UNIQUENESS AND LONG TIME ASYMPTOTIC FOR THE PARABOLIC-PARABOLIC KELLER-SEGEL EQUATION

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ABSTRACT. The present paper deals with the parabolic-parabolic Keller-Segel equation in the plane in the general framework of weak (or "free energy") solutions associated to an initial datum with finite mass $M < 8\pi$, finite second log-moment and finite entropy. The aim of the paper is twofold:

(1) We prove the uniqueness of the "free energy" solution. The proof uses a DiPerna-Lions renormalizing argument which makes possible to get the "optimal regularity" as well as an estimate of the difference of two possible solutions in the critical $L^{4/3}$ Lebesgue norm similarly as for the $2d$ vorticity Navier-Stokes equation.

(2) We prove a radially symmetric and polynomial weighted L^2 exponential stability of the self-similar profile in the quasi parabolic-elliptic regime. The proof is based on a (singular) perturbation argument which takes advantage of the exponential stability of the self-similar profile for the parabolic-elliptic Keller-Segel equation as established in [9, 14].

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CONTENTS

1. INTRODUCTION

The Patlak-Keller-Segel (PKS) system for chemotaxis describes the collective motion of cells that are attracted by a chemical substance that they are able to emit ([31, 21]). In this paper we are concerned with the parabolic-parabolic PKS model in the plane which takes the form

$$(1.1) \qquad \begin{aligned} \partial_t f &= \Delta f - \nabla(f \, \nabla u) \quad \text{in} \quad (0, \infty) \times \mathbb{R}^2, \\ \varepsilon \partial_t u &= \Delta u + f - \alpha \, u \quad \text{in} \quad (0, \infty) \times \mathbb{R}^2, \end{aligned}$$

which is complemented with an initial condition

$$(1.2) \qquad f(0, \cdot) = f_0 \geq 0 \quad \text{and} \quad u(0, \cdot) = u_0 \geq 0 \quad \text{in} \quad \mathbb{R}^2.$$

Here $t \geq 0$ is the time variable, $x \in \mathbb{R}^2$ is the space variable, $f = f(t, x) \geq 0$ stands for the *mass density of cells* while $u = u(t, x) \geq 0$ is the *chemo-attractant concentration* and $\varepsilon > 0$, $\alpha \geq 0$

are constants. We refer to [7] and the references quoted therein for biological motivation and mathematical introduction.

The fundamental identities are that any solution to the Keller-Segel equation (1.1) satisfies at least formally the conservation of mass

(1.3) $$M(t) := \langle f(t, .)\rangle = \langle f_0 \rangle =: M, \quad \text{with} \quad \langle g \rangle := \int_{\mathbb{R}^2} g(x)\, dx,$$

and the free energy-dissipation of the free energy identity

(1.4) $$\mathcal{F}(t) + \int_0^t \mathcal{D}_{\mathcal{F}}(s)\, ds = \mathcal{F}_0,$$

where the free energy $\mathcal{F}(t) = \mathcal{F}(f(t), u(t))$, $\mathcal{F}_0 = \mathcal{F}(f_0, u_0)$ is defined by

(1.5) $$\mathcal{F} = \mathcal{F}(f, u) := \int_{\mathbb{R}^2} f \log f\, dx - \int_{\mathbb{R}^2} f u\, dx + \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\, dx + \frac{\alpha}{2}\int_{\mathbb{R}^2} u^2 dx,$$

and the dissipation of free energy by

(1.6) $$\mathcal{D}_{\mathcal{F}} = \mathcal{D}_{\mathcal{F}}(f, u) := \int_{\mathbb{R}^2} f\, |\nabla(\log f) - \nabla u|^2\, dx + \frac{1}{\varepsilon}\int_{\mathbb{R}^2} |\Delta u + f - \alpha\, u|^2\, dx.$$

Following [7], throughout this paper, we shall assume that the initial data f_0 and u_0 satisfy

(1.7) $$\begin{cases} f_0\,(1 + \log\langle x\rangle^2) \in L^1(\mathbb{R}^2) \quad \text{and} \quad f_0 \log f_0 \in L^1(\mathbb{R}^2); \\ u_0 \in H^1(\mathbb{R}^2) \text{ if } \alpha > 0 \quad \text{or} \quad u_0 \in L^1(\mathbb{R}^2),\ \nabla u_0 \in L^2(\mathbb{R}^2) \text{ if } \alpha = 0; \\ f_0\, u_0 \in L^1(\mathbb{R}^2), \end{cases}$$

where here and below we define the weight function $\langle x \rangle := (1 + |x|^2)^{1/2}$. We also make the important restriction of subcritical mass hypothesis

$$M := \langle f_0 \rangle \in (0, 8\pi),$$

because a suitable global existence theory is available in that case (see [7, 27]) and that there exists blow up (not global in time) solution when $M > 8\pi$ (see [19, 29, 28] and the discussion in [27, 1. Introduction]). We also refer to [4, 12] where a global existence theory is developed in the possible supercritical case $M > 0$ and the condition that ε is large enough (which corresponds to a case where the nonlinearity in (1.1) is small).

As in [7], we consider the following definition of weak solution.

Definition 1.1. *For any initial datum (f_0, u_0) satisfying (1.7) with $M < 8\pi$, we say that the couple (f, u) of nonnegative functions satisfying*

$$f \in L^\infty(0, T; L^1(\mathbb{R}^2)) \cap C([0, T]; \mathcal{D}'(\mathbb{R}^2)), \quad \forall T \in (0, \infty),$$

(1.8) $$\begin{cases} u \in L^\infty(0, T; H^1(\mathbb{R}^2)) & \text{if } \alpha > 0; \\ u \in L^\infty(0, T; L^1(\mathbb{R}^2)) \quad \text{and} \quad \nabla u \in L^\infty(0, T; L^2(\mathbb{R}^2)) & \text{if } \alpha = 0; \end{cases}$$

$$fu \in L^\infty(0, T; L^1(\mathbb{R}^2))$$

is a weak solution to the Keller-Segel equation associated to the initial condition (f_0, u_0) whenever (f, u) satisfies the mass conservation (1.3), the bound

(1.9) $$\sup_{[0,T]} \mathcal{F}(t) + \sup_{[0,T]} \int_{\mathbb{R}^2} f\, \log\langle x\rangle^2\, dx + \int_0^T \mathcal{D}_{\mathcal{F}}(t)\, dt \le C_T,$$

as well as the Keller-Segel system of equations (1.1)-(1.2) in the distributional sense, namely

(1.10) $$\int_{\mathbb{R}^2} f_0(x)\, \varphi(0, x)\, dx = \int_0^T \int_{\mathbb{R}^2} f\left\{(\nabla_x(\log f) - \nabla_x u) \cdot \nabla_x \varphi - \partial_t \varphi\right\} dx\, dt$$

(1.11) $$\varepsilon \int_{\mathbb{R}^2} u_0(x)\, \psi(0, x)\, dx = \int_0^T \int_{\mathbb{R}^2} \left\{u\,(-\Delta\psi + \alpha\psi - \varepsilon\partial_t\psi) - f(t, x)\psi\right\} dx\, dt$$

for any $T > 0$ and $\varphi, \psi \in C_c^2([0, T) \times \mathbb{R}^2)$.

It is worth emphasizing that thanks to the Cauchy-Schwarz inequality, we have

$$\int_{\mathbb{R}^2} f \, |\nabla_x (\log f) - \nabla_x u| \, dx \le M^{1/2} \, \mathcal{D}_{\mathcal{F}}^{1/2},$$

and the RHS of (1.10) is then well defined thanks to (1.3) and (1.9).

This framework is well adapted for a global existence theory in the subcritical mass case.

Theorem 1.2. *([7, Theorem 1]) For any initial datum (f_0, u_0) satisfying (1.7) and $M < 8\pi$ there exists at least one weak solution in the sense of Definition 1.1 to the Keller-Segel equation (1.1)-(1.2).*

Our first main result establishes that this framework is also well adapted for the well-posedness issue.

Theorem 1.3. *For any initial datum (f_0, u_0) satisfying (1.7) with $M < 8\pi$ there exists at most one weak solution in the sense of Definition 1.1 to the Keller-Segel equation (1.1)-(1.2). This one is furthermore a classical solution in the sense that*

$$(1.12) \qquad\qquad f, u \in C_b^2((0, \infty) \times \mathbb{R}^2)$$

and satisfies the accurate small time estimate

$$(1.13) \qquad\qquad \forall \, q \in [4/3, 2), \qquad t^{1 - \frac{1}{q}} \, \|f(t)\|_{L^q} \to 0 \quad as \quad t \to 0.$$

Finally, the free energy-dissipation of the free energy identity (1.4) holds.

Theorem 1.3 improves the uniqueness result proved in [11] in the class of solutions $f \in C([0, T]; L_2^1(\mathbb{R}^2)) \cap L^\infty((0, T) \times \mathbb{R}^2)$ which can be built under the additional assumption $f_0 \in L^\infty(\mathbb{R}^2)$ (see also [17] where a uniqueness result is established for a related model and the recent works [15, 4, 12] where the well-posedness is proved in some particular regimes). Our proof follows a strategy introduced in [16] for the 2D viscous vortex model and generalize a similar result obtained in [14] for the parabolic-elliptic model (which corresponds to the case $\varepsilon = 0$). It is based on a DiPerna-Lions renormalization process (see [13]) which makes possible to get the optimal regularity of solutions for small time (1.13) and then to follow the uniqueness argument introduced by Ben-Artzi for the 2D viscous vortex model (see [2, 6]). It is worth emphasizing that such an argument is also related to Kato's works on the Navier-Stokes equation (see e.g. [20]).

From now on in this introduction, we definitively restrict ourself to the case $\alpha = 0$ and we focus on the long time asymptotic of the solutions. For that last purpose it is convenient to work with self-similar variables. We introduce the rescaled functions g and v defined by

$$(1.14) \qquad f(t, x) := R(t)^{-2} g(\log R(t), R(t)^{-1} x), \quad u(t, x) := v(\log R(t), R(t)^{-1} x),$$

with $R(t) := (1 + t)^{1/2}$. For these new unknowns, the rescaled parabolic-parabolic Keller-Segel system reads

$$(1.15) \qquad\qquad \partial_t g = \Delta g + \nabla(\frac{1}{2} x \, g - g \, \nabla v) \quad in \quad (0, \infty) \times \mathbb{R}^2,$$

$$(1.16) \qquad\qquad \varepsilon \partial_t v = \Delta v + g + \frac{\varepsilon}{2} x \cdot \nabla v \quad in \quad (0, \infty) \times \mathbb{R}^2.$$

We are interested in self-similar solutions to the Keller-Segel parabolic-parabolic equation (1.1), that is solutions which write as

$$f(t, x) = \frac{1}{t} G_\varepsilon(\frac{x}{t^{1/2}}), \quad u(t, x) = V_\varepsilon(\frac{x}{t^{1/2}})$$

with

$$(1.17) \qquad\qquad \int_{\mathbb{R}^2} f(t, x) \, dx = \int_{\mathbb{R}^2} G_\varepsilon(y) \, dy = M \in (0, 8\pi).$$

Such a couple of functions (f, u) is a solution to (1.1) if and only if the associated "self-similar profile" $(G_\varepsilon, V_\varepsilon)$ satisfies the elliptic system

$$(1.18) \qquad \Delta G_\varepsilon - \nabla(G_\varepsilon \nabla V_\varepsilon - \frac{1}{2} x\, G_\varepsilon) = 0 \quad \text{in} \quad \mathbb{R}^2,$$

$$\Delta V_\varepsilon + \frac{\varepsilon}{2}\, x \cdot \nabla V_\varepsilon + G_\varepsilon = 0 \quad \text{in} \quad \mathbb{R}^2,$$

and thus corresponds to a stationary solution to the rescaled parabolic-parabolic Keller-Segel system (1.15). It is known that for any $\varepsilon \in (0, 1/2)$ and any $M \in (0, 8\pi)$ there exists a unique solution $(G_\varepsilon, V_\varepsilon)$ to (1.18) such that the mass of G_ε equals M which is furthermore radially symmetric and smooth (say $C^2(\mathbb{R}^2)$), see [30, 3].

Our second main result concerns the exponential nonlinear stability of the self-similar profile for any given mass $M \in (0, 8\pi)$ under the strong restriction of radial symmetry and closeness to the parabolic-elliptic regime. We define the norm

$$\||(g, v)\|| := \|g\|_{H_k^1} + \|v\|_{H^2}$$

where the weighted Lebesgue space $L_k^p(\mathbb{R}^2)$ for $1 \le p \le \infty$, $k \ge 0$, is defined by

$$L_k^p(\mathbb{R}^2) := \{f \in L_{loc}^1(\mathbb{R}^2);\ \|f\|_{L_k^p} := \|f\,\langle x \rangle^k\|_{L^p} < \infty\},$$

and higher-order Sobolev spaces $W_k^{\ell,p}(\mathbb{R}^2)$ are defined by the norm

$$\|f\|_{W_k^{\ell,p}}^p := \sum_{|\alpha| \le \ell} \|\langle x \rangle^k\, \partial^\alpha f\|_{L^p}^p.$$

Theorem 1.4. *For any given mass $M \in (0, 8\pi)$, there exist $\varepsilon^* > 0$ and $\eta^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*)$ and any initial datum (g_0, v_0) satisfying*

$$\||(g_0, v_0) - (G_\varepsilon, V_\varepsilon)\|| \le \eta^*, \quad \int_{\mathbb{R}^2} g_0\, dx = \int_{\mathbb{R}^2} G_\varepsilon\, dx = M,$$

the associated solution (g, v) to (1.15)-(1.16) satisfies

$$\||(g(t), v(t)) - (G_\varepsilon, V_\varepsilon)\|| \le C_a\, e^{at} \quad \forall a \in (-1/2, \infty),\ \forall t \ge 0,$$

for some constant $C_a = C_a(g_0, v_0)$.

That result extends to the parabolic-parabolic Keller-Segel equations similar results known on the parabolic-elliptic Keller-Segel equations, see [14]. To our knowledge, Theorem 1.4 is the first exponential stability result for the system (1.1) even under the two strong restrictions of radial symmetry and quasi parabolic-elliptic regime (we mean $\varepsilon > 0$ small). However, we refer again to the recent work [12, Section 4] where some results of convergence (without rate) of some solutions to the associated self-similar profile are established. We also refer to that work for further discussion and additional references.

Let us end the introduction by describing the plan of the paper. In Section 2 we present some functional inequalities which will be useful in the sequel of the paper and we establish several a posteriori bounds satisfied by any weak solution. Section 3 is dedicated to the proof of the uniqueness result stated in Theorem 1.3. In Section 4 we prove the exponential stability of the linearized problem associated to (1.15)-(1.16). Finally, in Section 5 we prove the long-time behaviour result as stated in Theorem 1.4.

2. LOCAL IN TIME A PRIORI AND A POSTERIORI ESTIMATES

2.1. **A priori estimates.** In that short paragraph, we follow [7] and we explain how to obtain the basic estimates which lead to the notion of weak solution as presented in Definition 1.1. We first observe that the following space logarithmic moment control holds true

$$\frac{d}{dt}\int_{\mathbb{R}^2} f\,(-\log H)dx \;=\; \int_{\mathbb{R}^2} f\,\nabla(\log f - u)\cdot\nabla(\log H)\,dx$$

$$\leq \; \frac{\delta}{2}\,\mathcal{D}_{\mathcal{F}}(f,u) + \frac{1}{2\delta}\int_{\mathbb{R}^2} f\,|\nabla\log H|^2\,dx,$$

where

$$H(x) := \frac{1}{\pi}\,\frac{1}{\langle x\rangle^4} \quad\text{and then}\quad |\nabla\log H(x)| \leq 2,$$

which together with (1.4) imply that the modified free energy functional

$$\mathcal{F}_H = \mathcal{F}(f,u) - \int_{\mathbb{R}^2} f\,\log H$$

satisfies

(2.1) $$\frac{d}{dt}\mathcal{F}_H(t) + \frac{1}{2}\mathcal{D}_{\mathcal{F}}(t) \leq M.$$

On the one hand, introducing the Laplace kernel $\kappa_0(z) := -\frac{1}{2\pi}\log|z|$ and the Bessel kernel $\kappa_\alpha(z) := \frac{1}{4\pi}\int_0^\infty t^{-1}\exp(-|z|^2/(4t) - \alpha t)\,dt$ for $\alpha > 0$, so that $u = \bar{u}_\alpha := \kappa_\alpha * f$ is a solution to the Laplace type equation

$$-\Delta u = f - \alpha u \quad\text{in}\quad \mathbb{R}^2,$$

and introducing as well the *chemical energy* and the modified entropy

$$F_\alpha(f,u) := \frac{1}{2}\int |\nabla u|^2 + \frac{\alpha}{2}\int u^2 - \int f\,u, \quad \mathcal{H}_H(f) := \int f\log(f/H)$$

one can easilly show (see e.g. [7, Lemma 2.2])

(2.2) $$\mathcal{F}_H(f,u) = \mathcal{H}_H(f) + F_\alpha(f,\bar{u}_\alpha) + \frac{1}{2}\int|\nabla(u-\bar{u}_\alpha)|^2 + \frac{\alpha}{2}\int(u-\bar{u}_\alpha)^2$$

and

(2.3) $$F_\alpha(f,\bar{u}_\alpha) = -\frac{1}{2}\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)\,f(y)\,\kappa_\alpha(x-y)\,dxdy.$$

On the other hand, we know from the classical logarithmic Hardy-Littlewood Sobolev inequality (see e.g. [1, 10]) or its generalization for the Bessel kernel (see [7, Lemma 4.2])

(2.4) $$\forall\, f \geq 0, \quad \int_{\mathbb{R}^2} f(x)\log f(x)\,dx \;-\; \frac{4\pi}{M}\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)\,f(y)\,\kappa_\alpha(x-y)\,dxdy$$

$$-\; \int_{\mathbb{R}^2} f(x)\log H(x)\,dx \geq -C_1(M),$$

where here and below $C_i(M)$ denotes a positive constant which only depends on the mass M.

Then from (2.2), (2.3) and (2.4) together with the very classical functional inequality (see e.g. [7, Lemma 2.4])

(2.5) $$\mathcal{H}^+ := \mathcal{H}^+(f) = \int f(\log f)_+ \leq \mathcal{H}_H(f) - \frac{1}{4}\int f\log\langle x\rangle^2 + C_2(M),$$

one immediately obtains for $M < 8\pi$

$$\mathcal{F}_H(f,u) \;\geq\; (1 - \frac{M}{8\pi})\,\mathcal{H}_H(f) + \frac{M}{8\pi}\Big(\mathcal{H}_H(f) - \frac{4\pi}{M}\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)\,f(y)\,\kappa_\alpha(x-y)\,dxdy\Big)$$

$$\geq\; C_3(M)\,\mathcal{H}_+(f) + C_4(M)\int f\log\langle x\rangle^2 - C_5(M).$$

One concludes that under the assumption (1.7) on the initial datum, the identity (1.3) and the inequality(2.1) provide a convenient a priori estimates in order to define weak solutions, namely

$$(2.6) \qquad C_3(M)\,\mathcal{H}^+(f(t)) + C_4(M)\int f(t)\log\langle x\rangle^2 + \frac{1}{2}\int_0^t \mathcal{D}_\mathcal{F}(f(s),u(s))\,ds \leq$$

$$\leq \mathcal{F}_H(0) + C_5(M) + M\,t,$$

and we remark that the RHS term is finite under assumption (1.7) on (f_0, u_0), because

$$\mathcal{F}_H(0) = \mathcal{F}(f_0, u_0) - \int f_0 \log H$$

$$= \mathcal{F}(f_0, u_0) + M\log\pi + 2\int f_0 \log\langle x\rangle^2 < +\infty.$$

It is worth emphasazing that in order to get the bounds announced in Definition 1.1 in the case $\alpha > 0$ one may use the inequality

$$(2.7) \qquad \mathcal{F}_H \geq C_6(M)\int |\nabla u|^2 + C_7(M)\alpha \int u^2 + C_8(M)\int f\,u - C_9(M)(1 + 1/\alpha)$$

which is establihed in [7, (3.5)].

2.2. **Local in time a posteriori estimates.** We start by presenting some elementary functional inequalities which will be of main importance in the sequel. The two first estimates are picked up from [16, Lemma 3.2] but are probably classical and the third one is a variant of the Gagliardo-Niremberg-Sobolev inequality.

Lemma 2.1. *For any $0 \leq f \in L^1(\mathbb{R}^2)$ with finite mass M and finite Fisher information*

$$I = I(f) := \int_{\mathbb{R}^2} \frac{|\nabla f|^2}{f},$$

there holds

$$(2.8) \qquad \forall\,p \in [1, \infty), \quad \|f\|_{L^p(\mathbb{R}^2)} \leq C_p\,M^{1/p}\,I(f)^{1-1/p},$$

$$(2.9) \qquad \forall\,q \in [1, 2), \quad \|\nabla f\|_{L^q(\mathbb{R}^2)} \leq C_q\,M^{1/q-1/2}\,I(f)^{3/2-1/q}.$$

For any $0 \leq f \in L^1(\mathbb{R}^2)$ with finite mass M, there holds

$$(2.10) \qquad \forall\,p \in [2, \infty) \quad \|f\|_{L^{p+1}(\mathbb{R}^2)} \leq C_p\,M^{1/(p+1)}\,\|\nabla(f^{p/2})\|_{L^2}^{2/(p+1)}.$$

We refer to [16, Lemma 3.2] and [14, Lemma 2.1] for a proof.

The proof of (1.12) in Theorem 1.3 is split in several steps that we present as some intermediate autonomous a posteriori bounds.

Proposition 2.2. *For any weak solution, we have*

$$I(f(t)) \in L^1(0,T), \quad \forall\,T > 0.$$

Proof of Proposition 2.2. We write

$$(2.11) \qquad \mathcal{D}_\mathcal{F}(f) \geq I(f) + 2\int f\,\Delta u.$$

Then by Young's inequality it follows

$$\int f\,\Delta u \;=\; \int f\,(\varepsilon\partial_t u - f + \alpha u)$$

$$\geq\; -(1 + \alpha/2 + \varepsilon/2)\int f^2 - \varepsilon/2\int(\partial_t u)^2 - \alpha/2\int u^2.$$

The second and third terms belong to $L^1(0,T)$ from (1.9), so we only need to estimate the first one.

For any $A > 1$, using the Cauchy-Schwarz inequality and the inequality (2.8) for $p = 3$, we have

$$\int f^2 \, \mathbf{1}_{f \geq A} \leq \left(\int f \, \mathbf{1}_{f \geq A} \right)^{1/2} \left(\int f^3 \right)^{1/2}$$

$$\leq \left(\int f \, \frac{(\log f)_+}{\log A} \right)^{1/2} \left(C_3^3 \, M \, I(f)^2 \right)^{1/2},$$

from what we deduce for $A = A(M, \mathcal{H}^+)$ large enough, and more precisely taking A such that $\log A = 16 \, \mathcal{H}_+ \, C_3^3 \, M (1 + \alpha/2 + \varepsilon/2)^2$,

$$(2.12) \qquad \int f^2 \, \mathbf{1}_{f \geq A} \leq C_3^{3/2} \, M^{1/2} \, \frac{\mathcal{H}_+(f)^{1/2}}{(\log A)^{1/2}} \, I(f) \leq \frac{(1 + \alpha/2 + \varepsilon/2)^{-1}}{4} \, I(f).$$

Denoting $\Phi(u) = \varepsilon \int (\partial_t u)^2 + \alpha \int u^2 \in L^1(0, T)$ and putting together the last estimate with (2.11), it follows

$$\frac{1}{2} I(f) \leq \mathcal{D}_{\mathcal{F}} + C \int f^2 \, \mathbf{1}_{f \leq A} + \Phi(u)$$

$$\leq \mathcal{D}_{\mathcal{F}} + 2 \, M \, \exp(C \, \mathcal{H}_+ \, M) + \Phi(u),$$

and we conclude thanks to (1.3)–(2.6). $\qquad \square$

Remark 2.3. *The logarithmic Hardy-Littlewood Sobolev inequality (2.4) in the supercritical case $M \geq 8\pi$ does not lead to a global estimate as for the subcritical case $M \in (0, 8\pi)$. However, introducing the function $\mathcal{M} := MH$ of mass M and the modified free energy*

$$\tilde{\mathcal{F}}_M(f, u) := \int_{\mathbb{R}^2} (f \log(f/\mathcal{M}) - f + \mathcal{M}) \, dx - \int_{\mathbb{R}^2} fu \, dx + \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \frac{\alpha}{2} \int_{\mathbb{R}^2} u^2 dx$$

one shows that any solution (f, u) to the Keller-Segel equation (1.1) formally satisfies

$$\frac{d}{dt} \tilde{\mathcal{F}}_M(f, u) \leq -\frac{1}{2} \mathcal{D}_{\mathcal{F}}(f, u) + M$$

$$\leq -\frac{1}{2} I(f) - \int f \Delta u - \frac{\varepsilon}{2} \int (\partial_t u)^2 + M$$

$$\leq -\frac{1}{2} I(f) - \frac{\varepsilon}{4} \int (\partial_t u)^2 + (1 + \varepsilon) \int f^2 + \frac{\alpha}{2} \int u^2 + M,$$

where we have just used (2.1) and the estimate (2.11) and the ones which follow at the beginning of the proof of Proposition 2.2. We also observe that from (2.5) and (2.7), we may deduce

$$\mathcal{H}_+(f) + \int |\nabla u|^2 + \alpha \int u^2 \leq K_1 \tilde{\mathcal{F}}_M(f, u) + K_2$$

where K_i, $i = 1, 2$, are constants which may depend on $M > 0$ and $\alpha \geq 0$. Arguing then as in the proof of Proposition 2.2, we easily get

$$\frac{d}{dt} \tilde{\mathcal{F}}_M(f, u) \leq -\frac{1}{2} I(f) - \frac{\varepsilon}{4} \int (\partial_t u)^2 + (1 + \varepsilon) \left\{ MA + C_3^{3/2} \, M^{1/2} \, \frac{(K_1 \tilde{\mathcal{F}}_M(f, u))^{1/2}}{(\log A)^{1/2}} \, I(f) \right\}$$

$$+ \frac{K_1}{2} \tilde{\mathcal{F}}_M(f, u) + \frac{K_2}{2} + M \qquad (\forall \, A > 0)$$

$$\leq -\frac{1}{4} I(f) - \frac{\varepsilon}{4} \int (\partial_t u)^2 + K_3 \exp(K_4 \tilde{\mathcal{F}}_M(f, u)) + K_5$$

by making the appropriate choice $\log A = K' \tilde{\mathcal{F}}_M(f, u)$ for A. This differential inequality provides a local a priori estimate on the modified free energy which can be used in order to prove local existence result for supercritical mass. Because we will prove in Theorem 1.3 that the above resulting bound is suitable in order to get the uniqueness of the solution, we classically can obtain the existence and

uniqueness of maximal solutions (in the weak sense of definition 1.1) $(f, u) \in C([0, T^*); \mathcal{D}'(\mathbb{R}^2) \times \mathcal{D}'(\mathbb{R}^2))$ *such that*

$$\sup_{[0,T)} \tilde{\mathcal{F}}_M(f(t), u(t)) + \int_0^T \left\{ I(f(t)) + \mathcal{D}_\mathcal{F}(f(t), u(t)) \right\} dt < \infty \quad \forall T \in (0, T^*)$$

and the alternative

$$T_* = +\infty \quad or \quad (T_* < \infty, \ \tilde{\mathcal{F}}_M(f(t), u(t)) \to \infty \ as \ t \to T^*).$$

As an immediate consequence of Lemma 2.1 and Proposition 2.2, we have

Lemma 2.4. *For any $T > 0$, any weak solution f satisfies*

(2.13) $f \in L^{p/(p-1)}(0, T; L^p(\mathbb{R}^2)), \quad \forall \, p \in (1, \infty),$

(2.14) $\nabla f \in L^{2p/(3p-2)}(0, T; L^p(\mathbb{R}^2)), \quad \forall \, p \in [1, 2),$

(2.15) $\Delta u \in L^2(0, T; L^2(\mathbb{R}^2)).$

Proof of Lemma 2.4. The bound (2.13) is a direct consequence of (2.8) and Proposition 2.2. The bound (2.14) is consequence of (2.9) and Proposition 2.2. The bound (2.15) is an immediate consequence of the equation $\Delta u = \varepsilon \partial_t u + \alpha u - f$ and the fact that each term of the right-hand side lies in $L^2(0, T; L^2(\mathbb{R}^2))$ thanks to (2.13) and (1.9). □

Lemma 2.5. *Any weak solution (f, u) satisfies*

(2.16)
$$\int_{\mathbb{R}^2} \beta(f_{t_1}) \, dx + \int_{t_0}^{t_1} \int_{\mathbb{R}^2} \beta''(f_s) \, |\nabla f_s|^2 \, dx ds$$

$$\leq \int_{\mathbb{R}^2} \beta(f_{t_0}) \, dx + \int_{t_0}^{t_1} \int_{\mathbb{R}^2} \{\beta(f_s) - f_s \beta'(f_s)\} \Delta u_s \, dx ds,$$

for any times $0 \leq t_0 \leq t_1 < \infty$ and any renormalizing function $\beta : \mathbb{R} \to \mathbb{R}$ which is convex, piecewise of class C^1 and such that

$$|\beta(\xi)| \leq C \, (1 + \xi \, (\log \xi)_+), \quad (\beta(\xi) - \xi \beta'(\xi))_+ \leq C \, (1 + \xi) \quad \forall \xi \in \mathbb{R}.$$

Proof of Lemma 2.5. Consider a weak solution (f, u), we write

$$\partial_t f = \Delta f - \nabla u \cdot \nabla f - (\Delta u) \, f,$$

and we split the proof into three steps.

Step 1. Continuity. Consider a mollifier sequence (ρ_n) on $\mathbb{R}^2$, that is $\rho_n(x) := n^2 \rho(nx)$, $0 \leq \rho \in \mathcal{D}(\mathbb{R}^2)$, $\int \rho = 1$, and introduce the mollified function $f_t^n := f_t *_x \rho_n$. Clearly, $f^n \in C([0, T); L^1(\mathbb{R}^2))$. Using (2.13) and (1.9), a variant of the commutation Lemma [13, Lemma II.1 and Remark 4] tells us that

(2.17) $\partial_t f^n = \Delta f^n - \nabla u \cdot \nabla f^n - (\Delta u) \, f^n + r^n,$

with $r^n = r_1^n + r_2^n$ given by

$$r_1^n := \nabla u \cdot \nabla f^n - (\nabla u \cdot \nabla f) * \rho_n \to 0 \quad in \quad L^1(0, T; L^1_{loc}(\mathbb{R}^2)),$$

$$r_2^n := (\Delta u) f^n - [(\Delta u) f] * \rho_n \to 0 \quad in \quad L^1(0, T; L^1_{loc}(\mathbb{R}^2)).$$

The important point here is that $f \in L^2(0, T; L^2(\mathbb{R}^2))$ thanks to (2.13) and $\nabla u \in L^2(0, T; W^{1,2}(\mathbb{R}^2))$ thanks to (2.15), hence the commutation lemma holds true.

As a consequence, the chain rule applied to the smooth function f^n reads

(2.18) $\partial_t \beta(f^n) = \Delta \beta(f^n) - \beta''(f^n) \, |\nabla f^n|^2 - \nabla u \cdot \nabla \beta(f^n) - (\Delta u) f^n \beta'(f^n) + \beta'(f^n) \, r^n,$

for any $\beta \in C^1(\mathbb{R}) \cap W^{2,\infty}_{loc}(\mathbb{R})$ such that β'' is piecewise continuous and vanishes outside of a compact set. Because the equation (2.17) with u fixed is linear, the difference $f^{n,k} := f^n - f^k$ satisfies (2.17) with r^n replaced by $r^{n,k} := r^n - r^k \to 0$ in $L^1(0, T; L^1_{loc}(\mathbb{R}^2))$ and then also (2.18)

(with again f^n and r^n changed in $f^{n,k}$ and $r^{n,k}$). For any non-negative function $\chi \in C_c^2(\mathbb{R}^d)$ we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^2} \beta(f^{n,k}(t))\,\chi \;=\;& \int_{\mathbb{R}^2} \beta(f^{n,k}(0))\,\chi - \int_0^t\!\!\int_{\mathbb{R}^2} \beta''(f^{n,k}(s))|\nabla f^{n,k}(s)|^2\,\chi \\
&+ \int_0^t\!\!\int_{\mathbb{R}^2} \beta(f^{n,k}(s))\,\nabla u(s)\cdot\nabla\chi + \int_0^t\!\!\int_{\mathbb{R}^2}\{\beta(f^{n,k}(s)) - f^{n,k}(s)\beta'(f^{n,k}(s))\}\,\Delta u(s)\,\chi \\
&+ \int_0^t\!\!\int_{\mathbb{R}^2} \beta'(f^{n,k}(s))\,r^{n,k}(s)\,\chi.
\end{aligned}
$$

In that last equation, we choose $\beta(\xi) = \beta_1(\xi)$ where $\beta_A(\xi) = \xi^2/2$ for $|\xi| \le A$, $\beta_A(\xi) = A\,|\xi| - A^2/2$ for $|\xi| \ge A$, and using $|\beta_1'| \le 1$ and $\beta_1'' \ge 0$ it follows

$$
\begin{aligned}
\int_{\mathbb{R}^2} \beta_1(f^{n,k}(t))\,\chi \;\le\;& \int_{\mathbb{R}^2} \beta_1(f^{n,k}(0))\,\chi + \int_0^t\!\!\int_{\mathbb{R}^2} \beta_1(f^{n,k}(s))\,|\nabla u(s)||\nabla\chi| \\
&+ \int_0^t\!\!\int_{\mathbb{R}^2} |\beta_1(f^{n,k}(s)) - f^{n,k}(s)\beta_1'(f^{n,k}(s))|\,|\Delta u(s)|\,\chi + \int_0^t\!\!\int_{\mathbb{R}^2} |r^{n,k}(s)|\,\chi,
\end{aligned}
$$

Since $f_0 \in L^1$, we have $f^{n,k}(0) \to 0$ in $L^1(\mathbb{R}^2)$ and we deduce from the previous inequality and the following convergences: $r^{n,k} \to 0$ in $L^1(0,T; L^1_{loc}(\mathbb{R}^2))$; $\beta_1(f^{n,k})|\nabla u| \to 0$ in $L^1(0,T; L^1_{loc}(\mathbb{R}^2))$, because $\beta_1(\xi) \le |\xi|$, $f^{n,k} \to 0$ in $L^2(0,T,L^2(\mathbb{R}^2))$ by (2.13) with $p=2$ and $\nabla u \in L^2(0,T;L^2(\mathbb{R}^2))$ by definition 1.1; $\beta_1(f^{n,k})|\Delta u| \to 0$ in $L^1(0,T;L^1(\mathbb{R}^2))$, because $\beta_1(\xi) \le |\xi|$, $f^{n,k} \to 0$ in $L^2(0,T,L^2(\mathbb{R}^2))$ and $\Delta u \in L^2(0,T;L^2(\mathbb{R}^2))$ by (2.15); and $f^{n,k}\beta_1'(f^{n,k})|\Delta u| \to 0$ in $L^1(0,T;L^1(\mathbb{R}^2))$, because $|\beta_1'| \le 1$, $f^{n,k} \to 0$ in $L^2(0,T,L^2(\mathbb{R}^2))$ and $\Delta u \in L^2(0,T;L^2(\mathbb{R}^2))$; that

$$
\sup_{t\in[0,T]} \int_{\mathbb{R}^2} \beta_1(f^{n,k}(t,x))\,\chi(x)\,dx \xrightarrow[n,k\to\infty]{} 0.
$$

Since χ is arbitrary, we deduce that there exists $\bar{f} \in C([0,\infty); L^1_{loc}(\mathbb{R}^2))$ so that $f^n \to \bar{f}$ in $C([0,T]; L^1_{loc}(\mathbb{R}^2))$, $\forall T > 0$. Together with the convergence $f^n \to f$ in $C([0,\infty); \mathcal{D}'(\mathbb{R}^2))$ and the bound (1.9), we deduce that $f = \bar{f}$ and

$$
(2.19) \qquad\qquad f^n \to f \quad \text{in} \quad C([0,T]; L^1(\mathbb{R}^2)), \quad \forall T > 0.
$$

Step 2. Linear estimates. We come back to (2.18), which implies, for all $0 \le t_0 < t_1$, all $\chi \in C_c^2(\mathbb{R}^2)$,

$$
\begin{aligned}
(2.20) \quad \int_{\mathbb{R}^2} \beta(f^n_{t_1})\,\chi + \int_{t_0}^{t_1}\!\int_{\mathbb{R}^2} \beta''(f^n_s)\,|\nabla_x f^n_s|^2\,\chi =\;& \int_{\mathbb{R}^2} \beta(f^n_{t_0})\,\chi + \int_{t_0}^{t_1}\!\int_{\mathbb{R}^2} \beta(f^n_s)\,\nabla u\cdot\nabla\chi \\
&+ \int_{t_0}^{t_1}\!\int_{\mathbb{R}^2} \{\beta(f^n_s) - f^n_s\beta'(f^n_s)\}\,\Delta u_s\,\chi + \int_{t_0}^{t_1}\!\int_{\mathbb{R}^2} \beta'(f^n_s)\,r^n\,\chi.
\end{aligned}
$$

Choosing $0 \le \chi \in C_c^2(\mathbb{R}^2)$ and $\beta \in C^1(\mathbb{R}) \cap W^{2,\infty}_{loc}(\mathbb{R})$ such that β'' is non-negative and vanishes outside of a compact set, and passing to the limit as $n \to \infty$, we get

$$
\begin{aligned}
(2.21) \quad \int_{\mathbb{R}^2} \beta(f_{t_1})\,\chi + \int_{t_0}^{t_1}\!\int_{\mathbb{R}^2} \beta''(f_s)\,|\nabla_x f_s|^2\,\chi \le\;& \int_{\mathbb{R}^2} \beta(f_{t_0})\,\chi + \int_{t_0}^{t_1}\!\int_{\mathbb{R}^2} \{\beta(f_s) - f_s\beta'(f_s)\}\,\Delta u_s\,\chi \\
&+ \int_{t_0}^{t_1}\!\int_{\mathbb{R}^2} \beta(f_s)\,\nabla u\cdot\nabla\chi.
\end{aligned}
$$

By approximating $\chi \equiv 1$ by the sequence (χ_R) with $\chi_R(x) = \chi(x/R)$, $0 \le \chi \in \mathcal{D}(\mathbb{R}^2)$, we see that the last term in (2.21) vanishes and we get (2.16) in the limit $R \to \infty$ for any renormalizing function β with linear growth at infinity.

Step 3. Superlinear estimates. Finally, for any β satisfying the growth condition as in the statement of the Lemma, we just approximate β by an increasing sequence of smooth renormalizing functions β_R with linear growth at infinity, and pass to the limit in (2.16) in order to conclude. $\qquad\square$

Lemma 2.6. *For any weak solution f and any $p \geq 2$, there exists a constant $C := C(M, \mathcal{H}_0, M_2(0), T, p)$ such that*

$$(2.22) \qquad \|f(t_1)\|_{L^p}^p + \frac{1}{2}\int_{t_0}^{t_1}\|\nabla_x f^{p/2}\|_{L^2}^2\, dt \leq C\|f(t_0)\|_{L^p}^p.$$

Proof of Lemma 2.6. We define the renormalizing function $\beta_K : \mathbb{R}_+ \to \mathbb{R}_+$, $K \geq 2$, by

$$\beta_K(\xi) := \xi^p \text{ if } \xi \leq K, \quad \beta_K(\xi) := \frac{K^{p-1}}{\log K}\xi\log\xi \text{ if } \xi \geq K,$$

so that β_K is convex and piecewise of class C^1, and moreover there holds

$$\beta_K'(\xi)\xi - \beta_K(\xi) = (p-1)\,\xi^p\,\mathbf{1}_{\xi<K} + \frac{K^{p-1}}{\log K}\xi\,\mathbf{1}_{\xi>K},$$

and

$$\beta_K''(\xi) = p(p-1)\,\xi^{p-2}\,\mathbf{1}_{\xi<K} + \frac{K^{p-1}}{\log K}\frac{1}{\xi}\,\mathbf{1}_{\xi>K}.$$

Thanks to Lemma 2.5, we may write

$$\int_{\mathbb{R}^2}\beta_K(f_{t_1})\,dx + \frac{4}{p'}\int_{t_0}^{t_1}\int_{\mathbb{R}^2}|\nabla_x(f^{p/2})|^2\,\mathbf{1}_{f\leq K}\,dxds + \frac{K^{p-1}}{\log K}\int_{t_0}^{t_1}\int_{\mathbb{R}^2}\frac{|\nabla_x f|^2}{f}\,\mathbf{1}_{f\geq K}\,dxds$$

$$\leq \int_{\mathbb{R}^2}\beta_K(f_{t_0})\,dx - (p-1)\int_{t_0}^{t_1}\int_{\mathbb{R}^2}\Delta u\,f^p\,\mathbf{1}_{f\leq K}\,dxds - \frac{K^{p-1}}{\log K}\int_{t_0}^{t_1}\int_{\mathbb{R}^2}\Delta u\,f\,\mathbf{1}_{f\geq K}\,dxds.$$

On the one hand, using the Gagliardo-Niremberg-Sobolev inequality

$$\int_{\mathbb{R}^2}g^4\,dx \leq C\int_{\mathbb{R}^2}g^2\,dx\int_{\mathbb{R}^2}|\nabla g|^2\,dx,$$

we have

$$\begin{aligned}
\mathcal{T}_1 &:= (p-1)\int_{\mathbb{R}^2}|\Delta u|\,f^p\,\mathbf{1}_{f\leq K}\,dxds\\
&\leq (p-1)\|\Delta u\|_{L^2}\left(\int_{\mathbb{R}^2}(f\wedge K)^{2p}\,dx\right)^{1/2}\\
&\leq C\,\|\Delta u\|_{L^2}\left(\int_{\mathbb{R}^2}(f\wedge K)^p\,dx\int_{\mathbb{R}^2}|\nabla(f\wedge K)^{p/2}|^2\,dx\right)^{1/2}\\
&\leq C\,\|\Delta u\|_{L^2}^2\int_{\mathbb{R}^2}\beta_K(f)\,dx + \frac{1}{p'}\int_{\mathbb{R}^2}|\nabla_x(f^{p/2})|^2\,\mathbf{1}_{f\leq K}\,dx.
\end{aligned}$$

On the other hand, thanks to the Sobolev inequality (line 2) and the Cauchy-Schwarz inequality (line 3), we have for K large enough

$$\begin{aligned}
\mathcal{T}_2 &:= \frac{K^{p-1}}{\log K}\int_{\mathbb{R}^2}|\Delta u|\,f\,\mathbf{1}_{f\geq K}\,dx \leq 4\frac{K^{p-1}}{\log K}\|\Delta u\|_{L^2}\left(\int_{\mathbb{R}^2}(f-K/2)_+^2\,dx\right)^{1/2}\\
&\leq 4\frac{K^{p-1}}{\log K}\|\Delta u\|_{L^2}\int_{\mathbb{R}^2}|\nabla(f-K/2)_+|\,dx = 4\frac{K^{p-1}}{\log K}\|\Delta u\|_{L^2}\int_{\mathbb{R}^2}|\nabla f|\,\mathbf{1}_{f\geq K/2}\,dx\\
&\leq 4\frac{K^{p-1}}{\log K}\|\Delta u\|_{L^2}\,dx\left(\int f\,\mathbf{1}_{f\geq K/2}\,dx\right)^{1/2}\left(\int_{\mathbb{R}^2}\frac{|\nabla f|^2}{f}\,\mathbf{1}_{f\geq K/2}\right)^{1/2}\\
&\leq C\,\|\Delta u\|_{L^2}^2\int\frac{K^{p-1}}{\log K}f\,\mathbf{1}_{f\geq K/2}\,dx + \frac{1}{2^p}\frac{K^{p-1}}{\log K}\int_{\mathbb{R}^2}\frac{|\nabla f|^2}{f}\,\mathbf{1}_{f\geq K/2}\\
&\leq C\,\|\Delta u\|_{L^2}^2\int\beta_K(f)\,dx + \frac{1}{2^p}\frac{K^{p-1}}{\log K}\left\{\frac{4}{p^2}\int_{\mathbb{R}^2}|\nabla(f^{p/2})|^2\left(\frac{2}{K}\right)^{p-1}\mathbf{1}_{f\leq K} + \int_{\mathbb{R}^2}\frac{|\nabla f|^2}{f}\,\mathbf{1}_{f\geq K}\right\}\\
&\leq C\,\|\Delta u\|_{L^2}^2\int\beta_K(f)\,dx + \frac{1}{p'}\int_{\mathbb{R}^2}|\nabla(f^{p/2})|^2\,\mathbf{1}_{f\leq K}\,dx + \frac{1}{2}\frac{K^{p-1}}{\log K}\int_{\mathbb{R}^2}\frac{|\nabla f|^2}{f}\,\mathbf{1}_{f\geq K}\,dx.
\end{aligned}$$

All together, we have proved that there holds for some numerical constant C

$$\int_{\mathbb{R}^2} \beta_K(f_{t_1}) \, dx + \frac{2}{p'} \int_{t_0}^{t_1} \int_{\mathbb{R}^2} |\nabla_x(f^{p/2})|^2 \, \mathbf{1}_{f \le K} \, dx ds$$

$$\le \int_{\mathbb{R}^2} \beta_K(f_{t_0}) \, dx + C \int_{t_0}^{t_1} \|\Delta u\|_{L^2}^2 \int_{\mathbb{R}^2} \beta_K(f) \, dx \, ds$$

We conclude to (2.22) by applying first the Gronwall lemma, using that $\|\Delta u\|_{L^2}^2 \in L^1(0,T)$, and passing next to the limit $K \to \infty$. $\qquad \square$

Lemma 2.7. *Any weak solution (f, u) satisfies*

$$\partial_t f, \partial_x f, \partial_{x_i x_j}^2 f, \partial_t u, \partial_x u, \partial_{x_i x_j}^2 u \in C_b((0,T] \times \mathbb{R}^2), \ \forall T > 0,$$

so that it is a "classical solution" for positive time.

Proof of Lemma 2.7. For any time $t_0 \in (0,T)$ and any exponent $p \in (1,\infty)$, there exists $t_0' \in (0, t_0)$ such that $f(t_0') \in L^p(\mathbb{R}^2)$ thanks to (2.13), from what we deduce using (2.22) on time interval (t_0', T) that

(2.23) $$f \in L^\infty(t_0, T; L^p(\mathbb{R}^2)) \quad \text{and} \quad \nabla_x f \in L^2((t_0, T) \times \mathbb{R}^2).$$

Since u satisfies the parabolic equation

$$\varepsilon \partial_t u - \Delta u + \alpha u = f,$$

the maximal regularity of the heat equation in L^p-spaces (see Theorem X.12 stated in [5] and the quoted references) and the fact that

$$u(t) = \gamma_{t/\varepsilon}^\alpha *_{t,x} f + \gamma_{t/\varepsilon}^\alpha *_x u_0 \quad \text{and} \quad \nabla u = \Gamma_{t/\varepsilon}^\alpha *_{t,x} f + \gamma_{t,\varepsilon}^\alpha *_x \nabla u_0$$

where we denote $\gamma_s^\alpha = e^{-\alpha s} \gamma_s$ and similarly for Γ, γ_t is the heat kernel given by

$$\gamma_t(x) := \frac{1}{4\pi t} \exp\left(-\frac{|x|^2}{4t}\right) \in L^{z_1}(0, T; L^{z_2}(\mathbb{R}^2)), \quad \forall z_1, z_2 \ge 1, \ 1/z_1 + 1/z_2 > 1,$$

and

$$\Gamma_t(x) := \nabla_x \gamma_t(x) \in L^{s_1}(0, T; L^{s_2}(\mathbb{R}^2)), \quad \forall s_1, s_2 \ge 1, \ 1/s_1 + 1/s_2 > 3/2,$$

provide the bound

(2.24) $$u \in L^\infty(t_0, T; L^p(\mathbb{R}^2)), \quad \nabla u \in L^\infty(t_0, T; L^p(\mathbb{R}^2)), \quad \partial_t u, D^2 u \in L^p((t_0, T) \times \mathbb{R}^2),$$

for all $t \in (0,T)$ and $p \in (1,\infty)$. Since now f satisfies the parabolic equation

$$\partial_t f - \Delta f = -\nabla u \cdot \nabla f - (\Delta u) f =: Z$$

with $Z \in L^2(t_0, T; L^q(\mathbb{R}^2))$ for all $t_0 \in (0,T)$ and all $q \in [1,2)$ from (2.23) and (2.24), the same maximal regularity of the heat equation in L^q-spaces (with the choice $s_1 = s_2 = (4/3)^-$) implies

$$\nabla f \in L^p(t_0, T; L^p(\mathbb{R}^2)), \forall p \in [2,4),$$

and then $Z \in L^p(t_0, T; L^p(\mathbb{R}^2)), \ \forall p \in [2,4)$. By a bootstrap argument of the regularity property of the heat equation we easily get

(2.25) $$f \in L^\infty(t_0, T; L^p(\mathbb{R}^2)), \quad \nabla f \in L^\infty(t_0, T; L^p(\mathbb{R}^2)), \quad \partial_t f, D^2 f \in L^p((t_0, T) \times \mathbb{R}^2),$$

for all $t \in (0,T)$ and $p \in (1,\infty)$. The Morrey inequality implies then $f, \nabla f, u, \nabla u \in C^{0,\alpha}((t_0, T) \times \mathbb{R}^2)$ for any $0 < \alpha < 1$, and any $t_0 > 0$. Finally the classical Holderian regularity result for the heat equation (see Theorem X.13 stated in [5] and the quoted references) implies first $u \in C^{2,\alpha}((t_0, T) \times \mathbb{R}^2)$ and next $f \in C^{2,\alpha}((t_0, T) \times \mathbb{R}^2)$, which concludes the proof. $\qquad \square$

We prove now the free energy-dissipation of the free energy identity (1.4) in Theorem 1.3.

Proof of the free energy identity in Theorem 1.3. We split the proof into two steps.

Step 1. We claim that the free energy functional $\mathcal{F}$ is lsc in the sense that for any sequence (f_n, u_n) of nonnegative functions such that (f_n) is bounded in $L^1 \cap L^1(\log\langle x\rangle^2)$ with same mass $M < 8\pi$, (u_n) is bounded in H^1 if $\alpha > 0$ or in $L^1 \cap \dot{H}^1$ if $\alpha = 0$, $(f_n u_n)$ is bounded in L^1 and such that $\mathcal{F}(f_n, u_n) \leq A$ and $(f_n, u_n) \rightharpoonup (f, u)$ in $\mathcal{D}'(\mathbb{R}^2) \times \mathcal{D}'(\mathbb{R}^2)$, there holds

$$(2.26) \qquad 0 \leq f \in L^1 \cap L^1(\log\langle x\rangle^2) \quad \text{and} \quad \mathcal{F}(f, u) \leq \liminf_{n\to\infty} \mathcal{F}(f_n, u_n).$$

Because of (??) and (2.5), we have $\mathcal{H}^+(f_n) \leq C$ and we may apply the Dunford-Pettis lemma which implies that $f_n \rightharpoonup f$ weakly in $L^1(\mathbb{R}^2)$.

We rewrite the free energy functional as

$$\mathcal{F}(f_n, u_n) = \mathcal{H}(f_n) + F_\alpha(f_n, u_n)$$

with

$$\mathcal{H}(f_n) := \int f_n \log f_n \quad \text{and} \quad F_\alpha(f_n, u_n) := \frac{1}{2}\int |\nabla u_n|^2 + \frac{\alpha}{2}\int u_n^2 - \int f_n\, u_n.$$

Case $\alpha > 0$. We denote $\bar{u}_n = \kappa_\alpha * f_n$ where $\kappa_\alpha(z) = \frac{1}{4\pi}\int_0^\infty \frac{1}{t} e^{-\frac{|z|^2}{4t} - \alpha t}\, dt$ is the Bessel kernel. Since $f_n \geq 0$, $f_n \in L^1 \cap L\log L$ and $u_n \in H^1$ [7, Lemma 2.2] implies that $\bar{u}_n \in H^1$ and also that the functional $F_\alpha(f_n, u_n)$ is finite and satisfies

$$F_\alpha(f_n, u_n) - F_\alpha(f_n, \bar{u}_n) = \frac{1}{2}\|\nabla(u_n - \bar{u}_n)\|_{L^2}^2 + \frac{\alpha}{2}\|u_n - \bar{u}_n\|_{L^2}^2.$$

Hence we can write

$$\mathcal{F}(f_n, u_n) = \mathcal{H}(f_n) - \frac{1}{2}\iint f_n(x)f_n(y)\kappa_\alpha(x-y) + \frac{1}{2}\|\nabla(u_n - \bar{u}_n)\|_{L^2}^2 + \frac{\alpha}{2}\|u_n - \bar{u}_n\|_{L^2}^2$$
$$=: \mathcal{H}(f_n) + \mathcal{V}(f_n) + \mathcal{U}_1(u_n - \bar{u}_n) + \mathcal{U}_2(u_n - \bar{u}_n),$$

where the functionals $\mathcal{U}_1$ and $\mathcal{U}_2$ are defined through the third and fourth term respectively. We clearly have that $\mathcal{U}_1 + \mathcal{U}_2$ is lsc for the weak H^1 convergence and $\mathcal{H}$ is lsc for the weak L^1 convergence, so we investigate the functional $\mathcal{V}$. For any $\epsilon \in (0, 1)$ we split $\mathcal{V} = \mathcal{V}_\epsilon + \mathcal{R}_\epsilon$ as

$$\mathcal{V}_\epsilon(g) := -\frac{1}{2}\iint f_n(x)f_n(y)\kappa_\alpha(x-y)\mathbf{1}_{|x-y|>\epsilon}$$

$$\mathcal{R}_\epsilon(g) := -\frac{1}{2}\iint f_n(x)f_n(y)\kappa_\alpha(x-y)\mathbf{1}_{|x-y|\leq\epsilon}.$$

The Bessel kernel κ_α is a positive radial decreasing function with a singularity at the origin: $\kappa_\alpha(z) = -\frac{1}{2\pi}\log|z| + O(1)$ when $|z| \to 0$. Hence $\mathcal{V}_\epsilon$ is continuous for the weak L^1 convergence and for the rest term we obtain, for any $\epsilon \in (0, 1)$ and $\lambda > 1$,

$$|\mathcal{R}_\varepsilon(g)| \leq C\iint g(x)g(y)\mathbf{1}_{|x-y|\leq\epsilon} + C\iint g(x)g(y)(\log|x-y|)_-\mathbf{1}_{|x-y|\leq\epsilon}$$

$$\leq C\iint g(x)\mathbf{1}_{g(x)\leq\lambda}g(y)\mathbf{1}_{|x-y|\leq\epsilon} + C\iint g(x)\mathbf{1}_{g(x)>\lambda}g(y)\mathbf{1}_{|x-y|\leq\epsilon}$$

$$+ C\iint g(x)\mathbf{1}_{g(x)\geq\lambda}g(y)(\log|x-y|)_-\mathbf{1}_{|x-y|\leq\epsilon} + C\iint g(x)\mathbf{1}_{g(x)>\lambda}g(y)\log(|x-y|^{-1})\mathbf{1}_{|x-y|\leq\epsilon}$$

$$\leq C\lambda\int_y g(y)\left\{\int_{|z|\leq\epsilon} dz\right\} + C\int_y g(y)\left\{\frac{1}{\log\lambda}\int_x g(x)\log g(x)\right\}$$

$$+ C\lambda\int_y g(y)\left\{\int_{|z|\leq\epsilon}(\log|z|)_-\, dz\right\} + C\int_x g(x)\mathbf{1}_{g(x)>\lambda}\int_y\{g(y)\log g(y) + |x-y|^{-1}\}\mathbf{1}_{|x-y|\leq\epsilon}$$

$$\lesssim M\lambda\epsilon^2 + M\frac{\mathcal{H}(g)}{\log\lambda} + M\lambda\epsilon^{3/2} + \frac{\mathcal{H}(g)}{\log\lambda}\{\mathcal{H}(g) + \epsilon\},$$

where we have used the convexity inequality $uv \le u \log u + e^v$ for all $u > 0$, $v \in \mathbb{R}$ and the elementary inequality $u \log u \ge -u^{1/2}$ for all $u \in (0,1)$. Hence $\sup_n |\mathcal{R}_\varepsilon(f_n)| \to 0$ as $\epsilon \to 0$ and we deduce that $\mathcal{F}$ is lsc.

Case $\alpha = 0$. We define $\bar{u}_n = \kappa_0(f_n - MH)$ where $H(x) = \langle x \rangle^{-4}/\pi$ and $\kappa_0(z) = -\frac{1}{2\pi} \log |z|$ is the Poisson kernel. Since $0 \le f_n \in L^1 \cap L^1(\log\langle x \rangle^2)$, $\mathcal{H}(f_n)$ is finite, $\int (f - MH) = 0$ and $u_n \in \dot{H}^1$, [7, Lemma 2.2] implies that $\bar{u}_n \in \dot{H}^1$ and also that the functional $F_0(f_n - MH, u_n)$ is finite and verifies

$$F_0(f_n - MH, u_n) - F_0(f_n - MH, \bar{u}_n) = \frac{1}{2}\|\nabla(u_n - \bar{u}_n)\|_{L^2}^2.$$

Now we argue as in the case $\alpha > 0$. First we write

$$\mathcal{F}(f_n, u_n) = \mathcal{H}(f_n) - \frac{1}{2}\iint f_n(x)f_n(y)\kappa_0(x-y) + M\iint f_n(x)H(y)\kappa_0(x-y)$$

$$+ \frac{1}{2}\|\nabla(u_n - \bar{u}_n)\|_{L^2}^2 - M\int H u_n - \frac{M^2}{2}\iint H(x)H(y)\kappa_0(x-y)$$

$$=: \mathcal{H}(f_n) + \mathcal{V}(f_n) + \mathcal{W}(f_n) + \mathcal{U}_1(u_n - \bar{u}_n) + \mathcal{U}_0(u_n) + \mathcal{Z}(H).$$

The functional $\mathcal{U}_1$ is lsc for the weak $\dot{H}^1$ convergence and $\mathcal{H}$ is lsc for the weak L^1 convergence. For $\mathcal{V}$ we just argue as in the preceding case $\alpha > 0$. In the same way (even simpler) we conclude that $\mathcal{W}$ is lsc for the weak L^1 convergence. Finally we conclude that $\mathcal{F}$ is lsc.

Step 2. Now, we easily deduce that the free energy identity (1.4) holds. Indeed, since f is smooth for positive time, for any fixed $t > 0$ and any given sequence (t_n) of positive real numbers which decreases to 0, we clearly have

$$\mathcal{F}(f(t_n)) = \mathcal{F}(t) + \int_{t_n}^t \mathcal{D}_\mathcal{F}(f(s))\,ds.$$

Then, thanks to the Lebesgue convergence theorem, the lsc property of $\mathcal{F}$ and the fact that $f(t_n) \rightharpoonup f_0$ weakly in $\mathcal{D}'(\mathbb{R}^2)$, we deduce from the above free energy identity for positive time that

$$\mathcal{F}(f_0) \le \liminf_{n\to\infty} \mathcal{F}(f(t_n)) \le \lim_{n\to\infty} \left\{ \mathcal{F}(t) + \int_{t_n}^t \mathcal{D}_\mathcal{F}(f(s))\,ds \right\} = \mathcal{F}(t) + \int_0^t \mathcal{D}_\mathcal{F}(f(s))\,ds.$$

Together with the reverse inequality (1.9) we conclude to (1.4). $\qquad\square$

3. Uniqueness - Proof of Theorem 1.3

In this section we prove the uniqueness part of Theorem 1.3. In order to do so we first prove some estimates in Lemmas 3.1 and 3.2.

Lemma 3.1. *Any weak solution f to the Keller-Segel equation satisfies that for any $p \in (1, \infty)$, $T \in (0, \infty)$ there exists a constant $K = K(f_0, p, T)$ such that*

$$(3.1) \qquad\qquad t^{p-1}\|f(t)\|_{L^p}^p \le K \qquad \forall\, t \in (0, T).$$

Proof of Lemma 3.1. Recall that we already know that $\|f\|_{L^p} \in C^1(0, T)$ for any $p > 1$ and $\|f\|_{L^p} \in L^\infty(t_0, T)$ for any $0 < t_0 < T$ and any $p \in [1, \infty]$. We have then for $p > 1$

$$\frac{d}{dt}\int f^p = -4(1 - 1/p)\int |\nabla(f^{p/2})|^2 + (p-1)\int f^{p+1} - (p-1)\int (\partial_t u + \alpha u)f^p$$

$$=: T_1 + T_2 + T_3.$$

Using the splitting $f = \min(f, A) + (f - A)_+$, for some $A > 0$, and denoting $h(u) := \partial_t u + \alpha u \in L^2(0, T; L^2(\mathbb{R}^2))$, we have

$$|T_3| \le C\int |h(u)| \min(f, A)^p + C\int |h(u)|(f - A)_+^p =: T_{31} + T_{32}.$$

For the term T_{31} we have

$$|T_{31}| \leq CA^{p-1/2} \int |h(u)| f \leq CA^{p-1/2} (\|h(u)\|_{L^2}^2 + M).$$

For T_{32}, using Gagliardo-Niremberg-Sobolev inequality $\|g\|_{L^4}^4 \leq C\|g\|_{L^2}^2 \|\nabla g\|_{L^2}^2$ with $g = (f - A)_+^{p/2}$ it follows

$$|T_{32}| \leq C \int |h(u)|(f - A)_+^p$$

$$\leq C\|h(c)\|_{L^2} \left(\int (f - A)_+^{2p} \right)^{1/2}$$

$$\leq C\|h(c)\|_{L^2} \left(\int (f - A)_+^p \right)^{1/2} \left(\int |\nabla (f - A)_+^{p/2}|^2 \right)^{1/2}$$

$$\leq C_\delta \|h(u)\|_{L^2}^2 \int (f - A)_+^p + \delta \int |\nabla (f^{p/2})|^2 \mathbf{1}_{f \geq A},$$

for any $\delta > 0$.

For the second term we have

$$|T_2| \leq C \int \min(f, A)^{p+1} + C \int (f - A)_+^{p+1},$$

the first part is easily bounded by $CA^p M$ and for the second one, using (2.10) we obtain

$$\int (f - A)_+^{p+1} \leq C \left(\int (f - A)_+ \right) \left(\int |\nabla (f - A)_+^{p/2}|^2 \right) \leq C \frac{\mathcal{H}^+(f)}{\log A} \left(\int |\nabla (f^{p/2})|^2 \mathbf{1}_{f \geq A} \right).$$

Gathering all the previous estimates and choosing $\delta > 0$ small enough and A big enough, it follows

(3.2) $$\frac{d}{dt} \int f^p \leq -C_0 \int |\nabla (f^{p/2})|^2 + C_1 \|h(u)\|_{L^2}^2 \int f^p + C_1 (M + \|h(u)\|_{L^2}^2).$$

Thanks to the following inequalities

$$(\|f\|_{L^p}^p)^{p/(p-1)} \leq C\|f\|_{L^1}^{1/(p-1)} \|f\|_{L^{p+1}}^{p+1} \quad \text{and} \quad \|f\|_{L^{p+1}}^{p+1} \leq C\|f\|_{L^1} \|\nabla (f^{p/2})\|_{L^2}^2$$

we obtain from (3.2)

$$\frac{d}{dt} X(t) \leq -C_0 X(t)^{\frac{p}{p-1}} + C_1 H(t) X(t) + C_2 (1 + H(t)), \qquad t \in (0, T),$$

where we denote $X(t) := \|f(t)\|_{L^p}^p$ and $H(t) := \|h(u)\|_{L^2}^2(t) \in L^1(0, T)$. By standard arguments (see e.g. [7]) we conclude to (3.1). $\qquad \square$

We (crucially) improve the preceding estimate by showing

Lemma 3.2. *Any weak solution f to the Keller-Segel equation satisfies that for any $p \in [2, \infty)$*

(3.3) $$t^{\frac{p-1}{2p}} \|f(t, .)\|_{L^{2p/(p+1)}} \to 0 \quad \text{as } t \to 0.$$

Proof of Lemma 3.2. We now prove (3.3) from (3.1) and an interpolation argument. On the one hand, we use the Holder inequality in order to get

$$\int f^{2p/(p+1)} = \int f^{p/(p+1)} \langle \log f \rangle^{p/(p+1)} f^{p/(p+1)} \langle \log f \rangle^{p/(p+1)}$$

$$\leq \left(\int f \langle \log f \rangle \right)^{p/(p+1)} \left(\int f^p \langle \log f \rangle^{-p} \right)^{1/(p+1)},$$

or in other words

(3.4) $$\|f\|_{L^{2p/(p+1)}} \leq C \left(\int f^p \langle \log f \rangle^{-p} \right)^{1/(2p)}.$$

On the other hand, we observe that

$$
\begin{aligned}
t^{p-1} \int f^p \left\langle \log f \right\rangle^{-p} &\leq t^{p-1} \int_{f \leq R} f^p \left\langle \log f \right\rangle^{-p} + t^{p-1} \int_{f \geq R} f^p \left\langle \log f \right\rangle^{-p} \\
&\leq t^{p-1} \frac{R^{p-1}}{\langle \log R \rangle^p} \int_{f \leq R} f + \frac{t^{p-1}}{\langle \log R \rangle^p} \int_{f \geq R} f^p \\
&\leq t^{p-1} \frac{M R^{p-1}}{\langle \log R \rangle^p} + \frac{K}{\langle \log R \rangle^p} \\
&\leq \frac{M + K}{\langle \log t^{-1} \rangle^2} \to 0,
\end{aligned}
$$

(3.5)

where we have used the mass conservation and the estimate (3.1) in the third line and we have chosen $R := t^{-1}$ in the last line. We conclude to (3.3) by gathering (3.4) and (3.5). □

We are now able to prove the uniqueness of solutions.

Proof of the uniqueness part in Theorem 1.3. We consider two weak solutions (f_1, u_1) and (f_2, u_2) to the Keller-Segel equation (1.1) that we write in the mild form

$$
\begin{aligned}
f_i(t) &= e^{t\Delta} f_i(0) - \int_0^t e^{(t-s)\Delta} \nabla \cdot (f_i(s) \nabla u_i(s)) \, ds \\
&= e^{t\Delta} f_i(0) - \int_0^t \nabla e^{(t-s)\Delta} (f_i(s) \nabla u_i(s)) \, ds
\end{aligned}
$$

and

$$
u_i(s) = e^{-\frac{\alpha}{\varepsilon} s} e^{\frac{s}{\varepsilon}\Delta} u_i(0) + \frac{1}{\varepsilon} \int_0^s e^{-\frac{\alpha}{\varepsilon}(s-\sigma)} e^{\frac{(s-\sigma)}{\varepsilon}\Delta} f_i(\sigma) \, d\sigma,
$$

from which we also obtain

$$
\nabla u_i(s) = e^{-\frac{\alpha}{\varepsilon} s} e^{\frac{s}{\varepsilon}\Delta} (\nabla u_i(0)) + \frac{1}{\varepsilon} \int_0^s e^{-\frac{\alpha}{\varepsilon}(s-\sigma)} (\nabla e^{\frac{(s-\sigma)}{\varepsilon}\Delta}) f_i(\sigma) \, d\sigma.
$$

When we assume $f_1(0) = f_2(0)$ and $u_1(0) = u_2(0) = u_0$, the difference $F := f_2 - f_1$ satisfies

$$
\begin{aligned}
F(t) = &- \int_0^t \nabla e^{(t-s)\Delta} \left\{ F(s) \left[e^{-\frac{\alpha}{\varepsilon} s} e^{\frac{s}{\varepsilon}\Delta} (\nabla u_0) \right] \right\} ds \\
&- \int_0^t \nabla e^{(t-s)\Delta} \left\{ F(s) \left[\frac{1}{\varepsilon} \int_0^s e^{-\frac{\alpha}{\varepsilon}(s-\sigma)} \nabla e^{\frac{(s-\sigma)}{\varepsilon}\Delta} f_2(\sigma) \, d\sigma \right] \right\} ds \\
&- \int_0^t \nabla e^{(t-s)\Delta} \left\{ f_1(s) \left[\frac{1}{\varepsilon} \int_0^s e^{-\frac{\alpha}{\varepsilon}(s-\sigma)} \nabla e^{\frac{(s-\sigma)}{\varepsilon}\Delta} F(\sigma) \, d\sigma \right] \right\} ds \\
=: &\, I_1(t) + I_2(t) + I_3(t).
\end{aligned}
$$

(3.6)

For any $t > 0$, we define

$$
Z_p^i(t) := \sup_{0 < s \leq t} s^{\frac{1}{2} - \frac{1}{2p}} \left\| f_i(s) \right\|_{L^{\frac{2p}{p+1}}}, \quad \Delta(t) := \sup_{0 < s \leq t} s^{\frac{1}{4}} \left\| F(s) \right\|_{L^{4/3}}.
$$

We recall the explicit formula for the heat semigroup

$$
e^{t\Delta} g = \gamma(t, \cdot) *_x g, \qquad \gamma(t, x) := \frac{1}{4\pi t} \exp\left(-\frac{|x|^2}{4t} \right),
$$

and the following inequalities that will be useful in the sequel

$$
\left\| K * g \right\|_{L^r} \leq \left\| K \right\|_{L^q} \left\| g \right\|_{L^p}, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1, \quad 1 \leq p, q, r \leq \infty
$$

and

$$
\left\| \gamma(t, \cdot) \right\|_{L^q(\mathbb{R}^2)} \leq C_q \, t^{\frac{1}{q} - 1}, \quad \left\| \nabla \gamma(t, \cdot) \right\|_{L^q(\mathbb{R}^2)} \leq C_q \, t^{\frac{1}{q} - \frac{3}{2}}.
$$

We fix $p > 2$ and we shall compute the quantity $t^{\frac{1}{4}} \| \cdot \|_{L^{4/3}}$ for each term of (3.6).

For the second term we compute

$$
\begin{aligned}
t^{\frac{1}{4}}\|I_2(t)\|_{L^{4/3}} &\leq C(\alpha,\varepsilon)\, t^{\frac{1}{4}} \int_0^t \left\| \nabla e^{(t-s)\Delta} \left\{ F(s) \int_0^s \nabla e^{\frac{(s-\sigma)}{\varepsilon}\Delta} f_2(\sigma)\, d\sigma \right\} \right\|_{L^{4/3}} ds \\
&\leq C\, t^{\frac{1}{4}} \int_0^t \|\nabla e^{(t-s)\Delta}\|_{L^{4/3}} \left\| F(s) \int_0^s \nabla e^{\frac{(s-\sigma)}{\varepsilon}\Delta} f_2(\sigma)\, d\sigma \right\|_{L^1} ds \\
&\leq C\, t^{\frac{1}{4}} \int_0^t (t-s)^{-\frac{3}{4}} \|F(s)\|_{L^{4/3}} \int_0^s \|\nabla e^{\frac{(s-\sigma)}{\varepsilon}\Delta} f_2(\sigma)\|_{L^4}\, d\sigma\, ds \\
&\leq C\, t^{\frac{1}{4}} \int_0^t (t-s)^{-\frac{3}{4}} \|F(s)\|_{L^{4/3}} \int_0^s \|\nabla e^{\frac{(s-\sigma)}{\varepsilon}\Delta} f_2(\sigma)\|_{L^4}\, d\sigma\, ds,
\end{aligned}
\tag{3.7}
$$

where we have used Young's inequality for convolution in the second line and Holder's inequality in the third line. Now we can estimate the integral over $d\sigma$ using Young's inequality with $1/4+1 = 1/a + (p+1)/(2p)$, i.e. $1/a = 3/4 - 1/(2p)$, by

$$
\begin{aligned}
\int_0^s \left\| \nabla e^{\frac{(s-\sigma)}{\varepsilon}\Delta} f_2(\sigma) \right\|_{L^4} d\sigma &\leq \int_0^s \|\nabla e^{\frac{(s-\sigma)}{\varepsilon}\Delta}\|_{L^a} \|f_2(\sigma)\|_{L^{\frac{2p}{p+1}}}\, d\sigma \\
&\leq C \int_0^s (s-\sigma)^{\frac{3}{4}-\frac{1}{2p}-\frac{3}{2}} \|f_2(\sigma)\|_{L^{\frac{2p}{p+1}}}\, d\sigma \\
&\leq C\, Z_p^2(s) \int_0^s (s-\sigma)^{-\frac{3}{4}-\frac{1}{2p}} \sigma^{-\frac{1}{2}+\frac{1}{2p}}\, d\sigma \\
&\leq C\, Z_p^2(s)\, s^{-\frac{1}{4}} \int_0^1 (1-y)^{-\frac{3}{4}-\frac{1}{2p}} y^{-\frac{1}{2}+\frac{1}{2p}}\, dy \\
&\leq C\, Z_p^2(s)\, s^{-\frac{1}{4}}
\end{aligned}
$$

since the last integral is bounded thanks to $-\frac{3}{4} - \frac{1}{2p} > -1$ from $p > 2$.

Gathering last estimate with (3.7) it follows

$$
\begin{aligned}
t^{\frac{1}{4}}\|I_2(t)\|_{L^{4/3}} &\leq C\, t^{\frac{1}{4}} \int_0^t (t-s)^{-\frac{3}{4}} s^{-\frac{1}{4}} \|F(s)\|_{L^{4/3}}\, Z_p^2(s)\, ds \\
&\leq C\, Z_p^2(t)\, \Delta(t) \int_0^t (t-s)^{-\frac{3}{4}} t^{\frac{1}{4}} s^{-\frac{1}{2}}\, ds \\
&\leq C\, Z_p^2(t)\, \Delta(t) \int_0^1 (1-y)^{-\frac{3}{4}} y^{-\frac{1}{2}}\, dy \\
&\leq C\, Z_p^2(t)\, \Delta(t).
\end{aligned}
\tag{3.8}
$$

For the term I_3 we compute

$$
\begin{aligned}
t^{\frac{1}{4}}\|I_3(t)\|_{L^{4/3}} &\leq C\, t^{\frac{1}{4}} \int_0^t \left\| \nabla e^{(t-s)\Delta} \left\{ f_1(s) \int_0^s \nabla e^{\frac{(s-\sigma)}{\varepsilon}\Delta} F(\sigma)\, d\sigma \right\} \right\|_{L^{4/3}} ds \\
&\leq C\, t^{\frac{1}{4}} \int_0^t \|\nabla e^{(t-s)\Delta}\|_{L^{4/3}} \left\| f_1(s) \int_0^s \nabla e^{\frac{(s-\sigma)}{\varepsilon}\Delta} F(\sigma)\, d\sigma \right\|_{L^1} ds \\
&\leq C\, t^{\frac{1}{4}} \int_0^t (t-s)^{-\frac{3}{4}} \|f_1(s)\|_{L^{\frac{2p}{p+1}}} \int_0^s \|\nabla e^{\frac{(s-\sigma)}{\varepsilon}\Delta} F(\sigma)\|_{L^{\frac{2p}{p-1}}}\, d\sigma\, ds.
\end{aligned}
\tag{3.9}
$$

We compute the integral over $d\sigma$, we have

$$\int_0^s \|\nabla e^{\frac{(s-\sigma)}{\varepsilon}\Delta} F(\sigma)\|_{L^{\frac{2p}{p-1}}} d\sigma \leq \int_0^s \|\nabla e^{\frac{(s-\sigma)}{\varepsilon}\Delta}\|_{L^{\frac{4p}{3p-2}}} \|F(\sigma)\|_{L^{4/3}} d\sigma$$

$$\leq C \int_0^s (s-\sigma)^{-\frac{3}{4}-\frac{1}{2p}} \|F(\sigma)\|_{L^{4/3}} d\sigma$$

$$\leq C \Delta(s) \int_0^s (s-\sigma)^{-\frac{3}{4}-\frac{1}{2p}} \sigma^{-\frac{1}{4}} d\sigma$$

$$\leq C \Delta(s) s^{-\frac{1}{2p}},$$

since the last integral is bounded because $p > 2$. Putting together this estimate with (3.9) we obtain

(3.10)
$$t^{\frac{1}{4}} \|I_3(t)\|_{L^{4/3}} \leq C t^{\frac{1}{4}} \int_0^t (t-s)^{-\frac{3}{4}} \|f_1(s)\|_{L^{\frac{2p}{p+1}}} \Delta(s) s^{-\frac{1}{2p}} ds$$

$$\leq C Z_p^1(t) \Delta(t) \int_0^t (t-s)^{-\frac{3}{4}} t^{\frac{1}{4}} s^{-\frac{1}{2}}$$

$$\leq C Z_p^1(t) \Delta(t).$$

For the term I_1 we compute

(3.11)
$$t^{\frac{1}{4}} \|I_1(t)\|_{L^{4/3}} \leq C t^{\frac{1}{4}} \int_0^t \left\|\nabla e^{(t-s)\Delta} \left\{ F(s) e^{\frac{s}{\varepsilon}\Delta} \nabla u_0 \right\}\right\|_{L^{4/3}} ds$$

$$\leq C t^{\frac{1}{4}} \int_0^t \|\nabla e^{(t-s)\Delta}\|_{L^{4/3}} \left\| F(s) e^{\frac{s}{\varepsilon}\Delta} \nabla u_0 \right\|_{L^1} ds$$

$$\leq C t^{\frac{1}{4}} \int_0^t (t-s)^{-\frac{3}{4}} \|F(s)\|_{L^{4/3}} \left\| e^{\frac{s}{\varepsilon}\Delta} \nabla u_0 \right\|_{L^4} ds,$$

where we have used Young's and Holder's inequality, respectively. Let $K > 0$ to be chosen later, we estimate

(3.12)
$$\|e^{\frac{s}{\varepsilon}\Delta} \nabla u_0\|_{L^4} \leq \|e^{\frac{s}{\varepsilon}\Delta} \nabla u_0 \, \mathbf{1}_{\{|\nabla u_0| \leq K\}}\|_{L^4} + \|e^{\frac{s}{\varepsilon}\Delta} \nabla u_0 \, \mathbf{1}_{\{|\nabla u_0| \geq K\}}\|_{L^4}.$$

Using Young's inequality we can write

$$\|e^{\frac{s}{\varepsilon}\Delta} \nabla u_0 \, \mathbf{1}_{\{|\nabla u_0| \leq K\}}\|_{L^4} \leq \|e^{\frac{s}{\varepsilon}\Delta}\|_{L^1} \|\nabla u_0 \, \mathbf{1}_{\{|\nabla u_0| \leq K\}}\|_{L^4} \leq C K^{\frac{1}{2}} \|\nabla u_0\|_{L^2}^{1/2}.$$

For the second term in (3.12) we have (again by Young's inequality)

$$\|e^{\frac{s}{\varepsilon}\Delta} \nabla u_0 \, \mathbf{1}_{\{|\nabla u_0| \geq K\}}\|_{L^4} \leq \|e^{\frac{s}{\varepsilon}\Delta}\|_{L^{4/3}} \|\nabla u_0 \, \mathbf{1}_{\{|\nabla u_0| \geq K\}}\|_{L^2} \leq C s^{-\frac{1}{4}} \varphi(K)$$

where

$$\varphi(K) := \|\nabla u_0 \, \mathbf{1}_{\{|\nabla u_0| \geq K\}}\|_{L^2} \to 0 \quad \text{as} \quad K \to +\infty,$$

by the dominated convergence theorem. Putting together that last estimates in (3.12) and choosing $K = s^{-\frac{1}{4}}$ it follows

$$\|e^{\frac{s}{\varepsilon}\Delta} \nabla u_0\|_{L^4} \leq C s^{-\frac{1}{4}} \epsilon(s) \quad \text{with} \quad \epsilon(s) := s^{\frac{1}{8}} + \varphi(s^{-\frac{1}{4}}) \xrightarrow[s\to 0]{} 0,$$

from which, coming back to (3.11), we obtain

(3.13)
$$t^{\frac{1}{4}} \|I_1(t)\|_{L^{4/3}} \leq C t^{\frac{1}{4}} \int_0^t (t-s)^{-\frac{3}{4}} \|F(s)\|_{L^{4/3}} s^{-\frac{1}{4}} \epsilon(s) ds$$

$$\leq C \left(\sup_{0 < s \leq t} \epsilon(s) \right) \Delta(t) \int_0^t t^{\frac{1}{4}} (t-s)^{-\frac{3}{4}} s^{-\frac{1}{2}} ds$$

$$\leq C \left(\sup_{0 < s \leq t} \epsilon(s) \right) \Delta(t).$$

Gathering (3.8), (3.10) and (3.13) we conclude

$$\Delta(t) \le C \left[\sup_{0 < s \le t} \epsilon(s) + Z_p^1(t) + Z_p^2(t) \right] \Delta(t) \le \frac{1}{2} \Delta(t)$$

for $t \in (0, T)$, $T > 0$ small enough, which in turn implies $\Delta(t) \equiv 0$ on $[0, T)$. We may then repeat the argument for later times and conclude to the uniqueness of the solution.   □

4. SELF-SIMILAR SOLUTIONS AND LINEAR STABILITY

4.1. **Convergence of the stationary solutions.** First for a given mass $M \in (0, 8\pi)$ and a given parameter $\varepsilon > 0$, we consider the associated self-similar profile solution $(G_\varepsilon, V_\varepsilon)$ which is the unique solution of the system of elliptic equations (1.17)-(1.18), as well as the unique positive solutions (G, V) to the limit case $\varepsilon = 0$ system of equations

$$(4.1) \qquad \Delta G - \nabla(G \nabla V - \frac{1}{2} x G) = 0 \quad \text{in} \quad \mathbb{R}^2, \quad \int_{\mathbb{R}^2} G \, dx = M,$$

$$\Delta V + G = 0 \quad \text{in} \quad \mathbb{R}^2.$$

It is worth emphasizing that (G, V) is the unique self-similar profile associated to the parabolic-elliptic Keller-Segel equation, see [9, 14].

Lemma 4.1. *There exists a constant C such that for any $\varepsilon \in (0, 1/4]$*

$$(4.2) \qquad 0 \le G_\varepsilon(x) \le C \, e^{-|x|^2/4},$$

$$(4.3) \qquad \sup_{x \in \mathbb{R}^2} \left(\frac{1}{|x|} + \langle x \rangle \right) |\nabla V_\varepsilon(x)| \le C,$$

and

$$(4.4) \qquad \sup_{x \in \mathbb{R}^2} |\Delta V_\varepsilon(x)| \le C.$$

Proof of Lemma 4.1.

Step 1. The estimate (4.2) has been proved in [3]. More precisely it is a consequence of equations (26) and (49) in [3], and

$$G(0) = b, \quad 0 \le M(\varepsilon, b) \le 4\pi \min(2, b).$$

Here the parametrization of G is made in function of ε and $b = G(0)$ instead of ε and M because this dependence is more tractable. Observe that the estimate above guarantees that the mass is subcritical, i.e. $M(\varepsilon, b) \le 8\pi$.

Step 2. Since V_ε and G_ε are radially symmetric functions the equation on V_ε writes

$$(4.5) \qquad V_\varepsilon'' + \left(\frac{1}{r} + \frac{1}{2} \varepsilon r \right) V_\varepsilon' + G_\varepsilon = 0 \quad \forall r > 0,$$

where we abuse notations in writing $V_\varepsilon(r) = V_\varepsilon(x)$, $G_\varepsilon(r) = G_\varepsilon(|x|)$, $r = |x|$. The function V_ε is smooth and the equation is complemented with the boundary conditions $V_\varepsilon'(0) = V_\varepsilon'(\infty) = 0$. Defining $w := (r V_\varepsilon')^2$, we find

$$\frac{w'}{2} = -\frac{1}{2} \varepsilon r w - G_\varepsilon V_\varepsilon' r^2 \le C \sqrt{w} \frac{r}{\langle r \rangle^3}, \quad C := \sup_{r > 0} G_\varepsilon \langle r \rangle^3.$$

As a consequence

$$\frac{d}{dr} \sqrt{w} \le C \frac{r}{\langle r \rangle^3},$$

and then

$$\sqrt{w} \le C (1 \wedge r)^2,$$

from which the inequality $\sup_x [\langle x \rangle |\nabla V_\varepsilon(x)|] \le C$ of (4.3) follows.   □

Step 3. We rewrite (4.5) as

$$\frac{1}{r}(V_\varepsilon' r)' = w := -G_\varepsilon - \frac{1}{2}\varepsilon r V_\varepsilon' \in L^\infty,$$

which implies

$$|V_\varepsilon'(r)r| = \left|\int_0^r s w(s)\,ds\right| \le C\,r^2.$$

This completes the estimate (4.3) and coming back to (4.5) we also obtain

$$|V_\varepsilon''(r)| \le C,$$

which gives (4.4) and completes the proof.

Corollary 4.2. *There hold*

$$G_\varepsilon \to G \quad in \ \ W^{2,p} \quad \forall\, p \in (1,\infty),$$

and

$$\nabla V_\varepsilon \to \nabla V \quad in \ \ L_1^\infty, \quad \Delta V_\varepsilon \to \Delta V \quad in \ \ L^\infty$$

Proof of Corollary 4.2. Coming back to (1.18) and using Lemma 4.1 it follows, for any $p \in (1,\infty)$, that

$$LG_\varepsilon = \nabla G_\varepsilon \cdot \nabla V_\varepsilon + G_\varepsilon \Delta V_\varepsilon \in L^p,$$

where L denotes the operator $LG_\varepsilon := \Delta G_\varepsilon - \nabla \cdot (\frac{1}{2}x G_\varepsilon)$. By elliptic regularity we obtain that G_ε is unifomly bounded (with respect to $\varepsilon \in (0,1/2)$) in $W^{2,p}$.

Thanks to previous estimates and Lemma 4.1 there exists $(\bar{G},\bar{V})$ and a subsequence (still denoted as $(G_\varepsilon, V_\varepsilon)$) such that $G_\varepsilon \to \bar{G}$, $V_\varepsilon \to \bar{V}$. We may pass to the limit (in the weak sense) in the system of equations, and we find

$$\bar{V}'' + \frac{1}{r}\bar{V}' + \bar{G} = 0, \quad \bar{V}'(0) = \bar{V}'(\infty) = 0.$$

We conclude that $(\bar{G},\bar{V})$ is a solution to the stationary equation (4.1) and complete the proof. $\quad\square$

4.2. **Splitting structure for the linearized operator.** The evolution equation in self-similar variables writes (see (1.15) and (1.16))

(4.6)
$$\begin{cases} \partial_t g = \Delta g + \nabla(\frac{1}{2}\,x\,g - g\,\nabla v), \\ \partial_t v = \frac{1}{\varepsilon}(\Delta v + g) + \frac{1}{2}\,x \cdot \nabla v, \end{cases}$$

and the associated linearized equation is given by

(4.7)
$$\begin{cases} \partial_t f = \Lambda_{1,\varepsilon}(f,u) = \Delta f + \nabla(\frac{1}{2}\,x\,f - f\,\nabla V_\varepsilon - G_\varepsilon\,\nabla u), \\ \partial_t u = \Lambda_{2,\varepsilon}(f,u) = \frac{1}{\varepsilon}(\Delta u + f) + \frac{1}{2}\,x \cdot \nabla u, \end{cases}$$

which we also denote $\partial_t(f,u) = \Lambda_\varepsilon(f,u) = \big(\Lambda_{1,\varepsilon}(f,u), \Lambda_{2,\varepsilon}(f,u)\big)$. We restrict ourself to a radially symmetric setting.

We introduce the Hilbert space

(4.8) $$X := X_1 \times X_2, \quad X_1 := L_{rad}^2 \cap L_{k,0}^2 \subset L_{k,1}^2, \quad k > 7, \quad X_2 = L_{rad}^2,$$

associated to the norm

(4.9) $$\|(f,u)\|_X^2 := \|f\|_{L_k^2}^2 + \|u\|_{L^2}^2.$$

We now state a property of the spectrum of Λ_ε in X that is the main result of this subsection.

Proposition 4.3. *There exist $\varepsilon^*, r^* > 0$ such that in X*

$$\forall\,\varepsilon \in (0,\varepsilon^*) \qquad \Sigma(\Lambda_\varepsilon) \cap \Delta_{-1/3} \subset \Sigma_d(\Lambda_\varepsilon) \cap B(0,r^*).$$

We define the bounded operator $\mathcal{A}$ by

$$(4.10) \qquad \mathcal{A}_1(f, u) := N\chi_R[f] := N(\chi_R f - \chi_1 \langle \chi_R f \rangle), \quad \mathcal{A}_2(f, u) := 0,$$

for some constants $N, R > 0$ to be chosen later and a smooth non-negative and radially symmetric cut-off function $\chi_R(x) := \chi(x/R)$ with $\chi \equiv 1$ on $B_{1/2}$, $\operatorname{Supp} \chi \subset B_2$ and $\langle \chi_1 \rangle = 1$. We can split the operator $\Lambda_\varepsilon = \mathcal{A} + \mathcal{B}_\varepsilon$ and we shall investigate some properties of $\mathcal{A}$ and $\mathcal{B}_\varepsilon$ in the next lemmas before proving Proposition 4.3.

Lemma 4.4. *In the above splitting, we may choose N_* and R_* large enough in such a way that for any $N \geq N_*$, $R \geq R_*$, the operator $\mathcal{B}_\varepsilon$ is a-hypo-dissipative in X for any $a \in (-1/2, 0)$ in the sense that*

$$\|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathscr{B}(X)} \leq C_a \, e^{at} \quad \forall t \geq 0,$$

for some constant $C_a > 0$.

Proof of Lemma 4.4. First of all, thanks to (A.1) we see that in X the norm of $L_k^2 \times L^2$ is equivalent to the norm defined by

$$(4.11) \qquad \|(f, u)\|_{X_*}^2 := \|f\|_{L_k^2}^2 + \eta \, \|u - \kappa_f\|_{L^2}^2,$$

for any fixed $\eta > 0$. We now consider the equation $\partial_t(f, u) = \mathcal{B}_\varepsilon(f, u) = \Lambda_\varepsilon(f, u) - \mathcal{A}(f, u)$ and split the proof in several steps.

Step 1. First of all, we write the equation satisfied by f as

$$\partial_t f = \Delta f + \nabla \cdot (\frac{1}{2} x \, f - f \, \nabla V_\varepsilon - G_\varepsilon \, \nabla u) - N\chi_R[f].$$

Then we compute, using that $\langle f \rangle = 0$ and the notation $\chi_R^c = 1 - \chi_R$,

$$
\begin{aligned}
(4.12) \quad \frac{1}{2} \frac{d}{dt} \int f^2 \langle x \rangle^{2k} &= \int \Delta f \, f \langle x \rangle^{2k} + \frac{1}{2} \int \nabla \cdot (x f) \, f \langle x \rangle^{2k} - \int \nabla \cdot (f \nabla V_\varepsilon) \, f \langle x \rangle^{2k} \\
&\quad - \int \nabla \cdot (G_\varepsilon \nabla u) \, f \langle x \rangle^{2k} - \int N\chi_R[f] f \langle x \rangle^{2k} \\
&= - \int |\nabla f|^2 \langle x \rangle^{2k} + \int \{ \varphi(x) - N\chi_R(x) \} \, f^2 \langle x \rangle^{2k} \\
&\quad - \int \nabla \cdot (G_\varepsilon \nabla u) \, f \langle x \rangle^{2k} - N \int f \, \chi_1 \, \langle x \rangle^{2k} dx \, \langle \chi_R^c f \rangle
\end{aligned}
$$

where

$$
\begin{aligned}
\varphi(x) &= \left(\frac{1}{2} \Delta \langle x \rangle^{2k} - \frac{1}{2} x \cdot \nabla \langle x \rangle^{2k} + \frac{1}{4} \nabla \cdot (x \langle x \rangle^{2k}) - \frac{1}{2} \nabla \cdot (\nabla V_\varepsilon \langle x \rangle^{2k}) + \nabla V_\varepsilon \cdot \nabla \langle x \rangle^{2k} \right) \langle x \rangle^{-2k} \\
&= -\frac{1}{2}(k-1) + k(2k+1/2)\langle x \rangle^{-2} - k(2k-2)\langle x \rangle^{-4} - \frac{1}{2} \Delta V_\varepsilon + k(\nabla V_\varepsilon \cdot x)\langle x \rangle^{-2}.
\end{aligned}
$$

Thanks to Lemma 4.1 we have $(\nabla V_\varepsilon \cdot x)\langle x \rangle^{-2} \to 0$ as $|x| \to \infty$ and from (1.18) we have that

$$-\frac{1}{2} \Delta V_\varepsilon = \frac{1}{2} G_\varepsilon + \frac{\varepsilon}{4} x \cdot \nabla V_\varepsilon$$

with $G_\varepsilon \to 0$ as $|x| \to \infty$ from (4.2) and $|x \cdot \nabla V_\varepsilon| \leq C_{V_\varepsilon}$ from (4.3). It follows then

$$(4.13) \qquad \varphi(x) \underset{|x| \to \infty}{\approx} -\frac{1}{2}(k - 1 - \varepsilon \, C_{V_\varepsilon}/2).$$

For the third term in (4.12), for any $\delta > 0$, thanks to Hölder's inequality and using that $G_\varepsilon(x) \leq Ce^{-|x|^2/4} \leq C\langle x \rangle^{-\alpha}$ (see (4.2)), we get

$$
\begin{aligned}
-\int \nabla \cdot (G_\varepsilon \nabla u) f \, \langle x \rangle^{2k} &= \int \nabla f \cdot \nabla u \, G_\varepsilon \, \langle x \rangle^{2k} + \int G_\varepsilon \nabla u \cdot \nabla(\langle x \rangle^{2k}) \, f \\
&\leq \delta \|\nabla f\|_{L^2}^2 + C(\delta)\|\nabla u\|_{L^2}^2 + C\|f\|_{L^2}^2 \\
&\leq \delta \|\nabla f\|_{L^2}^2 + C(\delta)\|\nabla(u - \kappa_f)\|_{L^2}^2 + C(\delta)\|f\|_{L_\ell^2}^2 + C\|f\|_{L^2}^2,
\end{aligned}
$$

for some $\ell \in (3, k)$. For the fourth terme, we have

$$-N \int f \chi_1 \langle x \rangle^{2k} dx \, \langle \chi_R^c f \rangle \leq N \, \|\chi_1\|_{L^2_{2k-\ell}} \, \|\chi_R^c \langle x \rangle^{-\ell}\|_{L^2} \, \|f\|_{L^2_\ell}^2$$

$$\leq N \, R^{1-\ell} \, C_\ell \, \|f\|_{L^2_\ell}^2.$$

We conclude this step by gathering the previous estimates to obtain

(4.14)
$$\frac{1}{2}\frac{d}{dt}\|f\|_{L^2_k}^2 \leq -(1-\delta)\|\nabla f\|_{L^2_k}^2 + C(\delta)\|\nabla(u-\kappa_f)\|_{L^2}^2$$
$$+ \int \left\{ \varphi(x) + (C(\delta) + C\,N\,R^{1-\ell})\langle x \rangle^{2(\ell-k)} - N\chi_R(x) \right\} f^2 \langle x \rangle^{2k}.$$

Step 2. On the other hand, from the second equation in (4.7), we get
(4.15)
$$\frac{1}{2}\frac{d}{dt}\int (u-\kappa_f)^2 = \int (u-\kappa_f)\left\{ \frac{1}{\varepsilon}(\Delta u + f) + \frac{1}{2}x\cdot\nabla(u-\kappa_f) + \frac{1}{2}x\cdot\nabla\kappa_f - \partial_t\kappa_f \right\}$$
$$= -\frac{1}{\varepsilon}\int |\nabla(u-\kappa_f)|^2 - \frac{1}{2}\int (u-\kappa_f)^2 + \int (u-\kappa_f)\left\{ \frac{1}{2}x\cdot\nabla\kappa_f - \partial_t\kappa_f \right\},$$

and we shall estimate the last integral. Since $\partial_t\kappa_f = \kappa * \partial_t f$, we write

$$\int (u-\kappa_f)\left\{ \frac{1}{2}x\cdot\nabla\kappa_f - \partial_t\kappa_f \right\} = \frac{1}{2}\int (u-\kappa_f)\{x\cdot\nabla\kappa_f\}$$
$$- \int (u-\kappa_f)\kappa * \left\{ \Delta f + \frac{1}{2}\nabla(xf) - \nabla(f\nabla V_\varepsilon) - \nabla(G_\varepsilon\nabla u) - N\chi_R[f] \right\},$$

and we estimate each of these terms. It follows that

$$I_1 := \frac{1}{2}\int (u-\kappa_f)\{x\cdot\nabla\kappa_f\} = \frac{1}{2}\int (u-\kappa_f)\{x\cdot(\mathcal{K}*f)\}$$
$$\leq \delta\|(u-\kappa_f)\|_{L^2}^2 + C(\delta)\|\mathcal{K}*f\|_{L^2_1}^2$$
$$\leq \delta\|(u-\kappa_f)\|_{L^2}^2 + C(\delta)\|f\|_{L^2_\ell}^2,$$

where we have used Lemma A.2 in the last line with $3 < \ell < k$ since $f \in X_1 \subset L^2_{k,1}$. Moreover we get

$$I_2 := -\int (u-\kappa_f)\kappa * \{\Delta f\} = \int (u-\kappa_f)f$$
$$\leq \delta\,\|(u-\kappa_f)\|_{L^2}^2 + C(\delta)\,\|f\|_{L^2}^2.$$

Arguing as for the term I_1 and using Lemma A.2, we also obtain (we denote here $\mathcal{K} = (\mathcal{K}_i)_{i=1,2}$ and use the convention of summation of repeated indices)

$$I_3 := -\frac{1}{2}\int (u-\kappa_f)\kappa * \nabla(xf) = -\frac{1}{2}\int (u-\kappa_f)\,\mathcal{K}_i * (x_i f)$$
$$\leq \delta\|(u-\kappa_f)\|_{L^2}^2 + C(\delta)\|f\|_{L^2_\ell}^2,$$

with $3 < \ell < k$. Furthermore, since $f\nabla V_\varepsilon \in L^2_{k,0}$, we can apply Lemma A.2 with the same $\ell \in (2,k)$ and use that $\nabla V_\varepsilon \in L^\infty$ to obtain

$$I_4 := \int (u-\kappa_f)\kappa * \nabla(f\nabla V_\varepsilon) = \int (u-\kappa_f)\,\mathcal{K}_i * (f\partial_i V_\varepsilon)$$
$$\leq \delta\|(u-\kappa_f)\|_{L^2}^2 + C(\delta)\|f\nabla V_\varepsilon\|_{L^2_\ell}^2$$
$$\leq \delta\|(u-\kappa_f)\|_{L^2}^2 + C(\delta)\|f\|_{L^2_\ell}^2.$$

For the next term we get, using (A.2) since $G_\varepsilon \nabla u \in L^2_{k,0}$ and the bound $G_\varepsilon \in L^\infty_k$,

$$I_5 := \int (u - \kappa_f)\kappa * \nabla(G_\varepsilon \nabla u) = \int (u - \kappa_f)\,\mathcal{K}_i * (G_\varepsilon\, \partial_i u)$$
$$\leq \delta\|(u - \kappa_f)\|^2_{L^2} + C(\delta)\|\mathcal{K} * (G_\varepsilon \nabla u)\|^2_{L^2}$$
$$\leq \delta\|(u - \kappa_f)\|^2_{L^2} + C(\delta)\|G_\varepsilon \nabla u\|^2_{L^2_k}$$
$$\leq \delta\|(u - \kappa_f)\|^2_{L^2} + C(\delta)\|\nabla u\|^2_{L^2}$$
$$\leq \delta\|(u - \kappa_f)\|^2_{L^2} + C(\delta)\|\nabla(u - \kappa_f)\|^2_{L^2} + C(\delta)\|f\|^2_{L^2_\ell}.$$

For the last term and thanks to Lemma A.1

$$I_6 := N \int (u - \kappa_f)\kappa * \{\chi_R[f]\} \leq C\,\|u - \kappa_f\|_{L^2}\, N\,\|\chi_R[f]\|_{L^2_{\ell,1}}$$
$$\leq \delta\|u - \kappa_f\|^2_{L^2} + C(\delta)N^2\|f\|^2_{L^2_\ell}.$$

Putting together all the estimates of this step it follows

(4.16) $\dfrac{1}{2}\dfrac{d}{dt}\|u - \kappa_f\|^2_{L^2} \leq -\Big(\dfrac{1}{\varepsilon} - C(\delta)\Big)\|\nabla(u - \kappa_f)\|^2_{L^2} - \Big(\dfrac{1}{2} - \delta\Big)\|u - \kappa_f\|^2_{L^2} + C(\delta)N^2\|f\|^2_{L^2_\ell}.$

Step 3. Conclusion. Gathering (4.14)-(4.16), we obtain

(4.17)
$$\frac{1}{2}\frac{d}{dt}\|(f,u)\|^2_{X_*} \leq \int \Big\{\varphi(x) + \Big[(1 + \eta N^2)C(\delta) + \frac{CN}{R^{\ell-1}}\Big]\langle x\rangle^{2(\ell-k)} - N\chi_R(x)\Big\}\,|f|^2\langle x\rangle^{2k}$$
$$- \eta\Big(\frac{1}{2} - \delta\Big)\|u - \kappa_f\|^2_{L^2} - (1 - \delta)\|\nabla f\|^2_{L^2_k} - \eta\Big(\frac{1}{\varepsilon} - C(\delta)\Big)\|\nabla(u - \kappa_f)\|^2_{L^2}$$

Taking then first $\delta \in (0,1)$ small enough and next $\varepsilon \in (0,1)$ small enough, it follows that for $\eta = N^{-3}$ and $R = N$ we have

$$\frac{1}{2}\frac{d}{dt}\|(f,u)\|^2_{X_*} \leq \int \{\bar\varphi_N(x) - N\chi_R(x)\}\,|f|^2\langle x\rangle^{2k} + a\|\nabla f\|^2_{L^2}$$
$$+ a\,\eta\,\|u - \kappa_f\|^2_{L^2} + a\,\eta\,\|\nabla(u - \kappa_f)\|^2_{L^2}$$

for any $a > -1/2$ and where $\bar\varphi_N(x) = \varphi(x) + C(1 + N^{-1} + N^{2-\ell})\langle x\rangle^{2(\ell-k)}$ has the same asymptotic behavior as $\varphi(x)$ when $|x| \to \infty$ and $\bar\varphi_N$ decreases as N increases. We can choose N large enough such that

$$\bar\varphi_N(x) - N\chi_R(x) \leq a, \quad \forall\, x \in \mathbb{R}^2,$$

which yields that $\mathcal{B}_\varepsilon$ is a-hypo-dissipative for any $a > -1/2$.   □

We introduce the space

(4.18) $Y := Y_1 \times Y_2, \quad Y_1 := H^1_k \cap L^2_{k,0} \cap L^2_{rad}, \quad Y_2 := H^1 \cap L^2_{rad},$

endowed with the norm

(4.19) $\|(f,u)\|^2_Y := \|(f,u)\|^2_X + \|\nabla f\|^2_{L^2_k} + \|\nabla u\|^2_{L^2}.$

A consequence of the definition of the operator $\mathcal{A}$ in (4.10) is the following result.

Lemma 4.5. *There hold $\mathcal{A} \in \mathscr{B}(X)$ and $\mathcal{A} \in \mathscr{B}(Y)$.*

Proof. The proof is straightforward so we omit it.   □

Lemma 4.6. *We can choose N and R large enough such that $\mathcal{B}_\varepsilon$ is a-hypo-dissipative in Y for any $a \in (-1/2, 0)$, i.e.*

$$\|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathscr{B}(Y)} \leq Ce^{at}.$$

Moreover we also have

$$\|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathscr{B}(X,Y)} \leq C\,t^{-1/2}\,e^{at}.$$

Proof. We introduce the following norm,

(4.20) $$\|(f,u)\|_{Y_*}^2 := \|(f,u)\|_{X_*}^2 + \eta_1\|\nabla f\|_{L_k^2}^2 + \eta_1\|\nabla(u-\kappa_f)\|_{L^2}^2,$$

which is equivalent to (4.19) thanks to (A.2), for any $\eta_1 > 0$. Observe that

$$\|\nabla u\|_{L^2} \le \|\nabla(u-\kappa_f)\|_{L^2} + C\|f\|_{L_\ell^2},$$

with the same $\ell \in (2,k)$ of Lemma 4.4, and also that

$$\|\nabla^2 u\|_{L^2} \le \|\nabla^2(u-\kappa_f)\|_{L^2} + \|\nabla^2\kappa_f\|_{L^2} \le \|\nabla^2(u-\kappa_f)\|_{L^2} + \|f\|_{L^2}.$$

Consider now the equation $\partial_t(f,u) = \mathcal{B}_\varepsilon(f,u) = \Lambda_\varepsilon(f,u) - \mathcal{A}(f,u)$.

Step 1. For $i = 1,2$, $\partial_i u$ verifies

$$\partial_t(\partial_i u) = \frac{1}{\varepsilon}\Delta(\partial_i u) + \frac{1}{\varepsilon}\partial_i f + \frac{1}{2}x\cdot\nabla(\partial_i u) + \frac{1}{2}\partial_i u.$$

We have then

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\partial_i(u-\kappa_f)\|_{L^2}^2 &= \int \partial_i(u-\kappa_f)\partial_t\{\partial_i u - \partial_i\kappa_f\} \\
&= \frac{1}{\varepsilon}\int \partial_i(u-\kappa_f)\Delta\{\partial_i(u-\kappa_f)\} + \frac{1}{2}\int \partial_i(u-\kappa_f)\partial_i u \\
&\quad + \frac{1}{2}\int \partial_i(u-\kappa_f)x\cdot\nabla\{\partial_i(u-\kappa_f)\} + \frac{1}{2}\int \partial_i(u-\kappa_f)x\cdot\nabla\{\partial_i\kappa_f\} \\
&\quad - \int \partial_i(u-\kappa_f)\mathcal{K}_i * (\partial_t f) \\
&=: T_1 + T_2 + T_3 + T_4 + T_5.
\end{aligned}$$

For the first term we easily get

$$T_1 = -\frac{1}{\varepsilon}\|\nabla\{\partial_i(u-\kappa_f)\}\|_{L^2}^2,$$

moreover for the second one we obtain

$$T_2 \le C\|\nabla(u-\kappa_f)\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 \le C\|\nabla(u-\kappa_f)\|_{L^2}^2 + C\|f\|_{L_\ell^2}^2.$$

We easily see that

$$T_3 = -\frac{1}{2}\|\partial_i(u-\kappa_f)\|_{L^2}^2 \le 0,$$

and also that, for the fourth term,

$$T_4 \le C\|\nabla(u-\kappa_f)\|_{L^2}^2 + C\|D^2\kappa_f\|_{L_1^2}^2 \le C\|\nabla(u-\kappa_f)\|_{L^2}^2 + C\|f\|_{L_1^2}^2.$$

For the last term T_5 we use the equation satisfied by f to write

$$\begin{aligned}
T_5 &= -\int \partial_i(u-\kappa_f)\mathcal{K}_i * \left\{\Delta f + \frac{1}{2}\nabla(xf) - \nabla(f\nabla V_\varepsilon) - \nabla(G_\varepsilon\nabla u) - N\chi_R[f]\right\} \\
&=: T_{51} + T_{52} + T_{53} + T_{54} + T_{55},
\end{aligned}$$

and we estimate each term. We have

$$T_{51} \le C\|\nabla(u-\kappa_f)\|_{L^2}^2 + C\|\nabla f\|_{L^2}^2$$

and

$$T_{52} \le C\|\nabla(u-\kappa_f)\|_{L^2}^2 + C\|\nabla^2\kappa * (xf)\|_{L^2}^2 \le C\|\nabla(u-\kappa_f)\|_{L^2}^2 + C\|f\|_{L_1^2}^2.$$

Moreover, using that $\nabla V_\varepsilon \in L^\infty$ we get

$$\begin{aligned}
T_{53} &\le C\|\nabla(u-\kappa_f)\|_{L^2}^2 + C\|\nabla^2\kappa * (f\nabla V_\varepsilon)\|_{L^2}^2 \\
&\le C\|\nabla(u-\kappa_f)\|_{L^2}^2 + C\|f\nabla V_\varepsilon\|_{L^2}^2 \\
&\le C\|\nabla(u-\kappa_f)\|_{L^2}^2 + C\|f\|_{L^2}^2,
\end{aligned}$$

and arguing as above with $G_\varepsilon \in L^\infty$ we also obtain

$$\begin{aligned}
T_{54} &\le C\|\nabla(u - \kappa_f)\|_{L^2}^2 + C\|\nabla^2\kappa * (G_\varepsilon \nabla u)\|_{L^2}^2 \\
&\le C\|\nabla(u - \kappa_f)\|_{L^2}^2 + C\|G_\varepsilon \nabla u\|_{L^2}^2 \\
&\le C\|\nabla(u - \kappa_f)\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 \\
&\le C\|\nabla(u - \kappa_f)\|_{L^2}^2 + C\|f\|_{L^2_\ell}^2.
\end{aligned}$$

For the last term, using Lemma A.2 we have

$$\begin{aligned}
T_{55} &\le C\|\nabla(u - \kappa_f)\|_{L^2}^2 + CN^2\|\mathcal{K} * \chi_R[f]\|_{L^2}^2 \\
&\le C\|\nabla(u - \kappa_f)\|_{L^2}^2 + CN^2\|f\|_{L^2_\ell}^2
\end{aligned}$$

Gathering previous estimates we finally obtain
(4.21)
$$\frac{1}{2}\frac{d}{dt}\|\nabla(u - \kappa_f)\|_{L^2}^2 \le -\frac{1}{\varepsilon}\|\nabla^2(u - \kappa_f)\|_{L^2}^2 + C(1 + N^2)\|f\|_{L^2_\ell}^2 + C\|\nabla f\|_{L^2}^2 + C\|\nabla(u - \kappa_f)\|_{L^2}^2.$$

Step 2. The equation satisfied by $\partial_i f$ is

$$\partial_t(\partial_i f) = \mathcal{B}_{\varepsilon,1}(\partial_i f, \partial_i u) - \frac{1}{2}\partial_i f - \nabla(f\nabla(\partial_i V_\varepsilon)) - \nabla(\partial_j G_\varepsilon \nabla u) - N(\partial_i \chi_R)f + N\langle \chi_R f\rangle \partial_i \chi_1,$$

hence it follows

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\partial_i f\|_{L^2_k}^2 = &\int \mathcal{B}_{\varepsilon,1}(\partial_i f, \partial_i u)\,\partial_i f\,\langle x\rangle^{2k} - \frac{1}{2}\|\partial_i f\|_{L^2_k}^2 - \int \nabla(f\nabla(\partial_i V_\varepsilon))\,\partial_i f\,\langle x\rangle^{2k} - \int \nabla(\partial_j G_\varepsilon \nabla u)\,\partial_i f\,\langle x\rangle^{2k} \\
&- N\int (\partial_i \chi_R)f\partial_i f\langle x\rangle^{2k} + N\langle \chi_R f\rangle\int (\partial_i \chi_1)\,\partial_i f\,\langle x\rangle^{2k} \\
=: &A_1 + A_2 + A_3 + A_4 + A_5 + A_6.
\end{aligned}$$

Arguing as in Lemma 4.4 (step 1) we obtain for any $\delta > 0$,

$$\begin{aligned}
A_1 \le &-(1 - \delta)\|\nabla(\partial_i f)\|_{L^2_k}^2 + C(\delta)\|\nabla^2(u - \kappa_f)\|_{L^2}^2 \\
&+ \int \left\{\varphi(x) + [C(\delta) + C\,N\,R^{1-\ell}]\langle x\rangle^{2(\ell-k)} - N\chi_R(x)\right\}|\partial_i f|^2\langle x\rangle^{2k}.
\end{aligned}$$

Next we compute

$$\begin{aligned}
A_3 := &\int f\nabla(\partial_i V_\varepsilon)\cdot\nabla(\partial_i f)\langle x\rangle^{2k} + \int f\nabla(\partial_i V_\varepsilon)\cdot\nabla\langle x\rangle^{2k}\,\partial_i f \\
&\le \varepsilon C(\delta)\|f\|_{L^2_k}^2 + \delta\|\nabla f\|_{L^2_k}^2 + \delta\|\nabla^2 f\|_{L^2_k}^2,
\end{aligned}$$

using that $\Delta V_\varepsilon = -G_\varepsilon - (\varepsilon/2)x \cdot \nabla V_\varepsilon$ and Lemma 4.3. We also have

$$\begin{aligned}
A_4 = &\int \partial_i G_\varepsilon \nabla u \cdot \nabla(\partial_i f)\langle x\rangle^{2k} + \int \partial_i G_\varepsilon \nabla u \cdot \nabla\langle x\rangle^{2k}\,\partial_j f \\
&\le C(\delta)\|\nabla u\|_{L^2}^2 + \delta\|\nabla f\|_{L^2_{k-1/2}}^2 + \delta\|\nabla^2 f\|_{L^2_k}^2 \\
&\le C(\delta)\|\nabla(u - \kappa_f)\|_{L^2}^2 + C(\delta)\|f\|_{L^2_\ell}^2 + \delta\|\nabla f\|_{L^2_{k-1/2}}^2 + \delta\|\nabla^2 f\|_{L^2_k}^2,
\end{aligned}$$

and we easily get

$$A_5 \le N\frac{C}{R}\int \mathbf{1}_{R/2 \le |x| \le 2R}\,f^2\langle x\rangle^{2k} + N\frac{C}{R}\int \mathbf{1}_{R/2 \le |x| \le 2R}\,|\partial_i f|^2\langle x\rangle^{2k}.$$

For the last term

$$\begin{aligned}
A_6 &\le N\langle \chi_R f\rangle\int (\partial_i \chi_1)\,\partial_i f\,\langle x\rangle^{2k} \\
&\le CN\,\|f\|_{L^2}\,\|\partial_i f\|_{L^2_k} \le C(\delta)N^2\,\|f\|_{L^2}^2 + \delta\|\partial_i f\|_{L^2_k}^2.
\end{aligned}$$

Finally, putting together the above estimates, we obtain
(4.22)
$$\frac{1}{2}\frac{d}{dt}\|\nabla f\|_{L_k^2}^2 \leq -(1-\delta)\|\nabla^2 f\|_{L_k^2}^2 + C(\delta)\|\nabla(u-\kappa_f)\|_{L^2}^2 + C(\delta)\|\nabla^2(u-\kappa_f)\|_{L^2}^2$$

$$+ \int \left\{ \varepsilon C(\delta) + N\frac{C}{R}\mathbf{1}_{R/2\leq|x|\leq 2R} + C(\delta)\langle x\rangle^{2(\ell-k)} + C(\delta)N^2\langle x\rangle^{-2k} \right\} |f|^2\langle x\rangle^{2k}$$

$$+ \int \left\{ \varphi(x) - \frac{1}{2} + \delta + N\frac{C}{R}\mathbf{1}_{R/2\leq|x|\leq 2R} + [C(\delta)+CNR^{1-\ell}]\langle x\rangle^{2(\ell-k)} - N\chi_R \right\} |\nabla f|^2\langle x\rangle^{2k}.$$

Step 3. Gathering previous estimates (4.21) and (4.22) together with (4.17) it follows that
(4.23)
$$\frac{1}{2}\frac{d}{dt}\|(f,u)\|_{Y_*}^2$$

$$\leq \int \left\{ \varphi(x) + \eta_1\varepsilon C(\delta) + \eta_1 C\frac{N}{R}\mathbf{1}_{R/2\leq|x|\leq 2R} + \eta_1 C(\delta)N^2\langle x\rangle^{-2k} \right.$$

$$\left. + [\eta_1 C(\delta) + \eta_1 C(1+N^2) + C(\delta)(1+\eta N^2) + C\frac{N}{R^{\ell-1}}]\langle x\rangle^{2(\ell-k)} - N\chi_R \right\} f^2\langle x\rangle^{2k}$$

$$+ \eta_1\int \left\{ \varphi(x) - \frac{1}{2} + \delta + C\frac{N}{R}\mathbf{1}_{R/2\leq|x|\leq 2R} + [C(\delta) + C\frac{N}{R^{\ell-1}}]\langle x\rangle^{2(\ell-k)} + C\langle x\rangle^{-2k} - N\chi_R \right\} |\nabla f|^2\langle x\rangle^{2k}$$

$$- (1-\delta)\|\nabla f\|_{L_k^2}^2 - \eta\left(\frac{1}{2}-\delta\right)\|u-\kappa_f\|_{L^2}^2 - \eta_1(1-\delta)\|\nabla^2 f\|_{L_k^2}^2$$

$$- \eta\left(\frac{1}{\varepsilon} - C(\delta) - \frac{\eta_1}{\eta}C(\delta)\right)\|\nabla(u-\kappa_f)\|_{L^2}^2 - \eta_1\left(\frac{1}{\varepsilon} - C(\delta)\right)\|\nabla^2(u-\kappa_f)\|_{L^2}^2.$$

Now we conclude as in step 3 of the proof of Lemma 4.4. We choose first $\delta \in (0,1)$ small enough and next $\varepsilon \in (0,1)$ small enough, then for $\eta_1 = \eta = N^{-3}$ and $R = N$ it follows

(4.24)
$$\frac{1}{2}\frac{d}{dt}\|(f,u)\|_{Y_*}^2 \leq \int \{\varphi_N^1(x) - N\chi_R\}f^2\langle x\rangle^{2k} + \eta_1\int \{\varphi_N^2(x) - N\chi_R\}|\nabla f|^2\langle x\rangle^{2k}$$

$$+ a\|\nabla f\|_{L_k^2}^2 + a\eta\|u-\kappa_f\|_{L^2}^2 + a\eta_1\|\nabla^2 f\|_{L_k^2}^2$$

$$+ a\eta_1\|\nabla(u-\kappa_f)\|_{L^2}^2 + a\eta_1\|\nabla^2(u-\kappa_f)\|_{L^2}^2,$$

for any $a > -1/2$, where

$$\varphi_N^1(x) := \varphi(x) + CN^{-3} + CN^{-1}\mathbf{1}_{R/2\leq|x|\leq 2R} + (C+CN^{-1}+CN^{-3}+CN^{2-\ell})\langle x\rangle^{2(\ell-k)} + CN^{-1}\langle x\rangle^{-2k}$$

and

$$\varphi_N^2(x) := \varphi(x) + C\mathbf{1}_{R/2\leq|x|\leq 2R} + (C+CN^{2-\ell})\langle x\rangle^{2(\ell-k)} + C\langle x\rangle^{-2k}$$

have the same asymptotic behaviour as $\varphi(x)$ when $|x| \to \infty$ and φ_N^i is decreasing as a function of N. Hence picking N large enough such that

$$\varphi_N^i(x) - N\chi_R(x) \leq a, \quad \forall x \in \mathbb{R}^2,$$

we deduce that, for some constant $K > 0$,

(4.25)
$$\frac{1}{2}\frac{d}{dt}\|(f,u)\|_{Y_*}^2 \leq a\|(f,u)\|_{Y_*}^2 - K(\|\nabla^2 f\|_{L_k^2}^2 + \|\nabla^2(u-\kappa_f)\|_{L^2}^2),$$

from which $\mathcal{B}_\varepsilon$ is a-hypo-dissipative in Y. Moreover, using the interpolation inequality

$$\|g\|_{H_k^1}^2 \lesssim \|g\|_{H_k^2}\|g\|_{L_k^2}$$

it follows from (4.25) that

$$\frac{1}{2}\frac{d}{dt}\|(f,u)\|_{Y_*}^2 \leq a\|(f,u)\|_{Y_*}^2 - K\|(f,u)\|_{Y_*}^4\|(f,u)\|_{X_*}^{-2},$$

hence by standard arguments we get the estimate

$$\|S_{\mathcal{B}_\varepsilon}(t)(f,u)\|_Y \leq C\,t^{-1/2}\,e^{at}\|(f,u)\|_X,$$

concluding the proof.   □

Proof of Proposition 4.3. The domain $D(\Lambda_\varepsilon)$ of the operator $\Lambda_\varepsilon : D(\Lambda_\varepsilon) \subset X \to X$ is given by

$$D(\Lambda_\varepsilon) = H_k^2 \cap L_{k,0}^2 \cap L_{rad}^2 \times H^2 \cap L_{rad}^2$$

and we recall that $X = L_{k,0}^2 \cap L_{rad}^2 \times L_{rad}^2$ and $Y = H_k^1 \cap L_{k,0}^2 \cap L_{rad}^2 \times H^1 \cap L_{rad}^2$ (see equations (4.8) and (4.18)). We define a family of interpolation spaces

$$X^\eta := H_k^{2\eta} \cap L_{k,0}^2 \cap L_{rad}^2 \times H^{2\eta} \cap L_{rad}^2, \quad \eta \in [0,1],$$

so that $X^0 = X$, $X^1 = D(\Lambda_\varepsilon)$ and $X^{1/2} = Y$. Thanks to classical interpolation results we have $Y = X^{1/2} \subset D(\Lambda_\varepsilon^\eta)$ for any $\eta \in [0, 1/2)$, see [22, 23, 26]. Now we fix some $\eta \in (0, 1/2)$ and we have $Y \subset D(\Lambda_\varepsilon^\eta) \subset X$.

Recalling the results from Lemma 4.4, Lemma 4.5, Lemma 4.6 and (4.10) we have, for any $a > -1/2$,

$$S_{\mathcal{B}_\varepsilon}(t) : X \to X, \quad \text{with} \quad \|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathscr{B}(X)} \le C\, e^{at},$$

$$S_{\mathcal{B}_\varepsilon}(t) : Y \to Y, \quad \text{with} \quad \|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathscr{B}(Y)} \le C\, e^{at},$$

$$S_{\mathcal{B}_\varepsilon}(t) : X \to Y, \quad \text{with} \quad \|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathscr{B}(X,Y)} \le C\, t^{-1/2}\, e^{at},$$

moreover $\mathcal{A} \in \mathscr{B}(X) \cap \mathscr{B}(Y)$ and

$$\mathcal{A}S_{\mathcal{B}_\varepsilon}(t) : X \to Y, \quad \text{with} \quad \|\mathcal{A}S_{\mathcal{B}_\varepsilon}(t)\|_{\mathscr{B}(X,Y)} \le C\, t^{-1/2}\, e^{at}.$$

First of all, we already obtain from previous estimates that

$$(4.26) \qquad \forall \ell \ge 0, \quad \|S_{\mathcal{B}_\varepsilon} * (\mathcal{A}S_{\mathcal{B}_\varepsilon})^{(*\ell)}(t)\|_{\mathscr{B}(X)} \le C\, e^{at},$$

Moreover, from [18, Lemma 2.17] there exists $n \in \mathbb{N}$ such that

$$\|(\mathcal{A}S_{\mathcal{B}_\varepsilon})^{*(n)}(t)\|_{\mathscr{B}(X,Y)} \le C\, e^{at},$$

which together with the fact $S_{\mathcal{B}_\varepsilon}(t) : D(\Lambda_\varepsilon^\eta) \to D(\Lambda_\varepsilon^\eta)$ with $\|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathscr{B}(D(\Lambda_\varepsilon^\eta))} \le C\, e^{at}$ (by interpolation of the same results in X and Y), yield

$$(4.27) \qquad \|S_{\mathcal{B}_\varepsilon} * (\mathcal{A}S_{\mathcal{B}_\varepsilon})^{(*n)}(t)\|_{\mathscr{B}(X,D(\Lambda_\varepsilon^\eta))} \le C\, e^{at}.$$

Gathering that last estimate with (4.26), we can apply [26, Theorem 2.1] which yields, for some $r^* > 0$,

$$\Sigma(\Lambda_\varepsilon) \cap \Delta_a \subset B(0, r^*) \quad \text{on } X.$$

Now we define $\overline{Y} = H_k^1 \cap L_{k,0}^2 \cap L_{rad}^2 \times H_k^1 \cap L_{rad}^2 \subset Y$. From previous estimates we also obtain

$$(4.28) \qquad \int_0^\infty \|(\mathcal{A}S_{\mathcal{B}_\varepsilon})^{*(n+1)}(t)\|_{\mathscr{B}(X,\overline{Y})}\, e^{-at}\, dt \le C,$$

where $\overline{Y} \subset X$ with compact embedding. Hence, thanks to (4.26)-(4.27)-(4.28), we are able to apply [26, Theorem 3.1] that implies

$$\Sigma(\Lambda_\varepsilon) \cap \Delta_a \subset \Sigma_d(\Lambda_\varepsilon),$$

and that concludes the proof.   □

4.3. **Localization of the spectrum for the linearized operator in a radially symmetric setting.** We recall that we consider a radially symmetric setting and we have already defined the space X in (4.8). We establish in this subsection the following localization of the spectrum of Λ_ε.

Theorem 4.7. *There exists $\varepsilon^* > 0$ such that in X there holds*

$$\Sigma(\Lambda_\varepsilon) \cap \Delta_{-1/3} = \emptyset \quad \text{for any} \quad \varepsilon \in (0, \varepsilon^*).$$

As a consequence, there exists C such that

$$\|S_{\Lambda_\varepsilon}(t)\|_{\mathscr{B}(X)} \leq C\, e^{-t/3} \quad \forall\, t \geq 0,\ \forall\, \varepsilon \in (0, \varepsilon^*).$$

The difficulty is that Λ_ε is not a perturbation of some fixed operator Λ and we cannot apply directly the perturbation theory developed in [25, 32]. However, we are able to identify the limit of $\mathcal{R}_{\Lambda_\varepsilon}$ as $\varepsilon \to 0$ which is enough to conclude.

We introduce the notations

$$Af := \Delta f + \nabla(\frac{1}{2}\, x\, f - f\, \nabla V_\varepsilon), \quad Bu := -\nabla(G_\varepsilon\, \nabla u)$$

$$Cu := \Delta u, \quad Du := \frac{1}{2}\, x \cdot \nabla u,$$

so that the linearized equation writes

(4.29) $$\partial_t f = Af + Bu, \qquad \partial_t u = \frac{1}{\varepsilon}(Cu + f) + Du.$$

The important point is that at a very formal level, the limit system (as $\varepsilon \to 0$) is the linearized parabolic-elliptic system

(4.30) $$\partial_t f = A_0 f + B_0 u, \qquad Cu = -f,$$

where

$$A_0 f := \Delta f + \nabla(\frac{1}{2}\, x\, f - f\, \nabla V), \quad B_0 u := -\nabla(G\, \nabla u),$$

with G and V defined in (4.1), which simplifies into a single equation

(4.31) $$\partial_t f = (A_0 + B_0(-C)^{-1})f =: \Omega f.$$

Observe that the last equation is nothing but the linearized equation associated to parabolic-elliptic Keller-Segel equation which has been studied in [8, 9, 14] and it has been proved therein that the associated semigroup is exponentially stable in several weighted Lebesgue spaces. In the sequel we explain why the linearized parabolic-parabolic system inherits that exponential stability at least for $\varepsilon > 0$ small enough.

We recall the following result which is an immediate consequence of [9, Section 6.1] and [14, Theorem 4.3].

Theorem 4.8. *There exists a constant C such that*

$$\forall\, h \in L^2_{k,0} \quad \|e^{\Omega t} h\|_{L^2_k} \leq C\, e^{-t}\|h\|_{L^2_k},$$

hence it follows

$$\mathcal{R}_\Omega \in \mathcal{H}(\Delta_{-1}; \mathscr{B}(X_1)) \quad \text{and then} \quad \Sigma(\Omega) \cap \Delta_{-1} = \emptyset.$$

In order to formalize the link between the linearized parabolic-parabolic equation and the linearized parabolic-elliptic equation we write the linearized parabolic-parabolic system (4.29) into the matrix form

$$\frac{d}{dt}\begin{pmatrix} f \\ u \end{pmatrix} = \Lambda_\varepsilon \begin{pmatrix} f \\ u \end{pmatrix}, \quad \Lambda_\varepsilon := \begin{pmatrix} A & B \\ \varepsilon^{-1} & \varepsilon^{-1}C + D \end{pmatrix}.$$

In order to analysis the spectrum of Λ_ε, for any $z \in \mathbb{C}$, we denote

$$\Lambda_\varepsilon(z) = \Lambda_\varepsilon - z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with

$$a = A(z) = A - z, \quad b = B, \quad c := \varepsilon^{-1}I, \quad d := \varepsilon^{-1}C + D(z), \quad D(z) = D - z.$$

One can readily verify that for $z \in \mathbb{C}$ such that $d = d(z)$ is invertible as well as its Schur's complement

$$s_\varepsilon = s_\varepsilon(z) := a - bd^{-1}c = A - B(C + \varepsilon D)^{-1}$$

is invertible, the resolvent of Λ_ε is given by

$$\mathcal{R}_{\Lambda_\varepsilon}(z) = \Lambda_\varepsilon(z)^{-1} = \begin{pmatrix} s_\varepsilon^{-1} & -s_\varepsilon^{-1}bd^{-1} \\ -d^{-1}cs_\varepsilon^{-1} & d^{-1} + d^{-1}cs_\varepsilon^{-1}bd^{-1} \end{pmatrix} =: \begin{pmatrix} \mathcal{R}_{11}^{\Lambda_\varepsilon} & \mathcal{R}_{12}^{\Lambda_\varepsilon} \\ \mathcal{R}_{21}^{\Lambda_\varepsilon} & \mathcal{R}_{22}^{\Lambda_\varepsilon} \end{pmatrix}.$$

Then at least formally, we see that

(4.32) $$\mathcal{R}_{\Lambda_\varepsilon}(z) \xrightarrow[\varepsilon \to 0]{} \begin{pmatrix} \mathcal{R}_\Omega(z) & 0 \\ -C^{-1}\mathcal{R}_\Omega(z) & 0 \end{pmatrix}.$$

Indeed, on the one hand, we have

$$\mathcal{R}_{11}^{\Lambda_\varepsilon} = s_\varepsilon^{-1} = \{A - z - B(\varepsilon^{-1}C + D - z)^{-1}\varepsilon^{-1}I\}^{-1}$$
$$= \{A - z - B(C + \varepsilon D - \varepsilon z)^{-1}\}^{-1} \xrightarrow[\varepsilon \to 0]{} (A_0 - B_0 C^{-1} - z)^{-1} = \mathcal{R}_\Omega(z).$$

and

$$\mathcal{R}_{21}^{\Lambda_\varepsilon} = -d^{-1}cs^{-1} = -(\varepsilon^{-1}C + D - z)^{-1}\varepsilon^{-1}I\{A - z - B(C + \varepsilon D - \varepsilon z)^{-1}\}^{-1}$$
$$= -(C + \varepsilon D - \varepsilon z)^{-1}\{A - z - B(C + \varepsilon(D - z))^{-1}\}^{-1}$$
$$\xrightarrow[\varepsilon \to 0]{} -C^{-1}(A_0 - B_0 C^{-1} - z)^{-1} = -C^{-1}\mathcal{R}_\Omega(z).$$

In the same way, we have

$$\mathcal{R}_{12}^{\Lambda_\varepsilon} = -s^{-1}bd^{-1} = -\varepsilon\{A - z - B(C + \varepsilon D - \varepsilon z)^{-1}\}^{-1}B(C + \varepsilon D - \varepsilon z)^{-1}$$

as well as

$$\mathcal{R}_{22}^{\Lambda_\varepsilon} = d^{-1} + d^{-1}cs^{-1}bd^{-1}$$
$$= \varepsilon(C + \varepsilon D - \varepsilon z)^{-1} + \varepsilon(C + \varepsilon D - \varepsilon z)^{-1}\{A - z - B(C + \varepsilon D - \varepsilon z)^{-1}\}^{-1}B(C + \varepsilon D - \varepsilon z)^{-1},$$

and then both last terms vanishes in the limit $\varepsilon \to 0$.

In fact, we will not try to prove that convergence (4.32) rigorously holds, but we will just prove the following result. We define

$$\mathcal{O}_\rho := \Delta_{-1/3} \cap B(0, \rho).$$

Proposition 4.9. *For any $\rho > 0$ there exists $\varepsilon_\rho^* > 0$ such that in X there holds*

$$\mathcal{R}_{\Lambda_\varepsilon} \in \mathcal{H}(\mathcal{O}_\rho; \mathscr{B}(X)) \quad \text{for any} \quad \varepsilon \in (0, \varepsilon_\rho^*).$$

Before proving Proposition 4.9 we establish some estimates on the terms involved in $\mathcal{R}_{\Lambda_\varepsilon}$.

Lemma 4.10. *Define*

$$\tilde{s}(z) := B(C + \varepsilon D(z))^{-1}(-D(z))C^{-1}.$$

For any $\rho > 0$, there exists $\varepsilon_\rho^* > 0$ such that

$$\sup_{z \in \mathcal{O}_\rho} \sup_{\varepsilon \in (0, \varepsilon_\rho^*)} \|\tilde{s}(z)\|_{\mathscr{B}(X_1)} \leq C.$$

Proof of Lemma 4.10. On the one hand, from Lemma A.1 and Lemma A.2 we have

$$(-D(z))C^{-1} : L_{k,0}^2 \cap L_{rad}^2 \to L_{rad}^2$$

is bounded uniformly for $z \in \mathcal{O}_\rho$. More precisely, for $f \in L_{k,0}^2$ we write

$$f = f_0 + f_1 + f_2; \quad f_i = \lambda_i F_i, \ i = 1, 2,$$

where we define the coefficients by

$$\lambda_1 = \int_0^\infty f(r)\, r^2\, r\, dr, \quad \lambda_2 = \int_0^\infty f(r)\, r^4\, r\, dr,$$

and the functions F_i by

$$F_1(r) = \left(-\frac{1}{8}r^4 + \frac{5}{4}r^2 - \frac{3}{2}\right)e^{-r^2/2}, \quad F_2(r) = \left(\frac{1}{64}r^4 - \frac{1}{8}r^2 + \frac{1}{8}\right)e^{-r^2/2},$$

so that it holds

$$\int_0^\infty F_1(r)\,(1, r^2, r^4)\,r\,dr = (0, 1, 0), \quad \int_0^\infty F_2(r)\,(1, r^2, r^4)\,r\,dr = (0, 0, 1),$$

and hence $f_0 \in L_{k,5}^2$.

We may solve the equation

$$Cu = \Delta u = f, \quad u \text{ radially symmetric}, \quad u'(0) = u'(\infty) = 0,$$

by writing

$$u = u_0 + u_1 + u_2, \quad \Delta u_i = f_i, \quad u_i'(0) = u_i'(\infty) = 0$$

where u_i, $i = 1, 2$, is defined by ther relation

$$r u_i'(r) := \int_0^r \sigma f_i(\sigma)\,d\sigma,$$

so that $\langle r \rangle |u_i'| \leq C$ and $|u| \leq C \log(1 + \langle r \rangle)$, and $u_0 \in L_{rad}^2 \cap L_5^2$, $\nabla u_0 \in L_5^2$ is the unique solution to the above Poisson equation as given by Lemma A.1 and Lemma A.2. As a consequence, $g_0 := D(z)u_0 \in L_{4,1}^2$ and $g_i := r u_i' + z u_i$ satisfy the estimates $g_i\,e^{-r/2} \in L^\infty$ for $i = 1, 2$. Thanks to Lemmas B.1 and B.2 and using the notation of appendix B, we have $v_i := L_\varepsilon^{-1} g_i$ which satisfy $\|v_0\|_{\dot{H}^1 \cap \dot{H}^2} \leq C\,\|g_0\|_{L_{4,1}^2}$ while v_i satisfy the estimates, for $i = 1, 2$,

$$\|v_i\,e^{-(1+\varepsilon|z|)r}\|_{L^\infty} + \|v_i'\,e^{-(1+\varepsilon|z|)r}\|_{L^\infty} \leq C\|g_i\,e^{-r/2}\|_{L^\infty}.$$

Finally we solve the equation $w_i := Bv_i$ which means

$$w_i = G_\varepsilon \Delta v_i + \nabla G_\varepsilon \cdot \nabla v_i = G_\varepsilon\left(v_i'' + \frac{1}{r}\,v_i'\right) + r\,G_\varepsilon' v_i',$$

then the previous estimates togheter with the bound (4.2) yield $w = w_0 + w_1 + w_2 \in L_{k,0}^2 \cap L_{rad}^2 = X_1$. $\qquad\square$

Lemma 4.11. *With the above notations, for any $\rho > 0$, there exists C_ρ such that*

$$(4.33) \qquad \sup_{z \in \mathcal{O}_\rho} \|d^{-1}(z)\|_{\mathscr{B}(X_2)} \leq C_\rho$$

Proof of Lemma 4.11. Consider the equation

$$(4.34) \qquad d(z)v = \varepsilon^{-1}\Delta v + \frac{1}{2}x \cdot \nabla v - z\,v = u,$$

for $z \in \Delta_a \cap B(0, \rho)$. Multiplying the equation by $\bar{v}$ and the conjugated equation by v, we find

$$(4.35) \qquad \frac{1}{\varepsilon}\int |\nabla v|^2 + \left(\frac{1}{2} + \Re e z\right)\int |v|^2 = -\frac{1}{2}\int v\bar{u} - \frac{1}{2}\int \bar{v}u$$
$$\leq \|u\|_{L^2}\,\|v\|_{L^2}$$

and then

$$\left(\frac{1}{2} + \Re e z\right)\|v\|_{L^2} \leq \|u\|_{L^2}.$$

$\qquad\square$

Lemma 4.12. *With the above notations, for any $\rho > 0$, there exists $C_{k,a,\rho} > 0$ such that*

$$(4.36) \qquad \sup_{z \in \mathcal{O}_\rho} \|bd^{-1}(z)\|_{\mathscr{B}(X_2, X_1)} \leq C_{k,a,\rho}\,\sqrt{\varepsilon} \to 0.$$

Proof of Lemma 4.12. Consider the equation (4.34) again. Coming back to (4.35), we have

$$(4.37) \qquad \|\nabla v\|_{L^2} \le \sqrt{\frac{\varepsilon}{\frac{1}{2} + \Re ez}}\, \|u\|_{L^2}.$$

Next, multiplying the equation (4.34) by $x \cdot \nabla \bar{v}$ and the conjugated equation by $x \cdot \nabla v$, we find

$$\int |x \cdot \nabla v|^2 = \int (u - zv)(x \cdot \nabla \bar{v}) + (\bar{u} - \bar{z}\bar{v})(x \cdot \nabla v)$$

which in turn implies

$$\|x \cdot \nabla v\|_{L^2} \le C_{a,r}.$$

Coming back to (4.34) and together with (4.37), we have proved

$$(4.38) \qquad \|\Delta v\|_{L^2} + \|\nabla v\|_{L^2} \le C_{a,r} \sqrt{\varepsilon} \|u\|_{L^2}.$$

We then immediately conclude to (4.36). $\qquad\square$

Lemma 4.13. *With the above notations, for any $\rho > 0$, there exists $C_{k,\rho} > 0$ such that*

$$(4.39) \qquad \sup_{z \in \mathcal{O}_\rho} \|cd^{-1}(z)\|_{\mathscr{B}(X_1, X_2)} \le C_{k,\rho}.$$

Proof of Lemma 4.13. We use appendix B. $\qquad\square$

Proof of Proposition 4.9. We split the proof into four steps.

Step 1. We prove that

$$(4.40) \qquad \forall \varepsilon \in (0, \varepsilon_\rho^*) \quad \mathcal{R}_{11}^{\Lambda_\varepsilon} = s_\varepsilon^{-1} \in \mathcal{H}(\mathcal{O}_\rho; \mathscr{B}(X_1)).$$

We write

$$s_\varepsilon(z) = A(z) - BC^{-1} - [B(C + \varepsilon D(z))^{-1} - BC^{-1}] =: s_0(z) - \varepsilon \tilde{s}(z)$$

and then

$$s_\varepsilon(z) - \Omega(z) = [s_0(z) - \Omega(z)] - \varepsilon \tilde{s}(z)$$

with

$$s_0(z) = A(z) - BC^{-1}, \quad \Omega(z) = \Omega - z = A_0(z) - B_0 C^{-1}, \quad \tilde{s}(z) := B(C + \varepsilon D(z))^{-1}(-D(z))\, C^{-1}.$$

Remark that

$$s_0(z) - \Omega(z) = s_0 - \Omega = -\nabla(\cdot\nabla(V_\varepsilon - V)) - \nabla((G_\varepsilon - G)\nabla(\Delta^{-1}\cdot))$$

does not depend on z, and thanks to Corollary 4.2 we easily get

$$\|s_0 - \Omega\|_{\mathscr{B}(Y_1, X_1)} \le \eta(\varepsilon) \quad \text{with} \quad \eta(\varepsilon) \to 0 \text{ as } \varepsilon \to 0,$$

where we recall that $Y_1 = H_k^1 \cap L_{k,0}^2 \cap L_{rad}^2$, see (4.18). Moreover, using Lemma 4.10 we get that

$$\sup_{z \in \mathcal{O}_\rho} \sup_{\varepsilon \in (0, \varepsilon_\rho^*)} \|\tilde{s}\|_{\mathscr{B}(X_1)} \le C$$

from which we deduce, for any $z \in \mathcal{O}_\rho$ and $\varepsilon \in (0, \varepsilon_\rho^*)$

$$\|s_\varepsilon(z) - \Omega(z)\|_{\mathscr{B}(Y_1, X_1)} \le \eta(\varepsilon) + C\varepsilon,$$

Then, arguing as is [32, Lemma 2.16], the operator

$$\mathcal{T}_\varepsilon(z) := (-1)^n (s_\varepsilon - \Omega)\mathcal{R}_\Omega(z)\, (\mathcal{A}_{11}\mathcal{R}_{11}^{\mathcal{B}_\varepsilon}(z))^n,$$

satisfies

$$\|\mathcal{T}_\varepsilon(z)\|_{\mathscr{B}(X_1)} \le \eta'(\varepsilon) \quad \forall z \in \mathcal{O}_\rho, \quad \eta'(\varepsilon) \to 0 \text{ as } \varepsilon \to 0,$$

where

$$\mathcal{R}_{\mathcal{B}_\varepsilon} =: \begin{pmatrix} \mathcal{R}_{11}^{\mathcal{B}_\varepsilon} & \mathcal{R}_{12}^{\mathcal{B}_\varepsilon} \\ \mathcal{R}_{21}^{\mathcal{B}_\varepsilon} & \mathcal{R}_{22}^{\mathcal{B}_\varepsilon} \end{pmatrix}, \quad \mathcal{A} =: \begin{pmatrix} \mathcal{A}_{11} & 0 \\ 0 & 0 \end{pmatrix},$$

and the integer n is defined in the proof of Proposition 4.3. As a consequence, the operators $I + \mathcal{T}_\varepsilon(z)$ and $s_\varepsilon(z)$ are invertible for any $z \in \mathcal{O}_\rho$, and furthermore

$$\mathcal{R}_{11}^{\Lambda_\varepsilon}(z) = s_\varepsilon(z)^{-1} = \mathcal{U}_\varepsilon(z)(1 + \mathcal{T}_\varepsilon(z))^{-1},$$

where

$$\mathcal{U}_\varepsilon(z) = \sum_{j=0}^{n-1}(-1)^j \mathcal{R}_{11}^{\mathcal{B}_\varepsilon}(z)(\mathcal{A}_{11}\mathcal{R}_{11}^{\mathcal{B}_\varepsilon}(z))^j + (-1)^n \mathcal{R}_\Omega(z)(\mathcal{A}_{11}\mathcal{R}_{11}^{\mathcal{B}_\varepsilon}(z))^n.$$

We immediately conclude to (4.40) because $\Sigma(\Omega) \cap \Delta_a = \emptyset$ on X_1 and $\|R_\Omega(z)\|_{\mathscr{B}(X_1)} \leq C$ for any $z \in \mathcal{O}_\rho$ from Theorem 4.8.

Step 2. We have

$$(4.41) \qquad\qquad \forall \varepsilon \in (0, \varepsilon_\rho^*) \quad \mathcal{R}_{12}^{\Lambda_\varepsilon} := -s_\varepsilon^{-1}bd^{-1} \in \mathcal{H}(\mathcal{O}_\rho; \mathscr{B}(X_2, X_1))$$

as an immediate consequence of Lemmas 4.10 and 4.12.

Step 3. We also have

$$(4.42) \qquad\qquad \forall \varepsilon \in (0, \varepsilon_\rho^*) \quad \mathcal{R}_{21}^{\Lambda_\varepsilon} := -d^{-1}cs_\varepsilon^{-1} \in \mathcal{H}(\mathcal{O}_\rho; \mathscr{B}(X_1, X_2))$$

as a consequence of Lemmas 4.10 and 4.13.

Step 4.   We finally have

$$(4.43) \qquad\qquad \forall \varepsilon \in (0, \varepsilon_\rho^*) \quad \mathcal{R}_{22}^{\Lambda_\varepsilon} := d^{-1} + d^{-1}cs_\varepsilon^{-1}bd^{-1} \in \mathcal{H}(\mathcal{O}_\rho; \mathscr{B}(X_2))$$

as an immediate consequence of Step 1 together with Lemmas 4.12 and 4.13. $\square$

Proof of Theorem 4.7. The proof is a consequence of Proposition 4.3, Proposition 4.9 and Theorem 4.8 togheter with [26, Theorem 2.1]. $\square$

5. NONLINEAR EXPONENTIAL STABILITY OF SELF-SIMILAR SOLUTIONS

5.1. **Linear stability in higher-order norms.** Define

$$(5.1) \qquad\qquad Z := Z_1 \times Z_2, \quad Z_1 := H_k^1 \cap L_{k,0}^2 \cap L_{rad}^2, \quad Z_2 := H^2 \cap L_{rad}^2,$$

associated to the norm

$$(5.2) \qquad\qquad \|(f, u)\|_Z^2 := \|(f, u)\|_Y^2 + \|\nabla^2 u\|_{L^2}^2.$$

We shall prove that the same linear stability estimate in X from Theorem 4.7 also holds in Z, as stated in the following result.

Proposition 5.1. *There exists $\varepsilon^* > 0$ such that there holds in Z*

$$\Sigma(\Lambda_\varepsilon) \cap \Delta_{-1/3} = \emptyset, \quad \forall \varepsilon \in (0, \varepsilon^*).$$

As a consequence we have

$$\|S_{\Lambda_\varepsilon}(t)\|_{\mathscr{B}(Z)} \leq Ce^{-t/3}, \quad \forall\, t \geq 0, \, \forall \varepsilon \in (0, \varepsilon^*).$$

Lemma 5.2. *(1) $\mathcal{A} \in \mathscr{B}(Z)$.*
(2) There exist N, R large enough such that the operator $\mathcal{B}_\varepsilon$ is a-hypodissipative in Z, i.e.

$$\|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathscr{B}(Z)} \leq Ce^{at}.$$

Moreover we have the following estimate

$$\|S_{\mathcal{B}_\varepsilon}(t)\|_{\mathscr{B}(X,Z)} \leq C\,t^{-1}\,e^{at}.$$

Proof. Point (1) is straightforward from (4.10) and we omit the proof. For point (2), consider the equation $\partial_t(f, u) = \mathcal{B}_\varepsilon(f, u)$. First of all, observe that the norm $\|\cdot\|_Z$ is equivalent to

$$(5.3) \qquad\qquad \|(f, u)\|_{Z_*}^2 := \|(f, u)\|_{Y_*}^2 + \eta_2 \|\nabla^2(u - \kappa_f)\|_{L^2}^2$$

for any $\eta_2 > 0$. We write

$$\partial_t(\partial_{ij}u) = \frac{1}{\varepsilon}\Delta(\partial_{ij}(u - \kappa_f)) + \partial_{ij}u + \frac{1}{2}x \cdot \nabla(\partial_{ij}u)$$

and then we compute

$$\frac{1}{2}\frac{d}{dt}\int |\partial_{ij}(u-\kappa_f)|^2 = \int \partial_{ij}(u-\kappa_f)\partial_t\{\partial_{ij}(u-\kappa_f)\}$$

$$= \int \partial_{ij}(u-\kappa_f)\frac{1}{\varepsilon}\Delta(\partial_{ij}(u-\kappa_f)) + \int \partial_{ij}(u-\kappa_f)\partial_{ij}u$$

$$+ \int \partial_{ij}(u-\kappa_f)\frac{1}{2}x\cdot\nabla(\partial_{ij}(u-\kappa_f)) + \int \partial_{ij}(u-\kappa_f)\frac{1}{2}x\cdot\nabla(\partial_{ij}\kappa_f)$$

$$- \int \partial_{ij}(u-\kappa_f)\partial_{ij}\kappa*(\partial_t f)$$

$$=: B_1 + \cdots + B_5.$$

We easily obtain

$$B_1 = -\frac{1}{\varepsilon}\|\nabla\{\partial_{ij}(u-\kappa_f)\}\|_{L^2}^2,$$

moreover

$$B_2 \le C\|\nabla^2(u-\kappa_f)\|_{L^2}^2 + C\|\nabla^2 u\|_{L^2}^2 \le C\|\nabla^2(u-\kappa_f)\|_{L^2}^2 + C\|f\|_{L^2}^2,$$

by integration by parts $B_3 \le 0$ and also

$$B_4 \le C\|\nabla^2(u-\kappa_f)\|_{L^2}^2 + C\|\nabla^3\kappa*f\|_{L_1^2}^2 \le C\|\nabla^2(u-\kappa_f)\|_{L^2}^2 + C\|\nabla f\|_{L_1^2}^2.$$

For the last term we get

$$B_5 = -\int \partial_{ij}(u-\kappa_f)\partial_{ij}\kappa*\left\{\Delta f + \frac{1}{2}\nabla(xf) - \nabla(f\nabla V_\varepsilon) - \nabla(G_\varepsilon\nabla u) - N\chi_R[f]\right\}$$

$$=: B_{51} + \cdots + B_{54} + B_{55}.$$

We have

$$B_{51} \le C(\delta)\|\nabla^2(u-\kappa_f)\|_{L^2}^2 + \delta\|\nabla^2 f\|_{L^2}^2$$

and

$$B_{52} \le C\|\nabla^2(u-\kappa_f)\|_{L^2}^2 + C\|\nabla^2\kappa*\nabla(xf)\|_{L^2}^2 \le C\|\nabla^2(u-\kappa_f)\|_{L^2}^2 + C\|\nabla f\|_{L_1^2}^2 + C\|f\|_{L^2}^2.$$

Moreover, using that $\nabla V_\varepsilon, \Delta V_\varepsilon \in L^\infty$ we get

$$B_{53} \le C\|\nabla^2(u-\kappa_f)\|_{L^2}^2 + C\|\nabla^2\kappa*\nabla(f\nabla V_\varepsilon)\|_{L^2}^2$$

$$\le C\|\nabla^2(u-\kappa_f)\|_{L^2}^2 + C\|\nabla(f\nabla V_\varepsilon)\|_{L^2}^2$$

$$\le C\|\nabla^2(u-\kappa_f)\|_{L^2}^2 + C\|\nabla f\|_{L^2}^2 + C\|f\|_{L^2}^2,$$

and arguing as above with $G_\varepsilon, \nabla G_\varepsilon \in L^\infty$ we also obtain

$$B_{54} \le C\|\nabla^2(u-\kappa_f)\|_{L^2}^2 + C\|\nabla^2\kappa*\nabla(G_\varepsilon\nabla u)\|_{L^2}^2$$

$$\le C\|\nabla^2(u-\kappa_f)\|_{L^2}^2 + C\|\nabla(G_\varepsilon\nabla u)\|_{L^2}^2$$

$$\le C\|\nabla^2(u-\kappa_f)\|_{L^2}^2 + C\|\nabla^2 u\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2$$

$$\le C\|\nabla^2(u-\kappa_f)\|_{L^2}^2 + C\|f\|_{L_\ell^2}^2,$$

where we recall $\ell \in (2, k)$ is the same as in Lemma 4.4. For the last term we easily obtain

$$B_{55} \le C\|\nabla^2(u-\kappa_f)\|_{L^2}^2 + C\|N\chi_R[f]\|_{L^2}^2 \le C\|\nabla^2(u-\kappa_f)\|_{L^2}^2 + CN^2\|f\|_{L^2}^2.$$

All the above estimates yield

(5.4)
$$\frac{1}{2}\frac{d}{dt}\|\nabla^2(u-\kappa_f)\|_{L^2}^2 \le -\frac{1}{\varepsilon}\|\nabla^3(u-\kappa_{\tilde{f}})\|_{L^2}^2 + C(\delta)\|\nabla^2(u-\kappa_{\tilde{f}})\|_{L^2}^2$$

$$+ C\|f\|_{L_\ell^2}^2 + CN^2\|f\|_{L^2}^2 + C\|\nabla f\|_{L_1^2}^2 + \delta\|\nabla^2 f\|_{L^2}^2.$$

Putting togheter last equation with (4.23) we get

(5.5)
$$\frac{1}{2}\frac{d}{dt}\|(f,u)\|_{Z_*}^2$$

$$\leq \int \Big\{ \varphi(x) + \eta_1 \varepsilon C(\delta) + \eta_1 C \frac{N}{R} \mathbf{1}_{R/2 \leq |x| \leq 2R} + \eta_1 C(\delta) N^2 \langle x \rangle^{-2k} + \eta_2 C N^2 \langle x \rangle^{-2k}$$

$$+ [\eta_1 C(\delta) + \eta_1 C(1+N^2) + C(\delta)(1+\eta N^2) + C\frac{N}{R^{\ell-1}} + \eta_2 C]\langle x \rangle^{2(\ell-k)} - N\chi_R \Big\} f^2 \langle x \rangle^{2k}$$

$$+ \eta_1 \int \Big\{ \varphi(x) - \frac{1}{2} + \delta + C\frac{N}{R} \mathbf{1}_{R/2 \leq |x| \leq 2R} + [C(\delta) + C\frac{N}{R^{\ell-1}}]\langle x \rangle^{2(\ell-k)}$$

$$+ C\langle x \rangle^{-2k} + \frac{\eta_2}{\eta_1} C \langle x \rangle^{2(1-k)} - N\chi_R \Big\} |\nabla f|^2 \langle x \rangle^{2k}$$

$$- (1-\delta)\|\nabla f\|_{L_k^2}^2 - \eta\left(\frac{1}{2}-\delta\right)\|u-\kappa_f\|_{L^2}^2 - \eta_1\left(1-\delta-\frac{\eta_2}{\eta_1}\delta\right)\|\nabla^2 f\|_{L_k^2}^2$$

$$- \eta\left(\frac{1}{\varepsilon}-C(\delta)-\frac{\eta_1}{\eta}C(\delta)\right)\|\nabla(u-\kappa_f)\|_{L^2}^2 - \eta_1\left(\frac{1}{\varepsilon}-C(\delta)-\frac{\eta_2}{\eta_1}C(\delta)\right)\|\nabla^2(u-\kappa_f)\|_{L^2}^2$$

$$- \frac{1}{\varepsilon}\|\nabla^3(u-\kappa_{\tilde{f}})\|_{L^2}^2.$$

We can now conclude exactly as in Lemma 4.4 and Lemma 4.6, and we obtain that for any $a > -1/2$,

(5.6)
$$\frac{1}{2}\frac{d}{dt}\|(f,u)\|_{Z_*}^2 \leq a\|(f,u)\|_{Z_*}^2 - K(\|\nabla^2 f\|_{L_k^2}^2 + \|\nabla^3(u-\kappa_f)\|_{L^2}^2),$$

for some constant $K > 0$, from which $\mathcal{B}_\varepsilon$ is a-hypodissipative in Z.

From (5.6) and the interpolation inequalities

$$\|f\|_{H_k^1}^2 \lesssim \|f\|_{H_k^2}\|f\|_{L_k^2}, \quad \|u\|_{H^2}^2 \lesssim \|u\|_{H^3}^{4/3}\|u\|_{L^2}^{2/3}$$

it follows that

$$\frac{1}{2}\frac{d}{dt}\|(f,u)\|_{Z_*}^2 \leq a\|(f,u)\|_{Z_*}^2 - K\|(f,u)\|_{X_*}^{-1}\|(f,u)\|_{Z_*}^3,$$

from which we obtain by standard arguments

$$\|S_{\mathcal{B}_\varepsilon}(t)(f,u)\|_Z \leq C\,t^{-1}\,e^{at}\,\|(f,u)\|_X.$$

$\square$

Proof of Proposition 5.1. From Lemma 4.4, Lemma 4.5, Lemma 5.2 and [18, Lemma 2.17] it follows that there is $n \in \mathbb{N}$ such that

$$\|(\mathcal{A}S_{\mathcal{B}_\varepsilon})^{*n}(t)\|_{\mathscr{B}(X,Z)} \leq C\,e^{at}.$$

Then the proof of the linear stability result in Z is a consequence of last estimate, Lemma 5.2, Theorem 4.7 and the "extension theorem" [24, Theorem 1.1].

$\square$

5.2. **Dissipative norm.** We define the new norm

$$\|\|(f,u)\|\|_Z^2 = \eta\|(f,u)\|_Z^2 + \int_0^\infty \|S_{\Lambda_\varepsilon}(\tau)(f,u)\|_Z^2\,d\tau$$

for some $\eta > 0$. Thanks to Proposition 5.1, the norm $\|\|\cdot\|\|_Z$ is equivalent to $\|\cdot\|_Z$ for any $\eta > 0$. Moreover, considering the equation $\partial_t(f,u) = \Lambda_\varepsilon(f,u)$ we obtain from Proposition 5.1, Lemma 5.2 and arguing as in [18, 14] that, for some $\eta > 0$ small enough, it holds

(5.7)
$$\frac{d}{dt}\|\|(f,u)\|\|_Z^2 \leq -K\|\|(f,u)\|\|_Z^2 - K\{\|\nabla^2 f\|_{L_k^2}^2 + \|\nabla^3 u\|_{L^2}^2\} =: -K\|(f,u)\|_{\tilde{Z}}^2,$$

for some constant $K > 0$.

5.3. **The nonlinear problem : Proof of theorem 1.4.** We focus now on the nonlinear parabolic-parabolic Keller-Segel system (1.15)-(1.16) in self-similar variables and we prove Theorem 1.4. Consider $g \in H_k^1 \cap L_{k,rad}^2$ and $v \in H^2 \cap L_{rad}^2$, then define $f := g - G_\varepsilon$ and $u := v - V_\varepsilon$, so that $(f, u) \in Z$ and satisfy

(5.8)
$$\partial_t f = \Lambda_{\varepsilon,1}(f, u) - \nabla \cdot (f \nabla u)$$
$$\partial_t u = \Lambda_{\varepsilon,2}(f, u),$$

with the initial condition $(f, u)_{|t=0} = (f_0, u_0) := (g_0, v_0) - (G_\varepsilon, V_\varepsilon)$, that satisfies moreover $\|(g_0, v_0) - (G_\varepsilon, V_\varepsilon)\|_Z \le \delta^*$.

We split the proof into three parts.

5.3.1. *A priori estimate.*

Lemma 5.3. *The solution (f_t, u_t) to (5.8) satisfies the following differential inequality*

(5.9)
$$\frac{d}{dt} \|(f, u)\|_Z^2 \le (-K + C \|(f, u)\|_Z) \|(f, u)\|_{\widetilde{Z}}^2,$$

where $\| \cdot \|_{\widetilde{Z}}$ is dedined in (5.7).

Proof. Since

$$\|(f, u)\|_Z^2 = \|f\|_{H_k^1}^2 + \|u\|_{H^2}^2,$$

we obtain from (5.8) that, denoting $Q(f, u) = (-\nabla \cdot (f \nabla u), 0)$,

$$\frac{1}{2} \frac{d}{dt} \|(f, u)\|_Z^2 = \eta \langle (f, u), \Lambda_\varepsilon(f, u) \rangle + \int_0^\infty \langle S(\tau)(f, u), S(\tau) \Lambda_\varepsilon(f, u) \rangle \, d\tau$$

$$+ \eta \langle (f, u), Q(f, u) \rangle + \int_0^\infty \langle S(\tau)(f, u), S(\tau) Q(f, u) \rangle \, d\tau =: I_1 + I_2.$$

For the first (linear) term we have already obtained in (5.7) that

$$I_1 := \eta \langle (f, u), \Lambda_\varepsilon(f, u) \rangle + \int_0^\infty \langle S(\tau)(f, u), S(\tau) \Lambda_\varepsilon(f, u) \rangle \, d\tau$$

$$\le -K \left\{ \|(f, u)\|_Z^2 + \|\nabla^2 f\|_{L_k^2}^2 + \|\nabla^3 u\|_{L^2}^2 \right\} =: -K \|(f, u)\|_{\widetilde{Z}}^2.$$

For the second (nonlinear) term we use the linear stability from Proposition 5.1 since $(f, u) \in Z$ and $Q(f, u) \in Z$ to obtain

$$I_2 := \eta \langle (f, u), Q(f, u) \rangle + \int_0^\infty \langle S(\tau)(f, u), S(\tau) Q(f, u) \rangle \, d\tau$$

$$\le \eta \|(f, u)\|_Z \|Q(f, u)\|_Z + \int_0^\infty \|S(\tau)(f, u)\|_Z \|S(\tau) Q(f, u)\|_Z \, d\tau$$

$$\le \eta \|(f, u)\|_Z \|Q(f, u)\|_Z + C \int_0^\infty e^{at} \|(f, u)\|_Z \, e^{at} \|Q(f, u)\|_Z \, d\tau$$

$$\le C \|(f, u)\|_Z \|Q(f, u)\|_Z.$$

Now we have to compute

$$\|Q(f, u)\|_Z^2 = \|(-\nabla \cdot (f \nabla u), 0)\|_Z^2 = \|\nabla \cdot (f \nabla u)\|_{L_k^2}^2 + \|\nabla \cdot (f \nabla u)\|_{\dot{H}_k^1}^2.$$

We split $\nabla \cdot (f \nabla u) = f \Delta u + \nabla f \cdot \nabla u$ and compute

$$\|f \Delta u\|_{L_k^2}^2 = \int f^2 |\Delta u|^2 \langle x \rangle^{2k} \lesssim \|\nabla^2 u\|_{L^2}^2 \|\langle x \rangle^k f\|_{L^\infty}^2$$

$$\lesssim \|\nabla^2 u\|_{L^2}^2 \|f\|_{H_k^2}^2 \lesssim \|(f, u)\|_Z^2 \|(f, u)\|_{\widetilde{Z}}^2,$$

where we have used the embedding $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$. Moreover we get

$$\|\nabla f \cdot \nabla u\|_{L_k^2}^2 = \int |\nabla f|^2 |\nabla u|^2 \langle x \rangle^{2k} \lesssim \|\nabla u\|_{L^\infty}^2 \|\nabla f\|_{L_k^2}^2$$

$$\lesssim \|u\|_{H^3}^2 \|\nabla f\|_{L_k^2}^2 \lesssim \|(f,u)\|_{\widetilde{Z}}^2 \|(f,u)\|_Z^2.$$

Furthermore we obtain

$$\|\nabla(f\Delta u)\|_{L_k^2}^2 \lesssim \int f^2 |\nabla^3 u|^2 \langle x \rangle^{2k} + \int |\nabla f|^2 |\nabla^2 u|^2 \langle x \rangle^{2k}$$

$$\lesssim \|\nabla^3 u\|_{L^2}^2 \|\langle x \rangle^k f\|_{L^\infty}^2 + \|\langle x \rangle^k \nabla f\|_{L^4}^2 \|\nabla^2 u\|_{L^4}^2$$

$$\lesssim \|\nabla^3 u\|_{L^2}^2 \|f\|_{H_k^2}^2 + \|f\|_{L_k^2} \|f\|_{H_k^2} \|\nabla u\|_{L^2} \|\nabla^3 u\|_{L^2}^2$$

$$\lesssim \|(f,u)\|_{\widetilde{Z}}^4,$$

where we have used Glagliardo-Niremberg-Sobolev inequality $\|h\|_{L^4(\mathbb{R}^2)}^2 \lesssim \|h\|_{L^2(\mathbb{R}^2)} \|\nabla h\|_{L^2(\mathbb{R}^2)}$. For the last term we get

$$\|\nabla(\nabla f \cdot \nabla u)\|_{L_k^2}^2 \lesssim \int |\nabla f|^2 |\nabla^2 u|^2 \langle x \rangle^{2k} + \int |\nabla^2 f|^2 |\nabla u|^2 \langle x \rangle^{2k}$$

$$\lesssim \|\langle x \rangle^k \nabla f\|_{L^4}^2 \|\nabla^2 u\|_{L^4}^2 + \|\nabla u\|_{L^\infty}^2 \|\nabla^2 f\|_{L_k^2}^2$$

$$\lesssim \|f\|_{L_k^2} \|f\|_{H_k^2} \|\nabla u\|_{L^2} \|\nabla^3 u\|_{L^2}^2 + \|u\|_{H^3}^2 \|\nabla^2 f\|_{H_k^2}^2$$

$$\lesssim \|(f,u)\|_{\widetilde{Z}}^4.$$

Finally, gathering the above estimates, the solution (f,u) of (5.8) verifies

$$\frac{d}{dt}\|(f,u)\|_Z^2 \leq -K\|(f,u)\|_{\widetilde{Z}}^2 + C\|(f,u)\|_Z \|(f,u)\|_{\widetilde{Z}}^2 \leq (-K + C\|(f,u)\|_Z) \|(f,u)\|_{\widetilde{Z}}^2,$$

which concludes the proof.

$\square$

5.3.2. *Existence.* In this section we prove the existence of solutions to (5.8). They are solutions to the Keller-Segel equation (1.1) which satsify some strong and uniform in time estimates.

Proposition 5.4. *There is $\delta > 0$ such that if $\|(f_0, u_0)\|_Z \leq \delta$ then there exists a solution $(f,u) \in C([0,T]; Z)$ to (5.8) that verifies, for any $t \in [0,T]$,*

$$(5.10) \qquad \sup_{t\in[0,T]} \|(f,u)(t)\|_Z^2 + K \int_0^t \|(f,u)(\tau)\|_{\widetilde{Z}}^2 \, d\tau \leq 2\delta^2.$$

Proof. We split the proof into several steps following the argument presented in [18, Theorem 5.3].

Step 1. Consider the iterative scheme

$$\partial_t(f^n, u^n) = \Lambda_\varepsilon(f^n, u^n) + Q(f^{n-1}, u^n), \quad \forall n \geq 1$$

and

$$\partial_t(f^0, u^0) = \Lambda_\varepsilon(f^0, u^0),$$

with initial condition $(f^n, u^n)_{|t=0} = (f_0, u_0)$ for all $n \in \mathbb{N}$ and $\|(f_0, u_0)\|_Z \leq \delta$ for some $\delta > 0$ small enough to be chosen later. For any $n \geq 0$, (f^n, u^n) is well-defined in $C([0,T]; Z)$ thanks to Proposition 5.1. For any $n \geq 0$ we claim that the following estimate holds true

$$(5.11) \qquad \forall t \geq 0, \quad A^n(t) := \|(f^n, u^n)(t)\|_Z^2 + K \int_0^t \|(f^n, u^n)(\tau)\|_{\widetilde{Z}}^2 \, d\tau \leq 2\delta^2.$$

It is true for $n = 0$ from (5.7). Now we assume that (5.11) holds for some $n-1$ and we shall prove that it implies (5.11) for n. From the proof of Lemma 5.3 we obtain

$$\frac{d}{dt}\|(f^n, u^n)\|_Z^2 \leq -K\|(f^n, u^n)\|_{\widetilde{Z}}^2 + C\|(f^n, u^n)\|_Z \|(f^n, u^n)\|_{\widetilde{Z}} \|(f^{n-1}, u^{n-1})\|_{\widetilde{Z}}$$

then integrating from 0 to t and using Holdër's inequality it follows that

$$A^n(t) \leq |\!|\!|(f_0, u_0)|\!|\!|_Z^2 + C \sup_{0 \leq \tau \leq t} |\!|\!|(f^n, u^n)(\tau)|\!|\!|_Z \left(\int_0^t \|(f^n, u^n)(\tau)\|_{\widetilde{Z}}^2 \, d\tau \right)^{1/2} \left(\int_0^t \|(f^{n-1}, u^{n-1})(\tau)\|_{\widetilde{Z}}^2 \, d\tau \right)^{1/2}$$

$$\leq |\!|\!|(f_0, u_0)|\!|\!|_Z^2 + CA^n(t) \, [A^{n-1}(t)]^{1/2}$$

$$\leq |\!|\!|(f_0, u_0)|\!|\!|_Z^2 + C\delta A^n(t),$$

from which we conclude if $\delta > 0$ is small enough such that $C\delta \leq 1/2$.

Step 2. Denote $(F^n, U^n) := (f^{n+1} - f^n, u^{n+1} - u^n)$ that verifies

$$\partial_t(F^n, U^n) = \Lambda_\varepsilon(F^n, U^n) + Q(F^{n-1}, u^{n+1}) + Q(f^{n-1}, U^n), \quad \forall n \geq 1$$

and

$$\partial_t(F^0, U^0) = \Lambda_\varepsilon(F^0, U^0) + Q(f^0, u^1),$$

with initial condition $(F_0, U_0) = (0, 0)$ for all $n \geq 0$. For any $n \geq 0$ we claim that

$$(5.12) \qquad \forall t \geq 0, \quad B^n(t) := |\!|\!|(F^n, U^n)(t)|\!|\!|_Z^2 + K \int_0^t \|(F^n, U^n)(\tau)\|_{\widetilde{Z}}^2 \, d\tau \leq (C\delta)^{2n}.$$

When $n = 0$, arguing as above we obtain

$$\frac{d}{dt} |\!|\!|(F^0, U^0)(t)|\!|\!|_Z^2 + K \|(F^0, U^0)\|_{\widetilde{Z}}^2 \leq C |\!|\!|(F^0, U^0)|\!|\!|_Z \, \|(f^1, u^1)\|_{\widetilde{Z}} \, \|(f^0, u^0)\|_{\widetilde{Z}}$$

which togheter with the bounds on (f^1, u^1) and (f^0, u^0) obtained before yield

$$B^0(t) \leq C\delta^2 \leq \delta, \quad \forall t \geq 0,$$

for $\delta > 0$ small enough. We now assume (5.12) for some $n - 1$ and prove it for n. Arguing in a similar way as before, we estimate

$$B^n(t) \leq |\!|\!|(F_0, U_0)(t)|\!|\!|_Z^2 + C \int_0^t \|(F^{n-1}, U^{n-1})(\tau)\|_{\widetilde{Z}} \, \|(f^{n+1}, u^{n+1})(\tau)\|_{\widetilde{Z}??} \, |\!|\!|(F^n, U^n)(\tau)|\!|\!|_Z \, d\tau$$

$$+ C \int_0^t \|(f^{n-1}, u^{n-1})(\tau)\|_{\widetilde{Z}} \, \|(F^n, U^n)(\tau)\|_{\widetilde{Z}} \, |\!|\!|(F^n, U^n)(\tau)|\!|\!|_Z \, d\tau$$

$$\leq C\delta \, [B^n(t)]^{1/2} \, [B^{n-1}(t)]^{1/2} + C\delta \, B^n(t),$$

and using the induction hypothesis

$$B^n(t) \leq C\delta^n \, [B^n(t)]^{1/2} + C\delta \, B^n(t),$$

from which we conclude to (5.12) if $\delta > 0$ is small enough.

Therefore the sequence $\{(f^n, u^n)\}_n$ is a Cauchy sequence in $C([0, T]; Z)$ and in $L^2(0, T; \widetilde{Z})$ for any $T > 0$. The limit $(f, u) := \lim_{n \to \infty}(f^n, u^n)$ is a solution of (5.8) in a weak sense. Passing to the limit $n \to \infty$ in (5.11) we get the stability estimate for the solution (f, u) constructed above

$$\sup_{t \geq 0} |\!|\!|(f, u)(t)|\!|\!|_Z^2 + K \int_0^t \|(f, u)(\tau)\|_{\widetilde{Z}}^2 \, d\tau \leq 2\delta^2.$$

$\square$

5.3.3. *Sharp exponential convergence to the equilibrium.* We now complete the proof of Theorem 1.4 (following [18]). Applying Lemma 5.3 to the solution (f, u) constructed above and using the estimate (5.10) we obtain

$$\frac{d}{dt} |\!|\!|(f, u)|\!|\!|_Z^2 \leq (-K + C|\!|\!|(f, u)|\!|\!|_Z) \, \|(f, u)\|_{\widetilde{Z}}^2 \leq (-K + C'\delta) \, \|(f, u)\|_{\widetilde{Z}}^2 \leq (-K + C'\delta) \, C'' |\!|\!|(f, u)|\!|\!|_Z^2,$$

and if $\delta > 0$ is small enough so that $-K + C'\delta \leq -K/2$, this implies an exponential decay to the equilibrium

$$|\!|\!|(f, u)(t)|\!|\!|_Z \leq e^{-\frac{KC''}{2}t} \, |\!|\!|(f_0, u_0)|\!|\!|_Z.$$

Finally, we can recover the optimal decay rate $O(e^{at})$ of the linearized semigroup in Proposition 5.1 by performing a bootstrap argument as in [18, Proof of Theorem 5.3], which concludes the proof.

APPENDIX A. ESTIMATES ON THE SOLUTIONS TO THE POISSON EQUATION

In this section, we give some technical (and we believe standard) estimates on the solutions to the Poisson equation in $\mathbb{R}^2$ that are useful in the paper. We recall the notations

$$\kappa(z) = -\frac{1}{2\pi} \log |z|, \quad \mathcal{K}(z) := \nabla \kappa(z) = -\frac{1}{2\pi} \frac{z}{|z|^2}$$

and

$$\kappa_f := \kappa * f, \quad \mathcal{K}_f := \mathcal{K} * f,$$

so that there holds

$$\kappa_f \in C^2(\mathbb{R}^2), \quad |\kappa_f| \le C \left(1 + \log\langle x \rangle\right), \quad -\Delta \kappa_f = f.$$

Lemma A.1. *For any $k > 3$, $j \ge 1$, $k > j + 2$, there exists $C_{k,j}$ such that*

$$(A.1) \qquad \qquad \|\kappa * f\|_{L_j^2} \le C\|f\|_{L_k^2} \quad \forall f \in L_{k,j}^2.$$

Proof of Lemma A.1. Consider $f \in L_{k,j}^2$, $k > 3$, so that $\hat{f} \in C^2$, and $\partial_\xi^\alpha \hat{f}(0) = 0$ for any $|\alpha| \le j$ thanks to the moments condition, hence

$$\hat{f}(\xi) = \sum_{|\alpha|=j+1} \frac{1}{\alpha!} \partial_\xi^\alpha \hat{f}(\eta) \, \xi^\alpha,$$

for some η between 0 and ξ. In Fourier variables the Laplace equation writes $|\xi|^2 \hat{\kappa}_f(\xi) = \hat{f}(\xi)$, and then

$$\int |\kappa_f|^2 = \int |\hat{\kappa}_f|^2 = \int \frac{|\hat{f}|^2}{|\xi|^4}.$$

All together, we have

$$\begin{aligned}
\int \frac{|\hat{f}|^2}{|\xi|^4} &\le \int_{B_1^c} \frac{|\hat{f}|^2}{|\xi|^4} + \int_{B_1} \sup_{\eta \in B_1} |D^{j+1}\hat{f}(\eta)|^2 \, d\xi \\
&\le \|f\|_{L^2}^2 + C\|f\|_{L_{j+1}^1}^2 \\
&\le \|f\|_{L^2}^2 + C\|f\|_{L_k^2}^2 \le C\|f\|_{L_k^2}^2,
\end{aligned}$$

where we used $\|g\|_{L^1} \lesssim \|g\|_{L_r^2}$ for $r > 1$ with $g = \langle x \rangle^{j+1} f$ in the last line, which gives $k > j+2$. $\square$

Lemma A.2. *For any $k > 2$, $j \ge 0$, $k > j + 2$, there exists $C_{k,j}$ such that*

$$(A.2) \qquad \qquad \|\mathcal{K} * f\|_{L_j^2} \le C\|f\|_{L_k^2} \quad \forall f \in L_{k,j}^2.$$

Proof of Lemma A.2. The proof is similar to the Lemma above, writing in Fourier variables $|\xi|^2 \hat{\kappa}_f(\xi) = \hat{f}(\xi)$, observing that

$$\int |\nabla \kappa_f|^2 = \int |\xi|^2 |\hat{\kappa}_f|^2 = \int \frac{|\hat{f}|^2}{|\xi|^2}$$

and using the moments conditions.

$\square$

Appendix B. Estimates on the $c^{-1}d$ operator

With the notations of section 4.3, we consider the equation

$$(\text{B.1}) \qquad L_\varepsilon u := c^{-1} du = \Delta u + \frac{\varepsilon}{2} x \cdot \nabla u - \varepsilon z u = f$$

with $z \in \Delta_{-1/2}$.

Lemma B.1. *For any $f \in L^2_{k,0}$, $k > 2$, $z \in \Delta_{-1/2}$, there exists a solution $u \in H^2$ to the equation $L_\varepsilon u_\varepsilon = f$ and there exists a constant C (which does not depend on $\varepsilon > 0$ and $z \in \Delta_{-1/2}\backslash\{0\}$) such that*

$$\|\nabla u\|_{L^2} + \|\Delta u\|_{L^2} \le C \|f\|_{L^2_k}.$$

Proof. Multiplying equation (B.1) by $\bar{u}$ and $\Delta\bar{u}$ and its conjugate by u and Δu, we find

$$\int |\nabla u|^2 + \varepsilon\Big(\frac{1}{2} + \Re ez\Big)\int |u|^2 = -\frac{1}{2}\int (f\bar{u} + \bar{f}u)$$

and

$$(\text{B.2}) \qquad \int |\Delta u|^2 + (\varepsilon \Re ez)\int |\nabla u|^2 = \frac{1}{2}\int f\Delta\bar{u} + \frac{1}{2}\int \bar{f}\Delta u.$$

Writing

$$u(x) = u(y) + \int_0^1 \nabla u(z_s)\,(x-y)\,ds, \quad z_s := y + s\,(y-x),$$

we have

$$u(x) = \langle u\rangle_1 + \int_{B(0,1)}\int_0^1 \nabla u(z_s)\,(x-y)\,ds\,dy, \quad \langle u\rangle_1 := \int_{B(0,1)} u(y)\,dy,$$

and then from the first equation and the moment condition

$$\begin{aligned}
\int |\nabla u|^2 \;\le\;& -\int_{\mathbb{R}^2}\int_{B(0,1)}\int_0^1 f(x)\nabla u(z_s)\cdot(x-y)\,ds\,dy\,dx \\
\le\;& \frac{1}{2\alpha}\int_{\mathbb{R}^2}\int_{B(0,1)}\int_0^1 |f(x)|^2\,\langle x\rangle^{2\ell}\,|x-y|^2\,ds\,dy\,dx \\
& +\frac{\alpha}{2}\int_{\mathbb{R}^2}\int_0^{1/2}\Big\{\int_{B(0,1)}|\nabla u(sx+(1-s)y)|^2\,dy\Big\}\,ds\,\frac{dx}{\langle x\rangle^{2\ell}} \\
& +\frac{\alpha}{2}\int_{B(0,1)}\int_{1/2}^1\Big\{\int_{\mathbb{R}^2}|\nabla u(sx+(1-s)y)|^2\,dx\Big\}\,ds\,dy \\
\le\;& \frac{1}{\alpha}\Big\{\int_{B(0,1)} dy\Big\}\int_{\mathbb{R}^2}|f(x)|^2\,\langle x\rangle^{2(\ell+1)}\,dx \\
& +\frac{\alpha}{2}\int_{\mathbb{R}^2}\frac{dx}{\langle x\rangle^{2\ell}}\int_0^{1/2}\frac{ds}{(1-s)^2}\int_{\mathbb{R}^2}|\nabla u(z)|^2\,dz \\
& +\frac{\alpha}{2}\Big\{\int_{B(0,1)} dy\Big\}\int_{1/2}^1\frac{ds}{s^2}\int_{\mathbb{R}^2}|\nabla u(z)|^2\,dz,
\end{aligned}$$

and we deduce that

$$\int |\nabla u|^2 \le C\int_{\mathbb{R}^2}|f(x)|^2\,\langle x\rangle^{2k}\,dx$$

by choosing $\ell = k-1$ and $\alpha > 0$ small enough. We conclude by gathering that last estimate together with (B.1). $\qquad\square$

Lemma B.2. *There exists a constant C such that for any $\varepsilon \in (0,1)$, any $z \in \Delta_{-1/2}$ and any radially symmetric function f the equation*

$$L_\varepsilon u = f, \quad u \text{ radially symmetric}, \quad u(0) = u'(0) = 0,$$

has a unique solution which furthermore satisfies

(B.3)
$$\|u\, e^{-(1+\varepsilon|z|)r}\|_{L^\infty} + \|u'\, e^{-(1+\varepsilon|z|)r}\|_{L^\infty} \le C \, \|f\, e^{-r/2}\|_{L^\infty}.$$

Proof. We may write the equation as

(B.4)
$$u'' + \Big(\frac{1}{r} + \varepsilon\, r\Big)\, u' + \varepsilon z u = f, \quad \forall\, r > 0,$$

with the additional boundary conditions $u(0) = u'(0) = 0$. Defining $U := |u|^2 + |u'|^2$, we have

$$
\begin{aligned}
U' &= u\bar{u}' + u'\bar{u} + u'\bar{u}'' + u''\bar{u}' \\
&\le 2(1+\varepsilon|z|)\,|u|\,|u'| - \Big(\frac{1}{r} + \varepsilon\, r\Big)\,|u'|^2 + 2\,|u'|\,|f| \\
&\le (2+\varepsilon|z|)\,U + |f|^2,
\end{aligned}
$$

from which we immediately get (B.3) thanks to Gronwall lemma. $\qquad\square$

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