{M,M'}$ and $\tilde{f} = f_{\tilde{M},\tilde{M}'}$. It is convenient to use the time variable τ instead of t throughout our argument. Note that the definition of τ in terms of t in (2.21) is different for the solutions f and $\tilde{f}$. We also define $g = f - \lambda(\tau) f_0$ and $\tilde{g} = \tilde{f} - \tilde{\lambda}(\tau) f_0$, where $\lambda = \lambda_{M,M'}$ and $\tilde{\lambda} = \tilde{\lambda}_{\tilde{M},\tilde{M}'}$. Note that functions g and $\tilde{g}$ both solve the problem (2.22), (2.23). Then

$$\frac{\partial(g - \tilde{g})}{\partial \tau} = \iint_{D(k)} W_{M,M'} \tilde{\ell}(f_0, g - \tilde{g}) \,\mathrm{d}k_3 \,\mathrm{d}k_4 + (\lambda_\tau - \tilde{\lambda}_\tau) f_0 + \mathcal{S}_1 + \mathcal{S}_2, \tag{5.37}$$

where

$$\mathcal{S}_1 = \iint_{D(k)} (W_{M,M'} - W_{\tilde{M},\tilde{M}'}) \tilde{\ell}(f_0, \tilde{g}) \,\mathrm{d}k_3 \,\mathrm{d}k_4,$$

$$\mathcal{S}_2 = \iint_{D(k)} \left(\frac{W_{M,M'}}{\lambda} s(f_0, g) - \frac{W_{\tilde{M},\widetilde{M'}}}{\tilde{\lambda}} s(f_0, \tilde{g}) \right) \mathrm{d}k_3 \, \mathrm{d}k_4 + \left(\frac{\mathcal{R}_1 + \mathcal{R}_2}{\lambda^2} - \frac{\tilde{\mathcal{R}}_1 + \tilde{\mathcal{R}}_2}{\tilde{\lambda}^2} \right),$$

and $\mathcal{R}_i$ and $\tilde{\mathcal{R}}_i$, $i = 1, 2$, are defined by means of (2.19) and (2.20) using the functions g and $\tilde{g}$, respectively.

LEMMA 5.11. *Let us define $m \equiv \min(M, M', \tilde{M}, \tilde{M}')$. Then, for some positive constant $C = C(A, B, D, \delta)$,*

$$\|\mathcal{S}_1(t)\|_{3/2-\delta/2,\beta} \leqslant C\mathrm{e}^{-DM/2}, \tag{5.38}$$

$$\|\mathcal{S}_2(t)\|_{3/2-\delta/2,\beta} \leqslant C\mathrm{e}^{-DM/2} + |||g - \tilde{g}|||_{7/6-\delta/2,\beta} + \|\lambda - \tilde{\lambda}\|_{L^\infty(0,T)}. \tag{5.39}$$

Proof of lemma 5.11. We assume without any loss of generality that $\tilde{M} \geqslant M$. The estimate (5.38) is a consequence of the exponential decay of the functions g, $\tilde{g}$ and the fact that the support of $W_{M,M'} - W_{\tilde{M},\tilde{M}'}$ is contained in the region where $k_3 > \frac{1}{2}M$, $k_4 > \frac{1}{2}M$. To estimate $\mathcal{S}_2$, we decompose it into the sum of different terms containing the differences $W_{M,M'} - W_{\tilde{M},\tilde{M}'}$, $g - \tilde{g}$ and $\lambda - \tilde{\lambda}$, by means of the usual triangular argument. □

LEMMA 5.12. *Under the assumptions of proposition 5.10*

$$|\lambda(\tau) - \tilde{\lambda}(\tau)| + |\lambda_\tau(\tau) - \tilde{\lambda}_\tau(\tau)| \leqslant C|||g - \tilde{g}|||_{7/6-\delta/2,\beta}, \quad 0 \leqslant \tau \leqslant T. \tag{5.40}$$

Proof of lemma 5.12. This result is a consequence of the estimates obtained for the derivatives of the solution of the integral equation (4.22) (cf. (4.38)). On the other hand, using (4.39), we have

$$\left| \frac{\mathrm{d}}{\mathrm{d}\tau}\lambda(\tau) - \frac{\mathrm{d}}{\mathrm{d}\tau}\tilde{\lambda}(\tau)\right| \leqslant \frac{1}{A}\int_0^\tau \left|\frac{\partial a}{\partial \tilde{\tau}}(\tau, \bar{\tau})\right| \left|(\lambda_\tau - \tilde{\lambda}_\tau)(\bar{\tau})\right| \mathrm{d}\bar{\tau} + \frac{1}{A}\int_0^\tau \left|\left(\frac{\partial a}{\partial \tilde{\tau}} - \frac{\partial a}{\partial \tilde{\tau}}\right)(\tau, \bar{\tau})\right| |\tilde{\lambda}_\tau| \, \mathrm{d}\tilde{\tau} + |b_\tau(\tau) - \tilde{b}_\tau(\tau)|. \tag{5.41}$$

The first term on the right-hand side of (5.41) is estimated, using (3.90), by

$$CT^{3\delta}\|\lambda_\tau - \tilde{\lambda}_\tau\|_{L^\infty(0,T)}.$$

The second term is estimated, applying theorem 3.1 to (3.85), (3.86), by

$$CT^{3\delta}\|\lambda - \tilde{\lambda}\|_{L^\infty(0,T)}.$$

Finally, the last term can be estimated, applying once more theorem 3.1 to (4.19), (4.20), by $CT^{3\delta/2}\|\lambda - \tilde{\lambda}\|_{L^\infty(0,T)} + C|||g - \tilde{g}|||_{7/6-\delta/2,\beta}$. A similar argument using the equation (4.22) shows that

$$\|\lambda - \tilde{\lambda}\|_{L^\infty(0,T)} \leqslant CT^{3\delta/2}\|\lambda - \tilde{\lambda}\|_{L^\infty(0,T)} + C|||g - \tilde{g}|||_{7/6-\delta/2,\beta}. \tag{5.42}$$

Combining these estimates, the lemma follows for T sufficiently small. □

We now complete the proof of proposition 5.10. Combining theorem 3.1 with lemmas 5.11 and 5.12, we obtain

$$\begin{aligned}\|\lambda-\tilde\lambda\|_{L^\infty(0,T)} &+ |||g-\tilde g|||_{7/6-\delta/2,\beta} \\ &\leqslant CT^{3\delta/2}(\|\lambda-\tilde\lambda\|_{L^\infty(0,T)} + |||g-\tilde g|||_{7/6-\delta/2,\beta}) + C\mathrm{e}^{-DM/2}, \quad (5.43)\end{aligned}$$

whence, for T sufficiently small, we obtain

$$\|\lambda-\tilde\lambda\|_{L^\infty(0,T)} + |||g-\tilde g|||_{7/6-\delta/2,\beta} \leqslant C\mathrm{e}^{-DM/2}$$

and the proposition follows. This shows the existence of $\bar f$ as defined in the statement of the proposition. Note that we may deduce from (2.10) and (2.12) that

$$f_{M,M'}(t,k) = f_0(k) + \int_0^t Q_{M,M'}(f_{M,M'})(s,k)\,\mathrm{d}s.$$

Taking the limit $M, M' \to \infty$, we deduce that

$$\bar f(t,k) = f_0(k) + \int_0^t Q(\bar f)(s,k)\,\mathrm{d}s. \quad (5.44)$$

Since the second term on the right-hand side of (5.44) is a differentiable function of time, we deduce that $\partial_t \bar f = Q(\bar f) \in \boldsymbol{Y}_{3/2,\beta}$. □

We now complete the proof of theorem 2.1.

PROPOSITION 5.13 (uniqueness of solutions). *Suppose that f_0 satisfies (2.3)–(2.5). Then there exists a unique solution of (2.1), (2.2) satisfying (5.1), (5.3) and (5.4).*

Proof of proposition 5.13. The proof is basically the same as that of proposition 5.10. Indeed, if f and $\tilde f$ are two solutions of (2.1), (2.2) satisfying (5.1), (5.3) and (5.4), then they are of the form $f = \lambda(\tau)f_0 + g$, $\tilde f = \tilde\lambda(\tau)f_0 + \tilde g$ with g and $\tilde g$ in the space $\boldsymbol{Y}_{7/6-\delta/2,\beta}$. Arguing exactly as in the proof of (5.43) we obtain

$$\|\lambda-\tilde\lambda\|_{L^\infty(0,T)} + |||g-\tilde g|||_{7/6-\delta/2,\beta} \leqslant CT^{3\delta/2}(\|\lambda-\tilde\lambda\|_{L^\infty(0,T)} + |||g-\tilde g|||_{7/6-\delta/2,\beta}),$$

which yields the desired uniqueness for T small enough. □

The proof of theorem 2.1 is just a consequence of propositions 5.10 and 5.13.

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