RATE OF CONVERGENCE OF THE NANBU PARTICLE SYSTEM FOR HARD POTENTIALS

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Abstract. We consider the (numerically motivated) Nanbu stochastic particle system associated to the spatially homogeneous Boltzmann equation for true hard potentials. We establish a rate of propagation of chaos of the particle system to the unique solution of the Boltzmann equation. More precisely, we estimate the expectation of the squared Wasserstein distance with quadratic cost between the empirical measure of the particle system and the solution. The rate we obtain is almost optimal as a function of the number of particles but is not uniform in time.

1. Introduction and main results

1.1. **The Boltzmann equation.** The Boltzmann equation predicts that the density $f(t,v)$ of particles with velocity $v \in \mathbb{R}^3$ at time $t \geq 0$ in a spatially homogeneous dilute gas solves

$$\partial_t f_t(v) = \frac{1}{2}\int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}^2} d\sigma B(|v-v_*|,\theta)\big[f_t(v')f_t(v'_*) - f_t(v)f_t(v_*)\big], \tag{1.1}$$

where the pre-collisional velocities are given by

$$v' = v'(v,v_*,\sigma) = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma, \quad v'_* = v'_*(v,v_*,\sigma) = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma \tag{1.2}$$

and $\theta = \theta(v,v_*,\sigma)$ is the *deviation angle* defined by $\cos\theta = \frac{(v-v_*)}{|v-v_*|}\cdot\sigma$. The *collision kernel* $B(|v-v_*|,\theta) \geq 0$ depends on the nature of the interactions between particles. See Cercignani [11], Desvillettes [12], Villani [41] and Alexandre [2] for physical and mathematical reviews on this equation. Conservation of mass, momentum and kinetic energy hold at least formally for solutions to (1.1) and we classically may assume without loss of generality that $\int_{\mathbb{R}^3} f_0(v)dv = 1$.

We will assume that the collision kernel is of the form

$$B(|v-v_*|,\theta)\,\sin\theta = \Phi(|v-v_*|)\,\beta(\theta) \quad \text{with} \quad \beta > 0 \ \text{ on } \ (0,\pi/2) \quad \text{and} \quad \beta = 0 \ \text{ on } \ [\pi/2,\pi]. \tag{1.3}$$

This last condition $\beta = 0$ on $(\pi/2,\pi]$ is not a restriction, since one can always reduce to this case for symmetry reasons, as noted in the introduction of Alexandre *et al.* [3].

When particles behave like hard spheres, it holds that $\Phi(z) = z$ and $\beta \equiv 1$. When particles interact through a repulsive force in $1/r^s$, with $s \in (2,\infty)$, one has

$$\Phi(z) = z^\gamma \ \text{ with } \ \gamma = \frac{s-5}{s-1} \in (-3,1) \ \text{ and } \ \beta(\theta) \overset{0}{\sim} \text{cst}\,\theta^{-1-\nu} \ \text{ with } \ \nu = \frac{2}{s-1} \in (0,2).$$

One classically names *hard potentials* the case when $\gamma \in (0,1)$ (*i.e.*, $s > 5$ and $\nu \in (0,1/2)$), *Maxwell molecules* the case when $\gamma = 0$ (*i.e.*, $s = 5$ and $\nu = 1/2$) and *soft potentials* the case when

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$\gamma \in (-3,0)$ (*i.e.*, $s \in (2,5)$ and $\nu \in (1/2,2)$). The present paper concerns Maxwell molecules, hard potentials as well as hard spheres, so that we always assume $\gamma \in [0,1]$.

1.2. **Stochastic particle systems.** As a step to the rigorous derivation of the Boltzmann equation, Kac [26] proposed to show the convergence of a stochastic particle system to the solution to (1.1). Kac's particle system is a $(\mathbb{R}^3)^N$-valued Markov process with infinitesimal generator $\tilde{\mathcal{L}}_N$ defined, for $\phi : (\mathbb{R}^3)^N \mapsto \mathbb{R}$ sufficiently regular and $\mathbf{v} = (v_1, \dots, v_N) \in (\mathbb{R}^3)^N$, by

$$\tilde{\mathcal{L}}_N \phi(\mathbf{v}) = \frac{1}{2(N-1)} \sum_{i \neq j} \int_{\mathbb{S}^2} [\phi(\mathbf{v} + (v'(v_i,v_j,\sigma) - v_i)\mathbf{e}_i + (v'_*(v_i,v_j,\sigma) - v_j)\mathbf{e}_j) - \phi(\mathbf{v})] B(|v_i - v_j|, \theta) d\sigma.$$

For $h \in \mathbb{R}^3$, we note $h\mathbf{e}_i = (0,\dots,0,h,0,\dots,0) \in (\mathbb{R}^3)^N$ with h at the i-th place. Roughly speaking, the system is constituted of N particles entirely characterized by their velocities $(v_1,\dots,v_N)$ and each couple of particles with velocities (v_i,v_j) are modified, for each $\sigma \in \mathbb{S}^2$, at rate $B(|v_i - v_j|,\theta)/(2(N-1))$ and are then replaced by particles with velocities $v'(v_i,v_j,\sigma)$ and $v'_*(v_i,v_j,\sigma)$.

In the present paper, we will consider a slightly modified and non-symmetric particle system introduced by Nanbu [34]. The Nanbu stochastic particle system corresponds to the generator $\mathcal{L}_N$ defined, for $\phi : (\mathbb{R}^3)^N \mapsto \mathbb{R}$ sufficiently regular and $\mathbf{v} = (v_1, \dots, v_N) \in (\mathbb{R}^3)^N$, by

$$\mathcal{L}_N \phi(\mathbf{v}) = \frac{1}{N} \sum_{i \neq j} \int_{\mathbb{S}^2} [\phi(\mathbf{v} + (v'(v_i,v_j,\sigma) - v_i)\mathbf{e}_i) - \phi(\mathbf{v})] B(|v_i - v_j|, \theta) d\sigma. \tag{1.4}$$

This system still describes N particles characterized by their velocities $(v_1,\dots,v_N)$, but now each couple of particles with velocities (v_i,v_j) are modified, for each $\sigma \in \mathbb{S}^2$, at rate $B(|v_i - v_j|,\theta)/N$ and are then replaced by particles with velocities $v'(v_i,v_j,\sigma)$ and v_j. Thus only one particle is modified at each "collision", but the rate of collision is multiplied by 2. All in all, the asymptotic behavior, as $N \to \infty$, should be the same.

1.3. **Aims.** Our aim is to prove that as N tends to ∞, the Nanbu stochastic system is asymptotically constituted of independent particles with identical law governed by the Boltzmann equation, and better, to quantify this convergence.

There are two main motivations for such a study. (i) From a physical point of view, we want to know how well the Boltzmann equation approximates true particles. Of course, true particles are subjected to classical (non random) dynamics. However, Kac's particle system can be seen as such a (strongly) simplified system. In some sense, the Nanbu non-symmetric particle system under study is again a simplification of Kac's dynamics. (ii) From a numerical point of view, we want to know how well the particle system approximates the Boltzmann equation. It is then important to get rates of convergence, to know how to choose the number of particles (and the cutoff parameter) to reach a given accuracy.

One might be surprised at first glance: why approximate a Boltzmann equation by a particle system which the Boltzmann equation is expected to approximate? The reason is of numerical computation order. Indeed, it is numerically interesting to approximate a particle system with a huge number of particles (as in a true gas) by a particle system with much lesser particles.

The main difficulty lies in the fact that even if the particle system is initially constituted of independent particles, they do not remain independent for later times, because of interactions. Hence to answer the convergence issue, we have to prove that particles asymptotically become independent and in the same time to identify their common law: we have to prove that the system is chaotic in the sense of Kac [26].

We are able to prove and quantify the chaotic property for Nanbu's particle system. Unfortunately, our study does really not seem to work for Kac's particle system. From the physical point of view, Nanbu's system is less pertinent. However, we believe that the behaviors of the two systems are very similar, so that our results should also hold true for Kac's particle system. From the numerical point of view, both systems are expected to approximate the solution to (1.1) with an error of the same order, so that the system under study is as interesting as Kac's system.

We will also study a cutoff version of Nanbu's system. This is motivated by two reasons. From a numerical point of view, the particle system with generator $\mathcal{L}_N$ cannot be directly simulated, because each particle collides with infinitely many others on each time interval (except for hard spheres). Thus we have to introduce a cutoff. From a technical point of view, we are not able to prove directly our estimates for the particle system without cutoff: we have to study first the particle system with cutoff and then to pass to the limit.

1.4. **Assumptions.** We assume that the collision kernel is of the form (1.3) with

$$\exists\, \gamma \in [0,1],\ \forall z \geq 0,\ \Phi(z) = z^\gamma, \tag{1.5}$$

and either

$$\forall \theta \in (0,\pi/2),\ \beta(\theta) = 1 \tag{1.6}$$

or

$$\exists\, \nu \in (0,1),\ \exists\, 0 < c_0 < c_1,\ \forall \theta \in (0,\pi/2),\ c_0\theta^{-1-\nu} \leq \beta(\theta) \leq c_1\theta^{-1-\nu}. \tag{1.7}$$

This work could probably be extended to $\nu \in (0,2)$, since the *important* computations on which it relies also hold in this case. However, this would introduce several technical difficulties. Since Maxwell molecules and hard potentials, which we study, satisfy (1.7) with $\nu \in (0,1)$, we decided to avoid these technical complications.

The propagation of exponential moments requires the following additional condition

$$\beta(\theta) = b(\cos\theta) \quad \text{with } b \text{ non-decreasing, convex and } C^1 \text{ on } [0,1). \tag{1.8}$$

In practice, all these assumptions are satisfied for Maxwell molecules ($\gamma = 0$ and $\nu = 1/2$), hard potentials ($\gamma \in (0,1)$ and $\nu \in (1,1/2)$) and hard spheres ($\gamma = 1$ and $\beta \equiv 1$).

1.5. **Notation.** For $\theta \in (0,\pi/2)$ and $z \in [0,\infty)$ we introduce

$$H(\theta) = \int_\theta^{\pi/2} \beta(x)dx \ \text{ and } \ G(z) = H^{-1}(z). \tag{1.9}$$

Under (1.7), H is a continuous decreasing bijection from $(0,\pi/2)$ into $(0,\infty)$, and its inverse function $G : (0,\infty) \mapsto (0,\pi/2]$ is defined by $G(H(\theta)) = \theta$, and $H(G(z)) = z$. It is immediately checked that under (1.7), there are some constants $0 < c_2 < c_3$ such that

$$\forall\, z > 0, \quad c_2(1+z)^{-1/\nu} \leq G(z) \leq c_3(1+z)^{-1/\nu} \tag{1.10}$$

and, as checked in [19, Lemma 1.1], there is a constant $c_4 > 0$ such that for all $x, y \in \mathbb{R}_+$,

$$\int_0^\infty (G(z/x) - G(z/y))^2\, dz \leq c_4 \frac{(x-y)^2}{x+y}. \tag{1.11}$$

Under (1.6), we have $G(z) = (\pi/2 - z)_+$ (with the common notation $x_+ = \max\{x, 0\}$) and a direct computation shows that (1.11) also holds true.

1.6. **Well-posedness.** Let $\mathcal{P}_k(\mathbb{R}^3)$ be the set of all probability measures f on $\mathbb{R}^3$ such that $\int_{\mathbb{R}^3} |v|^k f(dv) < \infty$. We first recall known well-posedness results for the Boltzmann equation, as well as some properties of solutions we will need. A precise definition of weak solutions is stated in the next section.

Theorem 1.1. *Assume* (1.3), (1.5) *and* (1.6) *or* (1.7). *Let* $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$.

(i) If $\gamma = 0$, *there exists a unique weak solution* $(f_t)_{t\geq 0} \in C([0,\infty), \mathcal{P}_2(\mathbb{R}^3))$ *to* (1.1). *If* $f_0 \in \mathcal{P}_p(\mathbb{R}^3)$ *for some* $p \geq 2$, *then* $\sup_{[0,\infty)} \int_{\mathbb{R}^3} |v|^p f_t(dv) < \infty$. *If* $\int_{\mathbb{R}^3} f_0(v) \log f_0(v) dv < \infty$ *or if* $f_0 \in \mathcal{P}_4(\mathbb{R}^3)$ *and is not a Dirac mass, then* f_t *has a density for all* $t > 0$.

(ii) If $\gamma \in (0,1]$, *assume additionally* (1.8) *and that*

$$\exists\, p \in (\gamma, 2), \quad \int_{\mathbb{R}^3} e^{|v|^p} f_0(dv) < \infty. \tag{1.12}$$

There is a unique weak solution $(f_t)_{t\geq 0} \in C([0,\infty), \mathcal{P}_2(\mathbb{R}^3))$ *to* (1.1) *such that*

$$\forall\, q \in (0,p), \quad \sup_{[0,\infty)} \int_{\mathbb{R}^3} e^{|v|^q} f_t(dv) < \infty. \tag{1.13}$$

Under (1.7) *and if* f_0 *is not a Dirac mass, then* f_t *has a density for all* $t > 0$. *Under* (1.6) *and if* f_0 *has a density, then* f_t *has a density for all* $t > 0$.

Concerning well-posedness, see Toscani-Villani [40] for Maxwell molecules, [22, 14] for hard potentials and [5, 33, 27, 15, 28] for hard spheres. The propagation of moments in the Maxwell case in standard, see e.g. Villani [41, Theorem 1 p 74]. The propagation of exponential moments for hard potentials and hard spheres, initiated by Bobylev [7], is checked in [22, 28]. Finally, the existence of a density for f_t has been proved in [17] (under (1.7) and when f_0 is not a Dirac mass and belongs to $\mathcal{P}_4(\mathbb{R}^3)$), in [33] (under (1.6) when f_0 has a density) and is very classical by monotonicity of the entropy when f_0 has a finite entropy, see e.g. Arkeryd [4].

We now introduce our particle system with cutoff.

Proposition 1.2. *Assume* (1.3), (1.5) *and* (1.6) *or* (1.7). *Let* $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ *and a number of particles* $N \geq 1$ *be fixed. Let* $(V_0^i)_{i=1,\dots N}$ *be i.i.d. with common law* f_0.

(i) For each cutoff parameter $K \in [1,\infty)$, *there exists a unique (in law) Markov process* $(V_t^{i,N,K})_{i=1,\dots,N,t\geq 0}$ *with values in* $(\mathbb{R}^3)^N$, *starting from* $(V_0^i)_{i=1,\dots N}$ *and with generator* $\mathcal{L}_{N,K}$ *defined, for all bounded measurable* $\phi : (\mathbb{R}^3)^N \mapsto \mathbb{R}$ *and any* $\mathbf{v} = (v_1, \dots, v_N) \in \mathbb{R}^3$, *by*

$$\mathcal{L}_{N,K}\phi(\mathbf{v}) = \frac{1}{N}\sum_{i\neq j} \int_{\mathbb{S}^2} [\phi(\mathbf{v} + (v'(v_i,v_j,\sigma) - v_i)\mathbf{e}_i) - \phi(\mathbf{v})] B(|v_i - v_j|,\theta) \mathbb{1}_{\{\theta \geq G(K/|v_i-v_j|^\gamma)\}} d\sigma,$$

with G *defined by* (1.9) *and, for* $h \in \mathbb{R}^3$, $h\mathbf{e}_i = (0,\dots,h,\dots,0) \in (\mathbb{R}^3)^N$ *with* h *at the i-th place.*

(ii) There exists a unique (in law) Markov process $(V_t^{i,N,\infty})_{i=1,\dots,N,t\geq 0}$ *with values in* $(\mathbb{R}^3)^N$, *starting from* $(V_0^i)_{i=1,\dots N}$ *and with generator* $\mathcal{L}_N$ *defined, for all Lipschitz bounded function* $\phi : (\mathbb{R}^3)^N \mapsto \mathbb{R}$ *and any* $\mathbf{v} = (v_1,\dots,v_N) \in \mathbb{R}^3$, *by* (1.4).

Let us emphasize that the cut-off used for defining the generator $\mathcal{L}_{N,K}$ is not the usual one since it depends not only on the deviation angle $\theta \in (0, 2\pi)$ but also of the relative velocity $|v - v_*|$. It is more convenient in order to perform the computations we want to do. It might also be convenient for practical simulations. Indeed, the total rate of collision of the particle system does not depend on the configuration of the velocities: it always equals $2\pi(N-1)K$. Hence, the (mean) simulation cost of the particle system on a time interval $[0,T]$ is proportional to $(N-1)KT$.

1.7. **Wasserstein distance.** For $g,\tilde g \in \mathcal{P}_2(\mathbb{R}^3)$, let $\mathcal{H}(g,\tilde g)$ be the set of probability measures on $\mathbb{R}^3\times\mathbb{R}^3$ with first marginal g and second marginal $\tilde g$. We then set

$$\mathcal{W}_2(g,\tilde g) \quad = \quad \inf\left\{\left(\int_{\mathbb{R}^3\times\mathbb{R}^3} |v-\tilde v|^2\,\eta(dv,d\tilde v)\right)^{1/2} ;\quad \eta\in\mathcal{H}(g,\tilde g)\right\}.$$

This is the Wasserstein distance with quadratic cost. It is well-known that the inf is reached. We refer to Villani [42, Chapter 2] for more details on this distance. A remarkable result, due to Tanaka [38, 39], is that in the case of Maxwell molecules, $t\mapsto \mathcal{W}_2(f_t,\tilde f_t)$ is non-increasing for each pair of reasonable solutions $f,\tilde f$ to the Boltzmann equation. The present work is strongly inspired by the ideas of Tanaka.

1.8. **Empirical law of large numbers.** For $f\in\mathcal{P}_2(\mathbb{R}^3)$ and $N\geq 1$, we define

$$\varepsilon_N(f) := \mathbb{E}\left[\mathcal{W}_2^2\left(f, N^{-1}\sum_1^N \delta_{X_i}\right)\right] \text{ with } X_1,\dots,X_N \text{ independent and } f\text{-distributed.} \tag{1.14}$$

Since a f-chaotic stochastic particle system is asymptotically constituted of i.i.d. f-distributed particles, $\varepsilon_N(f)$ is the best rate (as far as $\mathcal{W}_2^2$ is concerned) we can hope for such a system. We summarize in the following statement the best available estimates on $\varepsilon_N(f)$, essentially picked up from Rachev-Ruschendorf [35, Theorem 10.2.1], [31, Lemma 4.2] and Boissard-Le Gouic [8].

Theorem 1.3. *For all $A>0$, all $k>2$, all $f\in\mathcal{P}_k(\mathbb{R}^3)$ verifying $\int_{\mathbb{R}^3}|v|^k f(dv)\leq A$, all $N\geq 1$,*

$$\varepsilon_N(f) \leq \begin{cases} C_{A,k}N^{-2r} & \text{for } r=(k-2)/(5k) & \text{if } k\in(2,7),\\ C_{A,k,r}N^{-2r} & \text{for any } r<1/7 & \text{if } k=7,\\ C_{A,k}N^{-2r} & \text{for } r=1/7 & \text{if } k\in(7,20],\\ C_{A,k,r}N^{-2r} & \text{for any } r<(k-2)/(6k+6) & \text{if } k>20.\end{cases} \tag{1.15}$$

The proof of Theorem 1.3 will be discussed in Appendix A. We shall use different proofs for the different cases, which explains why the statement is awful. The above rates are clearly not optimal. If f has infinitely many moments, then we almost obtain $\varepsilon_N(f)\leq CN^{-1/3}$. This power $1/3$ might be optimal. See Barthe-Bordenave [6] and the references therein for a discussion concerning this issue. We also refer to [25, Theorem 2.13] (and the remarks which follow) for a general discussion about the rate of chaoticity for independent and dependent random arrays.

1.9. **Main result.** Our study concerns both the particle systems with and without cutoff. It is worth to notice that for true Maxwell molecules and hard potentials, $\nu\in(0,1/2]$ so that $1-2/\nu\leq -3$ and the contribution of the cut-off approximation vanishes rapidly in the limit $K\to\infty$.

Theorem 1.4. *Let B be a collision kernel satisfying* (1.3), (1.5) *and* (1.6) *or* (1.7) *and let $f_0\in\mathcal{P}_2(\mathbb{R}^3)$ not be a Dirac mass. If $\gamma>0$, assume additionally* (1.8) *and* (1.12). *Consider the unique weak solution $(f_t)_{t\geq 0}$ to* (1.1) *defined in Theorem 1.1 and, for each $N\geq 1$, $K\in[1,\infty]$, the unique Markov process $(V_t^{i,N,K})_{i=1,\dots,N,t\geq 0}$ defined in Proposition 1.2. Let $\mu_t^{N,K} := N^{-1}\sum_1^N \delta_{V_t^{i,N,K}}$.*

(i) Maxwell molecules. Assume that $\gamma=0$, (1.7) *and either $\int_{\mathbb{R}^3} f_0(v)\log f_0(v)dv<\infty$ or $f_0\in\mathcal{P}_4(\mathbb{R}^3)$. There is a constant C such that for all $T\geq 0$, all $N\geq 1$, all $K\in[1,\infty]$,*

$$\sup_{[0,T]}\mathbb{E}[\mathcal{W}_2^2(\mu_t^{N,K},f_t)] \leq C(1+T)^2\sup_{[0,T]}\varepsilon_N(f_t) + CTK^{1-2/\nu}. \tag{1.16}$$

If $f_0 \in \mathcal{P}_k(\mathbb{R}^3)$ for some $k>2$, we have $\sup_{[0,\infty)} \int_{\mathbb{R}^3} |v|^k f_t(dv) < \infty$ and we can use Theorem 1.3 to bound $\sup_{[0,T]} \varepsilon_N(f_t)$. In particular if $f_0 \in \mathcal{P}_k(\mathbb{R}^3)$ for all $k \geq 2$, then for all $\varepsilon > 0$, all $T \geq 0$, all $N \geq 1$, all $K \in [1,\infty]$,

$$\sup_{[0,T]} \mathbb{E}[\mathcal{W}_2^2(\mu_t^{N,K}, f_t)] \leq C_\varepsilon (1+T)^2 N^{-1/3+\varepsilon} + CTK^{1-2/\nu}. \tag{1.17}$$

(ii) Hard potentials. Assume that $\gamma \in (0,1)$ and (1.7). For all $\varepsilon \in (0,1)$, all $T \geq 0$, there is a constant $C_{\varepsilon,T}$ such that for all $N \geq 1$, all $K \in [1,\infty]$,

$$\sup_{[0,T]} \mathbb{E}[\mathcal{W}_2^2(\mu_t^{N,K}, f_t)] \leq C_{\varepsilon,T} \left(\sup_{[0,T]} \varepsilon_N(f_t) + K^{1-2/\nu} \right)^{1-\varepsilon}. \tag{1.18}$$

Consequently, for all $\varepsilon \in (0,1)$, all $T \geq 0$, there is $C_{\varepsilon,T}$ such that for all $N \geq 1$, all $K \in [1,\infty]$,

$$\sup_{[0,T]} \mathbb{E}[\mathcal{W}_2^2(\mu_t^{N,K}, f_t)] \leq C_{\varepsilon,T} (N^{-1/3} + K^{1-2/\nu})^{1-\varepsilon}. \tag{1.19}$$

(iii) Hard spheres. Assume finally that $\gamma = 1$, (1.6) and that f_0 has a density. For all $\varepsilon \in (0,1)$, all $T \geq 0$, all $q \in (1,p)$, there is a constant $C_{\varepsilon,q,T}$ such that for all $N \geq 1$, all $K \in [1,\infty)$,

$$\sup_{[0,T]} \mathbb{E}[\mathcal{W}_2^2(\mu_t^{N,K}, f_t)] \leq C_{\varepsilon,q,T} \left(\left(\sup_{[0,T]} \varepsilon_N(f_t) \right)^{1-\varepsilon} + e^{-K^q} \right) e^{C_{\varepsilon,q,T} K}. \tag{1.20}$$

Thus for all $\varepsilon \in (0,1)$, all $T \geq 0$, all $q \in (1,p)$, there is $C_{\varepsilon,q,T}$ such that for all $N \geq 1$, all $K \in [1,\infty)$,

$$\sup_{[0,T]} \mathbb{E}[\mathcal{W}_2^2(\mu_t^{N,K}, f_t)] \leq C_{\varepsilon,q,T} (N^{-1/3+\varepsilon} + e^{-K^q}) e^{C_{\varepsilon,q,T} K}. \tag{1.21}$$

Concerning the rate of convergence of the simulation algorithm, we have the following.

Remark 1.5. *Recall that the simulation cost per unit of time is proportional to $(N-1)K$.*

(i) For Maxwell molecules and hard potentials the error (for $\mathcal{W}_2$) is $(N^{-1/6} + K^{1/2-1/\nu})^{1-}$. For a given simulation cost τ, the best choices are $N \simeq \tau^{(6-3\nu)/(6-2\nu)}$ and $K \simeq \tau^{\nu/(6-3\nu)}$, which leads to an error in $\tau^{-(2-\nu)/(12-4\nu)+}$. For true hard potentials and Maxwell molecules, this is at worst $\tau^{-3/20+}$ and at best $\tau^{-1/6+}$.

(ii) For hard spheres, make the choice $K \simeq (\log N)^a$ with $a \in (1/q, 1)$. Then $e^{CK} << N^\varepsilon$ for any $\varepsilon \in (0,1)$ and $e^{-K^q} << N^{-r}$ for any $r > 1$. With this choice, we thus find an error in $N^{-1/6+\varepsilon}$ for a simulation cost in $N(\log N)^a$. Consequently, for a given simulation cost τ, we find an error in $\tau^{-1/6+}$.

We excluded the case where f_0 is a Dirac mass because we need that f_t has a density and because if $f_0 = \delta_{v_0}$, then the unique solution to (1.1) is given by $f_t = \delta_{v_0}$ and the Markov process of Proposition 1.2 is nothing but $V_t^{1,N,K} = (v_0, \dots v_0)$ (for any value of $K \in [1,\infty]$), so that $\mu_t^{N,K} = \delta_{v_0}$ and thus $\mathcal{W}_2(f_t, \mu_t^{N,K}) = 0$.

1.10. **Comments.** We thus show that the empirical law of the particle system converges to f_t as fast as i.i.d. f_t-distributed particles (up to an arbitrary small loss if $\gamma \neq 0$). This is thus almost optimal in some sense. However, this is optimal only as far as $\mathcal{W}_2$ is concerned: we would have preferred to work with another distance and to obtain a rate in $N^{-1/2}$ as is expected for laws of large numbers. Here we obtain a rate in $N^{-1/6}$, since $\mathcal{W}_2$ is squared. However, $\mathcal{W}_2$ enjoys several properties that make it quite convenient when studying the Boltzmann equation, mainly because of the role of the kinetic energy. Another default of this work is that we obtain a non-uniform (in time) bound. For Maxwell molecules, the bound is slowly increasing (as T^2) but for hard potentials, it is growing very fast.

Note also that for hard spheres, we are not able to treat the case where $K = \infty$: we need to let K and N go to infinity simultaneously, with some constraints. We believe that this is only a technical problem, but we were not able to solve it. However, we still obtain a very reasonable rate of convergence (as a function of the computational cost).

Our proof is based on a coupling argument: we couple the N-particle system with a family of N i.i.d. Boltzmann processes, in such a way that they remain as close as possible. We prove an accurate control on the increment of the distance between the two systems at each collision. This last computation is similar to those of [22, 19] concerning uniqueness of the solution to (1.1). However, we need to handle much more precise computations: in [22], when studying the distance between two solutions to (1.1), both were supposed to have exponential moments. Such exponential moments are known to propagate for solutions to (1.1) since the seminal work of Bobylev [7], but for the particle system under study, we are not even able to prove the finiteness of a moment of order $2+\varepsilon$, $\varepsilon > 0$! We thus need a very precise refinement of the computations of [22, 19].

All these problems do not appear when studying Maxwell molecules. Roughly, the collision operator is globally Lipschitz continuous for Maxwell molecules and only locally Lipschitz continuous for hard potentials (which explains why large velocities have to be controlled by using exponential moments). This is why we obtain a better result for Maxwell molecules.

Note that for the (physically more relevant) Kac particle system moments are known to propagate (uniformly in N), see Sznitman [36] and also [31], which would simplify greatly the proof at many places. However, we are not able to exhibit a suitable coupling. This is due to the fact that in Kac's system, each collision modifies the velocity of two particles. In Nanbu's system, the Poisson measures governing two different particles are independent, which is not the case for Kac's system (because each time a particle's velocity is modified, another one has to be also modified) although the larger is the number of particles, the lower the correlation is. As a consequence, it is more difficult to couple the N-particle symmetric Kac's system with N independent copies of the Boltzmann process and we did not succeed.

1.11. **Known results.** Such a chaos result for the Boltzmann equation with bounded cross section, or for related models, has been first established without any rate by Kac [26] (for the so-called Maxwell molecules Kac's model which is roughly a "toy one-dimensional" Boltzmann equation) and then by McKean [30] and Grünbaum [24]. For unbounded cross section, the chaos property has been proved by Sznitman [36] for hard spheres, still without rate.

For Maxwell molecules with Grad's cutoff, a nice rate of convergence (of order $1/N$ in total variation distance on the two-marginal) has been obtained by McKean [29] and improved by Graham-Méléard [23]. This was extended by Desvillettes-Graham-Méléard [13], see also [20], to true (without Grad's cutoff) Maxwell molecules, but with a rate in $N^{-1}e^{KT} + K^{1-2/\nu}$ (with the notation of the present paper). From a numerical point of view, this leads to a logarithmic convergence as a function of the computational cost.

More recently, a uniform in time rate of chaos convergence of Kac's stochastic particle system to the Boltzmann equation for two unbounded models has been established in [31, 10] (see also [32]), by taking up again and improving Grünbaum's approach. For true Maxwell molecules, uniform in time rate of convergence of order $N^{-1/(6+\delta)}$, for any $\delta > 0$, for a weak distance on the two-marginals has been proved in [31, Theorem 5.1] when the initial condition f_0 has a compact support. This result was improved and made more precise in [10, Step 3 of the proof of Theorem 8], where, still for true Maxwell molecules, uniform in time rate of convergence of order $N^{-1/177}$, for the same $\mathcal{W}_2$ Wasserstein distance as used in (1.16), has been proved for any initial condition f_0 satisfying (1.12). Hard spheres have also been studied in [31, Theorem 6.1]: a uniform in time rate of convergence of order $1/(\log N)^\alpha$ with $\alpha > 0$ small, for the $\mathcal{W}_1$ distance on the two-marginals has been proved. When applying the methods of [31, 32, 10] on finite time intervals, the previous rates can not be really improved. Finally, let us mention that the present work follows some of the ideas of [18], which concerns the Kac equation.

To summarize:

• We obtain the first rate of convergence for hard potentials and this rate is reasonable. Recall that hard potentials are twice unbounded (the velocity cross section is unbounded and the angular cross section is non-integrable), while Maxwell molecules enjoy a bounded velocity cross section and hard spheres an integrable angular cross section.

• For hard spheres and Maxwell molecules, we prove a much faster convergence than [31, 32, 10], but we are restricted to finite time-intervals and we cannot study Kac's system.

Let us finally mention that we use a coupling method, as is widely used since the famous *cours à l'école d'été de Saint-Flour* by Sznitman [37] for providing rate of chaos convergence for the so-called McKean-Vlasov model and that such methods have been recently adapted to non-globally Lipschitz coefficients by Bolley-Cañizo-Carrillo in [9], making use of exponential moments.

1.12. **Plan of the paper.** In Section 2, we make precise the notion of weak solutions, rewrite the collision operators in a suitable form and check a accurate version of a lemma due to Tanaka [39]. Section 3 is devoted to the cornerstone estimate on the collision integral. In Section 4 we prove the convergence of the particle system with cutoff. The cutoff is removed in Section 5. Finally, elements of the proof of Theorem 1.3 on quantitative law of large number for empirical measures are presented in Appendix A.

2. Preliminaries

2.1. **Rewriting equations.** We follow here [20]. For each $X \in \mathbb{R}^3$, we introduce $I(X), J(X) \in \mathbb{R}^3$ such that $(\frac{X}{|X|}, \frac{I(X)}{|X|}, \frac{J(X)}{|X|})$ is a direct orthonormal basis of $\mathbb{R}^3$ and, of course, in such a way that I, J are measurable functions. For $X, v, v_* \in \mathbb{R}^3$, for $\theta \in (0, \pi/2)$ and $\varphi \in [0, 2\pi)$, we set

$$\left\{ \begin{array}{l} \Gamma(X,\varphi) := (\cos\varphi) I(X) + (\sin\varphi) J(X), \\ \\ a(v,v_*,\theta,\varphi) := -\dfrac{1-\cos\theta}{2}(v-v_*) + \dfrac{\sin\theta}{2}\Gamma(v-v_*,\varphi), \\ \\ v'(v,v_*,\theta,\varphi) := v + a(v,v_*,\theta,\varphi), \end{array} \right. \tag{2.1}$$

which is a suitable parametrization of (1.2): write $\sigma \in \mathbb{S}^2$ as $\sigma = \frac{v-v_*}{|v-v_*|}\cos\theta + \frac{I(v-v_*)}{|v-v_*|}\sin\theta\cos\varphi + \frac{J(v-v_*)}{|v-v_*|}\sin\theta\sin\varphi$. Let us define, classically, weak solutions to (1.1).

Definition 2.1. *Assume* (1.3), (1.5) *and* (1.6) *or* (1.7). *A family* $(f_t)_{t\geq 0} \in C([0,\infty), \mathcal{P}_2(\mathbb{R}^3))$ *is called a weak solution to* (1.1) *if it preserves momentum and energy, i.e.*

$$\forall\, t \geq 0, \quad \int_{\mathbb{R}^3} v f_t(dv) = \int_{\mathbb{R}^3} v f_0(dv) \quad \textit{and} \quad \int_{\mathbb{R}^3} |v|^2 f_t(dv) = \int_{\mathbb{R}^3} |v|^2 f_0(dv) \tag{2.2}$$

and if for any $\phi : \mathbb{R}^3 \mapsto \mathbb{R}$ bounded and Lipschitz-continuous, any $t \in [0,T]$,

$$\int_{\mathbb{R}^3} \phi(v)\, f_t(dv) = \int_{\mathbb{R}^3} \phi(v)\, f_0(dv) + \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathcal{A}\phi(v,v_*) f_s(dv_*) f_s(dv) ds \tag{2.3}$$

where

$$\mathcal{A}\phi(v,v_*) = |v-v_*|^\gamma \int_0^{\pi/2} \beta(\theta) d\theta \int_0^{2\pi} d\varphi \left[\phi(v + a(v,v_*,\theta,\varphi)) - \phi(v)\right]. \tag{2.4}$$

Noting that $|a(v,v_*,\theta,\varphi)| \leq C\theta|v-v_*|$ and that $\int_0^{\pi/2} \theta\beta(\theta)d\theta$, we easily get $|\mathcal{A}\phi(v,v_*)| \leq C_\phi|v-v_*|^{1+\gamma} \leq C_\phi(1+|v-v_*|^2)$, so that everything makes sense in (2.3).

We next rewrite the collision operator in a way that makes disappear the velocity-dependence $|v-v_*|^\gamma$ in the *rate*. Such a trick was already used in [21] and [19].

Lemma 2.2. *Assume* (1.3)*,* (1.5) *and* (1.6) *or* (1.7)*. Recalling* (1.9) *and* (2.1)*, define, for $z \in (0,\infty)$, $\varphi \in [0,2\pi)$, $v, v_* \in \mathbb{R}^3$ and $K \in [1,\infty)$,*

$$c(v,v_*,z,\varphi) := a[v,v_*,G(z/|v-v_*|^\gamma),\varphi] \quad \textit{and} \quad c_K(v,v_*,z,\varphi) := c(v,v_*,z,\varphi)\mathbb{1}_{\{z\leq K\}}. \tag{2.5}$$

For any bounded Lipschitz $\phi : \mathbb{R}^3 \mapsto \mathbb{R}$, any $v, v_* \in \mathbb{R}^3$

$$\mathcal{A}\phi(v,v_*) \quad = \quad \int_0^\infty dz \int_0^{2\pi} d\varphi \Big(\phi[v + c(v,v_*,z,\varphi)] - \phi[v]\Big). \tag{2.6}$$

For any $N \geq 1$, $K \in [1,\infty)$, $\mathbf{v} = (v_1,\dots,v_N) \in (\mathbb{R}^3)^N$, any bounded measurable $\phi : (\mathbb{R}^3)^N \mapsto \mathbb{R}$,

$$\mathcal{L}_{N,K}\phi(\mathbf{v}) = \frac{1}{N}\sum_{i\neq j} \int_0^\infty dz \int_0^{2\pi} d\varphi [\phi(\mathbf{v} + c_K(v_i,v_j,z,\varphi)\mathbf{e}_i) - \phi(\mathbf{v})]. \tag{2.7}$$

For any $N \geq 1$, any $\mathbf{v} = (v_1,\dots,v_N) \in (\mathbb{R}^3)^N$, any bounded Lipschitz $\phi : (\mathbb{R}^3)^N \mapsto \mathbb{R}$,

$$\mathcal{L}_N\phi(\mathbf{v}) = \frac{1}{N}\sum_{i\neq j} \int_0^\infty dz \int_0^{2\pi} d\varphi [\phi(\mathbf{v} + c(v_i,v_j,z,\varphi)\mathbf{e}_i) - \phi(\mathbf{v})]. \tag{2.8}$$

Proof. To get (2.6), start from (2.4) and use the substitution $\theta = G(z/|v-v_*|^\gamma)$ or equivalently $H(\theta) = z/|v-v_*|^\gamma$, which implies $|v-v_*|^\gamma\beta(\theta)d\theta = dz$. The expressions (2.7) and (2.8) are checked similarly. □

2.2. **Accurate version of Tanaka's trick.** As was already noted by Tanaka [39], it is not possible to choose I in such a way that $X \mapsto I(X)$ is continuous. However, he found a way to overcome this difficulty, see also [20, Lemma 2.6]. Here we need the following accurate version of Tanaka's trick.

Lemma 2.3. *Recall* (2.1)*. There are some measurable functions $\varphi_0, \varphi_1 : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto [0,2\pi)$, such that for all $X, Y \in \mathbb{R}^3$, all $\varphi \in [0,2\pi)$,*

$$\Gamma(X,\varphi)\cdot\Gamma(Y,\varphi+\varphi_0(X,Y)) = X\cdot Y\cos^2(\varphi+\varphi_1(X,Y)) + |X||Y|\sin^2(\varphi+\varphi_1(X,Y)),$$
$$|\Gamma(X,\varphi) - \Gamma(Y,\varphi+\varphi_0(X,Y))| \leq |X-Y|.$$

Proof. First observe that the second claim follows from the first one: writing $\varphi_i = \varphi_i(X,Y)$

$$\begin{aligned}|\Gamma(X,\varphi) - \Gamma(Y,\varphi+\varphi_0)|^2 =& |\Gamma(X,\varphi)|^2 + |\Gamma(Y,\varphi+\varphi_0)|^2 - 2\Gamma(X,\varphi)\cdot\Gamma(Y,\varphi+\varphi_0)\\ =& |X|^2 + |Y|^2 - 2(X\cdot Y\cos^2(\varphi+\varphi_1) + |X||Y|\sin^2(\varphi+\varphi_1))\\ \leq& |X|^2 + |Y|^2 - 2X\cdot Y = |X-Y|^2.\end{aligned}$$

We next check the first claim. Let thus X and Y be fixed. Observe that $\Gamma(X,\varphi)$ goes (at constant speed) all over the circle C_X with radius $|X|$ lying in the plane orthogonal to X. Let $i_X \in C_X$ and $i_Y \in C_Y$ such that X, Y, i_X, i_Y belong to the same plane and $i_X \cdot i_Y = X \cdot Y$ (there are exactly two possible choices for the couple (i_X, i_Y) if X and Y are not collinear, infinitely many otherwise). Consider φ_X and φ_Y such that $i_X := \Gamma(X, \varphi_X)$ and $i_Y := \Gamma(Y, \varphi_Y)$. Define $j_X := \Gamma(X, \varphi_X + \pi/2)$ and $j_Y := \Gamma(Y, \varphi_Y + \pi/2)$. Then j_X and j_X are collinear (because both are orthogonal to the plane containing X, Y, i_X, i_Y), satisfy $j_X \cdot j_Y = |j_X||j_Y| = |X||Y|$ and $i_X \cdot j_Y = i_Y . j_X = 0$. Next, observe that $\Gamma(X, \varphi + \varphi_X) = i_X \cos\varphi + j_X \sin\varphi$ while $\Gamma(Y, \varphi + \varphi_Y) = i_Y \cos\varphi + j_Y \sin\varphi$. Consequently, $\Gamma(X, \varphi + \varphi_X) \cdot \Gamma(Y, \varphi + \varphi_Y) = i_X \cdot i_Y \cos^2\varphi + j_X \cdot j_Y \sin^2\varphi = X \cdot Y \cos^2\varphi + |X||Y| \sin^2\varphi$. The conclusion follows: choose $\varphi_0 := \varphi_Y - \varphi_X$ and $\varphi_1 := -\varphi_X$ (all this modulo 2π). ☐

3. Main computations of the paper

The following estimate is our central argument.

Lemma 3.1. *Recall that G was defined in* (1.9) *and that the deviation functions c and c_K were defined in* (2.5). *For any $v, v_*, \tilde v, \tilde v_* \in \mathbb{R}^3$, any $K \in [1,\infty)$,*

$$\int_0^\infty \int_0^{2\pi} \Big(\big| v + c(v, v_*, z, \varphi) - \tilde v - c_K(\tilde v, \tilde v_*, z, \varphi + \varphi_0(v - v_*, \tilde v - \tilde v_*)) \big|^2 - |v - \tilde v|^2 \Big) d\varphi dz$$
$$\leq A_1^K(v, v_*, \tilde v, \tilde v_*) + A_2^K(v, v_*, \tilde v, \tilde v_*) + A_3^K(v, v_*, \tilde v, \tilde v_*),$$

where, setting $\Phi_K(x) = \pi \int_0^K (1 - \cos G(z/x^\gamma)) dz$ and $\Psi_K(x) = \pi \int_K^\infty (1 - \cos G(z/x^\gamma)) dz$,

$$\begin{aligned} A_1^K(v, v_*, \tilde v, \tilde v_*) =& 2|v - v_*||\tilde v - \tilde v_*| \int_0^K \big[G(z/|v - v_*|^\gamma) - G(z/|\tilde v - \tilde v_*|^\gamma) \big]^2 dz, \\ A_2^K(v, v_*, \tilde v, \tilde v_*) =& - \big[(v - \tilde v) + (v_* - \tilde v_*) \big] \cdot \big[(v - v_*) \Phi_K(|v - v_*|) - (\tilde v - \tilde v_*) \Phi_K(|\tilde v - \tilde v_*|) \big], \\ A_3^K(v, v_*, \tilde v, \tilde v_*) =& (|v - v_*|^2 + 2|v - \tilde v||v - v_*|) \Psi_K(|v - v_*|). \end{aligned}$$

Proof. We need to shorten notation. We write $x = |v - v_*|$, $\tilde x = |\tilde v - \tilde v_*|$, $\varphi_0 = \varphi_0(v - v_*, \tilde v - \tilde v_*)$, $c = c(v, v_*, z, \varphi)$, $\tilde c = c(\tilde v, \tilde v_*, z, \varphi + \varphi_0)$ and $\tilde c_K = c_K(\tilde v, \tilde v_*, z, \varphi + \varphi_0) = \tilde c \mathbb{1}_{\{z \leq K\}}$. We start with

$$\begin{aligned} \Delta_K :=& \int_0^\infty \int_0^{2\pi} \Big(|v + c - \tilde v - \tilde c_K|^2 - |v - \tilde v|^2 \Big) d\varphi dz \\ =& \int_0^K \int_0^{2\pi} \Big(|c|^2 + |\tilde c|^2 - 2c \cdot \tilde c + 2(v - \tilde v) \cdot (c - \tilde c) \Big) d\varphi dz \\ &+ \int_K^\infty \int_0^{2\pi} \Big(|c|^2 + 2(v - \tilde v) \cdot c \Big) d\varphi dz. \end{aligned}$$

First, it holds that $|c|^2 = | - (1 - \cos G(z/x^\gamma))(v - v_*) + (\sin G(z/x^\gamma)) \Gamma(v - v_*, \varphi)|^2/4 = (1 - \cos G(z/x^\gamma))|v - v_*|^2/2$. We used that by definition, see (2.1), $\Gamma(v - v_*, \varphi)$ has the same norm as $v - v_*$ and is orthogonal to $v - v_*$ and that $(1 - \cos\theta)^2 + (\sin\theta)^2 = 2 - 2\cos\theta$. Consequently, we have

$$\int_0^K \int_0^{2\pi} |c|^2 d\varphi dz = \pi |v - v_*|^2 \int_0^K (1 - \cos G(z/x^\gamma)) dz = x^2 \Phi_K(x).$$

Similarly, we also have $\int_0^K \int_0^{2\pi} |\tilde c|^2 d\varphi dz = \tilde x^2 \Phi_K(\tilde x)$ and $\int_K^\infty \int_0^{2\pi} |c|^2 d\varphi dz = x^2 \Psi_K(x)$.

Next, using that $c = -(1-\cos G(z/x^\gamma))(v-v_*)/2 + (\sin G(z/x^\gamma))\Gamma(v-v_*,\varphi)/2$ and that $\int_0^{2\pi}\Gamma(v-v_*,\varphi)d\varphi = 0$,

$$\int_0^K \int_0^{2\pi} c d\varphi dz = -(v-v_*)\pi \int_0^K (1-\cos G(z/x^\gamma))dz = -(v-v_*)\Phi_K(x).$$

By the same way, $\int_0^K\int_0^{2\pi}\tilde c d\varphi dz = -(\tilde v-\tilde v_*)\Phi_K(\tilde x)$ and $\int_K^\infty\int_0^{2\pi} c d\varphi dz = -(v-v_*)\Psi_K(x)$.

Finally, $c\cdot\tilde c = [(1-\cos G(z/x^\gamma))(v-v_*) - (\sin G(z/x^\gamma))\Gamma(v-v_*,\varphi)]\cdot[(1-\cos G(z/\tilde x^\gamma))(\tilde v-\tilde v_*) - (\sin G(z/\tilde x^\gamma))\Gamma(\tilde v-\tilde v_*,\varphi+\varphi_0)]/4$. Since $\int_0^{2\pi}\Gamma(v-v_*,\varphi)d\varphi = \int_0^{2\pi}\Gamma(\tilde v-\tilde v_*,\varphi+\varphi_0)d\varphi = 0$, we get

$$\begin{aligned}\int_0^{2\pi} c\cdot\tilde c d\varphi =& \frac{\pi}{2}(1-\cos G(z/x^\gamma))(1-\cos G(z/\tilde x^\gamma))(v-v_*)\cdot(\tilde v-\tilde v_*)\\ &+\frac14(\sin G(z/x^\gamma))(\sin G(z/\tilde x^\gamma))\int_0^{2\pi}\Gamma(v-v_*,\varphi)\cdot\Gamma(\tilde v-\tilde v_*,\varphi+\varphi_0)d\varphi.\end{aligned}$$

Recalling Lemma 2.3 and using that $\int_0^{2\pi}\cos^2(\varphi+\varphi_1)d\varphi = \int_0^{2\pi}\sin^2(\varphi+\varphi_1)d\varphi = \pi$, we obtain

$$\begin{aligned}\int_0^{2\pi} c\cdot\tilde c d\varphi =& \frac{\pi}{2}(1-\cos G(z/x^\gamma))(1-\cos G(z/\tilde x^\gamma))(v-v_*)\cdot(\tilde v-\tilde v_*)\\ &+\frac{\pi}{4}(\sin G(z/x^\gamma))(\sin G(z/\tilde x^\gamma))\big[(v-v_*)\cdot(\tilde v-\tilde v_*)+|v-v_*||\tilde v-\tilde v_*|\big].\end{aligned}$$

But G takes values in $(0,\pi/2)$, so that, since $|v-v_*||\tilde v-\tilde v_*|\geq (v-v_*)\cdot(\tilde v-\tilde v_*)$,

$$\begin{aligned}&\int_0^{2\pi} c\cdot\tilde c d\varphi\\ \geq&\frac{\pi}{2}[(1-\cos G(z/x^\gamma))(1-\cos G(z/\tilde x^\gamma))+(\sin G(z/x^\gamma))(\sin G(z/\tilde x^\gamma))](v-v_*)\cdot(\tilde v-\tilde v_*)\\ =&\frac{\pi}{2}[(1-\cos G(z/x^\gamma))+(1-\cos G(z/\tilde x^\gamma))](v-v_*)\cdot(\tilde v-\tilde v_*)\\ &-\frac{\pi}{2}(1-\cos(G(z/x^\gamma)-G(z/\tilde x^\gamma)))(v-v_*)\cdot(\tilde v-\tilde v_*).\end{aligned}$$

Using that $\pi(1-\cos\theta)\leq 2\theta^2$, we thus get

$$\int_0^K\int_0^{2\pi} c\cdot\tilde c d\varphi dz \geq (v-v_*)\cdot(\tilde v-\tilde v_*)\frac{\Phi_K(x)+\Phi_K(\tilde x)}{2} - x\tilde x\int_0^K (G(z/x^\gamma)-G(z/\tilde x^\gamma))^2dz.$$

All in all, we find

$$\begin{aligned}\Delta_K \leq& x^2\Phi_K(x)+\tilde x^2\Phi_K(\tilde x)-(v-v_*)\cdot(\tilde v-\tilde v_*)[\Phi_K(x)+\Phi_K(\tilde x)]\\ &+2(v-\tilde v)\cdot[(\tilde v-\tilde v_*)\Phi_K(\tilde x)-(v-v_*)\Phi_K(x)]\\ &+2x\tilde x\int_0^K (G(z/x^\gamma)-G(z/\tilde x^\gamma))^2dz\\ &+x^2\Psi_K(x)-2(v-\tilde v)\cdot(v-v_*)\Psi_K(x).\end{aligned}$$

Recalling that $x=|v-v_*|$, $\tilde x = |\tilde v-\tilde v_*|$, we realize that the third line is nothing but $A_1^K(v,v_*,\tilde v,\tilde v_*)$ while the fourth one is bounded from above by $A_3^K(v,v_*,\tilde v,\tilde v_*)$. To conclude, it suffices to note that the sum of the terms on the two first lines equals

$$\begin{aligned}=&(v-v_*)\cdot[(v-v_*)-(\tilde v-\tilde v_*)-2(v-\tilde v)]\Phi_K(x)\\&+(\tilde v-\tilde v_*)\cdot[(\tilde v-\tilde v_*)-(v-v_*)+2(v-\tilde v)]\Phi_K(\tilde x)\\=&-(v-v_*)\cdot((v-\tilde v)+(v_*-\tilde v_*))\Phi_K(x)+(\tilde v-\tilde v_*)\cdot((v-\tilde v)+(v_*-\tilde v_*))\Phi_K(\tilde x)\end{aligned}$$

which is $A_2^K(v,v_*,\tilde v,\tilde v_*)$ as desired. □

Next, we study each term found in the previous inequality. We start with the Maxwell case.

Lemma 3.2. *Assume* (1.3), (1.5) *with* $\gamma=0$, (1.7) *and adopt the notation of Lemma 3.1. For all* $K\in[1,\infty)$, *all* $v,v_*,\tilde v,\tilde v_*\in\mathbb{R}^3$,
(i) $A_1^K(v,v_*,\tilde v,\tilde v_*)=0$,
(ii) $A_2^K(v,v_*,\tilde v,\tilde v_*)=\zeta_K[-|v-\tilde v|^2+|v_*-\tilde v_*|^2]$ *where* $\zeta_K=\pi\int_0^K(1-\cos G(z))dz$,
(iii) $A_3^K(v,v_*,\tilde v,\tilde v_*)\le C(|v|^2+|v_*|^2+|\tilde v|^2)K^{1-2/\nu}$.

Proof. Point (i) is obvious. Point (ii) immediately follows from the fact that $\Psi_k(x)=\zeta_K$ does not depend on x. Point (iii) holds true because $\Psi_K(x)=\pi\int_K^\infty(1-\cos G(z))dz\le\pi\int_K^\infty G^2(z)dz\le CK^{1-2/\nu}$ by (1.10). □

The case of hard potentials is much more complicated. The following result gives a possible and useful upper bound on the A_i^K functions.

Lemma 3.3. *Assume* (1.3), (1.5) *with* $\gamma\in(0,1)$, (1.7) *and adopt the notation of Lemma 3.1.*
(i) For all $q>0$, *there is* $C_q>0$ *such that for all* $M\ge1$, *all* $K\in[1,\infty)$, *all* $v,v_*,\tilde v,\tilde v_*\in\mathbb{R}^3$,

$$A_1^K(v,v_*,\tilde v,\tilde v_*)\le M(|v-\tilde v|^2+|v_*-\tilde v_*|^2)+C_qe^{-M^{q/\gamma}}e^{C_q(|v|^q+|v_*|^q)}.$$

(ii) There is $C>0$ *such that for all* $K\in[1,\infty)$, *all* $v,v_*,\tilde v,\tilde v_*\in\mathbb{R}^3$ *and all* $z_*\in\mathbb{R}^3$,

$$\begin{aligned}A_2^K(v,v_*,\tilde v,\tilde v_*)-A_2^K(v,z_*,\tilde v,\tilde v_*)\le&C\Big[|v-\tilde v|^2+|v_*-\tilde v_*|^2\\&+|v_*-z_*|^2(1+|v|+|v_*|+|z_*|)^{2\gamma/(1-\gamma)}\Big].\end{aligned}$$

(iii) There is $C>0$ *such that for all* $K\in[1,\infty)$, *all* $v,v_*,\tilde v,\tilde v_*\in\mathbb{R}^3$,

$$A_3^K(v,v_*,\tilde v,\tilde v_*)\le C(1+|v|^{4\gamma/\nu+2}+|v_*|^{4\gamma/\nu+2}+|\tilde v|^2+|\tilde v_*|^2)K^{1-2/\nu}.$$

This lemma is very technical. The reason is the following. The solution $(f_t)_{t\ge0}$ has bounded exponential moments while, on the contrary, the particle system has only a bounded energy (moment of order 2). If $K\in[1,\infty)$, the particle system has all moments finite, which makes all the computations licit, but the moments of order strictly greater than 2 are not uniformly bounded with respect to K (at least, we were not able to show it). We will use the previous estimates with v,v_* (and z_*) taken from the solution f_t and $\tilde v,\tilde v_*$ taken in the particle system. Thus, it is very important that these estimates do not involve powers greater than 2 of $\tilde v,\tilde v_*$. For example in point (i), only v,v_* appear in the exponential and this is crucial.

Proof. Using (1.11) and that $|x^\gamma-y^\gamma|\le2|x-y|/(x^{1-\gamma}+y^{1-\gamma})$, we get

$$\begin{aligned}A_1^K(v,v_*,\tilde v,\tilde v_*)\le&2c_4|v-v_*||\tilde v-\tilde v_*|\frac{(|v-v_*|^\gamma-|\tilde v-\tilde v_*|^\gamma)^2}{|v-v_*|^\gamma+|\tilde v-\tilde v_*|^\gamma}\\\le&8c_4\frac{|v-v_*|\wedge|\tilde v-\tilde v_*|}{(|v-v_*|\vee|\tilde v-\tilde v_*|)^{1-\gamma}}(|v-v_*|-|\tilde v-\tilde v_*|)^2.\end{aligned}\tag{3.1}$$

Now for any $M \geq 1$, this is bounded from above by

$$\begin{aligned}
&\frac{M}{2}(|v-v_*|-|\tilde v-\tilde v_*|)^2 + 8c_4(|v-v_*|\vee|\tilde v-\tilde v_*|)^{2+\gamma}\mathbb{1}_{\{8c_4\frac{|v-v_*|\wedge|\tilde v-\tilde v_*|}{(|v-v_*|\vee|\tilde v-\tilde v_*|)^{1-\gamma}}\geq \frac{M}{2}\}}\\
\leq&\frac{M}{2}(|v-\tilde v|+|v_*-\tilde v_*|)^2 + 8c_4\left[\frac{16c_4}{M}(|v-v_*|\wedge|\tilde v-\tilde v_*|)\right]^{\frac{2+\gamma}{1-\gamma}}\mathbb{1}_{\{\frac{|v-v_*|\wedge|\tilde v-\tilde v_*|}{(|v-v_*|\vee|\tilde v-\tilde v_*|)^{1-\gamma}}\geq \frac{M}{16c_4}\}}\\
\leq&M(|v-\tilde v|^2+|v_*-\tilde v_*|^2) + 8c_4\left[16c_4(|v-v_*|\wedge|\tilde v-\tilde v_*|)\right]^{\frac{2+\gamma}{1-\gamma}}\mathbb{1}_{\{(|v-v_*|\wedge|\tilde v-\tilde v_*|)^\gamma\geq \frac{M}{16c_4}\}}\\
\leq&M(|v-\tilde v|^2+|v_*-\tilde v_*|^2) + 8c_4\left[16c_4(|v|+|v_*|)\right]^{\frac{2+\gamma}{1-\gamma}}\mathbb{1}_{\{(|v|+|v_*|)^\gamma\geq \frac{M}{16c_4}\}}
\end{aligned}$$

Fix now $q>0$ and observe that

$$x^{\frac{2+\gamma}{1-\gamma}}\mathbb{1}_{\{x^\gamma\geq\frac{M}{16c_4}\}} \leq x^{\frac{2+\gamma}{1-\gamma}}e^{-M^{q/\gamma}}e^{(16c_4)^{q/\gamma}x^q} \leq C_q e^{-M^{q/\gamma}}e^{2(16c_4)^{q/\gamma}x^q}.$$

Point (i) follows.

Point (ii) is quite delicate. First, there is C such that for all $K\in[1,\infty)$, all $x,y>0$,

$$\Phi_K(x)\leq Cx^\gamma \quad \text{and} \quad |\Phi_K(x)-\Phi_K(y)|\leq C|x^\gamma-y^\gamma|.$$

Indeed, it is enough to prove that for $\Gamma_K(x)=\int_0^K(1-\cos G(z/x))dz$, $\Gamma_K(0)=0$ and $|\Gamma_K'(x)|\leq C$. But $\Gamma_K(x)=x\int_0^{K/x}(1-\cos G(z))dz \leq x\int_0^\infty G^2(z)dz$, so that $\Gamma_K(0)=0$ and $|\Gamma_K'(x)|\leq \int_0^\infty(1-\cos G(z))dz + x(K/x^2)(1-\cos G(K/x)) \leq \int_0^\infty G^2(z)dz+(K/x)G^2(K/x)$, which is uniformly bounded by (1.10). Consequently, for all $X,Y\in\mathbb{R}^3$,

$$|X\Phi_K(|X|)-Y\Phi_K(|Y|)| \leq C|X-Y|(|X|^\gamma+|Y|^\gamma)+C(|X|+|Y|)||X|^\gamma-|Y|^\gamma|.$$

Using again that $|x^\gamma-y^\gamma|\leq 2|x-y|/(x^{1-\gamma}+y^{1-\gamma})$, we easily conclude that

$$|X\Phi_K(|X|)-Y\Phi_K(|Y|)| \leq C|X-Y|(|X|^\gamma+|Y|^\gamma). \tag{3.2}$$

Now we write

$$\begin{aligned}
\Delta_2^K :=&A_2^K(v,v_*,\tilde v,\tilde v_*)-A_2^K(v,z_*,\tilde v,\tilde v_*)\\
=&-\big[(v-\tilde v)+(v_*-\tilde v_*)\big]\cdot\big[(v-v_*)\Phi_K(|v-v_*|)-(\tilde v-\tilde v_*)\Phi_K(|\tilde v-\tilde v_*|)\big]\\
&+\big[(v-\tilde v)+(z_*-\tilde v_*)\big]\cdot\big[(v-z_*)\Phi_K(|v-z_*|)-(\tilde v-\tilde v_*)\Phi_K(|\tilde v-\tilde v_*|)\big]\\
=&-\big[(v-\tilde v)+(v_*-\tilde v_*)\big]\cdot\big[(v-v_*)\Phi_K(|v-v_*|)-(v-z_*)\Phi_K(|v-z_*|)\big]\\
&+(z_*-v_*)\cdot\big[(v-z_*)\Phi_K(|v-z_*|)-(\tilde v-\tilde v_*)\Phi_K(|\tilde v-\tilde v_*|)\big].
\end{aligned}\tag{3.3}$$

By (3.2) and the Young inequality, we deduce that

$$\begin{aligned}
\Delta_2^K \leq& C(|v-\tilde v|+|v_*-\tilde v_*|)|v_*-z_*|(|v-v_*|^\gamma+|v-z_*|^\gamma)\\
&+C|z_*-v_*|(|v-\tilde v|+|z_*-\tilde v_*|)(|v-z_*|^\gamma+|\tilde v-\tilde v_*|^\gamma)\\
\leq& C[(|v-\tilde v|+|v_*-\tilde v_*|)^2+|v_*-z_*|^2(|v-v_*|^\gamma+|v-z_*|^\gamma)^2]\\
&+C|z_*-v_*|(|v-\tilde v|+|z_*-v_*|+|v_*-\tilde v_*|)(|v-z_*|^\gamma+(|v-\tilde v|+|v-v_*|+|v_*-\tilde v_*|)^\gamma).
\end{aligned}$$

The first term is clearly bounded by $C(|v-\tilde v|^2+|v_*-\tilde v_*|^2+|v_*-z_*|^2(1+|v|+|v_*|+|z_*|)^{2\gamma})$ which fits the statement, since $2\gamma\le 2\gamma/(1-\gamma)$. We next bound the second term by

$$\begin{aligned}&C|z_*-v_*|^2(|v-z_*|+|v-v_*|)^\gamma\\&+C|z_*-v_*|^2(|v-\tilde v|+|v_*-\tilde v_*|)^\gamma\\&+C|z_*-v_*|(|v-\tilde v|+|v_*-\tilde v_*|)(|v-z_*|+|v-v_*|)^\gamma\\&+C|z_*-v_*|(|v-\tilde v|+|v_*-\tilde v_*|)^{1+\gamma}.\end{aligned}$$

Using that $x^2y^\gamma\le x^{4/(2-\gamma)}+y^2$ (for the second line), that $xyz^\gamma\le(xz^\gamma)^2+y^2$ (for the third line) and that $xy^{1+\gamma}\le x^{2/(1-\gamma)}+y^2$, we obtain the upper-bound

$$\begin{aligned}&C|z_*-v_*|^2(1+|v|+|z_*|+|v_*|)^\gamma\\&+C(|v-\tilde v|+|v_*-\tilde v_*|)^2+|z_*-v_*|^{4/(2-\gamma)}\\&+C(|v-\tilde v|+|v_*-\tilde v_*|)^2+|z_*-v_*|^2(|v-z_*|+|v-v_*|)^{2\gamma}\\&+C(|v-\tilde v|+|v_*-\tilde v_*|)^2+|z_*-v_*|^{2/(1-\gamma)},\end{aligned}$$

which is bounded by

$$\begin{aligned}C(|v-\tilde v|^2+|v_*-\tilde v_*|^2)+C|z_*-v_*|^2\Big\{&(1+|v|+|z_*|+|v_*|)^\gamma+|z_*-v_*|^{4/(2-\gamma)-2}\\&+(|v-z_*|+|v-v_*|)^{2\gamma}+|z_*-v_*|^{2/(1-\gamma)-2}\Big\}.\end{aligned}$$

One easily concludes, using that $\max\{\gamma,4/(2-\gamma)-2,2\gamma,2/(1-\gamma)-2\}=2\gamma/(1-\gamma)$.

We finally check point (iii). Using (1.10), we deduce that $1-\cos(G(z/x^\gamma))\le G^2(z/x^\gamma)\le C(z/x^\gamma)^{-2/\nu}$, whence $\Psi_K(x)\le Cx^{2\gamma/\nu}\int_K^\infty z^{-2/\nu}dz=Cx^{2\gamma/\nu}K^{1-2/\nu}$. Thus

$$A_3^K(v,v_*,\tilde v,\tilde v_*)\le C(|v-v_*|^2+|v-v_*||\tilde v-\tilde v_*|)|v-v_*|^{2\gamma/\nu}K^{1-2/\nu}, \tag{3.4}$$

from which we easily conclude, using that $|\tilde v-\tilde v_*||v-v_*|^{1+2\gamma/\nu}\le|\tilde v-\tilde v_*|^2+|v-v_*|^{2+4\gamma/\nu}$. $\square$

We conclude with the hard spheres case.

Lemma 3.4. *Assume* (1.3), (1.5) *with* $\gamma=1$, (1.6) *and adopt the notation of Lemma 3.1.*
(i) For all $q>0$, there is $C_q>0$ such that for all $M\ge1$, all $K\in[1,\infty)$, all $v,v_*,\tilde v,\tilde v_*\in\mathbb{R}^3$,

$$A_1^K(v,v_*,\tilde v,\tilde v_*)\le M(|v-\tilde v|^2+|v_*-\tilde v_*|^2)+C_qK(|\tilde v|+|\tilde v_*|)e^{-M^q}e^{C_q(|v|^q+|v_*|^q)}.$$

(ii) For all $q>0$, there is $C_q>0$ such that for all $M\ge1$, all $K\in[1,\infty)$, all $v,v_*,\tilde v,\tilde v_*\in\mathbb{R}^3$,

$$\begin{aligned}A_2^K(v,v_*,\tilde v,\tilde v_*)-A_2^K(v,z_*,\tilde v,\tilde v_*)\le& M(|v-\tilde v|^2+|v_*-\tilde v_*|^2)+C|v_*-z_*|^2(1+|v|+|v_*|+|z_*|)^2\\&+C_q(1+|\tilde v|+|\tilde v_*|)Ke^{-M^q}e^{C_q(|v|^q+|v_*|^q+|z_*|^q)}\end{aligned}$$

(iii) For all $q>0$, there is $C_q>0$ such that for all $K\in[1,\infty)$, all $v,v_*,\tilde v,\tilde v_*\in\mathbb{R}^3$,

$$A_3^K(v,v_*,\tilde v,\tilde v_*)\le C_q(1+|\tilde v|)e^{-K^q}e^{C_q(|v|^q+|v_*|^q+|z_*|^q)}.$$

Proof. On the one hand, (1.11) implies

$$\begin{aligned}A_1^K(v,v_*,\tilde v,\tilde v_*)\le&2c_4|v-v_*||\tilde v-\tilde v_*|\frac{(|v-v_*|-|\tilde v-\tilde v_*|)^2}{|v-v_*|+|\tilde v-\tilde v_*|}\\\le&4c_4(|v-v_*|\wedge|\tilde v-\tilde v_*|)(|v-\tilde v|^2+|v_*-\tilde v_*|^2).\end{aligned}$$

On the other hand, since G takes values in $(0, \pi/2)$, we obviously have

$$A_1^K(v, v_*, \tilde{v}, \tilde{v}_*) \leq \frac{\pi^2}{2} K|v - v_*||\tilde{v} - \tilde{v}_*|.$$

Consequently, we may write

$$A_1^K(v, v_*, \tilde{v}, \tilde{v}_*) \leq M(|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2) + \frac{\pi^2}{2} K|v - v_*||\tilde{v} - \tilde{v}_*| \mathbb{1}_{\{4c_4(|v-v_*|\wedge|\tilde{v}-\tilde{v}_*|) \geq M\}}.$$

Point (i) easily follows, using that $|v - v_*|\mathbb{1}_{\{4c_4(|v-v_*|\wedge|\tilde{v}-\tilde{v}_*|)\geq M\}} \leq |v - v_*|\mathbb{1}_{\{4c_4|v-v_*|\geq M\}} \leq |v - v_*|e^{-M^q} e^{(4c_4|v-v_*|)^q} \leq C_q e^{-M^q} e^{2(4c_4|v-v_*|)^q} \leq C_q e^{-M^q} e^{2^{q+1}(4c_4)^q(|v|^q+|v_*|^q)}$.

Using all the computations of the proof of Lemma 3.3-(ii) except the one that makes appear the power $2/(1-\gamma)$, we see that for $\Delta_2^K := A_2^K(v, v_*, \tilde{v}, \tilde{v}_*) - A_2^K(v, z_*, \tilde{v}, \tilde{v}_*)$

$$\begin{aligned}\Delta_2^K \leq & C[|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2 + |v_* - z_*|^2(1 + |v| + |v_*| + |z_*|)^2 + |z_* - v_*|(|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2)] \\ \leq & C(1 + |z_* - v_*|)(|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2) + C|v_* - z_*|^2(1 + |v| + |v_*| + |z_*|)^2.\end{aligned}$$

On the other hand, starting from (3.3) and using that $\phi_K(x) \leq \pi K$, we realize that

$$\Delta_2^K \leq CK(1 + |\tilde{v}| + |\tilde{v}_*|)(1 + |v|^2 + |v_*|^2 + |z_*|^2).$$

Hence we can write, for any $M > 1$,

$$\begin{aligned}\Delta_2^K \leq & M(|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2) + C|v_* - z_*|^2(1 + |v| + |v_*| + |z_*|)^2 \\ & + CK(1 + |\tilde{v}| + |\tilde{v}_*|)(1 + |v|^2 + |v_*|^2 + |z_*|^2)\mathbb{1}_{\{C(1+|z_*-v_*|)\geq M\}}.\end{aligned}$$

But $(1 + |v|^2 + |v_*|^2 + |z_*|^2)\mathbb{1}_{\{C(1+|z_*-v_*|)\geq M\}} \leq (1 + |v| + |v_*| + |z_*|)^2 \mathbb{1}_{\{C(1+|v|+|v_*|+|z_*|)\geq M\}} \leq (1 + |v| + |v_*| + |z_*|)^2 e^{-M^q} e^{C^q(1+|v|+|v_*|+|z_*|)^q} \leq C_q e^{-M^q} e^{C_q(|v|^q+|v_*|^q+|z_*|^q)}$. Point (ii) is checked.

Finally, we observe that $\Psi_K(x) \leq \pi \int_K^\infty G^2(z/x)dz$. But here, $G(z) = (\pi/2 - z)_+$ whence $\Psi_K(x) \leq (\pi^4/24)x\mathbb{1}_{\{x\geq 2K/\pi\}} \leq 5x\mathbb{1}_{\{x\geq K/2\}}$. Thus for any $q > 0$, $\Psi_K(x) \leq 5xe^{-K^q}e^{2^q x^q}$, so that

$$\begin{aligned}A_3^K(v, v_*, \tilde{v}, \tilde{v}_*) \leq & C(1 + |\tilde{v}|)(1 + |v|^2 + |v_*|^2)e^{-K^q}|v - v_*|e^{2^q|v-v_*|^q} \\ \leq & C_q(1 + |\tilde{v}|)e^{-K^q}e^{C_q(|v|^q+|v_*|^q)}\end{aligned}$$

as desired. $\square$

4. Convergence of the particle system with cutoff

To build a suitable coupling between the particle system and the solution to (1.1), we need to introduce the (stochastic) paths associated to (1.1). To do so, we follow the ideas of Tanaka [38, 39] and make use of two probability spaces. The main one is an abstract $(\Omega, \mathcal{F}, \Pr)$, on which the random objects are defined when nothing is precised. But we will also need an auxiliary one, $[0, 1]$ endowed with its Borel σ-field and its Lebesgue measure. In order to avoid confusion, a random variable defined on this latter probability space will be called an α-random variable, expectation on $[0, 1]$ will be denoted by $\mathbb{E}_\alpha$, etc.

4.1. A SDE for the Boltzmann equation.

First, we recall the classical probabilistic interpretation of the Boltzmann equation initiated by Tanaka [38, 39] in the Maxwell molecules case.

Proposition 4.1. *Assume* (1.3), (1.5), (1.6) *or* (1.7) *and let* $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$. *If* $\gamma \in (0, 1]$, *assume additionally* (1.8) *and that* f_0 *satisfies* (1.12). *Let* $(f_t)_{t\geq 0}$ *be the corresponding unique weak solution to* (1.1). *Consider any* f_0*-distributed random variable* W_0 *and any independent Poisson measure* $M(ds, d\alpha, dz, d\varphi)$ *on* $[0, \infty) \times [0, 1] \times [0, \infty) \times [0, 2\pi)$ *with intensity measure* $dsd\alpha dzd\varphi$. *Consider* *also, for each $t \geq 0$, a f_t-distributed α-random variable W_t^*, in such a way that $(t,\alpha) \mapsto W_t^*(\alpha)$ is measurable. Then there is a unique (càdlàg adapted) strong solution to*

$$W_t = W_0 + \int_0^t \int_{\mathbb{R}^3} \int_0^\infty \int_0^{2\pi} c(W_{s-}, W_s^*(\alpha), z, \varphi) M(ds, d\alpha, dz, d\varphi). \tag{4.1}$$

Furthermore, W_t is f_t-distributed for each $t \geq 0$.

We will note $(W_t)_{t\geq 0}$ such a Boltzmann process. It can be viewed as the time-evolution of the velocity of a typical particle in the gas.

Proof. The proof is very similar to that of [17, Proposition 5.1], see also [19, Section 4] and is omitted. In [17, Proposition 5.1], the same Boltzmann equation is studied, with much less assumptions on f_0 (so that uniqueness is not known for (1.1)). But the formulation of the SDE is different (it is equivalent in law). The same proof as in [17, Proposition 5.1] works here, with several difficulties avoided due to the facts that f_0 has exponential moments and that uniqueness is known to hold for (1.1). □

4.2. **A SDE for the particle system.** Here we write down a Poisson stochastic differential equation corresponding to Nanbu's particle system and we prove Proposition 1.2-(i).

Proposition 4.2. *Assume* (1.3), (1.5), (1.6) *or* (1.7) *and let $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$, $N \geq 1$ and $K \in [1,\infty)$. Consider a family $(V_0^i)_{i=1,\dots N}$ of i.i.d. f_0-distributed random variables and an independent family $(O_i^N(ds,dj,dz,d\varphi))_{i=1,\dots,N}$ of Poisson measures on $[0,\infty)\times\{1,\dots,N\}\times[0,\infty)\times[0,2\pi)$ with intensity measures $ds\left(N^{-1}\sum_{k=1}^N \delta_k(dj)\right)dzd\varphi$. There exists a unique (càdlàg and adapted) strong solution to*

$$V_t^{i,N,K} = V_0^i + \int_0^t \int_j \int_0^\infty \int_0^{2\pi} c_K(V_{s-}^{i,N,K}, V_{s-}^{j,N,K}, z, \varphi) O_i^N(ds,dj,dz,d\varphi), \quad i = 1,\dots,N. \tag{4.2}$$

Furthermore, $(V_t^{i,N,K})_{i=1,\dots,N,t\geq 0}$ is Markov with generator $\mathcal{L}_{N,K}$. We have $\mathbb{E}\left[|V_t^{1,N,K}|^2\right] = \int_{\mathbb{R}^3}|v|^2 f_0(dv)$ and, if $\int_{\mathbb{R}^3}|v|^p f_0(dv)$ for some $p\geq 2$, $\sup_{[0,T]}\mathbb{E}\left[|V_t^{1,N,K}|^p\right] \leq C_{p,T,f_0,K}$.

Proof. First of all, observe that we actually deal with finite Poisson measures, since c_K vanishes for $z \geq K$. Thus, strong existence and uniqueness for (4.2) is trivial: it suffices to work recursively on the instants of jumps (which are discrete) of the family $(O_i^N(ds,dj,dz,d\varphi))_{i=1,\dots,N}$. Consequently, $\mathbf{V}_t^{N,K} = (V_t^{1,N,K},\dots,V_t^{N,N,K})$ is a Markov process, since it solves a well-posed time-homogeneous SDE. Its infinitesimal generator is classically defined by (2.7), with actually a sum over all couples $(i,j)\in\{1,\dots,N\}^2$, but this changes nothing since the terms with $i=j$ vanish because $c_K(v,v,z,\varphi)=0$ for all $v\in\mathbb{R}^3$. Next, a simple computation shows that

$$\begin{aligned}
\mathbb{E}[|V_t^{1,N,K}|^2] =& \mathbb{E}[|V_0^1|^2] + \frac{1}{N}\sum_{j=1}^N \int_0^t \int_0^\infty \int_0^{2\pi} \mathbb{E}\Big(|V_s^{1,N,K} + c_K(V_s^{1,N,K}, V_s^{j,N,K}, z, \varphi)|^2 \\
&\qquad\qquad - |V_s^{1,N,K}|^2\Big) d\varphi dz ds \\
=& \mathbb{E}[|V_0^1|^2] + \frac{N-1}{N}\int_0^t \int_0^K \int_0^{2\pi} \mathbb{E}\Big(|c(V_s^{1,N,K}, V_s^{2,N,K}, z, \varphi)|^2 \\
&\qquad\qquad + 2V_s^{1,N,K}\cdot c(V_s^{1,N,K}, V_s^{2,N,K}, z, \varphi)\Big) d\varphi dz ds
\end{aligned}$$

by exchangeability. But, as seen in the proof of Lemma 3.1,

$$\int_0^K \int_0^{2\pi} \Big(|c(v,v_*,z,\varphi)|^2 + 2v\cdot c(v,v_*,z,\varphi)\Big) d\varphi dz = [|v-v_*|^2 - 2v\cdot(v-v_*)]\Phi_K(|v-v_*|),$$

whence, using again exchangeability,

$$\begin{aligned}\mathbb{E}[|V_t^{1,N,K}|^2] =& \mathbb{E}[|V_0^1|^2] + \frac{N-1}{N}\int_0^t \mathbb{E}\Big(\Big[|V_s^{1,N,K}-V_s^{2,N,K}|^2 - 2V_s^{1,N,K}\cdot(V_s^{1,N,K}-V_s^{2,N,K})\Big]\\ &\qquad\qquad \Phi_K(|V_s^{1,N,K}-V_s^{2,N,K}|)\Big)ds\\ =& \mathbb{E}[|V_0^1|^2] + \frac{N-1}{N}\int_0^t \mathbb{E}\Big(\Big[|V_s^{1,N,K}-V_s^{2,N,K}|^2 - V_s^{1,N,K}\cdot(V_s^{1,N,K}-V_s^{2,N,K})\\ &\qquad - V_s^{2,N,K}\cdot(V_s^{2,N,K}-V_s^{1,N,K})\Big]\Phi_K(|V_s^{1,N,K}-V_s^{2,N,K}|)\Big)ds.\end{aligned}$$

In this last expression, the integrand is zero, so that, as claimed, $\mathbb{E}[|V_t^{1,N,K}|^2] = \mathbb{E}[|V_0^1|^2] = \int_{\mathbb{R}^3}|v|^2 f_0(dv)$. Recalling finally (2.1) and (2.5), we see that $|c(v,v_*,z,\varphi)| \le |v-v_*|$. Thus for $p\ge 2$,

$$\int_0^K \int_0^{2\pi} (|v+c(v,v_*,z,\varphi)|^p - |v|^p) d\varphi dz \le C_pK(|v| + |v_*|^p).$$

Consequently, we obtain as previously

$$\mathbb{E}[|V_t^{1,N,K}|^p] \le \mathbb{E}[|V_0^1|^p] + \frac{C_pK}{N}\sum_{j=1}^N \int_0^t \mathbb{E}[|V_s^{1,N,K}|^p + |V_s^{j,N,K}|^p]ds$$

and conclude, using again exchangeability, that $\mathbb{E}[|V_t^{1,N,K}|^p] \le \mathbb{E}[|V_0^1|^p]e^{2C_pKt}$ as desired. □

This allows us to deduce

Proof of Proposition 1.2-(i). The strong existence and uniqueness for the SDE (4.2) classically implies the existence and uniqueness of a Markov process with generator $\mathcal{L}_{N,K}$. □

4.3. **The coupling.** Here we explain how we couple our particle system with a family of i.i.d. Boltzmann processes. For example, we want to couple $V_t^{1,N,K}$ with a Boltzmann process W_t^1. The main difficulty is that at each collision, W_t^1 is collided by an independent particle (using W_t^*) while $V_t^{1,N,K}$ is collided by some $V_t^{j,N,K}$. We thus have to choose j in such a way that $V_t^{j,N,K}$ is as close as possible to W_t^*, but j has to remain uniformly chosen.

A technical problem obliges us to introduce the set $(\mathbb{R}^3)^N_\bullet := \{\mathbf{w}\in(\mathbb{R}^3)^N : \ w_i \ne w_j \ \forall\, i\ne j\}$.

Lemma 4.3. *Let $f_t \in C([0,\infty),\mathcal{P}_2(\mathbb{R}^3))$ be such that f_t has a density for all $t>0$. Let also $N\ge 1$ be fixed. For $\mathbf{v}=(v_1,\dots,v_N)\in(\mathbb{R}^3)^N$, we denote by $\mu^N_{\mathbf{v}} := N^{-1}\sum_1^N \delta_{v_i}$ the empirical measure associated to $\mathbf{v}$. There exists a measurable map $(t,\mathbf{w},\mathbf{v},\alpha)\mapsto (W_t^*(\alpha), Z_t^*(\mathbf{w},\alpha), V_t^*(\mathbf{v},\mathbf{w},\alpha))$ from $(0,\infty)\times(\mathbb{R}^3)^N_\bullet\times(\mathbb{R}^3)^N\times[0,1]$ into $\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3$ enjoying the following properties*

(a) for all $t\ge 0$, the α-law of W_t^* is f_t,
(b) for all $t\ge0$, $\mathbf{w}\in(\mathbb{R}^3)^N_\bullet$, the α-law of $Z_t^*(\mathbf{w},.)$ is $\mu^N_{\mathbf{w}}$,
(c) for all $t\ge0$, $\mathbf{w}\in(\mathbb{R}^3)^N_\bullet$, $\mathbf{v}\in(\mathbb{R}^3)^N$, the α-law of $V_t^*(\mathbf{v},\mathbf{w},.)$ is $\mu^N_{\mathbf{v}}$,
(d) for all $t\ge0$, $\mathbf{w}\in(\mathbb{R}^3)^N_\bullet$, $\mathbf{v}\in(\mathbb{R}^3)^N$, the α-law of $(Z_t^*(\mathbf{w},.),V_t^*(\mathbf{v},\mathbf{w},.))$ is $N^{-1}\sum_1^N\delta_{(w_i,v_i)}$,
(e) for all $t\ge0$, all $\mathbf{w}\in(\mathbb{R}^3)^N_\bullet$, $\int_0^1|W_t^*(\alpha)-Z_t^*(\mathbf{w},\alpha)|^2d\alpha = \mathcal{W}_2^2(f_t,\mu^N_{\mathbf{w}})$.

Proof. We first consider, for each $t>0$, W_t^* such that point (a) holds true and such that $(t,\alpha)\mapsto W_t^*(\alpha)$ is measurable.

Next, we recall that by Brenier's theorem (see e.g. Villani [42, Theorem 2.12 p 66]) for each $t>0$ and each $\mathbf{w}\in(\mathbb{R}^3)^N$, since f_t does does not charge small sets (because it has a density by [17]), there exists a unique map $F_{t,\mathbf{w}}:\mathbb{R}^3\mapsto\mathbb{R}^3$ such that, setting $Z_t^*(\mathbf{w},\alpha):=F_{t,\mathbf{w}}(W_t^*(\alpha))$, points (b) and (e) hold true. In other words, $(W_t^*(.),Z_t^*(\mathbf{w},.))$ is an optimal coupling for f_t and $\mu_{\mathbf{w}}^N$. Furthermore, Fontbona-Guérin-Méléard [16] have shown that $F_{t,\mathbf{w}}(x)$ is a measurable function of $(t,\mathbf{w},x)$. Consequently, $Z_t^*(\mathbf{w},\alpha)$ is a measurable function of $(t,\mathbf{w},\alpha)$.

Finally, we define, for any $\mathbf{w}\in(\mathbb{R}^3)^N_\bullet$ and any $\mathbf{v}\in(\mathbb{R}^3)^N$, the map $G_{\mathbf{w},\mathbf{v}}:\{w_1,\dots,w_N\}\mapsto\{v_1,\dots,v_N\}$ by $G_{\mathbf{w},\mathbf{v}}(w_i)=v_i$ (here we need that $\mathbf{w}\in(\mathbb{R}^3)^N_\bullet$). We then we put $V_t^*(\mathbf{v},\mathbf{w},\alpha)=G_{\mathbf{w},\mathbf{v}}(Z_t^*(\mathbf{w},\alpha))$, which is clearly measurable (in all its variables). Point (d) follows from (b) and the definition of $G_{\mathbf{w},\mathbf{v}}$ and finally (c) follows from (d). ☐

Here is the coupling we propose.

Lemma 4.4. *Assume* (1.3), (1.5), (1.6) *or* (1.7). *Let $f_0\in\mathcal{P}_2(\mathbb{R}^3)$. Assume additionally* (1.8) *and* (1.12) *if $\gamma\in(0,1]$. Let $(f_t)_{t\geq0}$ be the unique weak solution to* (1.1) *and assume that f_t has a density for all $t>0$ (see Theorem 1.1). Consider $N\geq1$ and $K\in[1,\infty)$ fixed. Let $(V_0^i)_{i=1,\dots N}$ be i.i.d. with common law f_0 and let $(M_i(ds,d\alpha,dz,d\varphi))_{i=1,\dots,N}$ be an i.i.d. family of Poisson measures on $[0,\infty)\times[0,1]\times[0,\infty)\times[0,2\pi)$ with intensity measures $dsd\alpha dzd\varphi$, independent of $(V_0^i)_{i=1,\dots N}$.*

(i) The following SDE's, for $i=1,\dots,N$, define N independent copies of the Boltzmann process:

$$W_t^i=V_0^i+\int_0^t\int_0^1\int_0^\infty\int_0^{2\pi}c(W_{s-}^i,W_s^*(\alpha),z,\varphi)M_i(ds,d\alpha,dz,d\varphi).$$

In particular, for each $t\geq0$, $(W_t^i)_{i=1,\dots,N}$ are i.i.d. with common law f_t. Consequently, since f_t has a density for all $t>0$, $(W_t^i)_{i=1,\dots,N}\in(\mathbb{R}^3)^N_\bullet$ a.s.

(ii) Next, we consider the system of SDE's, for $i=1,\dots,N$,

$$V_t^{i,N,K}=V_0^i+\int_0^t\int_0^1\int_0^\infty\int_0^{2\pi}c_K(V_{s-}^{i,N,K},V_s^*(\mathbf{V}_{s-}^{N,K},\mathbf{W}_{s-},\alpha),z,\varphi+\varphi_{i,\alpha,s})M_i(ds,d\alpha,dz,d\varphi),$$

where we used the notation $\mathbf{V}_{s-}^{N,K}=(V_{s-}^{1,N,K},\dots,V_{s-}^{N,N,K})\in(\mathbb{R}^3)^N$, $\mathbf{W}_{s-}=(W_{s-}^1,\dots,W_{s-}^N)\in(\mathbb{R}^3)^N_\bullet$ and where we have set $\varphi_{i,\alpha,s}:=\varphi_0(W_{s-}^i-W_s^*(s,\alpha),V_{s-}^{i,N,K}-V_s^*(\mathbf{V}_{s-}^{N,K},\mathbf{W}_{s-},\alpha))$ for simplicity. This system of SDEs has a unique solution, and this solution is a Markov process with generator $\mathcal{L}_{N,K}$ and initial condition $(V_0^i)_{i=1,\dots N}$.

(iii) The family $((W_t^1,V_t^{1,N,K})_{t\geq0},\dots,(W_t^N,V_t^{N,N,K})_{t\geq0})$ is exchangeable.

Proof. Point (i) is a direct consequence of Proposition 4.1 and point (iii) follows from the exchangeability of the family $(V_0^i,M_i)_{i=1,\dots,N}$ and from uniqueness (in law). In point (ii), the existence and uniqueness result is also immediate, since the Poisson measures under consideration are finite (or rather, are finite when z is restricted to $[0,K]$, which is the case since $c_K=c1\!\!1_{\{z\leq K\}}$). Finally $(V_t^{1,N,K},\dots,V_t^{N,N,K})_{t\geq0}$ is a Markov process with generator $\mathcal{L}_{N,K}$ due to the fact that for all $\mathbf{v}\in(\mathbb{R}^3)^N$, all $\mathbf{w}\in(\mathbb{R}^3)^N_\bullet$, all $s>0$, all $\varphi_{ij}\in[0,2\pi)$, for all bounded measurable function

$\phi : (\mathbb{R}^3)^N \mapsto \mathbb{R}$,

$$\begin{aligned}&\sum_{i=1}^N \int_0^1 \int_0^\infty \int_0^{2\pi} \Big(\phi(\mathbf{v} + c_K(v_i, V_s^*(\mathbf{v}, \mathbf{w}, \alpha), z, \varphi + \varphi_{ij}).\mathbf{e}_i) - \phi(\mathbf{v})\Big) d\varphi dz d\alpha \\ =& \sum_{i=1}^N \frac{1}{N} \sum_{j=1}^N \int_0^\infty \int_0^{2\pi} \Big(\phi(\mathbf{v} + c_K(v_i, v_j, z, \varphi + \varphi_{ij}).\mathbf{e}_i) - \phi(\mathbf{v})\Big) d\varphi dz \\ =& \frac{1}{N} \sum_{i \neq j} \int_0^\infty \int_0^{2\pi} \Big(\phi(\mathbf{v} + c_K(v_i, v_j, z, \varphi).\mathbf{e}_i) - \phi(\mathbf{v})\Big) d\varphi dz,\end{aligned}$$

which is nothing but $\mathcal{L}_{N,K}\phi(\mathbf{v})$, see (2.7). We used Lemma 4.3-(c) for the first equality and the 2π-periodicity of c_K (in φ) and the fact that $c_K(v_i, v_i, z, \varphi) = 0$ for the second one. $\square$

4.4. **Estimate of the Wasserstein distance.** We can now prove our main result in the case with cutoff. We first study hard potentials.

Proof of Theorem 1.4-(ii) when $K \in [1, \infty)$. We thus assume (1.3), (1.5) with $\gamma \in (0, 1)$ and (1.7). We consider $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ satisfying (1.12) for some $p \in (\gamma, 2)$ and fix $q \in (\gamma, p)$ for the rest of the proof. We also assume that f_0 is not a Dirac mass, so that f_t has a density for all $t > 0$. We fix $N \geq 1$ and $K \in [1, \infty)$ and consider the processes introduced in Lemma 4.4.

Step 1. A direct application of the Itô calculus for jump processes shows that

$$\begin{aligned}&\mathbb{E}[|W_t^1 - V_t^{1,N,K}|^2] \\ =& \int_0^t \int_0^1 \int_0^\infty \int_0^{2\pi} \mathbb{E}\Big[|W_s^1 - V_s^{1,N,K} + \Delta^1(s, \alpha, z, \varphi)|^2 - |W_s^1 - V_s^{1,N,K}|^2\Big] d\varphi dz d\alpha ds,\end{aligned}$$

where

$$\Delta^1(s, \alpha, z, \varphi) = c(W_s^1, W_s^*(s, \alpha), z, \varphi) - c_K(V_s^{1,N,K}, V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha), z, \varphi + \varphi_{i,\alpha,s}).$$

Using Lemma 3.1, we thus obtain

$$\mathbb{E}[|W_t^1 - V_t^{1,N,K}|^2] \leq \int_0^t [B_1^K(s) + B_2^K(s) + B_3^K(s)] ds,$$

where, for $i = 1, 2, 3$,

$$B_i^K(s) := \int_0^1 \mathbb{E}\Big[A_i^K(W_s^1, W_s^*(\alpha), V_s^{1,N,K}, V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha))\Big] d\alpha.$$

Step 2. Using Lemma 3.3-(i), we see that for all $M \geq 1$ (recall that $q \in (\gamma, p)$ is fixed).

$$\begin{aligned}B_1^K(s) \leq& M \int_0^1 \mathbb{E}\left[|W_s^1 - V_s^{1,N,K}|^2 + |W_s^*(\alpha) - V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha))|^2\right] d\alpha \\ &+ Ce^{-M^{q/\gamma}} \int_0^1 \mathbb{E}\left[\exp(C(|W_s^1|^q + |W_s^*(\alpha)|^q))\right] d\alpha \\ \leq& M \int_0^1 \mathbb{E}\left[|W_s^1 - V_s^{1,N,K}|^2 + |W_s^*(\alpha) - V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha))|^2\right] d\alpha + Ce^{-M^{q/\gamma}}.\end{aligned}$$

To get the last inequality, we used that W_s^1 and $W_s^*(.)$ are independent and satisfy $W_s^1 \sim f_s$ and $W_s^*(.) \sim f_s$, whence

$$\int_0^1 \mathbb{E}\left[\exp(C(|W_s^1|^q + |W_s^*(\alpha)|^q)\right] d\alpha \quad = \quad \Big(\int_{\mathbb{R}^3} e^{C|w|^q} f_s(dw)\Big)^2 < \infty$$

by (1.13).

Step 3. Roughly speaking, B_2^K should not be far to be zero for symmetry reasons. We claim that B_2^K would be zero if $W_s^*(\alpha)$ was replaced by $Z_s^*(\mathbf{W}_s, \alpha)$. More precisely, we check here that

$$\tilde{B}_2^K(s) := \int_0^1 \mathbb{E}\left[A_2^K(W_s^1, Z_s^*(\mathbf{W}_s,\alpha), V_s^{1,N,K}, V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha))\right] d\alpha = 0.$$

By Lemma 4.3-(d), we simply have

$$\tilde{B}_2^K(s) = \mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N A_2^K(W_s^1, W_s^i, V_s^{1,N,K}, V_s^{i,N,K})\right] = \frac{N-1}{N}\mathbb{E}\left[A_2^K(W_s^1, W_s^2, V_s^{1,N,K}, V_s^{2,N,K})\right]$$

by exchangeability and since $A_2^K(v, v, \tilde{v}, \tilde{v}) = 0$. Finally, we write, using again exchangeability,

$$\tilde{B}_2^K(s) = \frac{N-1}{2N}\mathbb{E}\left[A_2^K(W_s^1, W_s^2, V_s^{1,N,K}, V_s^{2,N,K}) + A_2^K(W_s^2, W_s^1, V_s^{2,N,K}, V_s^{1,N,K})\right].$$

This is zero by symmetry of A_2^K: it holds that $A_2^K(v, v_*, \tilde{v}, \tilde{v}_*) + A_2^K(\tilde{v}, \tilde{v}_*, v, v_*) = 0$.

Step 4. By Step 3, we thus have

$$\begin{aligned} B_2^K(s) = \int_0^1 \mathbb{E}\Big[&A_2^K(W_s^1, W_s^*(\alpha), V_s^{1,N,K}, V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha)) \\ &- A_2^K(W_s^1, Z_s^*(\mathbf{W}_s,\alpha), V_s^{1,N,K}, V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha))\Big] d\alpha. \end{aligned}$$

Consequently, Lemma 3.3-(ii) implies

$$\begin{aligned} B_2^K(s) \le & C\int_0^1 \mathbb{E}\Big[|W_s^1 - V_s^{1,N,K}|^2 + |W_s^*(\alpha) - V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha)|^2 \\ &+ |W_s^*(\alpha) - Z_s^*(\mathbf{W}_s,\alpha)|^2(1 + |W_s^1| + |W_s^*(\alpha)| + |Z_s^*(\mathbf{W}_s,\alpha)|)^{2\gamma/(1-\gamma)}\Big] d\alpha. \end{aligned}$$

Step 5. Finally, we use Lemma 3.3-(iii) to obtain

$$B_3^K(s) \le CK^{1-2/\nu}\int_0^1 \mathbb{E}\Big[1 + |W_s^1|^{4\gamma/\nu+2} + |W_s^*(\alpha)|^{4\gamma/\nu+2} + |V_s^{1,N,K}|^2 + |V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha))|^2\Big] d\alpha.$$

Since $W_s^1 \sim f_s$, we deduce from (1.13) that $\mathbb{E}[|W_s^1|^{4\gamma/\nu+2}] = \int_{\mathbb{R}^3} |v|^{4\gamma/\nu+2} f_s(dv) \le C$. By Lemma 4.3-(a), we also have $W_s^*(.) \sim f_s$, whence $\int_0^1 |W_s^*(\alpha)|^{4\gamma/\nu+2} d\alpha = \int_{\mathbb{R}^3} |v|^{4\gamma/\nu+2} f_s(dv) \le C$. Proposition 4.2 shows that $\mathbb{E}[|V_s^{1,N,K}|^2] = \int_{\mathbb{R}^3} |v|^2 f_0(dv)$. We next infer from Lemma 4.3-(c) that $\int_0^1 |V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha))|^2 d\alpha = N^{-1}\sum_1^N |V_s^{i,N,K}|^2$. Consequently, $\mathbb{E}[\int_0^1 |V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha))|^2 d\alpha] = \mathbb{E}[|V_s^{1,N,K}|^2] = \int_{\mathbb{R}^3} |v|^2 f_0(dv)$. As a conclusion,

$$B_3^K(s) \le CK^{1-2/\nu}.$$

Step 6. We set $u_t^{N,K} := \mathbb{E}[|W_t^1 - V_t^{1,N,K}|^2]$. Using the previous steps, we see that for all $M \geq 1$,

$$u_t^{N,K} \leq Cte^{-M^{q/\gamma}} + CtK^{1-2/\nu} + (M+C)\int_0^t \Big[u_s^{N,K} + \int_0^1 \mathbb{E}[|W_s^*(\alpha) - V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha)|^2]d\alpha\Big]ds$$
$$+ C\int_0^t \int_0^1 \mathbb{E}\Big[|W_s^*(\alpha) - Z_s^*(\mathbf{W}_s, \alpha)|^2(1 + |W_s^1| + |W_s^*(\alpha)| + |Z_s^*(\mathbf{W}_s,\alpha)|)^{2\gamma/(1-\gamma)}\Big]d\alpha ds.$$

We now write, using Minkowski's inequality and Lemma 4.3-(d) and (e),

$$\Big[\int_0^1 \mathbb{E}\Big[|W_s^*(\alpha) - V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha)|^2\Big]d\alpha\Big]^{1/2} \tag{4.3}$$
$$\leq \Big[\int_0^1 \mathbb{E}\Big[|W_s^*(\alpha) - Z_s^*(\mathbf{W}_s, \alpha)|^2\Big]d\alpha\Big]^{1/2} + \Big[\int_0^1 \mathbb{E}\Big[|Z_s^*(\mathbf{W}_s, \alpha) - V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha)|^2\Big]d\alpha\Big]^{1/2}$$
$$= \mathbb{E}[\mathcal{W}_2^2(f_s, \mu_{\mathbf{W}_s}^N)]^{1/2} + \Big[\frac{1}{N}\sum_1^N \mathbb{E}[|W_s^i - V_s^{i,N,K}|^2]\Big]^{1/2}$$
$$= \mathbb{E}[\mathcal{W}_2^2(f_s, \mu_{\mathbf{W}_s}^N)]^{1/2} + (u_s^{N,K})^{1/2}$$

by exchangeability. We deduce that

$$\int_0^1 \mathbb{E}[|W_s^*(\alpha) - V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha)|^2]d\alpha \leq 2\mathbb{E}[\mathcal{W}_2^2(f_s, \mu_{\mathbf{W}_s}^N)] + 2u_s^{N,K}. \tag{4.4}$$

Next, a simple computation shows that for all $\varepsilon \in (0,1)$,

$$\int_0^1 \mathbb{E}\Big[|W_s^*(\alpha) - Z_s^*(\mathbf{W}_s, \alpha)|^2(1 + |W_s^1| + |W_s^*(\alpha)| + |Z_s^*(\mathbf{W}_s,\alpha)|)^{\frac{2\gamma}{1-\gamma}}\Big]d\alpha \tag{4.5}$$
$$\leq \int_0^1 \mathbb{E}\Big[|W_s^*(\alpha) - Z_s^*(\mathbf{W}_s, \alpha)|^{2-\varepsilon}(1 + |W_s^1| + |W_s^*(\alpha)| + |Z_s^*(\mathbf{W}_s,\alpha)|)^{\frac{2\gamma}{1-\gamma}+\varepsilon}\Big]d\alpha$$
$$\leq \Big(\int_0^1 \mathbb{E}\Big[|W_s^*(\alpha) - Z_s^*(\mathbf{W}_s, \alpha)|^2\Big]d\alpha\Big)^{\frac{2-\varepsilon}{2}}$$
$$\times \Big(\int_0^1 \mathbb{E}\Big[(1 + |W_s^1| + |W_s^*(\alpha)| + |Z_s^*(\mathbf{W}_s,\alpha)|)^{\frac{4\gamma}{\varepsilon(1-\gamma)}+2}\Big]d\alpha\Big)^{\frac{\varepsilon}{2}}$$
$$\leq C_\varepsilon \left(\mathbb{E}[\mathcal{W}_2^2(f_s, \mu_{\mathbf{W}_s}^N)]\right)^{\frac{2-\varepsilon}{2}}.$$

For the last inequality, we used Lemma 4.3-(e), the fact that by (1.13),

$$\mathbb{E}\Big[|W_s^1|^{\frac{4\gamma}{\varepsilon(1-\gamma)}+2}\Big] = \int_0^1 |W_s^*(\alpha)|^{\frac{4\gamma}{\varepsilon(1-\gamma)}+2}d\alpha = \int_{\mathbb{R}^3} |v|^{\frac{4\gamma}{\varepsilon(1-\gamma)}+2} f_s(dv) \leq C_\varepsilon$$

and that, by Lemma 4.3-(b)

$$\int_0^1 \mathbb{E}\Big[|Z_s^*(\mathbf{W}_s, \alpha)|^{\frac{4\gamma}{\varepsilon(1-\gamma)}+2}\Big]d\alpha = \mathbb{E}\Big[\frac{1}{N}\sum_1^N |W_s^i|^{\frac{4\gamma}{\varepsilon(1-\gamma)}+2}\Big] = \mathbb{E}\Big[|W_s^1|^{\frac{4\gamma}{\varepsilon(1-\gamma)}+2}\Big] \leq C_\varepsilon.$$

We end up with: for all $\varepsilon \in (0,1)$, all $M \geq 1$,

$$u_t^{N,K} \leq Cte^{-M^{q/\gamma}} + CtK^{1-2/\nu} + 3(M+C)\int_0^t \left[u_s^{N,K} + \mathbb{E}[\mathcal{W}_2^2(f_s, \mu^N_{\mathbf{W}_s})]\right] ds$$
$$+ C_\varepsilon \int_0^t (\mathbb{E}[\mathcal{W}_2^2(f_s, \mu^N_{\mathbf{W}_s})])^{1-\varepsilon/2} ds.$$

Now we observe that $\mathbb{E}[\mathcal{W}_2^2(f_s, \mu^N_{\mathbf{W}_s})] = \varepsilon_N(f_t)$, recall (1.14), because $W_t^1, \dots, W_t^N$ are i.i.d. and f_t-distributed. Since $\varepsilon_N(f_t) \leq 2\int_{\mathbb{R}^3} |v|^2 f_t(dv) = 2\int_{\mathbb{R}^3} |v|^2 f_0(dv)$, since $M \geq 1$ and $K \in [1,\infty)$, we get

$$u_t^{N,K} \leq C_\varepsilon \left(te^{-M^{q/\gamma}} + Mt\delta_{N,K,t}^{1-\varepsilon/2} + M\int_0^t u_s^{N,K} ds \right).$$

where we have set

$$\delta_{N,K,t} := K^{1-2/\nu} + \sup_{[0,t]} \varepsilon_N(f_s).$$

Hence by Grönwall's lemma,

$$\sup_{[0,T]} u_t^{N,K} \leq C_\varepsilon T \left(e^{-M^{q/\gamma}} + M\delta_{N,K,T}^{1-\varepsilon/2} \right) e^{C_\varepsilon MT},$$

this holding for any value of $M \geq 1$. We easily conclude that

$$\sup_{[0,T]} u_t^{N,K} \leq C_{\varepsilon,T} \delta_{N,K,T}^{1-\varepsilon},$$

by choosing $M = 1$ if $\delta_{N,K,T} \geq 1/e$ and $M = |\log \delta_{N,K,T}|^{\gamma/q}$ otherwise, which gives

$$\sup_{[0,T]} u_t^{N,K} \leq C_\varepsilon \left(T\delta_{N,K,T} + \delta_{N,K,T}^{1-\varepsilon/2} |\log \delta_{N,K,T}|^{\gamma/q} \right) e^{C_\varepsilon |\log \delta_{N,K,T}|^{\gamma/q} T} \leq C_{\varepsilon,T} \delta_{N,K,T}^{1-\varepsilon},$$

the last inequality following from the fact that $\gamma/q < 1$.

Final step. We now recall that $\mu_t^{N,K} = \mu^N_{\mathbf{V}_t^{N,K}}$ and write

$$\mathbb{E}[\mathcal{W}_2^2(\mu_t^{N,K}, f_t)] \leq 2\mathbb{E}[\mathcal{W}_2^2(\mu^N_{\mathbf{V}_t^{N,K}}, \mu^N_{\mathbf{W}_t})] + 2\mathbb{E}[\mathcal{W}_2^2(\mu^N_{\mathbf{W}_t}, f_t)].$$

But $\mathbb{E}[\mathcal{W}_2^2(\mu^N_{\mathbf{V}_t^{N,K}}, \mu^N_{\mathbf{W}_t})] \leq \mathbb{E}[N^{-1}\sum_1^N |V_t^{i,N,K} - W_t^i|^2] = \mathbb{E}[|V_t^{1,N,K} - W_t^1|^2] = u_t^{N,K}$ by exchangeability, and we have already seen that $\mathbb{E}[\mathcal{W}_2^2(\mu^N_{\mathbf{W}_t}, f_t)] = \varepsilon_N(f_t)$. Consequently, for all $\varepsilon \in (0,1)$, all $t \in [0,T]$,

$$\mathbb{E}[\mathcal{W}_2^2(\mu_t^N, f_t)] \leq C_{\varepsilon,T}\delta_{N,K,T}^{1-\varepsilon} + 2\varepsilon_N(f_t) \leq C_{\varepsilon,T} \left(K^{1-2/\nu} + \sup_{[0,T]} \varepsilon_N(f_t) \right)^{1-\varepsilon}$$

and this proves (1.18). Using finally (1.13) and applying finally Theorem 1.3 (with any choice of $k > 2$), (1.19) easily follows. $\square$

We next study the case of Maxwell molecules.

Proof of Theorem 1.4-(i) when $K \in [1,\infty)$. We thus assume (1.3), (1.5) with $\gamma = 0$ and (1.7). We consider $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ not being a Dirac mass. We also assume that $f_0 \in \mathcal{P}_4(\mathbb{R}^3)$ or that $\int_{\mathbb{R}^3} f_0(v)\log f_0(v) dv < \infty$, so that f_t has a density for all $t > 0$. We fix $N \geq 1$ and $K \in [1,\infty)$ and consider the processes introduced in Lemma 4.4.

Step 1. Exactly as in the case of hard potentials, we find that

$$\mathbb{E}[|W_t^1 - V_t^{1,N,K}|^2] \le \int_0^t [B_1^K(s) + B_2^K(s) + B_3^K(s)]ds,$$

where $B_i^K(s) := \int_0^1 \mathbb{E}\Big[A_i^K(W_s^1, W_s^*(\alpha), V_s^{1,N,K}, V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha))\Big]d\alpha$ for $i = 1, 2, 3$.

Step 2. By Lemma 3.2-(i), we have $B_1^K(s) = 0$.

Steps 3 and 4. By Lemma 3.2-(ii), it holds that for $\zeta_K = \pi \int_0^K (1 - \cos G(z))dz$,

$$B_2^K(s) = \zeta_K \int_0^1 \mathbb{E}\Big[-|W_s^1 - V_s^{1,N,K}|^2 + |W_s^*(\alpha) - V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha)|^2\Big]d\alpha.$$

Step 5. By Lemma 3.2-(iii)

$$B_3^K(s) \le CK^{1-2/\nu} \int_0^1 \mathbb{E}\Big[|W_s^1|^2 + |W_s^*(\alpha)|^2 + |V_s^{1,N,K}|^2\Big]d\alpha \le CK^{1-2/\nu},$$

since, as usual, $\mathbb{E}[|W_s^1|^2] = \int_0^1 |W_s^*(\alpha)|^2 d\alpha = \mathbb{E}[|V_s^{1,N,K}|^2] = \int_{\mathbb{R}^3} |v|^2 f_0(dv)$.

Step 6. Setting $u_t^{N,K} := \mathbb{E}[|W_t^1 - V_t^{1,N,K}|^2]$, we thus have

$$\begin{aligned} u_t^{N,K} \le & CK^{1-2/\nu}t + \zeta_K \int_0^t \left(-u_s^{N,K} + \int_0^1 \mathbb{E}\Big[|W_s^*(\alpha) - V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha)|^2\Big]d\alpha\right) ds \\ \le & CK^{1-2/\nu}t + \zeta_K \int_0^t \left(2\sqrt{u_s^{N,K}}\sqrt{\mathbb{E}\left[\mathcal{W}_2^2(f_s, \mu_{\mathbf{W}_s}^N)\right]} + \mathbb{E}\left[\mathcal{W}_2^2(f_s, \mu_{\mathbf{W}_s}^N)\right]\right) ds \end{aligned}$$

by (4.3). Next we recall that $\varepsilon_N(f_t) = \mathbb{E}[\mathcal{W}_2^2(f_s, \mu_{\mathbf{W}_s}^N)]$, we set $\varepsilon_{N,T} = \sup_{[0,T]} \varepsilon_N(f_t)$ and we recall that $\zeta_K \le \int_0^\infty (1 - \cos G(z))dz < \infty$. We thus may write, for all $t \in [0, T]$,

$$u_t^{N,K} \le C(K^{1-2/\nu} + \varepsilon_{N,T}T)T + C\varepsilon_{N,T}^{1/2} \int_0^t (u_s^{N,K})^{1/2} ds =: v_t^{N,K}.$$

Then we have $(v_t^{N,K})' \le C\varepsilon_{N,T}^{1/2}(v_t^{N,K})^{1/2}$, so that $(v_t^{N,K})^{1/2} \le (C(K^{1-2/\nu} + \varepsilon_{N,T}T)T)^{1/2} + C\varepsilon_{N,T}^{1/2}t$. We conclude that

$$\sup_{[0,T]} u_t^{N,K} \le C(K^{1-2/\nu} + \varepsilon_{N,T}T)T + CT^2\varepsilon_{N,T} \le CK^{1-2/\nu}T + C(T + T^2)\varepsilon_{N,T}.$$

Final step. Exactly as in the case of hard potentials, for $t \in [0, T]$,

$$\mathbb{E}[\mathcal{W}_2^2(\mu_t^{N,K}, f_t)] \le 2\varepsilon_N(f_t) + 2u_t^{N,K} \le CK^{1-2/\nu}T + C(1 + T)^2 \sup_{[0,T]} \varepsilon_N(f_t)$$

whence (1.16). If finally $f_0 \in \mathcal{P}_k(\mathbb{R}^3)$ for all $k \ge 2$, then we know that $\sup_{[0,\infty)} \int_{\mathbb{R}^3} |v|^k f_t(dv) < \infty$, so that (1.17) follows by application of Theorem 1.3. $\square$

We conclude with hard spheres.

Proof of Theorem 1.4-(iii). We thus assume (1.3), (1.5) with $\gamma = 1$ and (1.6). We consider $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ satisfying (1.12) for some $p \in (\gamma, 2)$ and fix $q \in (\gamma, p)$ for the rest of the proof. We also assume that f_0 has a density, so that f_t has a density for all $t > 0$. We fix $N \ge 1$ and $K \in [1, \infty)$ and consider the processes introduced in Lemma 4.4.

Step 1. Exactly as in the case of hard potentials, we find that

$$u_t^{N,K} := \mathbb{E}[|W_t^1 - V_t^{1,N,K}|^2] \le \int_0^t [B_1^K(s) + B_2^K(s) + B_3^K(s)]ds,$$

where $B_i^K(s) := \int_0^1 \mathbb{E}\Big[A_i^K(W_s^1, W_s^*(\alpha), V_s^{1,N,K}, V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha))\Big]d\alpha$ for $i = 1, 2, 3$.

Steps 2, 3, 4, 5, 6. Following the case of hard potentials, using Lemma 3.4 instead of Lemma 3.3, we deduce that for all $M > 1$,

$$\begin{aligned}\sum_1^3 B_i^K(s) \le& 2M \int_0^1 \mathbb{E}\left[|W_s^1 - V_s^{1,N,K}|^2 + |W_s^*(\alpha) - V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha)|^2\right] d\alpha \\ &+ C(Ke^{-M^q} + e^{-K^q}) \int_0^1 \mathbb{E}\Big[(1 + |V_s^{1,N,K}| + |V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha)|) \\ &\qquad \times e^{C(|W_s^1|^q + |W_s^*(\alpha)|^q + |Z_s^*(\mathbf{W}_s,\alpha)|^q)}\Big]d\alpha \\ &+ C\int_0^1 \mathbb{E}\Big[|W_s^*(\alpha) - Z_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha)|^2 \\ &\qquad \times (1 + |W_s^1| + |W_s^*(\alpha)| + |Z_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha)|)^2\Big]d\alpha\end{aligned}$$

Proceeding as in (4.5), we deduce that the last line is bounded, for all $\varepsilon \in (0, 1)$, by

$$C_\varepsilon \left(\mathbb{E}\left[\mathcal{W}_2^2(f_s, \mu_{\mathbf{W}_s}^N)\right]\right)^{\frac{2-\varepsilon}{\varepsilon}}$$

and using (4.4), the first term is bounded by

$$4M\mathbb{E}\left[\mathcal{W}_2^2(f_s, \mu_{\mathbf{W}_s}^N)\right] + 6Mu_s^{N,K}.$$

Using finally the Cauchy-Schwarz inequality, that, thanks to Lemma 4.3-(c) and by exchangeability, $\mathbb{E}[\int_0^1 |V_s^*(\mathbf{V}_s^{N,K}, \mathbf{W}_s, \alpha)|^2 d\alpha] = \mathbb{E}[N^{-1}\sum_1^N |V_s^{i,N,K}|^2] = \mathbb{E}[|V_s^{1,N,K}|^2] = \int_{\mathbb{R}^3} |v|^2 f_0(dv) < \infty$ and (1.13), we easily bound the second line by $C(Ke^{-M^q} + e^{-K^q})$ (recall that $W_s^i \sim f_s$, that $W_s^*(.) \sim f_s$ and that, by Lemma 4.4-(b), $\int_0^1 e^{C|Z_s^*(\mathbf{W}_s,\alpha)|^q} d\alpha = N^{-1}\sum_1^N e^{C|W_s^i|^q}$).

Recalling that $\mathbb{E}\left[\mathcal{W}_2^2(f_s, \mu_{\mathbf{W}_s}^N)\right] = \varepsilon_N(f_s)$ and setting $\varepsilon_{N,t} = \sup_{[0,t]} \varepsilon_N(f_s)$, we thus have, for any $M > 1$, any $\varepsilon \in (0, 1)$,

$$u_t^{N,K} \le 6M \int_0^t u_s^{N,K} ds + Ct(Ke^{-M^q} + e^{-K^q}) + C_\varepsilon t \varepsilon_{N,t}^{1-\varepsilon/2}.$$

Thus by Grönwall's Lemma,

$$u_t^{N,K} \le C_\varepsilon t(Ke^{-M^q} + e^{-K^q} + \varepsilon_{N,t}^{1-\varepsilon/2})e^{6Mt}.$$

Choosing $M = 2K$ and using that $Ke^{-(2K)^q} \le Ce^{-K^q}$, we deduce that

$$\sup_{[0,T]} u_t^{N,K} \le C_\varepsilon T(e^{-K^q} + \varepsilon_{N,T}^{1-\varepsilon/2})e^{12KT} = C_\varepsilon T(e^{-K^q} + (\sup_{[0,T]} \varepsilon_N(f_s))^{1-\varepsilon/2})e^{12KT}.$$

Final step. We conclude as usual, using that $\mathbb{E}[\mathcal{W}_2^2(\mu_t^{N,K}, f_t)] \le 2\varepsilon_N(f_t) + 2u_t^{N,K}$ to obtain (1.20) and then (1.13) and Theorem 1.3 to deduce (1.21). $\square$

5. Extension to the particle system without cutoff

It remains to check that the particle system without cutoff is well-posed and that we can pass to the limit as $K \to \infty$ in the convergence estimates (1.16)-(1.17)-(1.18)-(1.19). We will need the following rough computations.

Lemma 5.1. *Assume* (1.3), (1.5) *and* (1.6) *or* (1.7). *Adopt the notation of Lemma 3.1. There are $C > 0$, $\kappa > 0$ and $\delta > 0$ (depending on γ, ν) such that for all $K \in [1, \infty)$, all $v, v_*, \tilde{v}, \tilde{v}_* \in \mathbb{R}^3$,*

$$\sum_{i=1}^{3} A_i^K(v, v_*, \tilde{v}, \tilde{v}_*) \leq C(1 + |v| + |v_*| + |\tilde{v}| + |\tilde{v}_*|)^\kappa (|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2 + K^{-\delta}).$$

Proof. Concerning A_1^K, we start from (3.1) (this is valid for all $\gamma \in [0, 1]$) and we deduce that

$$\begin{aligned} A_1^K(v, v_*, \tilde{v}, \tilde{v}_*) \leq & 8c_4(|v - \tilde{v}| \wedge |v_* - \tilde{v}_*|)^\gamma (|v - \tilde{v}| + |v_* - \tilde{v}_*|)^2 \\ \leq & C(1 + |v| + |v_*| + |\tilde{v}| + |\tilde{v}_*|)^\gamma (|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2). \end{aligned}$$

We then make use of (3.2) (also valid for all $\gamma \in [0, 1]$) to write

$$\begin{aligned} A_2^K(v, v_*, \tilde{v}, \tilde{v}_*) \leq & C(|v - \tilde{v}| + |v_* - \tilde{v}_*|)^2 (|v - v_*|^\gamma + |\tilde{v} - \tilde{v}_*|^\gamma) \\ \leq & C(1 + |v| + |v_*| + |\tilde{v}| + |\tilde{v}_*|)^\gamma (|v - \tilde{v}|^2 + |v_* - \tilde{v}_*|^2). \end{aligned}$$

For A_3^K, we separate two cases. Under hypothesis (1.7), we immediately deduce from (3.4) that

$$A_3^K(v, v_*, \tilde{v}, \tilde{v}_*) \leq C(1 + |v| + |v_*| + |\tilde{v}| + |\tilde{v}_*|)^{2+2\gamma/\nu} K^{1-2/\nu}.$$

Under hypothesis (1.6), we have seen (when $\gamma = 1$, at the end of the proof of Lemma 3.4) that $\Psi_K(x) \leq 5x^\gamma \mathbb{1}_{\{x^\gamma \geq K/2\}}$, whence $\Psi_K(x) \leq 10x^{2\gamma}/K$ and thus

$$A_3^K(v, v_*, \tilde{v}, \tilde{v}_*) \leq C(|v - v_*| \vee |\tilde{v} - \tilde{v}_*|)^{2+2\gamma} K^{-1} \leq C(1 + |v| + |v_*| + |\tilde{v}| + |\tilde{v}_*|)^{2+2\gamma} K^{-1}.$$

The conclusion follows, choosing $\kappa = 2 + 2\gamma/\nu$ and $\delta = 2/\nu - 1$ under (1.7) and $\kappa = 2 + 2\gamma$ and $\delta = 1$ under (1.6). $\square$

Now we can give the

Proof of Proposition 1.2-(ii). We only sketch the proof, since it is quite standard. In the whole proof, $N \geq 2$ is fixed, as well as $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ and a family of i.i.d. f_0-distributed random variables $(V_0^{i,N})_{i=1,\dots,N}$.

Step 1. Recall (2.8). Classically, $(V_t^{i,N,\infty})_{i=1,\dots,N,t\geq 0}$ is a Markov process with generator $\mathcal{L}_N$ starting from $(V_0^{i,N})_{i=1,\dots,N}$ if it solves

$$V_t^{i,N,\infty} = V_0^i + \int_0^t \int_j \int_0^\infty \int_0^{2\pi} c(V_{s-}^{i,N,\infty}, V_{s-}^{j,N,\infty}, z, \varphi) O_i^N(ds, dj, dz, d\varphi), \quad i = 1, \dots, N \tag{5.1}$$

for some i.i.d. Poisson measures $O_i^N(ds, dj, dz, d\varphi))_{i=1,\dots,N}$ on $[0, \infty) \times \{1, \dots, N\} \times [0, \infty) \times [0, 2\pi)$ with intensity measures $ds \left(N^{-1} \sum_{k=1}^N \delta_k(dj)\right) dz d\varphi$.

Step 2. The existence of a solution (in law) to (5.1) is easily checked, using martingale problems methods (tightness and consistency), by passing to the limit in (4.2). The main estimates to be used are that, uniformly in $K \in [1, \infty)$ (and in $N \geq 1$ but this is not the point here),

$$\mathbb{E}[|V_t^{1,N,K}|^2] = \int_{\mathbb{R}^3} |v|^2 f_0(dv) \quad \text{and} \quad \mathbb{E}\left[\sup_{[0,T]} |V_t^{1,N,K}|\right] \leq C_T$$

for all $T > 0$. This second estimate is immediately deduced from the first one and the fact that $\int_0^\infty \int_0^{2\pi} |c(v, v_*, z, \varphi)| \le C|v - v_*|^{1+\gamma} \le C(1 + |v| + |v_*|)^2$. The tightness is easily checked by using Aldous's criterion [1].

Step 3. Uniqueness (in law) for (5.1) is more difficult. Consider a (càdlàg and adapted) solution $(V_t^{i,N,\infty})_{i=1,\dots,N,t\ge0}$ to (5.1). For $K \in [1,\infty)$, consider the solution to

$$V_t^{i,N,K} = V_0^i + \int_0^t \int_j \int_0^\infty \int_0^{2\pi} c_K(V_{s-}^{i,N,K}, V_{s-}^{j,N,K}, z, \varphi + \varphi_{s,i,j}) O_i^N(ds, dj, dz, d\varphi), \quad i = 1, \dots, N$$

where $\varphi_{s,i,j} := \varphi_0(V_{s-}^{i,N,\infty} - V_{s-}^{j,N,\infty}, V_{s-}^{i,N,K} - V_{s-}^{j,N,K})$. Such a solution obviously exists and is unique, because the involved Poisson measures are finite (recall that $c_K(v, v_*, z, \varphi) = 0$ for $z \ge K$). Furthermore, this solution $(V_t^{i,N,K})_{i=1,\dots,N,t\ge0}$ is a Markov process with generator $\mathcal{L}_{N,K}$ starting from $(V_0^{i,N})_{i=1,\dots,N}$ (because the only difference with (4.2) is the presence of $\varphi_{s,i,j}$ which does not change the law of the particle system, see Lemma 4.4-(ii) for a similar claim). Hence Proposition 1.2-(i) implies that the law of $(V_t^{i,N,K})_{i=1,\dots,N,t\ge0}$ is uniquely determined.

We next introduce $\tau_{N,K,A} = \inf\{t \ge 0 \,:\, \exists\, i \in \{1, \dots, N\},\ |V_t^{i,N,\infty}| + |V_t^{i,N,K}| \ge A\}$. Using, on the one hand, the fact that $(V_t^{i,N,\infty})_{i=1,\dots,N,t\ge0}$ is a.s. càdlàg (and thus locally bounded) and, on the other hand, the (uniform in K) estimate established in Step 2, one easily gets convinced that

$$\forall\, T > 0, \quad \lim_{A\to\infty} \sup_{K\ge1} \Pr[\tau_{N,K,A} \le T] = 0. \tag{5.2}$$

Next, a simple computation shows that

$$\mathbb{E}[|V^{1,N,\infty}_{t\wedge\tau_{N,K,A}} - V^{1,N,K}_{t\wedge\tau_{N,K,A}}|^2] \le \frac{1}{N}\sum_{j=1}^N \mathbb{E}\Big[\int_0^{t\wedge\tau_{N,K,A}} \int_0^\infty \int_0^{2\pi} \Big(\big|V_{s-}^{1,N,\infty} - V_{s-}^{1,N,K} + \Delta_{s-}^{1,j,N,K}(z,\varphi)\big|^2 - \big|V_{s-}^{1,N,\infty} - V_{s-}^{1,N,K}\big|^2\Big) d\varphi dz\Big]$$

where

$$\Delta_{s-}^{1,j,N,K}(z,\varphi) := c(V_{s-}^{1,N,\infty}, V_{s-}^{1,N,\infty}, z, \varphi) - c_K(V_{s-}^{1,N,K}, V_{s-}^{1,N,K}, z, \varphi + \varphi_{s,i,j}).$$

Using Lemmas 3.1 and 5.1 and the fact that all the velocities are bounded by A until $\tau_{N,K,A}$, we easily deduce that

$$\begin{aligned}
&\mathbb{E}[|V^{1,N,\infty}_{t\wedge\tau_{N,K,A}} - V^{1,N,K}_{t\wedge\tau_{N,K,A}}|^2]\\
\le& \frac{C(1+A)^\kappa}{N}\sum_{j=1}^N \mathbb{E}\Big[\int_0^{t\wedge\tau_{N,K,A}} (|V_s^{1,N,\infty} - V_s^{1,N,K}|^2 + |V_s^{j,N,\infty} - V_s^{j,N,K}|^2 + K^{-\delta}) ds\Big]\\
\le& C_T(1+A)^\kappa K^{-\delta} + C(1+A)^\kappa \int_0^t \mathbb{E}[|V^{1,N,\infty}_{s\wedge\tau_{N,K,A}} - V^{1,N,K}_{s\wedge\tau_{N,K,A}}|^2] ds
\end{aligned}$$

by exchangeability. We now use the Grönwall lemma and then deduce that for any $A > 0$,

$$\lim_{K\to\infty} \sup_{[0,T]} \mathbb{E}[|V^{1,N,\infty}_{t\wedge\tau_{N,K,A}} - V^{1,N,K}_{t\wedge\tau_{N,K,A}}|^2] = 0. \tag{5.3}$$

Gathering (5.2) and (5.3), we easily conclude that for all $t \ge 0$, $V_t^{1,N,K}$ tends in probability to $V_t^{1,N,\infty}$ as $K \to \infty$. Thus for any finite family $0 \le t_1 \le \dots \le t_l$, $(V_{t_j}^{i,N,K})_{i=1,\dots,N,j=1,\dots,l}$ goes in probability to $(V_{t_j}^{i,N,\infty})_{i=1,\dots,N,j=1,\dots,l}$, of which the law is thus uniquely determined. This is classically sufficient to characterize the whole law of the process $(V_t^{i,N,\infty})_{i=1,\dots,N,t\ge0}$.

Conclusion. We thus have the existence of a unique Markov process $(V_t^{i,N,\infty})_{i=1,\dots,N,t\geq 0}$ with generator $\mathcal{L}_N$ starting from $(V_0^{i,N})_{i=1,\dots,N}$, and it holds that for each $t \geq 0$, each $N \geq 2$, $(V_t^{i,N,\infty})_{i=1,\dots,N}$ is the limit in law, as $K \to \infty$, of $(V_t^{i,N,K})_{i=1,\dots,N}$. ☐

To conclude, we will need the following lemma.

Lemma 5.2. *Let $N \geq 2$ be fixed. Let $(X^{i,N,K})_{i=1,\dots,N}$ be a sequence of $(\mathbb{R}^3)^N$-valued random variable going in law, as $K \to \infty$, to some $(\mathbb{R}^3)^N$-valued random variable $(X^{i,N})_{i=1,\dots,N}$. Consider the associated empirical measures $\nu^{N,K} := N^{-1}\sum_{i=1}^N \delta_{X^{i,N,K}}$ and $\nu^N := N^{-1}\sum_{i=1}^N \delta_{X^{i,N}}$. Then for any $g \in \mathcal{P}_2(\mathbb{R}^3)$,*

$$\mathbb{E}\left[\mathcal{W}_2^2\left(\nu^N, g\right)\right] \leq \liminf_{K\to\infty} \mathbb{E}\left[\mathcal{W}_2^2\left(\nu^{N,K}, g\right)\right].$$

Proof. First observe that the map $(x_1, \dots, x_N) \mapsto \mathcal{W}_2(N^{-1}\sum_1^N \delta_{x_i}, g)$ is continuous on $(\mathbb{R}^3)^N$. Indeed, it suffices to use the triangular inequality for $\mathcal{W}_2$ and the easy estimate

$$\mathcal{W}_2^2\left(\frac{1}{N}\sum_1^N \delta_{x_i}, \frac{1}{N}\sum_1^N \delta_{y_i}\right) \leq \frac{1}{N}\sum_1^N |x_i - y_i|^2.$$

Consequently, $\mathcal{W}_2^2\left(\nu^{N,K}, g\right)$ goes in law to $\mathcal{W}_2^2\left(\nu^N, g\right)$. Thus for any $A > 1$, we have

$$\mathbb{E}\left[\mathcal{W}_2^2\left(\nu^N, g\right)\wedge A\right] = \lim_{K\to\infty} \mathbb{E}\left[\mathcal{W}_2^2\left(\nu^{N,K}, g\right)\wedge A\right] \leq \liminf_{K\to\infty} \mathbb{E}\left[\mathcal{W}_2^2\left(\nu^{N,K}, g\right)\right].$$

It then suffices to let A increase to infinity and to use the monotonic convergence theorem. ☐

This allows us to conclude the proof of our main results.

Proof of Theorem 1.4-(i)-(ii) when $K = \infty$. Recall that (1.16)-(1.17)-(1.18)-(1.19) have already been established when $K \in [1, \infty)$. Since $(V_t^{i,N,\infty})_{i=1,\dots,N}$ is the limit (in law) of $(V_t^{i,N,K})_{i=1,\dots,N}$ as $K \to \infty$ for each $t \geq 0$ and each $N \geq 2$ (see the conclusion of the proof of Proposition 1.2-(ii)), we can let $K \to \infty$ in (1.16)-(1.17)-(1.18)-(1.19) using Lemma 5.2. ☐

Appendix A. Quantitative empirical law of large number

We conclude the paper with the

Proof of Theorem 1.3. We thus assume that $f \in \mathcal{P}_k(\mathbb{R}^d)$ for some $k > 2$, with $d = 3$. We denote by μ^N the empirical measure associated to N i.i.d. f-distributed random variables. We denote by $M_k(f) := \int_{\mathbb{R}^d} |v|^k f(dv)$ and observe that $\mathbb{E}(M_k(\mu^N)) = M_k(f)$.

Step 1: $k > 20$. Consider the Sobolev norm defined, for $f, g \in \mathcal{P}(\mathbb{R}^d)$, by

$$\|f - g\|_{H^{-s}}^2 := \int_{\mathbb{R}^d} (1 + |\xi|^2)^{-s}\, |\hat{f}(\xi) - \hat{g}(\xi)|^2\, d\xi,$$

where $\hat{f}$ and $\hat{g}$ denote the Fourier transform of f and g. We have (see [31, Lemma 4.2])

$$\mathbb{E}(\|\mu^N - f\|_{H^{-s}}^2) = \frac{1}{N}\int_{\mathbb{R}^d} (1 + |\xi|^2)^{-s}(1 - |\hat{f}(\xi)|^2)d\xi \leq \frac{C_{d,s}}{N} \quad \text{if } s > d/2,$$

as well as the classical inequality (see e.g. [25, Lemma 2.1])

$$\mathcal{W}_2(f, g) \leq C_{d,s}(M_k(f + g))^{\frac{4s+d}{4ks+2d}}\|f - g\|_{H^{-s}}^{\frac{k-2}{2ks+d}} \quad \text{if } s > d/2.$$

In dimension $d = 3$ with the choice $s = 3/2$ (which is forbidden), this gives

$$\mathcal{W}_2(f, g) \leq C(M_k(f + g))^{\frac{3}{2k+2}}\|f - g\|_{H^{-3/2}}^{\frac{k-2}{3k+3}} \quad \text{if } s > d/2.$$

Consequently, by Hölder's inequality,

$$\begin{aligned}
\mathbb{E}\left(\mathcal{W}_2^2(\mu^N,f)\right) \leq & C\mathbb{E}\left((M_k(f+\mu^N))^{\frac{3}{k+1}}\|\mu_V^N-f\|_{H^{-3/2}}^{\frac{2k-4}{3k+3}}\right)\\
\leq & C\mathbb{E}\left(M_k(f+\mu^N)\right)^{\frac{3}{k+1}}\mathbb{E}\left(\|\mu_V^N-f\|_{H^{-3/2}}^{\frac{2}{3}}\right)^{\frac{k-2}{k+1}}\\
\leq & CM_k(f)^{\frac{3}{k+1}}\mathbb{E}\left(\|\mu_V^N-f\|_{H^{-3/2}}^2\right)^{\frac{k-2}{3k+3}}\\
\leq & CM_k(f)^{\frac{3}{k+1}}N^{-\frac{k-2}{3k+3}}.
\end{aligned}$$

Since the choice $s=d/2$ is forbidden, we only get an estimate in N^{-2r} for any $r<(k-2)/(6k+6)$.

Step 2: $k\in(7,20]$. We sketch the proof of Rachev-Ruschendorf result [35, Theorem 10.2.1] to which we refer for details and that we will slightly modify in the next step. Let

$$\mu^{N,\varepsilon}:=\mu^N*g^\varepsilon,\quad f^\varepsilon:=f*g^\varepsilon,\quad g^\varepsilon(x):=\varepsilon^{-d}g(x/\varepsilon),\quad g(x)=(2\pi)^{-d/2}\,e^{-|x|^2/2}.$$

Using a standard L^1-upperbound of $\mathcal{W}_2^2$ and the Cauchy-Schwarz inequality,

$$\begin{aligned}
\mathcal{W}_2^2(\mu^N,f) \leq & 2\mathcal{W}_2^2(\mu^N,\mu^{N,\varepsilon})+2\mathcal{W}_2^2(\mu_V^{N,\varepsilon},f^\varepsilon)\\
\leq & 2d\varepsilon^2+2\||x|^2(\mu^{N,\varepsilon}(x)-f^\varepsilon(x))\|_{L^1}\\
\leq & 2d\varepsilon^2+2\|(1+|x|)^{2+(d+\alpha)/2}(1+|x|)^{-(d+\alpha)/2}(\mu^{N,\varepsilon}(x)-f^\varepsilon(x))\|_{L^1}\\
\leq & 2d\varepsilon^2+C_\alpha\|(1+|x|)^{4+d+\alpha}(\mu^{N,\varepsilon}(x)-f^\varepsilon(x))^2\|_{L^1}^{1/2}
\end{aligned}$$

for any $\alpha>0$. By Jensen's inequality, we get, for any $\varepsilon>0$, any $\alpha>0$,

$$\mathbb{E}(\mathcal{W}_2^2(\mu^N,f))\leq 2d\varepsilon^2+C_\alpha\mathbb{E}\Big(\|(1+|x|)^{4+d+\alpha}(\mu^{N,\varepsilon}(x)-f^\varepsilon(x))^2\|_{L^1}\Big)^{1/2}.$$

But this last quadratic quantity can be easily computed and one gets

$$\mathbb{E}\left(\|(1+|x|)^{4+d+\alpha}(\mu^{N,\varepsilon}(x)-f^\varepsilon(x))^2\|_{L^1}\right)=\frac{1}{N}\int_{\mathbb{R}^d}(1+|x|)^{4+d+\alpha}(g_\varepsilon^2*f)(x)dx.$$

Since $d=3$ and since $f\in\mathcal{P}_k(\mathbb{R}^3)$ with $k>7=4+d$, we may choose $\alpha=k-7$ and we easily deduce that for all $\varepsilon\in(0,1)$, $\int_{\mathbb{R}^d}(1+|x|)^{4+d+\alpha}(g_\varepsilon^2*f)(x)dx\leq C\varepsilon^{-3}(1+M_k(f))$. We thus have

$$\mathbb{E}(\mathcal{W}_2^2(\mu^N,f))\leq 6\varepsilon^2+C\frac{(1+M_k(f))^{1/2}}{N^{1/2}\varepsilon^{3/2}}.$$

for all $\varepsilon\in(0,1)$. Choosing $\varepsilon=N^{-1/7}$, we find $\mathbb{E}(\mathcal{W}_2^2(\mu^N,f))\leq CN^{-2/7}$ as desired.

Step 3: $k\in(2,7)$. We start as in the previous step and write

$$\begin{aligned}
\mathcal{W}_2^2(\mu_V^N,f) \leq & 2d\varepsilon^2+2\||x|^2(\mu^{N,\varepsilon}(x)-f^\varepsilon(x))\|_{L^1}\\
\leq & 2d\varepsilon^2+2\|(1+|x|)^2\mathbb{1}_{\{|x|\leq R\}}(\mu^{N,\varepsilon}(x)-f^\varepsilon(x))\|_{L^1}+2\frac{M_k(f^\varepsilon+\mu^{N,\varepsilon})}{R^{k-2}}\\
\leq & 2d\varepsilon^2+2\|(1+|x|)^{2-k/2}\mathbb{1}_{\{|z|\leq R\}}(1+|x|)^{k/2}(\mu^{N,\varepsilon}(x)-f^\varepsilon(x))\|_{L^1}+2\frac{M_k(f^\varepsilon+\mu^{N,\varepsilon})}{R^{k-2}}\\
\leq & 2d\varepsilon^2+CR^{(4+d-k)/2}\|(1+|x|)^k(\mu^{N,\varepsilon}(x)-f^\varepsilon(x))^2\|_{L^1}^{1/2}+2\frac{M_k(f^\varepsilon+\mu^{N,\varepsilon})}{R^{k-2}}.
\end{aligned}$$

Proceeding as in Step 2 for the middle term and noting that for $\varepsilon \in (0,1)$, $\mathbb{E}[M_k(f^\varepsilon + \mu^{N,\varepsilon})] \leq C(1+M_k(f))$, we obtain, with $d=3$,

$$\mathbb{E}(\mathcal{W}_2^2(\mu^N, f)) \leq 6\varepsilon^2 + C\frac{R^{(7-k)/2}}{N^{1/2}\varepsilon^{3/2}}(1+M_k(f))^{1/2} + C\frac{1+M_k(f)}{R^{k-2}}.$$

Choosing $\varepsilon = N^{-(k-2)/5k}$ and $R = N^{2/5k}$, we get a bound in $N^{-(2k-4)/5k}$ as desired.

Step 4: $k=7$. It suffices to use the bounds we proved for any $k<7$. $\square$

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