EXPONENTIAL STABILITY OF SLOWLY DECAYING SOLUTIONS TO THE KINETIC-FOKKER-PLANCK EQUATION

S. MISCHLER, C. MOUHOT

Abstract. The aim of the present paper is twofold:

(1) We carry on with developing an abstract method for deriving growth estimates on the semigroup associated to non-symmetric operators in Banach spaces as introduced in [8]. We extend the method so as to consider the *shrinkage* of the functional space. Roughly speaking, we consider a class of operators writing as a dissipative part plus a mild perturbation, and we prove that if the associated semigroup satisfies a growth estimate in some reference space then it satisfies the same growth estimate in another – smaller or larger – Banach space under the condition that a certain iterate of the regularizing part of the operator combined with the dissipative part of the semigroup maps the larger space to the smaller space in a bounded way. The cornerstone of our approach is a factorization argument, reminiscent of the Dyson series.

(2) We apply this method to the kinetic Fokker-Planck equation when the spatial domain is either the torus with periodic boundary conditions, or the whole space with a confinement potential. We then obtain spectral gap for the associated semigroup for various metrics, including Lebesgue norms, negative Sobolev norms, and the Monge-Kantorovich-Wasserstein distance W_1.

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Contents

1. Introduction

1.1. **The question at hand.** This paper deals with the study of decay properties of linear semigroups and their link with spectral properties as well as some applications to the Fokker-Planck equations with various types of confinement. It continues the program of research in [15, 8] where quantitative methods for enlarging the functional space of spectral gap estimates were developped with application to kinetic equations; specifically in [8] spectral gap estimate were obtained in Lebesgue spaces for Boltzmann and Fokker-Planck equations in the spatially homogeneous and spatially periodic frameworks.

Our approach is based on the following abstract question: consider two Banach spaces $E \subset \mathcal{E}$ with E is dense in $\mathcal{E}$, and two unbounded closed linear operators L and $\mathcal{L}$ respectively on E and $\mathcal{E}$ with spectrum $\Sigma(L), \Sigma(\mathcal{L}) \subset \mathbb{C}$ which generate some semigroups $(\mathscr{S}_L(t))_{t\geq 0}$ on E and $(\mathscr{S}_{\mathcal{L}}(t))_{t\geq 0}$ on $\mathcal{E}$ respectively and so that $\mathcal{L}_{|E} = L$; can one deduce quantitative informations on $\Sigma(\mathcal{L})$ and $\mathscr{S}_{\mathcal{L}}(t)$ in terms of informations on $\Sigma(L)$ and $\mathscr{S}_L(t)$ (*enlargement* issue), or can one deduce quantitative informations on $\Sigma(L)$ and $\mathscr{S}_L(t)$ in terms of informations on $\Sigma(\mathcal{L})$ and $\mathscr{S}_{\mathcal{L}}(t)$ (*shrinkage* issue)?

We prove under some assumptions (i) that the spectral gap property of L in E (resp. of $\mathcal{L}$ in $\mathcal{E}$) can be shown to hold for $\mathcal{L}$ in the space $\mathcal{E}$ (resp. for L in E) and (ii) explicit estimates on the rate of decay of the semigroup $\mathscr{S}_{\mathcal{L}}(t)$ (resp. the semigroup $\mathscr{S}_L(t)$) can be computed from the ones on $\mathscr{S}_L(t)$ (resp. $\mathscr{S}_{\mathcal{L}}(t)$). This holds for a class of operators $\mathcal{L}$ which split as $\mathcal{L} = \mathcal{A}+\mathcal{B}$, where $\mathcal{B}$'s spectrum is well localized and some appropriate combination of $\mathcal{A}$ and the semigroup $\mathscr{S}_{\mathcal{B}}(t)$ of $\mathcal{B}$ has some regularising properties. This last condition is reminiscent of Hörmander's *commutator conditions* [13].

The Fokker-Planck equations we consider are then shown to belong to this general class of operators by extending the hypocoercivity results –usually obtained in L^2 spaces with inverse Gaussian type tail– to a larger class of Lebesgue and Sobolev spaces.

1.2. **The abstract result.** We denote $\mathscr{C}(E)$ the set of closed operators on a Banach space E, $\mathscr{B}(E)$ the set of bounded operators on E, and $\mathscr{B}(E, \mathcal{E})$ the set of bounded operators between two Banach spaces. We say that $P \in \mathscr{C}(E)$ is *hypodissipative* if it is *dissipative* for some norm equivalent to the canonical norm of E and we say that P is dissipative for the norm $\|\cdot\|$ on E if

$$\forall f \in \mathrm{Domain}(P),\ f^* \in E^*\,;\ \langle f, f^*\rangle = \|f\|_E^2, \quad \Re e\,\langle Pf, f^*\rangle \le 0$$

where the $\langle\cdot,\cdot\rangle$ denotes the duality bracket between E and its dual E^*. Finally we denote $\Delta_a := \{z \in \mathbb{C};\, \Re e\, z > a\}$.

Theorem 1.1 (Change of the functional space of the semigroup decay)**.** *Given E, $\mathcal{E}$, L, $\mathcal{L}$ defined as above, assume that there are $A, B \in \mathscr{C}(E)$, $\mathcal{A}, \mathcal{B} \in \mathscr{C}(\mathcal{E})$ so that*

$$\mathcal{L} = \mathcal{A} + \mathcal{B},\ \ L = A + B,\ \ A = \mathcal{A}_{|E},\ \ B = \mathcal{B}_{|E},$$

and a real number $a \in \mathbb{R}$ such that

- **(i)** *$(B - a)$ is hypodissipative on E, $(\mathcal{B} - a)$ is hypodissipative on $\mathcal{E}$;*
- **(ii)** *$A \in \mathscr{B}(E)$, $\mathcal{A} \in \mathscr{B}(\mathcal{E})$;*
- **(iii)** *there is $n \ge 1$ and $C_a > 0$ such that*

$$\left\|(\mathcal{A}\mathscr{S}_{\mathcal{B}})^{(*n)}(t)\right\|_{\mathscr{B}(\mathcal{E},E)} + \left\|(\mathscr{S}_{\mathcal{B}}\mathcal{A})^{(*n)}(t)\right\|_{\mathscr{B}(\mathcal{E},E)} \le C_a\, e^{at}.$$

Then the following two properties are equivalent:

- **(1)** *There are distinct $\xi_1, \dots, \xi_k \in \Delta_a$ and finite rank projectors $\Pi_{j,L} \in \mathscr{B}(E)$, $1 \le j \le k$, which commute with L and satisfy $\Sigma(L_{|\Pi_{j,L}}) = \{\xi_j\}$, so that the semigroup $\mathscr{S}_L(t)$ satisfies for any $a' > a$*

$$\forall\, t \ge 0, \quad \left\| \mathscr{S}_L(t) - \sum_{j=1}^k \mathscr{S}_L(t)\,\Pi_{j,L} \right\|_{\mathscr{B}(E)} \le C_{L,a'}\, e^{a'\,t} \tag{1.1}$$

  with some constant $C_{L,a'} > 0$.
- **(2)** *There are distinct $\xi_1, \dots, \xi_k \in \Delta_a$ and finite rank projectors $\Pi_{j,\mathcal{L}} \in \mathscr{B}(\mathcal{E})$, $1 \le j \le k$, which commute with $\mathcal{L}$ and satisfy $\Sigma(\mathcal{L}_{|\Pi_{j,\mathcal{L}}}) = \{\xi_j\}$, so that the semigroup $\mathscr{S}_{\mathcal{L}}(t)$ satisfies for any $a' > a$*

$$\forall\, t \ge 0, \quad \left\| \mathscr{S}_{\mathcal{L}}(t) - \sum_{j=1}^k \mathscr{S}_{\mathcal{L}}(t)\,\Pi_{j,\mathcal{L}} \right\|_{\mathscr{B}(\mathcal{E})} \le C_{\mathcal{L},a'}\, e^{a'\,t} \tag{1.2}$$

  with some constant $C_{\mathcal{L},a'} > 0$.

Remarks 1.2. **(a)** The constants in this statement can be estimated explicitly from the proof.

- **(b)** The same result holds in the case $\{\xi_1, \dots, \xi_k\} = \emptyset$, that we denote as a convention as the case $k = 0$.
- **(c)** The condition "$E \subset \mathcal{E}$" can be replaced by "$E \cap \mathcal{E}$ is dense in E and $\mathcal{E}$ with continuous embedding".
- **(d)** Note that one of the two terms of the LHS in condition **(iii)** can be omitted, because it can be deduced (at order $n+1$) from the other one and the assumptions **(i)** and **(ii)**.

1.3. **The main PDE results.** Let us briefly present the evolution PDEs of Fokker-Planck types for which we will be able to make use of the above abstract Theorem to establish exponential asymptotic stability of the associated familly of steady states.

(a) "Flat" confinement. Consider the kinetic Fokker-Planck equation

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (\nabla_v f + \nabla_v \Phi \, f) \,, \tag{1.3}$$

on the density $f = f(t,x,v)$, $t \geq 0$, $x \in \mathbb{T}^d$ the flat d-dimensional torus, $v \in \mathbb{R}^d$, for a friction potential $\Phi = \Phi(v)$ satisfying $\Phi \approx |v|^\gamma$, $\gamma \geq 1$, for large velocities.

Remark 1.3. Observe that this model contains as a subcase the (spatially homogeneous) Fokker-Planck equation

$$\partial_t f = \Delta_v f + \mathrm{div}_v(\nabla_v \Phi \, f), \tag{1.4}$$

when the probability density $f = f(t,v)$ is independent of the space variable, $t \geq 0$, $v \in \mathbb{R}^d$, for a friction potential $\Phi = \Phi(v)$ satisfying the same assumptions as above.

(b) Confinement by a potential. Consider second the kinetic Fokker-Planck equation in the whole space with a space confinement potential

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \Psi \cdot \nabla_v f = \nabla_v \cdot (\nabla_v f + v \, f) \,, \tag{1.5}$$

on the density $f = f(t,x,v)$, $t \geq 0$, $x \in \mathbb{R}^d$, $v \in \mathbb{R}^d$, for a confinement potential $\Psi = \Psi(x)$ which behaves like $|x|^\beta$, $\beta \geq 1$, for large values of the vector position.

For these models, we prove semigroup exponential decay estimates in weighted Sobolev spaces with weight function increasing like polynomial function or a stretch exponential function, so much slower than the usual inverse Gaussian used in previous works.

Theorem 1.4. *Consider $\mathcal{L}$ the Fokker-Planck operator as defined above in (a) or (b), and μ the unique positive associated steady state with mass 1. Consider the weighted Sobolev space $\mathcal{E} := W^{\sigma,p}(m)$ with $\sigma \in \{-1,0,1\}$ and $p \in [1,\infty]$, where the precise conditions on the weight m are given in Theorems 3.1 and 4.1.*

Then there exist $a < 0$ and $C_a > 0$ so that

$$\forall\, f \in X, \quad \|\mathscr{S}_\mathcal{L}(t) f - \mathscr{S}_\mathcal{L}(t) g\|_\mathcal{E} \leq C_a \, e^{at} \, \|f - g\|_\mathcal{E} \tag{1.6}$$

between two solutions with same mass; this implies the exponential convergence towards the projection on the first eigenspace $\mathbb{R}\mu$ (higher eigenvalue 0).

In the case (a) (periodic confinement), we also establish a similar decay estimate in Monge-Kantorovich-Wasserstein distance:

$$W_1 \left(\mathscr{S}_\mathcal{L}(t) f_0, \mathscr{S}_\mathcal{L}(t) g_0 \right) \leq C_a \, e^{a\,t} \, W_1(f_0, g_0)$$

between two solutions with same mass.

This theorem is proved by combining:

- the spectral gap property of the Fokker-Planck semigroup which is classically known in the space of self-adjointness $L^2(\mu^{-1/2})$ in a spatially homogeneous setting (Poincaré inequality) and has been recently proved in a series of works about *"hypocoercivity"* in the spaces $L^2(\mu^{-1/2})$ or $H^1(\mu^{-1/2})$ for the kinetic Fokker-Planck semigroup with periodic or potential confinements [12, 20, 6];
- an appropriate decomposition of the operator with:

– some additional dissipativity estimates adapted to each cases for the target functional spaces, for the "dissipative part" of the decomposition (this is the main difficulty in the case of confinement by a potential and we introduce specifically weight multipliers inspired from commutor-like estimates at the level of weights);
– some additional regularisation estimates adapted to each cases in the usual $L^2(\mu^{-1/2})$ space (inspired from ultracontractivity estimates in the spirit of Nash's regularity estimate [18] in the spatially homogeneous case and Hérau-Villani's quantitative global hypoellipticity estimate [11]–[20, section A.21.3];
– an application of Theorem 1.1 (whose assumptions are established by the previous estimates) which establishes the decay estimates of the semigroup in the target functional space;
– finally the W_1 estimate is obtained by some additional technical efforts in estimating the decay in weighted $W^{-1,1}$ type spaces.

Remark 1.5. Some decay estimates for kinetic Fokker-Planck semigroups with flat confinement had been already established in [8]. In this setting this new paper improves on these previous paper as follows: we use new integral identity in order to deal with any integrability exponent $p \in [1, \infty]$ and we introduce an appropriate duality argument in order to deal with the regularity exponent $\sigma = -1$.

1.4. **Plan of the paper.** The outline of the paper is as follows. We prove the main abstract theorem in Section 2. We prove the decay estimates on kinetic Fokker-Planck semigroups with periodic (or spatially homogeneous) confinements in Section 3. Finally we prove the decay estimates on kinetic Fokker-Planck semigroups with confinement by a potential in Section 4.

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2. Factorisation of semigroups in Banach spaces and applications

The section is devoted to the proof of Theorem 1.1. After having recalled some notation, we present the proof of Theorem 1.1 that we split into two steps, namely the analysis of the spectral problem and the semigroup decay.

2.1. **Notations and definitions.** We denote by $\mathscr{G}(E) \subset \mathscr{C}(E)$ the space of semigroup generators and for $\Lambda \in \mathscr{G}(E)$ we denote by $\mathscr{S}_\Lambda(t) = e^{\Lambda t}$, $t \geq 0$, its semigroup, by $\mathscr{D}(\Lambda)$ its domain, by $\mathscr{N}(\Lambda)$ its null space, by

$$\mathscr{M}(\Lambda) = \cup_{\alpha \geq 1} \mathscr{N}(\Lambda^\alpha)$$

its algebraic null space, and by $\mathscr{R}(\Lambda)$ its range. We also denote by $\Sigma(\Lambda)$ its spectrum, so that for any $\xi \in \mathbb{C} \backslash \Sigma(\Lambda)$ the operator $\Lambda - \xi$ is invertible and the resolvent operator

$$R_\Lambda(\xi) := (\Lambda - \xi)^{-1}$$

is well-defined, belongs to $\mathscr{B}(E)$ and has range equal to $\mathscr{D}(\Lambda)$.

We recall that $\xi \in \Sigma(\Lambda)$ is said to be an *eigenvalue* if $\mathscr{N}(\Lambda-\xi) \neq \{0\}$. Moreover an eigenvalue $\xi \in \Sigma(\Lambda)$ is said to be *isolated* if

$$\Sigma(\Lambda) \cap \{z \in \mathbb{C},\ |z-\xi| < r\} = \{\xi\} \quad \text{for some } r > 0.$$

In the case when ξ is an isolated eigenvalue we may define $\Pi_{\Lambda,\xi} \in \mathscr{B}(E)$ the spectral projector by

$$\Pi_{\Lambda,\xi} := \frac{i}{2\pi} \int_{|z-\xi|=r'} (\Lambda - z)^{-1}\, dz \tag{2.1}$$

with $0 < r' < r$. Note that this definition is independent of the value of r' by Cauchy's theorem as the application

$$\mathbb{C} \setminus \Sigma(\Lambda) \to \mathscr{B}(E), \quad z \mapsto R_\Lambda(z)$$

is holomorphic in $B(z, r)$. It is well-known [14, III-(6.19)] that $\Pi^2_{\Lambda,\xi} = \Pi_{\Lambda,\xi}$ is a projector, and its range $\mathscr{R}(\Pi_{\Lambda,\xi})$ is the closure of the algebraic eigenspace associated to ξ. Moreover the range of the spectral projector is finite-dimensional if and only if there exists $\alpha_0 \in \mathbb{N}^*$ such that

$$\dim \mathscr{N}(\Lambda-\xi)^{\alpha_0} < \infty, \quad \mathscr{N}(\Lambda-\xi)^{\alpha} = \mathscr{N}(\Lambda-\xi)^{\alpha_0} \ \text{ for any } \ \alpha \geq \alpha_0,$$

so that

$$\overline{\mathscr{M}(\Lambda-\xi)} = \mathscr{M}(\Lambda-\xi) = \mathscr{N}((\Lambda-\xi)^{\alpha_0}).$$

In that case, we say that ξ is a *discrete eigenvalue*, written as $\xi \in \Sigma_d(\Lambda)$. Observe that R_Λ is meromorphic on $(\mathbb{C} \setminus \Sigma(\Lambda)) \cup \Sigma_d(\Lambda)$ (with non-removable finite-order poles). Finally for any $a \in \mathbb{R}$ such that $\Sigma(\Lambda) \cap \Delta_a = \{\xi_1, \dots, \xi_k\}$ where $\xi_1, \dots, \xi_k$ are distinct discrete eigenvalues, we define without any risk of ambiguity

$$\Pi_{\Lambda,a} := \Pi_{\Lambda,\xi_1} + \cdots + \Pi_{\Lambda,\xi_k}.$$

We need the following definition on the convolution of semigroup (corresponding to composition at the level of the resolvent operators).

Definition 2.1 (Convolution of semigroups)**.** Consider some Banach spaces X_1, X_2, X_3. For two given functions

$$\mathscr{S}_1 \in L^1(\mathbb{R}_+; \mathscr{B}(X_1, X_2)) \ \text{ and } \ \mathscr{S}_2 \in L^1(\mathbb{R}_+; \mathscr{B}(X_2, X_3)),$$

we define the convolution $\mathscr{S}_2 * \mathscr{S}_1 \in L^1(\mathbb{R}_+; \mathscr{B}(X_1, X_3))$ by

$$\forall\, t \geq 0, \quad (\mathscr{S}_2 * \mathscr{S}_1)(t) := \int_0^t \mathscr{S}_2(s)\, \mathscr{S}_1(t-s)\, \mathrm{d}s.$$

When $\mathscr{S} = \mathscr{S}_1 = \mathscr{S}_2$ and $X_1 = X_2 = X_3$, we define inductively $\mathscr{S}^{(*1)} = \mathscr{S}$ and $\mathscr{S}^{(*\ell)} = \mathscr{S} * \mathscr{S}^{(*(\ell-1))}$ for any $\ell \geq 2$.

2.2. **Factorization and spectral analysis when changing space.**

Theorem 2.2. *Consider $E, \mathcal{E}, L, A, B, \mathcal{L}, \mathcal{A}, \mathcal{B}$ as above and assume that*

- **(i′)** $\Sigma(B) \cap \Delta_a = \Sigma(\mathcal{B}) \cap \Delta_a = \emptyset$ *for some $a \in \mathbb{R}$;*
- **(ii)** $A \in \mathscr{B}(E)$ *and* $\mathcal{A} \in \mathscr{B}(\mathcal{E})$*;*
- **(iii′)** *there is $n \geq 1$ such that for any $\xi \in \Delta_a$, the operators $(\mathcal{A}\, R_{\mathcal{B}}(\xi))^n$ and $(R_{\mathcal{B}}(\xi)\, \mathcal{A})^n$ are bounded from $\mathcal{E}$ to E.*

Then the following two properties are equivalent, with the same family of distinct complex numbers and the convention $\{\xi_1, \dots, \xi_k\} = \emptyset$ if $k = 0$:

(1) $\Sigma(L) \cap \Delta_a = \{\xi_1, \dots, \xi_k\} \subset \Sigma_d(L)$ *(distinct discrete eigenvalues).*

(2) $\Sigma(\mathcal{L}) \cap \Delta_a = \{\xi_1, \dots, \xi_k\} \subset \Sigma_d(\mathcal{L})$ *(distinct discrete eigenvalues).*

Moreover, in both cases, there hold

(3) *For any $z \in \Delta_a \setminus \{\xi_1, \dots, \xi_k\}$ the resolvent operators R_L and $R_\mathcal{L}$ satisfy:*

$$R_\mathcal{L}(z) = \sum_{\ell=0}^{n-1} (-1)^\ell R_\mathcal{B}(z) \left(\mathcal{A} R_\mathcal{B}(z)\right)^\ell + (-1)^n R_L(z) \left(\mathcal{A} R_\mathcal{B}(z)\right)^n \tag{2.2}$$

$$R_L(z) = \sum_{\ell=0}^{n-1} (-1)^\ell \left(R_B(z) A\right)^\ell R_B(z) + (-1)^n \left(R_\mathcal{B}(z) \mathcal{A}\right)^n R_\mathcal{L}(z). \tag{2.3}$$

(4) *For any $j = 1, \dots, k$, we have*

$$\begin{cases} \mathscr{N}(L-\xi_j)^\alpha & = & \mathscr{N}(\mathcal{L}-\xi_j)^\alpha, \quad \forall \, \alpha \geq 1 \\ \mathscr{M}(L-\xi_j) & = & \mathscr{M}(\mathcal{L}-\xi_j) \\ (\Pi_{\mathcal{L},\xi_j})_{|E} & = & \Pi_{L,\xi_j} \\ \mathscr{S}_{\mathcal{L},\xi_j}(t) & = & \mathscr{S}_\mathcal{L}(t) \Pi_{\mathcal{L},\xi_j} = \mathscr{S}_L(t) \Pi_{\mathcal{L},\xi_j}. \end{cases}$$

Remarks 2.3. (1) In this theorem, the implication **(1)** $\Rightarrow$ **(2)** has been established in [8, Theorem 2.1]; since $E \subset \mathcal{E}$, this is a recipe for *enlarging* the functional space where a property of localization of the discrete spectrum holds. The implication **(2)** $\Rightarrow$ **(1)** is a recipe for *shrinking* the functional space where a property of localization of the discrete spectrum holds.

(2) In the simplest case where $\mathcal{A} \in \mathscr{B}(\mathcal{E}, E)$, the assumption **(iii′)** is satisfied with $n = 1$.

(3) The hypothesis (i)-(ii)-(iii) (for some $a \in \mathbb{R}$) in Theorem 1.1 imply the hypothesis (i′)-(ii)-(iii′) above, for any $a' > a$.

(4) A similar result holds when we replace the assumption $E \subset \mathcal{E}$ by the assumption that $E \cap \mathcal{E}$ is dense in both E and $\mathcal{E}$.

Proof of Theorem 2.2. Because of Remark 2.3-(1), we only have to prove the implication **(2)** $\Rightarrow$ **(1)**. Let us denote $\Omega := \Delta_a \setminus \{\xi_1, \dots, \xi_k\}$ and define for $z \in \Omega$

$$U(z) := \sum_{\ell=0}^{n-1} (-1)^\ell \left(R_B(z) A\right)^\ell R_B(z) + (-1)^n \left(R_\mathcal{B}(z) \mathcal{A}\right)^n R_\mathcal{L}(z).$$

Observe that thanks to the assumptions **(i′)-(ii)-(iii′)** and **(2)**, the operator $U(z)$ is well-defined and bounded on E.

Step 1. $U(z)$ is a left-inverse of $(L-z)$ on Ω. For any $z \in \Omega$, we compute

$$\begin{aligned} U(z)(L-z) &= \sum_{\ell=0}^{n-1} (-1)^\ell \, \left(R_B(z) A\right)^\ell \, R_B(z) \left(A + (B-z)\right) \\ &\qquad + (-1)^n \, \left(R_\mathcal{B}(z) \mathcal{A}\right)^n \, R_\mathcal{L}(z) \left(L-z\right) \\ &= \sum_{\ell=0}^{n-1} (-1)^\ell \, \left(R_B(z) A\right)^{\ell+1} + \sum_{\ell=0}^{n-1} (-1)^\ell \, \left(R_B(z) A\right)^\ell \\ &\qquad + (-1)^n \, \left(R_B(z) A\right)^n = \mathrm{Id}_E. \end{aligned}$$

Step 2. $(L - z)$ is invertible on Ω. Consider $z_0 \in \Omega$. First observe that if the operator $(L - z_0)$ is bijective, then composing to the right the equation

$$U(z_0)(L - z_0) = \mathrm{Id}_E$$

by $(L - z_0)^{-1} = R_L(z_0)$ yields $R_L(z_0) = U(z_0)$ and we deduce that the inverse map is bounded (i.e. $(L - z_0)$ is an invertible operator in E) together with the desired formula for the resolvent.

Since $(L - z_0)$ has a left-inverse it is injective. Let us prove that it is surjective. Consider $g \in E$. Since $\mathcal{L} - z_0$ is invertible and therefore bijective there is $f \in \mathcal{E}$ so that

$$(\mathcal{L} - z_0)f = g \quad \text{and thus} \quad \mathrm{Id} + R_{\mathcal{B}}(z_0)\mathcal{A}f = R_{\mathcal{B}}(z_0)g = R_B(z_0)g.$$

We denote $\bar{g} := R_B(z_0)g \in E$ and $\mathcal{G}(z_0) := R_{\mathcal{B}}(z_0)\mathcal{A}$ and write

$$f = \bar{g} - \mathcal{G}(z_0)f = \sum_{\ell=0}^{n-1} (-1)^\ell \mathcal{G}(z_0)^\ell \bar{g} + (-1)^n \mathcal{G}(z_0)^n f.$$

Because of **(i′)-(ii)-(iii′)**, it implies that $f \in E$, and in fact since $\mathscr{D}(B) = \mathscr{D}(L)$, we further have $f \in \mathscr{D}(L) \subset E$. We conclude that $(L - z_0)f = g$ in E, and the proof of this step is complete.

Step 3. Spectrum, eigenspaces and spectral projectors. On the one hand, we have

$$\mathscr{N}(L - \xi_j)^\alpha \subset \mathscr{N}(\mathcal{L} - \xi_j)^\alpha, \quad j = 1, \dots, k, \quad \alpha \in \mathbb{N},$$

so that $\Sigma(L) \cap \Delta_a \subset \{\xi_1, \dots, \xi_k\}$. On the other hand, consider $\xi_j \in \Sigma(\mathcal{L}) \cap \Delta_a$, $\alpha \in \mathbb{N}^*$ and $f \in \mathscr{N}(\mathcal{L} - \xi_j)^\alpha$:

$$(\mathcal{L} - \xi_j)^\alpha f = 0.$$

Denote $g_\beta := (\mathcal{L} - \xi_j)^\beta f$, $\beta = 0, \dots, \alpha$ and argue by induction on β decreasingly to prove that $g_\beta \in E$. The initialisation $\beta = \alpha$ is clear. Assume $g_{\beta+1} \in E$ and write $(\mathcal{L} - \xi_j)\, g_\beta = g_{\beta+1}$. Using $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and composing to the left by $R_{\mathcal{B}}(\xi_j)$, we get

$$(\mathcal{G}(\xi_j) + \mathrm{Id})\, g_\beta = R_B(\xi_j) g_{\beta+1} \in E \quad \text{with} \quad \mathcal{G}(\xi_j) := R_{\mathcal{B}}(\xi_j)\mathcal{A}.$$

We deduce that

$$g_\beta = (-1)^n \mathcal{G}(\xi_j)^n g_{\beta+1} + \sum_{k=0}^{n-1} \mathcal{G}(\xi_j)^k R_B(\xi_j) g_{\beta+1}.$$

Since $\mathcal{G}(\xi_j)^n$ is bounded from $\mathcal{E}$ to E, and $\mathcal{G}(\xi_j)$ is bounded from E to E, with in each the range included in $\mathscr{D}(B) = \mathscr{D}(L)$, we deduce that $g_\beta \in \mathscr{D}(L) \subset E$, and the proof of the induction is complete. Finally $g_0 = f \in \mathscr{D}(L) \subset E$. Since the eigenvalues are discrete, this completes the proof of **(1)**.

Finally, the fact that $\Pi_{\mathcal{L},\xi_j|E} = \Pi_{L,\xi_j}$ is a straightforward consequence of $R_{\mathcal{L}}(z)f = R_L(z)f$ when $f \in E$ and the formula (2.1) for the projection operator. This concludes the proof of **(3)-(4)**. □

2.3. **Factorization and semigroup decay when changing spaces.** We now prove Theorem 1.1. First we notice that the assumptions of Theorem 2.2 are met since **(i′)** follows from **(i)** and **(ii)-(iii)** imply **(iii′)**. Because of Theorem 2.2 we know that $\mathscr{R}(\Pi_{\mathcal{L},a}) = \mathscr{R}(\Pi_{L,a}) \subset E$, and then for any $f_0 \in \mathscr{R}(\Pi_{L,a})$, there holds

$$\mathscr{S}_{\mathcal{L}}(t)\, f_0 = \mathscr{S}_L(t)\, f_0 = \sum_{j=1}^k e^{L_j t}\, \Pi_{L,\xi_j} f_0,$$

where $L_j := L_{|X_j}$, $X_j := \mathscr{R}(\Pi_{L,\xi_j})$. By linearity, it is enough to prove the equivalent estimates (1.1) and (1.2) in the supplementary space of the subspace $\mathscr{R}(\Pi_{L,a})$. We split the proof into two steps.

Step 1. Enlargement of the functional space. We give here an alternative presentation of the proof of **(1)** $\Rightarrow$ **(2)** in Theorem 1.1 which is in the spirit of [1] while the original (but similar) proof in [8] uses an iterate Duhamel formula. We assume (1.1) and denote $f_t := S_{\mathcal{L}}(t) f_0$ the solution to the evolution equation $\partial_t f = \mathcal{L} f$. We decompose

$$\begin{aligned} f &= \Pi_{L,a} f_t + g^1 + g^2 + \cdots + g^{n+1}, \\ \partial_t g^1 &= \mathcal{B} g^1, \quad g^1_0 = f_0 - \Pi_{L,a} f_0, \\ \partial_t g^k &= \mathcal{B} g^k + \mathcal{A} g^{k-1}, \quad g^k_0 = 0, \quad 2 \le k \le n, \\ \partial_t g^{n+1} &= \mathcal{L} g^{n+1} + \mathcal{A} g^n, \quad g^{n+1}_0 = 0, \end{aligned}$$

and we remark that this system of equations on g_k, $1 \le k \le n+1$, is compatible with the equation satisfied by f. Moreover, by induction

$$\mathcal{A} g_k(t) = (\mathcal{A} \mathcal{S}_{\mathcal{B}})^{(*k)}(t)(f_0 - \Pi_{L,a} f_0), \quad 1 \le k \le n,$$

so that $\mathcal{A} g_n(t) \in E$, because of assumption **(iii)**, and thus the equation on g_{n+1} is set in E and writes

$$\partial_t g_{n+1} = L g_n + \mathcal{A} g_n, \quad g_{n+1}(0) = 0.$$

We deduce successively the estimates (for $a' > a$)

$$\begin{aligned} \|g_k(t)\|_{\mathcal{E}} &\lesssim t^k e^{at} \|f_0 - \Pi_{L,a} f_0\|_{\mathcal{E}}, \quad 1 \le k \le n, \\ \|g_k(t)\|_{\mathcal{E}} &\lesssim_{a'} e^{a' t} \|f_0 - \Pi_{L,a} f_0\|_{\mathcal{E}}, \quad 1 \le k \le n, \\ \|\mathcal{A} g_n(t)\|_E &\lesssim t^n e^{at} \|f_0 - \Pi_{L,a} f_0\|_{\mathcal{E}}, \\ \|(\mathrm{Id} - \Pi_{L,a}) g_{n+1}(t)\|_{\mathcal{E}} &\lesssim \|(\mathrm{Id} - \Pi_{L,a}) g_{n+1}(t)\|_E \lesssim_{a'} e^{a' t} \|f_0 - \Pi_{L,a} f_0\|_{\mathcal{E}}, \end{aligned}$$

and since, from the definition of the decomposition,

$$\Pi_{L,a} g_{n+1} = -\Pi_{L,a} g_1 - \cdots - \Pi_{L,a} g_n$$

we have, using the previous decay estimates,

$$\|\Pi_{L,a} g_{n+1}(t)\|_{\mathcal{E}} \le \sum_{k=1}^n \|\Pi_{L,a} g_k(t)\|_{\mathcal{E}} \lesssim_{a'} e^{a' t} \|f_0 - \Pi_{\mathcal{L},a} f_0\|_{\mathcal{E}},$$

which concludes the proof of (1.2) by piling up these estimates on f.

Step 2. Shrinkage of the functional space. We assume (1.2) and $f_0 \in E$ and write the following family of operators depending on time on E through a factorization formula:

$$\begin{aligned} \mathscr{S}_*(t) = S_L(t) \Pi_{L,a} &+ \sum_{\ell=0}^{n-1} (-1)^\ell \left(\mathscr{S}_B(t) A\right)^{(*\ell)} \mathscr{S}_B(t)(\mathrm{Id} - \Pi_{L,a}) \\ &+ (-1)^n \left(\mathscr{S}_{\mathcal{B}}(t) \mathcal{A}\right)^{(*n)} \mathscr{S}_{\mathcal{L}}(t)(\mathrm{Id} - \Pi_{L,a}). \end{aligned}$$

Using the assumptions and (1.2) one gets

$$\begin{aligned}\|\mathscr{S}_{\mathcal{L}}(t)(\mathrm{Id}-\Pi_{L,a})\|_{\mathscr{B}(E,E)} &\lesssim_{a'} e^{a't}\\ \left\|(\mathscr{S}_{\mathcal{B}}(t)\mathcal{A})^{(*n)}\right\|_{\mathscr{B}(\mathcal{E},E)} &\lesssim_{a'} e^{a't}\\ \|\mathscr{S}_B(t)A\|_{\mathscr{B}(E,E)} &\lesssim_{a'} e^{a't}\end{aligned}$$

which proves that

$$\|\mathscr{S}_*(z)-S_L(t)\Pi_{L,a}\|_{\mathscr{B}(E,E)} \lesssim_{a'} e^{a't}.$$

Therefore the Laplace transform $U_*(z)$ of $t\mapsto(\mathscr{S}_*(t)-S_L(t)\Pi_{L,a})$ is well-defined on $\Re e\, z > a'$, and is

$$U_*(z)=\sum_{\ell=0}^{n-1}(-1)^\ell\,(R_B(z)A)^\ell\,R_B(z)(\mathrm{Id}-\Pi_{L,a})+(-1)^n\,(R_{\mathcal{B}}(z)\mathcal{A})^n\,R_{\mathcal{L}}(z)(\mathrm{Id}-\Pi_{L,a})$$

which is exactly $U_*(z)=R_L(z)(\mathrm{Id}-\Pi_{L,a})$ from Theorem 2.2. By uniqueness of the Laplace transform we deduce that $\mathscr{S}_*(t)=\mathscr{S}_L(t)$, and this proves the decay (1.1). ☐

2.4. **A practical criterion.** We finally prove a criterion implying both **(iii′)** in Theorem 2.2 and **(iii)** in Theorem 1.1.

Lemma 2.4. *Consider two Banach spaces E and $\mathcal{E}$ such that $E\cap\mathcal{E}$ is dense into E and $\mathcal{E}$ with continuous embedding. Consider $\mathcal{L}$ an operator on $E+\mathcal{E}$ so that there exist some operators $\mathcal{A}$ and $\mathcal{B}$ on $E+\mathcal{E}$ such that $\mathcal{L}$ splits as $\mathcal{L}=\mathcal{A}+\mathcal{B}$. Denoting with the same letter $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{L}$ the restriction of these operators on E and $\mathcal{E}$, we assume that there hold:*

- **(a)** *$(\mathcal{B}-a)$ is hypodissipative in E and $\mathcal{E}$ for some $a\in\mathbb{R}$;*
- **(b)** *$\mathcal{A}\in\mathscr{B}(E)$ and $\mathcal{A}\in\mathscr{B}(\mathcal{E})$;*
- **(c)** *for some $b\in\mathbb{R}$ and $\Theta\ge 0$ there holds $\|\mathcal{A}\mathscr{S}_{\mathcal{B}}(t)\|_{\mathscr{B}(\mathcal{E},E)}\le Ce^{bt}\,t^{-\Theta}$ and $\|\mathscr{S}_{\mathcal{B}}(t)\mathcal{A}\|_{\mathscr{B}(\mathcal{E},E)}\le Ce^{bt}t^{-\Theta}$.*

Then for any $a'>a$, there is some constructive $n\in\mathbb{N}$, $C_{a'}\ge 1$ such that

$$\forall\, t\ge 0,\quad \|(\mathcal{A}\mathscr{S}_{\mathcal{B}})^{(*n)}(t)\|_{\mathscr{B}(\mathcal{E},E)}+\|(\mathscr{S}_{\mathcal{B}}\mathcal{A})^{(*n)}(t)\|_{\mathscr{B}(\mathcal{E},E)}\le C_{a'}e^{a't}.$$

As a consequence, $(\mathcal{A}\,R_{\mathcal{B}}(z))^n$ and $(R_{\mathcal{B}}\mathcal{A})^n$ are bounded from $\mathcal{E}$ to E for any $z\in\Delta_a$.

Remark 2.5. It is necessary to include the non-integrable time factor in (c) for later application since (c) will be proved by hypoelliptic regularity which has this possibly non-integrable behavior at time zero.

Proof of Lemma 2.4. When $\Theta\ge 1$, we denote by J the integer such that $\Theta<J\le\Theta+1$ and we set $\theta:=\Theta/J\in[0,1)$. We define the family of intermediate complex interpolation spaces $\mathcal{E}_j=[E,\mathcal{E}]_{j/J}$. Thanks to the Riesz-Thorin interpolation theorem, we have

$$\mathcal{A}\mathscr{S}_{\mathcal{B}}(t)\ :\ \mathcal{E}_{\delta,\delta'}:=[[\mathcal{E}_0,\mathcal{E}_1]_\delta,\mathcal{E}_0]_{\delta'}\to\mathcal{E}^{\delta,\delta'}:=[[\mathcal{E}_0,\mathcal{E}_1]_\delta,\mathcal{E}_1]_{\delta'}$$

with the following estimate on the operator norm

$$\|\mathcal{A}\mathscr{S}_{\mathcal{B}}(t)\|_{\mathcal{E}_{\delta,\delta'}\to\mathcal{E}^{\delta,\delta'}}\le\|\mathcal{A}\mathscr{S}_{\mathcal{B}}(t)\|^{(1-\delta)(1-\delta')}_{\mathcal{E}_0\to\mathcal{E}_0}\,\|\mathcal{A}\mathscr{S}_{\mathcal{B}}(t)\|^{\delta(1-\delta')}_{\mathcal{E}_1\to\mathcal{E}_1}\,\|\mathcal{A}\mathscr{S}_{\mathcal{B}}(t)\|^{\delta'}_{\mathcal{E}_0\to\mathcal{E}_1}.$$

Since

$$\mathcal{E}_{\delta,\delta'} = [[\mathcal{E}_0, \mathcal{E}_1]_\delta, [\mathcal{E}_0, \mathcal{E}_1]_0]_{\delta'} = [\mathcal{E}_0, \mathcal{E}_1]_{(1-\delta')\delta}$$
$$\mathcal{E}^{\delta,\delta'} = [[\mathcal{E}_0, \mathcal{E}_1]_\delta, [\mathcal{E}_0, \mathcal{E}_1]_1]_{\delta'} = [\mathcal{E}_0, \mathcal{E}_1]_{(1-\delta')\delta+\delta'},$$

by taking $\delta' = 1/J$ and $\delta = j/(J-1)$, we get

$$\begin{aligned} \|\mathcal{A}\mathscr{S}_\mathcal{B}(t)\|_{\mathcal{E}_j \to \mathcal{E}_{j+1}} &\le \|\mathcal{A}\mathscr{S}_\mathcal{B}(t)\|^{1-(j+1)/J}_{\mathscr{B}(E)} \, \|\mathscr{S}_\mathcal{B}(t)\|^{j/J}_{\mathscr{B}(\mathcal{E})} \, \|\mathcal{A}\mathscr{S}_\mathcal{B}(t)\|^{1/J}_{\mathscr{B}(\mathcal{E},E)} \\ &\lesssim \frac{1}{t^\theta} \, e^{[(1-1/J)a + b/J]t}. \end{aligned}$$

We define now $n := \ell\, J$ so that $(\mathcal{A}\mathscr{S}_\mathcal{B})^{(*n)} = \mathscr{T}_J^{(*\ell)}$ with $\mathscr{T}_J := (\mathcal{A}\mathscr{S}_\mathcal{B})^{(*J)}$. From the assumptions and the previous estimate, for any $a'' > a$

$$\|\mathscr{T}_J(t)\|_{E\to E} \lesssim_{a''} e^{a' t}, \quad \|\mathscr{T}_J(t)\|_{\mathcal{E}\to E} \lesssim e^{bt}, \quad \|\mathscr{T}_J(t)\|_{\mathcal{E}\to\mathcal{E}} \lesssim_{a''} \, e^{a' t}.$$

As a consequence, we obtain

$$\|(\mathcal{A}\mathscr{S}_\mathcal{B})^{(*n)}(t)\|_{\mathcal{E}\to E} \lesssim_{a'} e^{[(1-1/\ell)a' + b/\ell]\, t},$$

which concludes the proof by fixing ℓ large enough so that $(1-1/\ell)a'' + b'/\ell < a'$. The estimate on $(\mathscr{S}_\mathcal{B}\mathcal{A})^{(*n)}$ is proved by the same argument. □

3. The kinetic Fokker-Planck equation with flat confinement

This section is dedicated to the proof of semigroup growth estimates for the kinetic Fokker-Planck equation confined either by spatial homogeneity (hence reducing the simpler "Fokker-Planck equation") or confined by spatial periodicity, in a large class of Banach spaces, including the case of negative Sobolev spaces. We deduce growth estimates in Wasserstein distance as well.

3.1. Main result.

Consider the Fokker-Planck equation

$$\partial_t f = \mathcal{L} f := \nabla_v \cdot (\nabla_v f + F f) - v \cdot \nabla_x f, \tag{3.1}$$

on the density $f = f(t,x,v)$, $t \ge 0$, $x \in \mathbb{T}^d$ (the torus' volume is normalised to one), $v \in \mathbb{R}^d$, where the (exterior) force field $F = F(v) \in \mathbb{R}^d$ takes the form

$$F = \nabla_v \Phi \quad \text{with} \quad \forall\, |v| \ge R_0, \ \Phi(v) = \frac{1}{\gamma}\, \langle v\rangle^\gamma + \Phi_0 \tag{3.2}$$

for some constants $R_0 \ge 0$ and $\gamma \ge 1$. Here and below, we denote $\langle v\rangle := (1+|v|^2)^{1/2}$. We define $\mu(v) := e^{-\Phi(v)}$ with $\Phi_0 \in \mathbb{R}$ such that μ is a probability measure. Observe that μ is a steady state for the evolution equation (3.1). We shall consider separately along this section the case where f does not depend on x, commenting on the simpler proofs and sharper estimates in this case.

Let us now introduce the key assumptions:

Assumptions on the functional spaces

Polynomial weights: For any $\gamma \geq 2$, $\sigma \in \{-1,0,1\}$ and $p \in [1,\infty]$, we introduce the weight functions

$$m := \langle v \rangle^k, \quad k > |\sigma| + |\sigma|\sqrt{d}(\gamma-2) + \left(1 - \frac{1}{p}\right)(d+\gamma-2) \tag{3.3}$$

and the abscissa

$$a_\sigma(p,m) := \begin{cases} |\sigma| + (1-1/p)d - k & \textit{if } \gamma = 2, \\ -\infty & \textit{if } \gamma > 2. \end{cases}$$

Stretched exponential weights: For any $\gamma \geq 1$, $\sigma \in \{-1,0,1\}$ and $p \in [1,\infty]$, we introduce the weight functions

$$m := e^{\kappa\, \langle v \rangle^s} \quad \begin{array}{l} \textit{with } s \in [2-\gamma,\gamma),\ \kappa > 0,\ s > 0, \\ \textit{or with } s = \gamma,\ \kappa \in (0, 1/\gamma), \end{array} \tag{3.4}$$

and the abscissa

$$a_\sigma(p,m) := \begin{cases} \kappa^2 - \kappa & \textit{if } \gamma = s = 1, \\ -\kappa s & \textit{if } \gamma + s = 2,\ s < \gamma, \\ -\infty & \textit{in the other cases.} \end{cases}$$

Definition of the spaces: For any weight function m, we define $L^p(m)$, $1 \leq p \leq \infty$, as the Lebesgue weighted space associated to the norm

$$\|f\|_{L^p(m)} := \|f\, m\|_{L^p},$$

and $W^{1,p}(m)$, $1 \leq p \leq \infty$, as the Sobolev weighted space associated to the norm

$$\|f\|_{W^{1,p}(m)} := \left(\|m\, f\|^p_{L^p} + \|m\, \nabla f\|^p_{L^p}\right)^{1/p}$$

when $p \in [1,\infty)$ and

$$\|f\|_{W^{1,\infty}(m)} := \max\left\{\|m\, f\|_{L^\infty}, \|m\, \nabla f\|_{L^\infty}\right\}.$$

We also define $W^{-1,p}(m)$, $p \in [1,\infty]$, as the weighted negative Sobolev space associated to the dual norm

$$\|f\|_{W^{-1,p}(m)} := \|f\, m\|_{W^{-1,p}} := \sup_{\|\phi\|_{W^{1,p'}} \leq 1} \langle f, \phi\, m\rangle, \quad p' := \frac{p}{p-1}, \tag{3.5}$$

where it is worth insisting that in this last equation the condition $\|\phi\|_{W^{1,p'}} \leq 1$ refers to the standard Sobolev space $W^{1,p'}$ (without weight).

Observe that for $\sigma \in \{-1,0,1\}$, $p \in [1,\infty]$ and m satisfying (3.3) or (3.4), the Sobolev space $W^{\sigma,p}(m)$ defined as above is such that $1 \in W^{\sigma',p'}(m^{-1})$ with $\sigma' = -\sigma \in \{-1,0,1\}$ and $p' := p/(p-1) \in [1,\infty]$. As a consequence, for any $f \in W^{\sigma,p}(m)$, we may define the "*mass of f*" by

$$\langle\langle f \rangle\rangle := \langle f, 1\rangle_{W^{\sigma,p}(m), W^{\sigma',p'}(m^{-1})} = \left\langle m\, f, \frac{1}{m}\right\rangle_{W^{\sigma,p}, W^{\sigma',p'}}$$

where the double bracket recalls that there are two variables x and v. In the case $\sigma = 0, 1$, there holds $W^{\sigma,p}(m) \subset L^1$ (here L^1 denotes the usual Lebesgue space without weight) and therefore the *"mass of f"* corresponds to the usual definition

$$\langle\langle f \rangle\rangle := \int_{\mathbb{T}^d \times \mathbb{R}^d} f(x,v) \, \mathrm{d}x \, \mathrm{d}v$$

and else this is the mass of the associated measure. Observe also that when f does not depend on x, this reduces thanks to the normalisation of the torus volume to

$$\langle\langle f \rangle\rangle = \langle f \rangle := \int_{\mathbb{R}^d} f(v) \, \mathrm{d}v.$$

We finally define the projector $\Pi_1^\perp$ on the orthogonal supplementary of the first eigenspace:

$$\forall\, f \in W^{\sigma,p}(m), \quad \Pi_1^\perp f := f - \langle\langle f \rangle\rangle \, \mu.$$

Theorem 3.1. *Consider $\sigma \in \{-1, 0, 1\}$ and m, $p \in [1, +\infty]$ that satisfy conditions (3.3) or (3.4) above (this implies $a_\sigma(p,m) < 0$). For any $a > \max\{a_\sigma(p,m), -\lambda\}$, there exists $C_a = C_a(\sigma, p, m)$ such that for any $f_0, g_0 \in W^{\sigma,p}(m)$ with the same mass, there holds*

$$\|\mathscr{S}_{\mathcal{L}}(t) f_0 - \mathscr{S}_{\mathcal{L}}(t) g_0\|_{W^{\sigma,p}(m)} \le C_a \, e^{at} \, \|f_0 - g_0\|_{W^{\sigma,p}(m)}, \tag{3.6}$$

which implies in particular the relaxation to equilibrium

$$\|\mathscr{S}_{\mathcal{L}}(t) f_0 - \langle f_0 \rangle \mu\|_{W^{\sigma,p}(m)} \le C_a \, e^{at} \, \|f_0 - \langle f_0 \rangle \mu\|_{W^{\sigma,p}(m)}, \tag{3.7}$$

where $\lambda := \lambda(d, \sigma, p, m) > 0$ is constructive from the proof.

Moreover, when $\gamma \in [2, 2 + 1/(d-1))$, there exists $\tilde{a}(\gamma) < 0$ and for any $a > \tilde{a}(\gamma)$ there exists $C_a \in (0, \infty)$ so that for any probability measures f_0, g_0 with bounded first moments, there holds

$$W_1\left(\mathscr{S}_{\mathcal{L}}(t) f_0, \mathscr{S}_{\mathcal{L}}(t) g_0\right) \le C_a \, e^{a\,t} \, W_1(f_0, g_0) \tag{3.8}$$

which implies the relaxation to equilibrium

$$W_1\left(\mathscr{S}_{\mathcal{L}}(t) f_0, \langle f_0 \rangle \mu\right) \le C_a \, e^{a\,t} \, W_1(f_0, \langle f_0 \rangle \mu). \tag{3.9}$$

Remarks 3.2. We first list the remarks in the spatially homogeneous case.

(1) For $m = \mu^{-1/2}$, $p = 2$ and $\sigma = 0$, (3.7) reduces to the classical spectral gap inequality for the Fokker-Planck semigroup in $L^2(\mu^{-1/2})$. In that case the semigroup spectral gap is equivalent to the Poincaré inequality. Denoting as λ_P the best constant in the Poincaré inequality, the estimate (3.7) holds with $a = -\lambda_P$ and $C_a = 1$.

(2) Our proof in the general case is based on the above mentioned semigroup spectral gap estimate in $L^2(\mu^{-1/2})$ and on the abstract extension Theorem 1.1. More precisely, our approach allows one to prove an equivalence between Poincaré's inequality and semigroup decay of the Fokker-Planck equation in Banach spaces, including the case of negative Sobolev spaces. The meaning of the sentence is that the functional inequality

$$\forall\, a' > a, \quad (\mathcal{L} f, f)_{L^2(\mu^{-1/2})} \le a' \, \|\Pi_1^\perp f\|^2_{L^2(\mu^{-1/2})} \tag{3.10}$$

is equivalent to the semigroup growth estimate (3.7) for a large class of weight function m.

(3) For $\gamma \geq 2$, it has been proved recently in [4] that a semigroup growth estimate similar to (3.7) holds for the Monge-Kantorovich-Wasserstein distance W_2, or in other words that for any probability measure f_0 with bounded second moment, there holds

$$W_2(\mathscr{S}_{\mathcal{L}}(t) f_0, \langle f_0 \rangle \mu) \leq C\, e^{\alpha\, t}\, W_2(f_0, \langle f_0 \rangle \mu). \tag{3.11}$$

In the above inequality $C = 1$ and $-\alpha$ is the optimal constant in the *"WJ inequality"* (introduced in [4, Definition 3.1]), which corresponds to the optimal constant in the *"log-Sobolev inequality"* for convex potential and in particular $-\alpha$ is smaller than the optimal constant λ_P in the Poincaré inequality (3.10). Our estimate (3.9) can be compared to (3.11).

(4) It is worth emphasizing that in Theorem 3.1, the function space can be chosen smaller in term of tail decay than the space of self-adjointness $L^2(\mu^{-1/2})$: one can choose for instance $L^2(\mu^{-\theta/2})$ with $\theta \in (1,2)$.

(5) Note that this statement implies in particular that for a strong enough weight function, so that the essential spectrum move far enough to the left, there holds

$$\Sigma\,(\mathcal{L}) \subset \{z \in \mathbb{C} \mid \Re e(z) \leq -\lambda_P\} \cup \{0\}$$

and that the null space of $\mathcal{L}$ is exactly $\mathbb{R}\mu$.

(6) Moreover, thanks to Weyl's Theorem, we know that in the $L^2(\mu^{-1/2})$ space the spectrum is constituted of discrete eigenvalues denotes as ξ_ℓ, $\ell \in \mathbb{N}$, with $\ell \mapsto \Re e \xi_\ell$ decreasing. In any Banach space $W^{\sigma,p}(m)$, exactly the same proof as for Theorem 3.1 (same splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and same application of the abstract extension Theorem 1.1) yields to the more accurate description of the spectrum

$$\Sigma\,(\mathcal{L}) \cap \Delta_{a_\sigma(p,m)} = \{\xi_\ell;\ \Re e(\xi_\ell) > a_\sigma(p,m)\}$$

as well as the more accurate estimate (1.2) for any $a > a_\sigma(p,m)$ and with k defined by $k = \sup\{\ell;\ \Re e(\xi_\ell) > a_\sigma(p,m)\}$.

(7) As a consequence of the preceding point, we may improve the intermediate asymptotic for the heat equation established in [3]. Consider g the solution to the heat equation

$$\partial_t g = \Delta_v g, \quad g(0) = g_0,$$

with $g_0 \in L^p(m)$, $m = \langle v \rangle^k$, $k > d/p' + n - 1$, $n \in \mathbb{N}^*$. Assume furthermore that

$$\forall\, \ell \in \mathbb{N}^d,\ |\ell| \leq n-1, \qquad \int_{\mathbb{R}^d} g_0\, H_\ell \,\mathrm{d}x = 0,$$

where (H_ℓ) stands for the family of Hermite polynomials (see [3] and the references therein). In particular $\langle g_0 \rangle = 0$ since $H_0 = 1$. We observe that the function f defined thanks to

$$g(t,x) = R^{-d}\, f(\log R, v/R), \quad R = R(t) = \sqrt{1+2t},$$

is a solution to the harmonic Fokker Planck equation

$$\partial_t f = \mathcal{L} f = \Delta_v f + \mathrm{div}_v(v f), \quad f(0) = g_0,$$

and that (H_ℓ) is an orthogonal family of eigenfunctions associated to the adjoint operator $\mathcal{L}^*$ (H_ℓ is associated to the eigenvalue $|\ell| = \ell_1 + \cdots + \ell_d$

for any $\ell \in \mathbb{N}^d$). An immediate application of our method implies

$$\|f_t\|_{L^p(m)} \le C_{d,p,n}\, e^{-nt}\, \|g_0\|_{L^p(m)} \quad \forall\, t \ge 0,$$

which improves (3.7) (which holds in that context with $a = -\lambda_P = -1$) whenever $n \ge 2$. Coming back to the function g we obtain the optimal intermediate asymptotic estimate

$$\|g_t\|_{L^p(m)} \le \frac{C_{d,p,n}}{(1+t)^{n/2+d/(2p')}}\, \|g_0\|_{L^p(m)} \quad \forall\, t > 0.$$

That last estimate improves [3, Corollary 4] because the range of initial data is larger and the rate in time is better (it is in fact optimal).

Remarks 3.3. We now list the remarks specific to the spatially periodic case.

(1) The value of λ in our quantitative estimate is related to the hypocoercivity estimate in $L^2(\mu^{-1})$ setting. However the best rate in general is the real part of the second eigenvalue defined by

$$\lambda := \sup_{\|\cdot\| \sim \|\cdot\|_{W^{\sigma,p}(m)}} \inf_{f \in C_c^\infty(\mathbb{R}^d)} \left(- \frac{\langle \mathcal{L} f, f^* \rangle}{\|\Pi_1^\perp f\|} \right) \tag{3.12}$$

where $C_c^\infty(\mathbb{R}^d)$ denotes the smooth compactly supported functions, and where the supremum is taken over all norms $\|\cdot\|$ on $W^{\sigma,p}(m)$ equivalent to the ambiant norm, and where $f^* \in W^{\sigma',p'}(m)$ is the unique element in $W^{\sigma',p'}(m)$ such that $\|f\|^2 = \|f^*\|_*^2 = \langle f^*, f\rangle$, where $\|\cdot\|_*$ is the corresponding dual norm.

(2) Our result partially generalize to a spacially unhomogeneous setting the estimate on the Monge-Kantorovich-Wasserstein distance obtained recently in [4].

(3) Our proof is based on the semigroup spectral gap estimate in $H^1(\mu^{-1/2})$ established in [16, 20] and on the abstract extension Theorem. As a consequence, it gives an alternative proof for the semigroup spectral gap estimate obtained in [7, 6] for the Lebesgue space $L^2(\mu^{-1/2})$.

(4) Again, the proof holds for $F := \nabla\Phi + U$ which fulfills the conditions of [8, section 3]. In particular the associated Fokker-Planck operator does not take the $AA^* + B$ structure of [20] (where the term $\nabla_v(Uf)$ is included in the "B" part).

The proof of Theorem 3.1 is split into several steps:

(1) We recall existing results for proving (3.7) in the space of $E = H^1(\mu^{-1/2})$:

Lemma 3.4. ([16, Theorem 1.1]) *The result in Theorem 3.1 is true in the Hilbert space $H^1(\mu^{-1/2})$ associated to the norm*

$$\|f\|_{H^1(\mu^{-1/2})} := \left(\|f\|^2_{L^2(\mu^{-1/2})} + \|\nabla_x f\|^2_{L^2(\mu^{-1/2})} + \|\nabla_v f\|^2_{L^2(\mu^{-1/2})} \right)^{1/2}.$$

Such a result has been proved in [16], see also [12, 11, 9, 7, 6].

(2) We devise an appropriate decomposition $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with $\mathcal{B} = \mathcal{L} - M\chi_R$ where χ_R is a smooth characteristic function of the set $|v| \le R$ with $M|\nabla_v \chi_R|$ small.

(3) We need then to establish the dissipativity of $\mathcal{B}$ in the spaces $W^{\sigma,p}(m)$ and of $B := \mathcal{B}_{|E}$ in E. The coercivity of $\mathcal{B}$ in these spaces is established in Lemma 3.8, 3.9 and 3.10. The coercivity of B in E follows also from the same Lemma since the weight $m = \mu^{-1/2}$ is allowed. The latter could be proved by adapting the proof of [16, Theorem 1.1]. Or finally it could be checked more generally that the coercivity of B in E follows from that of L combined with the strenghtened Poincaré inequality as described below.
(4) We prove that the semigroup $\mathscr{S}_{\mathcal{B}}(t)$ is regularizing in $L^2(\mu^{-1/2})$.
(5) We conclude by applying Theorem 1.1.

Remark 3.5. Observe that since we need only applying regularization estimates for the semigroup of $\mathcal{B}$ after a composition by the operator $\mathcal{A}$, it is enough to prove these regularisation estimates with the usual weight $\mu^{-1/2}$.

3.2. **Simplifications in the spatially homogeneous case.** Let us start by pointing out the simplifications in the spatially homogeneous case. First the decay (3.7) in the space $E = L^2(\mu^{-1/2})$ follows from the Poincaré inequality:

Lemma 3.6. *There exists a constant $\lambda_P > 0$ so that for $f \in \mathcal{D}(\mathbb{R}^d)$ with $\langle f \rangle = 0$*

$$\int_{\mathbb{R}^d} \left| \nabla_v \left(\frac{f}{\mu} \right) \right|^2 \mu(v)\, \mathrm{d}v \geq \lambda_P \int_{\mathbb{R}^d} f^2 \, \mu^{-1}(v)\, \mathrm{d}v \tag{3.13}$$

and moreover for $\lambda < \lambda_P$, there is $\varepsilon(\lambda) > 0$ so that

$$\begin{aligned} \int_{\mathbb{R}^d} \left| \nabla_v \left(\frac{f}{\mu} \right) \right|^2 \mu(v)\, \mathrm{d}v \quad &\geq \quad \lambda \int_{\mathbb{R}^d} f^2 \, \mu^{-1}(v)\, \mathrm{d}v \\ &\quad + \varepsilon \int_{\mathbb{R}^d} \left(f^2 \, |\nabla_v \Phi|^2 + |\nabla_v f|^2 \right) \mu^{-1}(v)\, \mathrm{d}v. \end{aligned}$$

Proof of Lemma 3.6. The proof of Lemma 3.6 is classical. We refer to [2] for a comprehensive proof of (3.13). For the sake of completeness, we present a quantitative proof of (3.14) as a consequence of (3.13) in the spirit of [17].

On the one hand, by developing the LHS term, we find

$$T := \int_{\mathbb{R}^d} \left| \nabla_v \left(\frac{f}{\mu} \right) \right|^2 \mu(v)\, \mathrm{d}v = \int_{\mathbb{R}^d} |\nabla_v f|^2 \, \mu^{-1}\, \mathrm{d}v - \int_{\mathbb{R}^d} f^2 \, (\Delta_v \Phi)\, \mu^{-1}\, \mathrm{d}v.$$

On the other hand, a similar computation leads to the following identity

$$\begin{aligned} T &= \int_{\mathbb{R}^d} \left| \nabla_v (f\mu^{-1/2})\, \mu^{1/2} + (f\mu^{-1/2})\, \nabla_v \mu^{1/2} \right|^2 \mu(v)\, \mathrm{d}v \\ &= \int_{\mathbb{R}^d} \left| \nabla_v (f\, \mu^{-1/2}) \right|^2 \mathrm{d}v + \int_{\mathbb{R}^d} f^2 \left(\frac{1}{4} |\nabla_v \Phi|^2 - \frac{1}{2} \Delta_v \Phi \right) \mu^{-1}\, \mathrm{d}v. \end{aligned}$$

The two above identities together with (3.13) imply that for any $\theta \in (0,1)$

$$\begin{aligned} T &\geq (1-\theta)\lambda_P \int_{\mathbb{R}^d} f^2 \, \mu^{-1}\, \mathrm{d}v + \theta \int_{\mathbb{R}^d} f^2 \left(\frac{1}{16} |\nabla_v \Phi|^2 - \frac{3}{4} \Delta_v \Phi \right) \mu^{-1}\, \mathrm{d}v \\ &+ \frac{\theta}{16} \int_{\mathbb{R}^d} f^2 \, |\nabla_v \Phi|^2 \, \mu^{-1}\, \mathrm{d}v + \frac{\theta}{2} \int_{\mathbb{R}^d} |\nabla_v f|^2 \, \mu^{-1}\, \mathrm{d}v. \end{aligned}$$

Observe that $|\nabla \Phi|^2 - 12 \Delta \Phi \geq 0$ for v large enough, and we can choose $\theta > 0$ small enough to conclude the proof. $\square$

We define

$$\mathcal{A}f := M\chi_R f, \qquad \mathcal{B}f := \mathcal{L}f - M\chi_R f \tag{3.14}$$

where $M > 0$, $\chi_R(v) = \chi(v/R)$, $R > 1$, and $0 \le \chi \in \mathcal{D}(\mathbb{R}^d)$ is such that $\chi(v) = 1$ for any $|v| \le 1$. The dissipativity estimates are proved as in the spatially periodic case in Lemmata 3.8–3.9–3.10–3.11. Finally the regularisation estimates are proved by using Nash's inequality:

Lemma 3.7. *For any $1 \le p \le q \le \infty$ and for any R, M as in the definition* (3.14) *of $\mathcal{B}$, there exists $b = b(R,M) > 0$ so that for any $\sigma \in \{-1, 0, 1\}$*

$$\forall\, t \in [0,1], \quad \|\mathscr{S}_{\mathcal{B}}(t) f\|_{W^{\sigma,q}(m)} \lesssim \frac{e^{bt}}{t^{\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}}\, \|f\|_{W^{\sigma,p}(m)} \tag{3.15}$$

and for any $-1 \le \sigma < s \le 1$

$$\forall\, t \in [0,1], \quad \|\mathscr{S}_{\mathcal{B}}(t) f\|_{H^{s}(m)} \lesssim \frac{e^{bt}}{t^{s-\sigma}}\, \|f\|_{H^{\sigma}(m)}. \tag{3.16}$$

Proof of Lemma 3.7. The proof is classical and is a variation around Nash's inequality, together with Riesz-Thorin interpolation Theorem. We refer for instance to [8, Lemma 3.9] for some similar results. □

3.3. **Dissipativity property of $\mathcal{B}$.** We define

$$\mathcal{A}f := M\chi_R f, \qquad \mathcal{B}f := \mathcal{L}f - M\chi_R f \tag{3.17}$$

where $M > 0$, $\chi_R(v) = \chi(v/R)$, $R > 1$, and $0 \le \chi \in \mathcal{D}(\mathbb{R}^d)$ is such that $\chi(v) = 1$ for any $|v| \le 1$.

Lemma 3.8. *For any exponents $\gamma \ge 1$, $p \in [1, \infty]$, for any weight function m given by* (3.3) *or* (3.4) *and for any $a > a_0(m,p)$, we can choose R, M large enough in the definition* (3.17) *of $\mathcal{B}$ such that the operator $\mathcal{B} - a$ is dissipative in $L^p(m)$.*

Proof of Lemma 3.8. We start by establishing an identity satisfied by the operator $\mathcal{L}$. For any smooth, rapidly decaying and positive function f, we make the splitting

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} (\mathcal{L}\, f)\, f^{p-1}\, m^p \,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbb{T}^d \times \mathbb{R}^d} f^{p-1}\, m^p \left(\Delta_v f + \mathrm{div}_v(F\, f)\right) \mathrm{d}x\,\mathrm{d}v =: T_1 + T_2.$$

For the second term T_2, we use integration by part in v:

$$\begin{aligned}
T_2 &= \int_{\mathbb{T}^d \times \mathbb{R}^d} f^{p-1}\, m^p\, \mathrm{div}_v(F\, f)\,\mathrm{d}x\,\mathrm{d}v \\
&= \int_{\mathbb{T}^d \times \mathbb{R}^d} f^{p-1}\, m^p \left(\mathrm{div}_v F\, f + F \cdot \nabla_v f\right) \mathrm{d}x\,\mathrm{d}v \\
&= \int_{\mathbb{T}^d \times \mathbb{R}^d} f^{p}\, (\mathrm{div}_v F)\, m^p \,\mathrm{d}v - \frac{1}{p} \int_{\mathbb{T}^d \times \mathbb{R}^d} f^{p}\, \mathrm{div}_v(F\, m^p)\,\mathrm{d}x\,\mathrm{d}v \\
&= \int_{\mathbb{T}^d \times \mathbb{R}^d} f^{p} \left[\left(1 - \frac{1}{p}\right) \mathrm{div}_v F - F \cdot \frac{\nabla_v m}{m}\right] m^p \,\mathrm{d}x\,\mathrm{d}v.
\end{aligned}$$

For the first term T_1, we use integrations by part in v and the identity $m\nabla m^{-1} + m^{-1}\nabla m = 0$ in order to get with the notation $h = fm$

$$
\begin{aligned}
T_1 &= \int_{\mathbb{T}^d\times\mathbb{R}^d} h^{p-1}\, m\, \Delta_v \left(hm^{-1}\right) \mathrm{d}x\,\mathrm{d}v \\
&= -\int_{\mathbb{T}^d\times\mathbb{R}^d} \nabla_v \left(h^{p-1}\right)\cdot\left(\nabla_v h + hm\nabla_v m^{-1}\right) \mathrm{d}x\,\mathrm{d}v \\
&\quad -\int_{\mathbb{T}^d\times\mathbb{R}^d} h^{p-1}\nabla_v m\cdot\left(\nabla_v h\, m^{-1} + h\nabla_v m^{-1}\right) \mathrm{d}x\,\mathrm{d}v \\
&= -\int_{\mathbb{T}^d\times\mathbb{R}^d} \nabla_v h^{p-1}\cdot\nabla_v h\,\mathrm{d}x\,\mathrm{d}v + \left(1-\frac{2}{p}\right)\int_{\mathbb{T}^d\times\mathbb{R}^d} \left(\nabla_v h^p\cdot\nabla_v m\right) m^{-1}\,\mathrm{d}x\,\mathrm{d}v \\
&\quad -\int_{\mathbb{T}^d\times\mathbb{R}^d} h^p \left(\nabla_v m\cdot\nabla_v m^{-1}\right) \mathrm{d}x\,\mathrm{d}v \\
&= -(p-1)\int_{\mathbb{T}^d\times\mathbb{R}^d} |\nabla_v h|^2\, h^{p-2}\,\mathrm{d}x\,\mathrm{d}v \\
&\quad +\int_{\mathbb{T}^d\times\mathbb{R}^d} h^p \left[\left(\frac{2}{p}-1\right)\nabla_v\left(\frac{\nabla_v m}{m}\right) + \frac{|\nabla_v m|^2}{m^2}\right] \mathrm{d}x\,\mathrm{d}v \\
&= -(p-1)\int_{\mathbb{T}^d\times\mathbb{R}^d} |\nabla_v (fm)|^2\, (fm)^{p-2}\,\mathrm{d}x\,\mathrm{d}v \\
&\quad +\int_{\mathbb{T}^d\times\mathbb{R}^d} (fm)^p \left[\left(\frac{2}{p}-1\right)\frac{\Delta_v m}{m} + 2\left(1-\frac{1}{p}\right)\frac{|\nabla_v m|^2}{m^2}\right] \mathrm{d}x\,\mathrm{d}v.
\end{aligned}
$$

All together, we then have established

$$
\begin{aligned}
(3.18)\qquad &\int_{\mathbb{T}^d\times\mathbb{R}^d} (\mathcal{B}\, f)\, f^{p-1}\, m^p\,\mathrm{d}x\,\mathrm{d}v \\
&= -(p-1)\int_{\mathbb{T}^d\times\mathbb{R}^d} |\nabla_v (mf)|^2\, (mf)^{p-2}\,\mathrm{d}x\,\mathrm{d}v + \int_{\mathbb{T}^d\times\mathbb{R}^d} f^p\, m^p\, \psi^0_{m,p}\,\mathrm{d}x\,\mathrm{d}v,
\end{aligned}
$$

with

$$
\psi^0_{m,p} := \left(\frac{2}{p}-1\right)\frac{\Delta_v m}{m} + 2\left(1-\frac{1}{p}\right)\frac{|\nabla_v m|^2}{m^2} + \left(1-\frac{1}{p}\right)\mathrm{div}_v F - F\cdot\frac{\nabla_v m}{m} - M\,\chi_R.
$$

Introducing the notation $s := 0$, $\kappa := 1$ when $m = \langle v\rangle^k$ and $k := s$ when $m := e^{\kappa\,\langle v\rangle^s}$, we have

$$
\begin{aligned}
\psi^0_{m,p} &= \left(\frac{2}{p}-1\right)\left(\kappa k d\langle v\rangle^{s-2} + \kappa k(k-2)|v|^2\langle v\rangle^{s-4} + \kappa^2 s^2|v|^2\langle v\rangle^{2s-4}\right) \\
&\quad + \left(2-\frac{2}{p}\right)\kappa^2 k^2\,|v|^2\,\langle v\rangle^{2s-4} + \left(1-\frac{1}{p}\right)\left(d\langle v\rangle^{\gamma-2} + (\gamma-2)\,|v|^2\,\langle v\rangle^{\gamma-4}\right) \\
&\quad -\kappa k\,|v|^2\,\langle v\rangle^{\gamma+s-4} - M\,\chi_R
\end{aligned}
$$

which gives the asymptotic behaviors

$$
\psi^0_{m,p}(v) \underset{|v|\to\infty}{\sim} \kappa^2 s^2 |v|^{2s-2} - \kappa s\,|v|^{\gamma+s-2} \qquad\qquad \text{if } s>0
$$

$$
\psi^0_{m,p}(v) \underset{|v|\to\infty}{\sim} \left[\left(1-\frac{1}{p}\right)(d+\gamma-2) - k\right]|v|^{\gamma-2} \qquad\qquad \text{if } s=0.
$$

As a consequence, when $m = e^{\kappa \langle v\rangle^s}$, $\gamma \ge s > 0$, $\gamma + s \ge 2$, $\kappa > 0$ (with $\kappa < 1/\gamma$ if $s = \gamma$) we obtain

$$\psi^0_{m,p} \xrightarrow[v\to\infty]{} \kappa^2 - \kappa \qquad \text{if } \gamma = s = 1,$$

$$\psi^0_{m,p} \xrightarrow[v\to\infty]{} -\kappa s \qquad \text{if } \gamma + s = 2,\ s < \gamma,$$

$$\psi^0_{m,p} \xrightarrow[v\to\infty]{} -\infty \qquad \text{in the other cases.}$$

When $\gamma \ge 2$ and $m = \langle v\rangle^k$, we get

$$\psi^0_{m,p} \xrightarrow[v\to\infty]{} \left(1 - \frac{1}{p}\right) d - k \qquad \text{if } \gamma = 2,$$

$$\psi^0_{m,p} \xrightarrow[v\to\infty]{} -\infty \qquad \text{if } \gamma > 2 \ \text{ and } \ k > \left(1 - \frac{1}{p}\right)(d + \gamma - 2).$$

Observe that in all cases when $\gamma + s > 2$, we have

$$\psi^0_{m,p} \underset{|v|\to\infty}{\sim} -\theta \langle v\rangle^{\gamma+s-2}, \quad \text{for some constant } \theta > 0. \tag{3.19}$$

We have then proved the following estimate: for any $a > a_{p,m}$, $\theta' \in (0, a - a_0(p,m))$ small enough and $p \in [1,\infty)$, we then can choose R, M large enough in such a way that $\psi^0_{m,p}(v) \le a - \theta'$ for any $v \in \mathbb{R}^d$, and

$$\begin{aligned} \int_{\mathbb{T}^d\times\mathbb{R}^d} (\mathcal{B} f)\, f^{p-1}\, m^p \,\mathrm{d}x\,\mathrm{d}v \le a \int_{\mathbb{T}^d\times\mathbb{R}^d} |f|^p\, m^p \,\mathrm{d}x\,\mathrm{d}v \\ - \theta' \int_{\mathbb{T}^d\times\mathbb{R}^d} |f|^p\, m^p \langle v\rangle^{\gamma+s-2} \,\mathrm{d}x\,\mathrm{d}v - (1-p) \int_{\mathbb{T}^d\times\mathbb{R}^d} |\nabla_v (fm)|^2 (fm)^{p-2} \,\mathrm{d}x\,\mathrm{d}v. \end{aligned} \tag{3.20}$$

As a consequence and in particular, throwing out the two last terms, we have

$$\forall\, f \in L^p(m), \quad \|\mathscr{S}_{\mathcal{B}}(t) f\|_{L^p(m)} \le e^{at}\, \|f\|_{L^p(m)}.$$

Since $p \mapsto a_0(p,m)$ is increasing, we may pass to the limit as $p \to \infty$ in the above inequality and we thus conclude that $\mathcal{B} - a$ is dissipative in $L^p(m)$ for any $p \in [1,\infty]$ and any $a > a_0(p,m)$. $\square$

Lemma 3.9. *For any exponents $\gamma \ge 1$, $p \in [1,\infty]$, for any weight function m given by (3.3) or (3.4) and for any $a > a_1(m,p)$, we can choose R, M large enough in the definition (3.17) of $\mathcal{B}$ such that the operator $\mathcal{B} - a$ is hypodissipative in $W^{1,p}(m)$.*

Proof of Lemma 3.9. The decay of $\nabla_x \mathscr{S}_{\mathcal{B}}(t) f = \mathscr{S}_{\mathcal{B}}(t) \nabla_x f$ is poved as in Lemma 3.8 since x-derivatives commute with the equation. We hence have

$$\int_{\mathbb{T}^d\times\mathbb{R}^d} (\mathcal{B} f)\, f^{p-1}\, m^p \,\mathrm{d}x\,\mathrm{d}v \le a \int_{\mathbb{T}^d\times\mathbb{R}^d} |f|^p\, m^p \,\mathrm{d}x\,\mathrm{d}v,$$

$$\int_{\mathbb{T}^d\times\mathbb{R}^d} \partial_{x_i} (\mathcal{B} f)\, \partial_{x_i} f |\partial_{x_i} f|^{p-2}\, m^p \,\mathrm{d}x\,\mathrm{d}v \le a \int_{\mathbb{T}^d\times\mathbb{R}^d} |\partial_{x_i} f|^p\, m^p \,\mathrm{d}x\,\mathrm{d}v.$$

For any $i \in \{1,\dots,d\}$, we compute

$$\begin{aligned}
&\int_{\mathbb{T}^d\times\mathbb{R}^d} (\partial_{v_i}\mathcal{L}\, f)\, \partial_{v_i} f |\partial_{v_i} f|^{p-2}\, m^p \,\mathrm{d}x\,\mathrm{d}v \\
&\qquad = \int_{\mathbb{T}^d\times\mathbb{R}^d} \partial_{v_i} f |\partial_{v_i} f|^{p-2}\, m^p \,\left(\Delta_v \partial_{v_i} f + \sum_{j=1}^d \partial_{v_i}\partial_{v_j}\,(F_j\, f)\right) \mathrm{d}x\,\mathrm{d}v \\
&\qquad\qquad\qquad - \int_{\mathbb{T}^d\times\mathbb{R}^d} \partial_{x_i} f \partial_{v_i} f |\partial_{v_i} f|^{p-1} m^p \,\mathrm{d}x\,\mathrm{d}v =: T_1 + T_2 + T_3.
\end{aligned}$$

For the first term T_1, proceeding exactly as in the proof of Lemma 3.8, we find

$$\begin{aligned}
T_1 &= -(p-1) \int_{\mathbb{T}^d\times\mathbb{R}^d} |\nabla_v (m \partial_{v_i} f)|^2 \ |m \partial_{v_i} f|^{p-2} \ \mathrm{d}x\,\mathrm{d}v \\
&\quad + \int_{\mathbb{T}^d\times\mathbb{R}^d} |\partial_{v_i} f|^p\, m^p \,\left\{ \left(\frac{2}{p}-1\right) \frac{\Delta_v m}{m} + 2\left(1-\frac{1}{p}\right) \frac{|\nabla_v m|^2}{m^2} \right\} \mathrm{d}x\,\mathrm{d}v.
\end{aligned}$$

For the second term T_2, we have

$$\begin{aligned}
T_2 &= \int_{\mathbb{T}^d\times\mathbb{R}^d} \sum_{j=1}^d \left(\partial_{v_i}\partial_{v_j} F_j\, f + \partial_{v_i} F_j\, \partial_{v_j} f\right)\, \partial_{v_i} f |\partial_{v_i} f|^{p-2} m^p \,\mathrm{d}x\,\mathrm{d}v \\
&\qquad + \int_{\mathbb{T}^d\times\mathbb{R}^d} |\partial_{v_i} f|^p \,\left[(\mathrm{div}_v F)\, \left(1-\frac{1}{p}\right) - F\cdot \frac{\nabla_v m}{m}\right] m^p \,\mathrm{d}x\,\mathrm{d}v.
\end{aligned}$$

For the third term T_3, we use Young inequality to split it as

$$T_3 \le \varepsilon^{-1} \int_{\mathbb{T}^d\times\mathbb{R}^d} |\partial_{x_i} f|^p\, m^p \,\mathrm{d}x\,\mathrm{d}v + \varepsilon \int_{\mathbb{T}^d\times\mathbb{R}^d} |\partial_{v_i} f|^p\, m^p \,\mathrm{d}x\,\mathrm{d}v$$

where ε will later be chosen small.

Using the Young inequality, we get

$$\begin{aligned}
&\sum_i \int_{\mathbb{T}^d\times\mathbb{R}^d} (\partial_{v_i}\mathcal{B}\, f)\, \partial_{v_i} f |\partial_{v_i} f|^{p-2}\, m^p \,\mathrm{d}x\,\mathrm{d}v \\
&= \sum_i \left\{ T_1 + T_2 + T_3 - \int_{\mathbb{T}^d\times\mathbb{R}^d} \partial_{v_i} (M\, \chi_R\, f)\, \partial_{v_i} f |\partial_{v_i} f|^{p-2}\, m^p \,\mathrm{d}x\,\mathrm{d}v \right\} \\
&\le \int_{\mathbb{T}^d\times\mathbb{R}^d} \frac{|f|^p}{p'}\, Z\, m^p + \int_{\mathbb{T}^d\times\mathbb{R}^d} \psi^1_{m,p} \,\left(\sum_{i=1}^d |\partial_{v_i} f|^p\right) m^p \,\mathrm{d}x\,\mathrm{d}v \\
&\quad + \varepsilon^{-1} \int_{\mathbb{T}^d\times\mathbb{R}^d} \left(\sum_{i=1}^d |\partial_{x_i} f|^p\right) m^p \,\mathrm{d}x\,\mathrm{d}v,
\end{aligned}$$

with

$$Z := \sum_{i,j=1}^d |\partial_{v_i}\partial_{v_j} F_j| + (M/R)\, |(\mathrm{div}_v \chi)_R|$$

and

$$\psi^1_{m,p} := \frac{1}{p}\, Z + \frac{1}{p'} \sup_i \sum_j |\partial_{v_i} F_j| + \frac{1}{p} \sup_j \sum_i |\partial_{v_i} F_j| + \psi^0_{m,p} + \varepsilon.$$

On the one hand, the function Z is always negligible with respect to the dominant term in $\psi^0_{m,p}$ (which is $F\cdot\nabla_v \ln m$). On the other hand, we compute

$$\sup_i \sum_j |\partial_{v_i} F_j| \le \Big(1+\sqrt{d}\,(\gamma-2)\Big)\,\langle v\rangle^{\gamma-2},$$
$$\sup_j \sum_i |\partial_{v_i} F_j| \le \Big(1+\sqrt{d}\,(\gamma-2)\Big)\,\langle v\rangle^{\gamma-2}.$$

We deduce

$$\limsup \psi^1_{m,p} \le \limsup \tilde\psi^1_{m,p}$$

with

$$\tilde\psi^1_{m,p} := \Big(1+\sqrt{d}\,(\gamma-2)\Big)\,\langle v\rangle^{\gamma-2} + \psi^0_{m,p} + \varepsilon.$$

When $m = e^{\kappa\langle v\rangle^s}$, $\gamma\ge s>0$, $\gamma+s\ge 2$, $\gamma\ge 1$, $\kappa>0$, we observe that $\tilde\psi^1_{m,p}\sim_{v\to\infty}\psi^0_{m,p}$, and when $m=\langle v\rangle^k$, $\gamma\ge 2$, we observe that

$$\limsup_{v\to\infty}\tilde\psi^1_{m,p} \le 1 + \left(1-\frac{1}{p}\right) d - k + \varepsilon \quad \text{if } \gamma = 2,$$
$$\limsup_{v\to\infty}\tilde\psi^1_{m,p} = -\infty \quad \text{if } \gamma>2 \ \text{ and } \ k > 1+\sqrt{d}(\gamma-2) + \left(1-\frac{1}{p}\right)(d+\gamma-2).$$

Summing up, for any $a > a_1(p,m)$, $\eta\in(0,a-a_1(p,m))$ and $p\in[1,\infty)$, we can choose R, M large enough and ε small enough in such a way that $\psi^1_{m,p}(v)\le a-\eta$ for any $v\in\mathbb{R}^d$. We then have established the following estimate

$$\begin{aligned}
\sum_{i=1}^d \int_{\mathbb{T}^d\times\mathbb{R}^d} (\partial_{v_i}\mathcal{B}f)\,\partial_{v_i} f|\partial_{v_i} f|^{p-2}\, m^p \,\mathrm{d}x\,\mathrm{d}v \le&\\
\le C\int_{\mathbb{T}^d\times\mathbb{R}^d} |f|^p\, m^p\,\langle v\rangle^{\gamma-2}\,\mathrm{d}x\,\mathrm{d}v &+ C\int_{\mathbb{T}^d\times\mathbb{R}^d}\left(\sum_{i=1}^d |\partial_{x_i} f|^p\right) m^p\,\mathrm{d}x\,\mathrm{d}v\\
&+ a\int_{\mathbb{T}^d\times\mathbb{R}^d}\left(\sum_{i=1}^d |\partial_{v_i} f|^p\right) m^p\,\mathrm{d}x\,\mathrm{d}v\\
&- \frac{\theta}{2}\int_{\mathbb{T}^d\times\mathbb{R}^d}\left(\sum_{i=1}^d |\partial_{v_i} f|^p\right) m^p\,\langle v\rangle^{\gamma+s-2}\,\mathrm{d}x\,\mathrm{d}v\\
&- (p-1)\sum_{i,j=1}^d \int_{\mathbb{T}^d\times\mathbb{R}^d} |\partial_{v_i}((\partial_j f)m)|^2 \partial_{v_i} f|\partial_{v_i} f|^{p-2} m^p\,\mathrm{d}x\,\mathrm{d}v
\end{aligned}$$

where C depends on M and R.

As a consequence, any solution f to the linear evolution equation

$$\partial_t f = \mathcal{B}\, f, \quad f(0) = f_0 \in W^{1,p}(m)$$

satisfies

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d\times\mathbb{R}^d}\left(\sum_{i=1}^d |\partial_{v_i} f|^p\right)\frac{m^p}{p}\,\mathrm{d}x\,\mathrm{d}v &\le C\int_{\mathbb{T}^d\times\mathbb{R}^d} |f|^p\, m^p\,\langle v\rangle^{\gamma-2}\,\mathrm{d}x\,\mathrm{d}v\\
&+ C\int_{\mathbb{T}^d\times\mathbb{R}^d}\left(\sum_{i=1}^d |\partial_{x_i} f|^p\right) m^p\,\mathrm{d}x\,\mathrm{d}v + a\int_{\mathbb{T}^d\times\mathbb{R}^d}\left(\sum_{i=1}^d |\partial_{v_i} f|^p\right) m^p\,\mathrm{d}x\,\mathrm{d}v.
\end{aligned}$$

Defining the equivalent norm $\|\cdot\|_{\tilde W^{1,p}(m)}$ thanks to

$$\|f\|^p_{\tilde W^{1,p}(m)} := \|f\|^p_{L^p(m)} + \sum_{i=1}^d \|\partial_{x_i} f\|^p_{L^p(m)} + \zeta \sum_{i=1}^d \|\partial_{v_i} f\|^p_{L^p(m)}$$

and choosing $\zeta > 0$ small enough, we conclude thanks to Lemma 3.8 and the estimate (3.19) that $(\mathcal{B} - a)$ is dissipative in $\tilde W^{1,p}(m)$ for any $a > a_1(p,m)$ and $p \in (1,\infty)$, and therefore in $W^{1,p}(m)$ for any $a > a_1(p,m)$ and $p \in [1,\infty]$. $\square$

Lemma 3.10. *For any $p \in [1,\infty]$, for any force F given by* (3.2)*, any weight function m given by* (3.3) *or* (3.4)*, and for any $a > a_{-1}(m,p)$, we can choose R, M large enough in the definition* (3.17) *of $\mathcal{B}$ such that the operator $\mathcal{B} - a$ is hypodissipative in $W^{-1,p}(m)$.*

Proof of Lemma 3.10. We split the proof into three steps.

Step 1. We first observe that if

$$\mathcal{C} f := A\, f + B \cdot \nabla_v f + \Delta_v\, f - v \cdot \nabla_x f,$$

and we make the change of unknown $h := fm$ with $m = m(v)$, then the corresponding operator $\mathcal{C}_m h = m\, \mathcal{C}(m^{-1} h)$ writes

$$\mathcal{C}_m h := A_m\, h + B_m \cdot \nabla_v h + \Delta_v\, h - v \cdot \nabla_x h$$

with

$$A_m := \left[-\frac{\Delta_v m}{m} + 2\, \frac{|\nabla_v m|^2}{m^2} + A - B \cdot \frac{\nabla_v m}{m} \right], \qquad B_m := \left[B - 2\, \frac{\nabla_v m}{m} \right].$$

We also observe that the dual operator $\mathcal{C}^*$ writes

$$\mathcal{C}^* \phi := A^*\, \phi + B^* \cdot \nabla_v \phi + \Delta_v\, \phi + v \cdot \nabla_x \phi$$

with

$$A^* := (A - \mathrm{div}_v\, B), \ \ B^* := -B.$$

Defining

$$\mathcal{B} f := (\mathrm{div}_v F - M \chi_R)\, f + F \cdot \nabla_v f + \Delta_v f - v \cdot \nabla_x f$$

and using the two above identities, we get

$$\mathcal{B}^*_m \phi = \left[\frac{\Delta_v m}{m} - M \chi_R - F \cdot \frac{\nabla_v m}{m} \right] \phi - \left[F - 2\, \frac{\nabla_v m}{m} \right] \cdot \nabla_v \phi + \Delta_v \phi + v \cdot \nabla_x \phi. \tag{3.21}$$

Besides, any solution g to the equation

$$\partial_t g = \alpha\, g + \beta \cdot \nabla_v g + \Delta_v g \pm v \cdot \nabla_x g$$

satisfies at least formally (by performing two integrations by parts) the identity

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}^d \times \mathbb{R}^d} \frac{|g|^p}{p} \,\mathrm{d}x\,\mathrm{d}v = -(p-1) \int_{\mathbb{T}^d \times \mathbb{R}^d} |\nabla_v g|^2\, |g|^{p-2} \,\mathrm{d}x\,\mathrm{d}v + \int_{\mathbb{T}^d \times \mathbb{R}^d} \left(\alpha - \frac{\mathrm{div}_v \beta}{p} \right) |g|^p \,\mathrm{d}x\,\mathrm{d}v.$$

As a consequence, for ϕ solution to the equation

$$\partial_t \phi = \mathcal{B}^*_m \phi, \tag{3.22}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d\times\mathbb{R}^d}\frac{|\phi|^p}{p}\,\mathrm{d}x\,\mathrm{d}v \le \int_{\mathbb{T}^d\times\mathbb{R}^d}|\phi|^p\,\psi_{p,m}^2\,\mathrm{d}x\,\mathrm{d}v,$$

with

$$\psi_{p,m}^2 := \left(1-\frac{2}{p}\right)\frac{\Delta_v m}{m} + \frac{1}{p}\,\mathrm{div}_v F + \frac{2}{p}\,\frac{|\nabla_v m|^2}{m^2} - F\cdot\frac{\nabla_v m}{m} - M\chi_R. \tag{3.23}$$

Recalling that

$$\frac{\Delta_v m}{m} \underset{v\to\infty}{\sim} k\kappa\,(d+s-2)|v|^{s-2} + k^2\kappa^2\,|v|^{2s-2}, \qquad \mathrm{div}_v F \underset{v\to\infty}{\sim} (d+\gamma-2)\,|v|^{\gamma-2},$$

$$\frac{|\nabla_v m|^2}{m^2} \underset{v\to\infty}{\sim} \kappa^2 k^2\,|v|^{2s-2}, \qquad F\cdot\frac{\nabla_v m}{m} \underset{v\to\infty}{\sim} k\kappa|v|^{\gamma+s-2},$$

we have for an exponential weight function (so that $s>0$ and $k=s$)

$$\psi_{p,m}^2 \underset{v\to\infty}{\sim} \kappa^2 s^2\,|v|^{2s-2} - s\kappa|v|^{\gamma+s-2},$$

and for a polynomial weight function (so that $s=0$), we have

$$\psi_{p,m}^2 \underset{v\to\infty}{\sim} \left(\frac{d+\gamma-2}{p}-k\right)|v|^{\gamma-2},$$

with again $\psi_{p,m}^2(v)\sim -\theta\langle v\rangle^{\gamma+s-2}$ for large v when $\gamma+s>2$. In both case, we conclude that for any $a>a_{p,m}$

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\phi\|_{L^p(\mathbb{R}^d)}^p \le a\|\phi\|_{L^p(\mathbb{R}^d)}^p - \theta'\left\|\phi\langle\cdot\rangle^{(\gamma+s-2)/p}\right\|_{L^p(\mathbb{R}^d)}^p \tag{3.24}$$

for some small θ', uniformly when $p\to\infty$.

Step 2. Now, we write

$$\begin{aligned}
\partial_t(\partial_{v_i}\phi) &= \partial_{v_i}\mathcal{B}_m^*\phi \\
&= \Delta_v(\partial_{v_i}\phi) - \sum_{j=1}^d \partial_{v_i}\left(F_j - 2\,\frac{\partial_{v_j}m}{m}\right)\partial_{v_j}\phi - \left[F - 2\,\frac{\nabla_v m}{m}\right]\cdot\nabla_v(\partial_{v_i}\phi) \\
&\quad + \left[\frac{\Delta_v m}{m} - F\cdot\frac{\nabla_v m}{m} - M\,\chi_R\right]\partial_{v_i}\phi + \partial_{v_i}\left[\frac{\Delta_v m}{m} - F\cdot\frac{\nabla_v m}{m} - M\,\chi_R\right]\phi \\
&\quad - v\cdot\nabla_x\,(\partial_{v_i}\phi) - \partial_{x_i}\phi \\
&=: \Delta_v(\partial_{v_i}\phi) - \sum_{j=1}^d \partial_{v_i}B_{m,j}^*\,(\partial_{v_j}\phi) - B_m^*\cdot\nabla_v(\partial_{v_i}\phi) + A_m^*\,(\partial_{v_i}\phi) + (\partial_{v_i}A_m^*)\,\phi \\
&\quad - v\cdot\nabla_x\,(\partial_{v_i}\phi) - \partial_{x_i}\phi.
\end{aligned}$$

By integration by parts, we deduce

$$\begin{aligned}
\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d\times\mathbb{R}^d}\left(\sum_{i=1}^d |\partial_{v_i}\phi|^p\right)\mathrm{d}x\,\mathrm{d}v &= \sum_{i=1}^d \int_{\mathbb{T}^d\times\mathbb{R}^d} (\partial_t\partial_{v_i}\phi)\,\partial_{v_i}\phi|\partial_{v_i}\phi|^{p-2}\,\mathrm{d}x\,\mathrm{d}v \\
&\le \int_{\mathbb{T}^d\times\mathbb{R}^d}\left(A^*_m + \frac{1}{p}\,\mathrm{div}_v B^*_m\right)\left(\sum_{i=1}^d|\partial_{v_i}\phi|^p\right)\mathrm{d}x\,\mathrm{d}v \\
&\quad + \sum_{i=1}^d\int_{\mathbb{T}^d\times\mathbb{R}^d}\Big[(\partial_{v_i}A^*_m)\,\phi - (\partial_{v_i}B^*_{m,j})\,\partial_{v_j}\phi\Big]\,\partial_{v_i}\phi|\partial_{v_i}\phi|^{p-2} \\
&\quad + \sum_{i=1}^d\int_{\mathbb{T}^d\times\mathbb{R}^d}\partial_{x_i}\phi\,\partial_{v_i}\phi|\partial_{v_i}\phi|^{p-2}\,\mathrm{d}x\,\mathrm{d}v \\
&\le \int_{\mathbb{T}^d\times\mathbb{R}^d}\left(\frac{\varepsilon^p}{p} + \psi^3_{p,m} + \frac{1}{p}\sup_{i=1,\dots,d}|\partial_{v_i}A^*_m|\right)\left(\sum_{i=1}^d|\partial_{v_i}\phi|^p\right)\mathrm{d}x\,\mathrm{d}v \\
&\quad + \frac{1}{p'\varepsilon^{p'}}\int_{\mathbb{T}^d\times\mathbb{R}^d}\left(\sum_{i=1}^d|\partial_{x_i}\phi|^p\right)\mathrm{d}x\,\mathrm{d}v + \frac{1}{p'}\int_{\mathbb{T}^d\times\mathbb{R}^d}\left(\sum_i|\partial_{v_i}A^*_m|\right)|\phi|^p\,\mathrm{d}x\,\mathrm{d}v
\end{aligned}$$

where

$$\psi^3_{p,m} := \sup_{i=1,\dots,d}\sum_{j=1}^d|\partial_{v_j}B^*_{m,i}| + A^*_m + \frac{1}{p}\,\mathrm{div}_v B^*_m.$$

We have

$$\sup_{i=1,\dots,d}\sum_{j=1}^d|\partial_j B^*_{m,i}| \le \left(1+(\gamma-2)\sqrt{d}\right)\,|v|^{\gamma-2} + 2k\kappa\left(1+(s-2)\sqrt{d}\right)\,|v|^{s-2},$$

as well as

$$A^*_m(v) + \frac{1}{p}\,\mathrm{div}_v B^*_m(v) \sim \left(1-\frac{2}{p}\right)\frac{\Delta m}{m} + \frac{2}{p}\frac{|\nabla m|^2}{m^2} + \frac{1}{p}\,\mathrm{div}F - F\cdot\frac{\nabla m}{m} \sim \psi^0_{p,m}(v)$$

and

$$\partial_{v_i}A^*_m \sim \begin{cases} k(\gamma-2)(k+d-3) \\ \qquad\text{when } \gamma\ge 2 \text{ and } m(v)=\langle v\rangle^k, \\ \left[2\kappa^2 s^2 + \kappa^2 s^2(2s-4)\right]v_i|v|^{2s-4} - \left[2\kappa s + \kappa s(\gamma+s-4)\right]v_i|v|^{\gamma+s-4} \\ \qquad\text{when } \gamma\ge 1 \text{ and } m(v)=e^{\kappa\langle v\rangle^s}, \end{cases}$$

which yields

$$|\partial_{v_i}A^*_m| \lesssim W(v), \quad W(v) := \langle v\rangle^{\max\{2s,s+\gamma\}-3}.$$

For an exponential weight function (so that $s>0$), we have thus

$$\psi^3_{p,m}(v) \sim \psi^2_{p,m}(v) \sim \kappa^2 k^2\,|v|^{2s-2} - k\kappa|v|^{\gamma+s-2}$$

and for a polynomial weight function (so that $s=0$), we have

$$\limsup\psi^3_{p,m} \le \left(1+(\gamma-2)\sqrt{d} + \frac{d+\gamma-2}{p} - k\right)|v|^{\gamma-2}.$$

In both case, we conclude that for any $a > a_1(p,m)$ and for M, R large enough

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^d \|\partial_{v_i}\phi\|_{L^p}^p\right) \le a\left(\sum_{i=1}^d \|\partial_{v_i}\phi\|_{L^p}^p\right) + C\left(\sum_{i=1}^d \|\partial_{x_i}\phi\|_{L^p}^p\right) + C\|\phi\|_{L^p(W)}^p$$

for some C depending on a, uniformly when $p\to\infty$. Defining again the norm

$$\|\phi\|_{\tilde W^{1,p}(m)} := \|\phi\|_{L^p(m)} + \sum_{i=1}^d \|\partial_{x_i}\phi\|_{L^p(m)} + \zeta\sum_{i=1}^d \|\partial_{v_i}\phi\|_{L^p(m)}$$

for ζ small enough, equivalent to $W^{1,p}(m)$, and using that $W \le C\langle v\rangle^{\gamma+s-2}$, we obtain the following differential inequality

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\|\phi\|_{\tilde W^{1,p}(m)}^p \le a\,\|\phi\|_{\tilde W^{1,p}(m)}^p$$

uniformly as $p\to\infty$. We have thus proved

$$\forall\, t\ge 0,\ \forall\,\phi\in W^{1,p},\quad \|\mathscr{S}_{\mathcal{B}_m^*}(t)\phi\|_{W^{1,p}} \le C\,e^{at}\|\phi\|_{W^{1,p}}$$

for some $C>0$ (depending on a), uniformly as $p\to\infty$.

Step 3. For any $h\in W^{-1,p}$ and $\phi\in W^{1,p'}$, we have

$$\begin{aligned}\langle \mathscr{S}_{\mathcal{B}_m}(t)h,\phi\rangle &= \langle h, \mathscr{S}_{\mathcal{B}_m^*}(t)\phi\rangle\\ &\le \|h\|_{W^{-1,p}}\,\|\mathscr{S}_{\mathcal{B}_m^*}(t)\phi\|_{W^{1,p'}} \le C\,e^{at}\,\|h\|_{W^{-1,p}}\,\|\phi\|_{W^{1,p'}},\end{aligned}$$

so that

$$\forall\, h\in W^{-1,p},\quad \|\mathscr{S}_{\mathcal{B}_m}(t)\,h\|_{W^{-1,p}} \le C\,e^{at}\,\|h\|_{W^{-1,p}}.$$

Then, coming back to the operator $\mathcal{B}$, we conclude with

$$\|\mathscr{S}_{\mathcal{B}}(t)\,f\|_{W^{-1,p}(m)} \le Ce^{a\,t}\,\|f\|_{W^{-1,p}(m)},$$

so that $\mathcal{B}-a$ is hypodissipative in $W^{-1,p}(m)$ for any $1\le p\le\infty$. $\square$

We introduce for $\zeta>0$ the norm

$$\|\psi\|_{\mathcal{F}_\infty} := \max\left\{\|\psi\,\langle v\rangle^{-1}\|_{L^\infty}\,;\ \sup_{i=1,\dots,d}\|\partial_{x_i}\psi\|_{L^\infty}\,;\ \zeta\sup_{i=1,\dots,d}\|\partial_{v_i}\psi\|_{L^\infty}\right\},$$

and the associated space

$$\mathcal{F}_\infty := \left\{\psi\in W^{1,\infty}_{\mathrm{loc}};\ \|\psi\|_{\mathcal{F}_\infty}<\infty\right\}$$

and its dual $(\mathcal{F}_\infty)'$. Observe that

$$\begin{aligned}\|f\|_{L^1(\langle v\rangle)} &:= \sup_{\phi\in L^\infty,\,\|\phi\|_{L^\infty}\le 1}\int_{\mathbb{T}^d\times\mathbb{R}^d}\langle v\rangle\, f\,\phi\,\mathrm{d}x\,\mathrm{d}v\\ &= \sup_{\psi\in L^\infty_{\mathrm{loc}};\|\psi\langle v\rangle^{-1}\|_{L^\infty}\le 1}\int_{\mathbb{T}^d\times\mathbb{R}^d} f\,\psi\,\mathrm{d}x\,\mathrm{d}v,\end{aligned}$$

so that $L^1(\langle v\rangle)\subset(\mathcal{F}_\infty)'$.

Lemma 3.11. *Assume that $\gamma\in[2,2+1/(d-1))$, then for any*

$$a > \tilde a_\gamma := (d-1)(\gamma-2)-1,$$

(observe that $\tilde a_\gamma<0$ from the assumptions), we can choose R, M large enough in the definition (3.17) *of $\mathcal{B}$ such that the operator $\mathcal{B}-a$ is dissipative in $(\mathcal{F}_\infty)'$.*

Proof of Lemma 3.11. The proof is an adaptation of the proof of Lemma 3.10, and we sketch it briefly, writing only the needed formal a priori estimates.

Step 1. For any $\psi \in L^\infty_{\rm loc}$, we denote by $\psi_t := \mathscr{S}_{\mathcal{B}^*}(t)\psi$ the solution (when it exists) to the dual evolution equation

$$\partial_t \psi_t = \mathcal{B}^* \psi_t, \quad \psi_0 = \psi, \tag{3.25}$$

with

$$\mathcal{B}^* \psi := \Delta_v \psi - F \cdot \nabla_v \psi - M\, \chi_R \psi + v \cdot \nabla_x \psi.$$

Introducing the new unknown $\phi := \psi \langle v \rangle^{-1}$, we observe that when ψ_t is a solution to (3.25), then the associated function ϕ_t is a solution to the rescaled equation

$$\partial_t \phi_t = \langle v \rangle^{-1} \partial_t \psi_t = \langle v \rangle^{-1}\, \mathcal{B}^*(\langle v \rangle \phi_t) =: \mathcal{B}^*_{\langle \cdot \rangle} \phi_t, \quad \psi_0 = \psi, \tag{3.26}$$

where $\mathcal{B}^*_{\langle \cdot \rangle}$ is defined by (3.21).

Step 2. We calculate

$$\begin{aligned}\partial_t \partial_{v_i} \psi = \partial_{v_i} \mathcal{B}^* \psi = \Delta_v \partial_{v_i} \psi - (\partial_{v_i} F_j)\, \partial_{v_j} \psi - F_j\, \partial_{v_i} \partial_{v_j} \psi \\ - M\, \chi_R\, \partial_{v_i} \psi + M\, (\partial_{v_i} \chi_R)\, \psi + v \cdot \nabla_x (\partial_{v_i} \psi) + \partial_{x_i} \psi,\end{aligned}$$

with $F_j \sim v_j \langle v \rangle^{\gamma-2}$ and $\partial_{v_i} F_j \sim \delta_{ij} \langle v \rangle^{\gamma-2} + (\gamma-2)\, v_i v_j\, \langle v \rangle^{\gamma-4}$. We deduce

$$\begin{aligned}&\frac{1}{p} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}^d \times \mathbb{R}^d} \left(\sum_{i=1}^d |\partial_{v_i} \psi|^p \right) \mathrm{d}x\, \mathrm{d}v \\ &\quad \le \int_{\mathbb{T}^d \times \mathbb{R}^d} \Big[- \left(\delta_{ij}\, \langle v \rangle^{\gamma-2} + (\gamma-2)\, v_i v_j\, \langle v \rangle^{\gamma-4} \right)\, \partial_{v_j} \psi - v_j \langle v \rangle^{\gamma-2}\, \partial_{v_j} \partial_{v_i} \psi \\ &\quad - M\, \chi_R\, \partial_{v_i} \psi + M\, (\partial_{v_i} \chi_R)\, \psi + \partial_{x_i} \psi \Big]\, \partial_{v_i} \psi |\partial_{v_i} \psi|^{p-2}\, \mathrm{d}x\, \mathrm{d}v =: T_1 + \cdots + T_5,\end{aligned}$$

with the convention of summation of repeated indices. We compute

$$\begin{aligned}T_1 &= - \int_{\mathbb{T}^d \times \mathbb{R}^d} \langle v \rangle^{\gamma-2} \left(\sum_{i=1}^d |\partial_{v_i} \psi|^p \right) \mathrm{d}x\, \mathrm{d}v \\ T_2 &\le - \sum_{i=1}^d (\gamma-2) \int_{\mathbb{T}^d \times \mathbb{R}^d} \langle v \rangle^{\gamma-4}\, |v_i|^2\, |\partial_{v_i} \psi|^p\, \mathrm{d}x\, \mathrm{d}v \\ &\quad + \sum_{i \neq j} (\gamma-2) \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 \langle v \rangle^{\gamma-4} \left(\frac{1}{p} |\partial_{v_j} \psi|^p + \frac{1}{p'}\, |\partial_{v_i} \psi|^p \right) \mathrm{d}x\, \mathrm{d}v \\ &\le (d-1)(\gamma-2) \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 \langle v \rangle^{\gamma-4} \left(\sum_{i=1}^d |\partial_{v_i} \psi|^p \right) \mathrm{d}x\, \mathrm{d}v \\ T_3 &= -\frac{1}{p} \sum_{i,j=1}^d \int_{\mathbb{T}^d \times \mathbb{R}^d} v_j \langle v \rangle^{\gamma-2}\, \partial_{v_j} |\partial_{v_i} \psi|^p\, \mathrm{d}x\, \mathrm{d}v \\ &= \frac{1}{p} \sum_{i,j=1}^d \int_{\mathbb{T}^d \times \mathbb{R}^d} \left(\langle v \rangle^{\gamma-2} + (\gamma-2)\, v_i^2\, \langle v \rangle^{\gamma-4} \right) |\partial_{v_i} \psi|^p\, \mathrm{d}x\, \mathrm{d}v \\ &\le \frac{d}{p} (\gamma-1) \int_{\mathbb{T}^d \times \mathbb{R}^d} \langle v \rangle^{\gamma-2} \left(\sum_{i=1}^d |\partial_{v_i} \psi|^p \right) \mathrm{d}x\, \mathrm{d}v\end{aligned}$$

and

$$\begin{aligned} T_4 &= \int_{\mathbb{T}^d\times R^d} \left[-M\,\chi_R\,\partial_{v_i}\psi + M\,[(\partial_{v_i}\chi)(v/R)]\,\frac{\langle v\rangle}{R}\,\frac{\psi}{\langle v\rangle} \right] \partial_{v_i}\psi |\partial_{v_i}\psi|^{p-2}\,\mathrm{d}x\,\mathrm{d}v \\ &\le C\,M\,\|\psi\langle v\rangle^{-1}\|_{L^p} \left(\sum_{i=1}^d \|\partial_{v_i}\psi\|_{L^p}^{p-1} \right) \end{aligned}$$

and

$$T_5 \le \frac{\varepsilon^p}{p} \int_{\mathbb{T}^d\times\mathbb{R}^d} \left(\sum_{i=1}^d |\partial_{v_i}\psi|^p \right) \mathrm{d}x\,\mathrm{d}v + \frac{1}{p'\varepsilon^{p'}} \int_{\mathbb{T}^d\times R^d} \left(\sum_{i=1}^d |\partial_{x_i}\psi|^p \right) \mathrm{d}x\,\mathrm{d}v.$$

All in all, we have proved

$$\begin{aligned} &\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{T}^d\times\mathbb{R}^d} \left(\sum_{i=1}^d |\partial_{v_i}\psi|^p \right) \mathrm{d}x\,\mathrm{d}v \\ &\qquad \le \left[\frac{d(\gamma-1)}{p} + \frac{\varepsilon^p}{p} + (d-1)(\gamma-2) - 1 \right] \int_{\mathbb{T}^d\times\mathbb{R}^d} \left(\sum_{i=1}^d |\partial_{v_i}\psi|^p \right) \mathrm{d}x\,\mathrm{d}v \\ &\quad + \frac{1}{p'\varepsilon^{p'}} \int_{\mathbb{T}^d\times\mathbb{R}^d} \left(\sum_{i=1}^d |\partial_{x_i}\psi|^p \right) \mathrm{d}x\,\mathrm{d}v + C\,M\, \left\|\psi\langle v\rangle^{-1}\right\|_{L^p} \left(\sum_{i=1}^d \|\partial_{v_i}\psi\|_{L^p}^{p-1} \right). \end{aligned}$$

We recall that any solution ϕ_t of (3.26) satisfies (3.24). Fixing $a > (d-1)(\gamma-2)-1$, next $\zeta_0 > 0$ so that $a - \zeta_0 > (d-1)(\gamma-2)-1$, and then fixing M and R so that (3.24) holds with the choice $a-\zeta_0$, M, R, we have for any $\zeta \in (0,\zeta_0)$ and $K \ge 1$ the differential inequality

$$\begin{aligned} \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t} \left[\|\phi\|_{L^p}^p + \sum_{i=1}^d \|\partial_{x_i}\phi\|_{L^p}^p + \zeta \sum_{i=1}^d \|\partial_{v_i}\psi\|_{L^p}^p \right] &\le (a-\zeta_0) \left(\|\phi\|_{L^p}^p + \sum_{i=1}^d \|\partial_{x_i}\phi\|_{L^p}^p \right) \\ + \zeta \left[\left(\frac{d(\gamma-1)}{p} + (d-1)(\gamma-2) - 1 \right) \left(\sum_{i=1}^d \|\partial_{v_i}\psi\|_{L^p}^p \right) \right. &+ C\,M\,\frac{K^p}{p}\,\|\phi\|_{L^p}^p \\ &\left. + \frac{C\,M}{K} \left(\sum_{i=1}^d \|\partial_{x_i}\psi\|_{L^p}^p \right) \right]. \end{aligned}$$

Taking K, p large enough and then ζ small enough, we deduce

$$\begin{aligned} &\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t} \left[\|\phi\|_{L^p}^p + \sum_{i=1}^d \|\partial_{x_i}\phi\|_{L^p}^p + \zeta \sum_{i=1}^d \|\partial_{v_i}\psi\|_{L^p}^p \right] \\ &\qquad\qquad \le a \left[\|\phi\|_{L^p}^p + \sum_{i=1}^d \|\partial_{x_i}\phi\|_{L^p}^p + \zeta \sum_{i=1}^d \|\partial_{v_i}\psi\|_{L^p}^p \right] \end{aligned}$$

uniformly for p large. As a consequence, we get by Gronwall lemma and then passing to the limit $p \to \infty$

$$\|\mathscr{S}_{\mathcal{B}^*}(t)\psi\|_{\mathcal{F}_\infty} = \|\psi_t\|_{\mathcal{F}_\infty} \le e^{at}\,\|\psi\|_{\mathcal{F}_\infty}.$$

We conclude the proof by duality. $\square$

3.4. **Regularisation in the spatially periodic case.** We prove a regularization property of the kinetic Fokker-Planck equation related to the theory of hypoellipticity. It can be considered well-known and "folklore", but we include a sketch of proof for clarity and in order to make explicit the estimate. The argument follows closely the methods and discussions in [11] and [20, Section A.21].

Lemma 3.12. *The semigroup $\mathscr{S}_\mathcal{B}$ satisfies (with no claim of optimality on the exponents) first (gain of derivative in L^2 spaces)*

$$(\mathbf{1}) \quad \forall\, t \in [0,1],\ \forall\, k \in \mathbb{N}^* \quad \begin{cases} \|\mathscr{S}_\mathcal{B}(t) f\|_{H^k(\mu^{-1/2})} \lesssim \dfrac{1}{t^{3k/2}}\, \|f\|_{L^2(\mu^{-1/2})}, \\[2ex] \|\mathscr{S}_\mathcal{B}(t) f\|_{L^2(\mu^{-1/2})} \lesssim \dfrac{1}{t^{3k/2}}\, \|f\|_{H^{-k}(\mu^{-1/2})}. \end{cases}$$

second (gain of integrability at order zero)

$$(\mathbf{2}) \quad \forall\, t \in [0,1], \quad \begin{cases} \|\mathscr{S}_\mathcal{B}(t) f\|_{L^2(\mu^{-1/2})} \lesssim \dfrac{1}{t^{(5d+1)/2}}\, \|f\|_{L^1(\mu^{-1/2})}, \\[2ex] \|\mathscr{S}_\mathcal{B}(t) f\|_{L^\infty(\mu^{-1/2})} \lesssim \dfrac{1}{t^{(5d+1)/2}}\, \|f\|_{L^2(\mu^{-1/2})} \end{cases}$$

third (gain of integrability at order one)

$$(\mathbf{3}) \quad \forall\, t \in [0,1], \quad \begin{cases} \|\nabla \mathscr{S}_\mathcal{B}(t) f\|_{L^2(\mu^{-1/2})} \lesssim \dfrac{1}{t^{(5d+1)/2}}\, \|\nabla f\|_{L^1(\mu^{-1/2})}, \\[2ex] \|\nabla \mathscr{S}_\mathcal{B}(t) f\|_{L^\infty(\mu^{-1/2})} \lesssim \dfrac{1}{t^{(5d+1)/2}}\, \|\nabla f\|_{L^2(\mu^{-1/2})} \end{cases}$$

fourth (gain of integrability at ordre minus one)

$$(\mathbf{4}) \quad \forall\, t \in [0,1], \quad \begin{cases} \|\mathscr{S}_\mathcal{B}(t) f\|_{W^{-1,\infty}(\mu^{-1/2})} \lesssim \dfrac{1}{t^{(5d+1)/2}}\, \|f\|_{W^{-1,2}(\mu^{-1/2})}, \\[2ex] \|\mathscr{S}_\mathcal{B}(t) f\|_{W^{-1,2}(\mu^{-1/2})} \lesssim \dfrac{1}{t^{(5d+1)/2}}\, \|f\|_{W^{-1,1}(\mu^{-1/2})}. \end{cases}$$

Proof of Lemma 3.12. We only sketch the proof which is similar to the arguments developed in [11], see also [20, A.21.2 Variants], and in Lemma 3.7.

Step 1. Proof of inequality **(1)**. We only prove the case $k = 1$, higher exponents k are obtained by differentiating the equation and applying the same argument. We write down the energy estimates for the solution f, its first derivatives, and the product of the first derivatives

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|f\|_{L^2(\mu^{-1/2})} &\leq -\int_{\mathbb{T}^d\times\mathbb{R}^d} |\nabla_v(f/\mu)|^2\,\mu\,\mathrm{d}x\,\mathrm{d}v \\
\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_{x_i}f\|_{L^2(\mu^{-1/2})} &\leq -\int_{\mathbb{T}^d\times\mathbb{R}^d} |\nabla_v(\partial_{x_i}f/\mu)|^2\,\mu\,\mathrm{d}x\,\mathrm{d}v \\
\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_{v_i}f\|_{L^2(\mu^{-1/2})} &\leq -\int_{\mathbb{T}^d\times\mathbb{R}^d} |\nabla_v(\partial_{v_i}f/\mu)|^2\,\mu\,\mathrm{d}x\,\mathrm{d}v \\
&\quad -\int_{\mathbb{T}^d\times\mathbb{R}^d} \partial_{v_i}f\partial_{x_i}f\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v + \int_{\mathbb{T}^d\times\mathbb{R}^d} |\partial_{v_i}f|^2\,\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v \\
&\quad + \frac{M}{2}\int_{\mathbb{T}^d\times\mathbb{R}^d} \left|\partial^2_{v_i}\chi_R\right| |f|^2\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v \\
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^d\times\mathbb{R}^d}\partial_{x_i}f\partial_{v_i}f\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v &\leq -\int_{\mathbb{T}^d\times\mathbb{R}^d} |\nabla_x f|^2\,\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v \\
&\quad -2\int_{\mathbb{T}^d\times\mathbb{R}^d} \nabla_v\left(\partial_{v_i}f/\mu\right)\cdot\nabla_v\left(\partial_{x_i}f/\mu\right)\mu\,\mathrm{d}x\,\mathrm{d}v \\
&\quad + 2M\int_{\mathbb{T}^d\times\mathbb{R}^d} \chi_R\partial_x f\partial_v f\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v \\
&\quad + M\int_{\mathbb{T}^d\times\mathbb{R}^d} |\partial_{v_i}\chi_R|\,|f||\partial_x f|\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v.
\end{aligned}
$$

Observe also that

$$
\int_{\mathbb{T}^d\times\mathbb{R}^d} |\nabla_v(g/\mu)|^2\,\mu\,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbb{T}^d\times\mathbb{R}^d} |\nabla_v g|^2\,\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v + \int_{\mathbb{T}^d\times\mathbb{R}^d} |g|^2\left(\frac{|v|^2}{2}-d\right)\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v.
$$

Define the energy functional

$$
\begin{aligned}
\mathcal{F}(t,f_t) := A\|f_t\|^2_{L^2(\mu^{-1/2})} &+ at\|\nabla_v f_t\|^2_{L^2(\mu^{-1/2})} \\
&+ 2ct^2\langle\nabla_v f_t,\nabla_x f_t\rangle_{L^2(\mu^{-1/2})} + bt^3\|\nabla_x f_t\|^2_{L^2(\mu^{-1/2})}
\end{aligned}
$$

with $a, b, c > 0$, $c < \sqrt{ab}$ (positive definite) and A large enough, and compute from above

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t,f_t) \leq & -A\int_{\mathbb{T}^d\times\mathbb{R}^d} |\nabla_v(f_t/\mu)|^2\,\mu\,\mathrm{d}x\,\mathrm{d}v + a\|\nabla_v f_t\|^2_{L^2(\mu^{-1/2})} \\
& + 4ct\langle \nabla_v f_t, \nabla_x f_t\rangle_{L^2(\mu^{-1/2})} + 3bt^2\|\nabla_x f_t\|^2_{L^2(\mu^{-1/2})} \\
& - bt^3\sum_{i=1}^d \int_{\mathbb{T}^d\times\mathbb{R}^d} |\nabla_v(\partial_{x_i} f/\mu)|^2\,\mu\,\mathrm{d}x\,\mathrm{d}v - at\sum_{i=1}^d \int_{\mathbb{T}^d\times\mathbb{R}^d} |\nabla_v(\partial_{v_i} f/\mu)|^2\,\mu\,\mathrm{d}x\,\mathrm{d}v \\
& - at\sum_{i=1}^d \int_{\mathbb{T}^d\times\mathbb{R}^d} \partial_{v_i} f\,\partial_{x_i} f\,\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v + at\int_{\mathbb{T}^d\times\mathbb{R}^d} |\nabla_v f|^2\,\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v \\
& + \frac{atM}{2}\sum_{i=1}^d \int_{\mathbb{T}^d\times\mathbb{R}^d} \left|\partial^2_{v_i}\chi_R\right| |f|^2\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v - 2ct^2\int_{\mathbb{T}^d\times\mathbb{R}^d} |\nabla_x f|^2\,\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v \\
& - 4ct^2\sum_{i=1}^d \int_{\mathbb{T}^d\times\mathbb{R}^d} \nabla_v\left(\partial_{v_i} f/\mu\right)\cdot\nabla_v\left(\partial_{x_i} f/\mu\right)\mu\,\mathrm{d}x\,\mathrm{d}v \\
& + 4cMt^2\sum_{i=1}^d \int_{\mathbb{T}^d\times\mathbb{R}^d} \chi_R\,\partial_{x_i} f\,\partial_{v_i} f\,\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v + 2cMt^2\sum_{i=1}^d \int_{\mathbb{T}^d\times\mathbb{R}^d} |\partial_{v_i}\chi_R|\,|f||\partial_x f|\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v.
\end{aligned}$$

which implies when the compatible conditions $c < \sqrt{ab}$, $2c > 3b$ and $A >> a, b, c, M$ are satisfied:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t,f) \leq -K\left(\|\nabla_v f_t\|^2_{L^2(\mu^{-1/2})} + t^2\|\nabla_x f_t\|^2_{L^2(\mu^{-1/2})}\right) + C\int_{\mathbb{T}^d\times\mathbb{R}^d} f^2\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v$$

for some constants $K, C > 0$. Since the $L^2(\mu^{-1/2})$ norm is decreasing over $t \in [0,1]$ we deduce that

$$\forall\, t \in [0,1], \quad \mathcal{F}(t,f_t) \leq \mathcal{F}(0,f_0) + C\|f_0\|_{L^2(\mu^{-1/2})} \lesssim \mathcal{F}(0,f_0)$$

which yields the first part of **(1)** by simple iteration of this gain.

For the second part of (1) we first establish in a similar manner as above

$$\|\mathscr{S}_{\mathcal{B}^*}(t)f\|_{H^k(\mu^{-1/2})} \lesssim \frac{1}{t^{3k/2}}\,\|f\|_{L^2(\mu^{-1/2})}$$

which means

$$\left\|\mathscr{S}_{\mu^{-1/2}\mathcal{B}^*(\mu^{1/2}\cdot)}(t)h\right\|_{H^k} \lesssim \frac{1}{t^{3k/2}}\,\|h\|_{L^2}$$

and by duality

$$\left\|\mathscr{S}_{\mu^{-1/2}\mathcal{B}(\mu^{1/2}\cdot)}(t)h\right\|_{L^2} \lesssim \frac{1}{t^{3k/2}}\,\|h\|_{H^{-k}}$$

which means (according to our definition of weighted dual spaces)

$$\|\mathscr{S}_{\mathcal{B}}(t)f\|_{L^2(\mu^{-1/2})} \lesssim \frac{1}{t^{3k/2}}\,\|f\|_{H^{-k}(\mu^{-1/2})}.$$

Proof of inequality **(2)**. Since the norms we consider are propagated by the flow it is no loss of generality to reduce to $t \in [0,\eta]$, $0 < \eta << 1$. We introduce the quantity

$$\begin{aligned}\mathcal{G}(t,f) &:= B\|f\|^2_{L^1(\mu^{-1/2})} + t^Z \bar{\mathcal{F}}(t,f_t)\\ \bar{\mathcal{F}}(t,f_t) &:= \Big(A\|f\|^2_{L^2(\mu^{-1/2})} + at^2\|\nabla_v f\|^2_{L^2(\mu^{-1/2})}\\ &\qquad +2ct^4\langle \nabla_x f, \nabla_v f\rangle_{L^2(\mu^{-1/2})} + bt^6\|\nabla_x f\|^2_{L^2(\mu^{-1/2})}\Big)\end{aligned}$$

with $B >> A >> a, b, c$ and $c < \sqrt{ab}$ and $Z = (d+3)/2$.

A similar calculation as above yields, for well-chosen $A, a, b, c > 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{\mathcal{F}}(t,f_t) \le -K\left(\|\nabla_v f_t\|^2_{L^2(\mu^{-1/2})} + t^4\|\nabla_x f_t\|^2_{L^2(\mu^{-1/2})}\right) + C\int_{\mathbb{T}^d\times\mathbb{R}^d} f^2\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v$$

and we deduce

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}(t,f) &\le \frac{dB}{2}\|f\|^2_{L^1(\mu^{-1/2})} + Zt^{Z-1}\bar{\mathcal{F}}(t,f_t)\\ &\quad - Kt^Z\left(\|\nabla_v f_t\|^2_{L^2(\mu^{-1/2})} + t^4\|\nabla_x f_t\|^2_{L^2(\mu^{-1/2})}\right) + Ct^Z\int_{\mathbb{T}^d\times\mathbb{R}^d} f^2\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v.\end{aligned}$$

We choose η small enough so that $Zt^{Z+1} << Kt^Z$, and deduce

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}(t,f) \le \frac{dB}{2}\|f\|^2_{L^1(\mu^{-1/2})} - \frac{K}{2}t^Z\left(\|\nabla_v f_t\|^2_{L^2(\mu^{-1/2})} + t^4\|\nabla_x f_t\|^2_{L^2(\mu^{-1/2})}\right)\\ + C' t^{Z-1}\int_{\mathbb{T}^d\times\mathbb{R}^d} f^2\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v.\end{aligned}$$

for some other constant $C' > 0$.

The Nash inequality implies

(3.27)

$$\int_{\mathbb{T}^d\times\mathbb{R}^d} f^2\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v \lesssim_d \left(\int_{\mathbb{T}^d\times\mathbb{R}^d}|f|\mu^{-1/2}\,\mathrm{d}x\,\mathrm{d}v\right)^{\frac{4}{2d+2}}\left(\int_{\mathbb{T}^d\times\mathbb{R}^d}|\nabla_{x,v}(f\mu^{-1/2})|^2\,\mathrm{d}x\,\mathrm{d}v\right)^{\frac{2d}{2d+2}}$$

and using the Young inequality we have

$$\|f\|^2_{L^2(\mu^{-1/2})} \le C_{\varepsilon,d}t^{-5d}\,\|f\|^2_{L^1(\mu^{-1/2})} + \varepsilon t^5\,\|\nabla_{x,v}f\|^2_{L^2(\mu^{-1/2})},$$

for ε small and $C_{\varepsilon,d}$ depending on ε and the dimension d. Taking ε small we deduce

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{G}(t,f) \le \frac{dB}{2}\|f\|^2_{L^1(\mu^{-1/2})} + C'' t^{Z-1-5d}\|f\|^2_{L^1(\mu^{-1/2})}$$

for some constant $C'' > 0$. Finally choosing $Z = 5d+1$ we conclude that

$$\forall\, t\in[0,\eta], \quad \mathcal{G}(t,f_t) \le \mathcal{G}(0,f_0) + C\|f_0\|^2_{L^1(\mu^{-1/2})} \lesssim \mathcal{G}(0,f_0)$$

which yields the first part of **(2)**. The second part can be proved either by duality, or by using the inequality **(1)** with $k = d$ and Sobolev embedding (the constant is then slightly better: $t^{-3d/2}$ which has no consequence for the rest of the paper).

Proof of inequality **(3)**. The proof of the first part is similar to the proof of the first part of inequality **(2)** after differentiating the equation to get

$$\begin{aligned}&\partial_t\partial_{x_i}f + v\cdot\nabla_x\partial_{x_i}f = \nabla_v\cdot(\nabla_v\partial_{x_i}f + v\partial_{x_i}f)\\ &\partial_t\partial_{v_i}f + v\cdot\nabla_x\partial_{v_i}f = \nabla_v\cdot(\nabla_v\partial_{v_i}f + v\partial_{v_i}f) - \partial_{x_i}f + \partial_{v_i}f\end{aligned}$$

(observe that it involves no term of order zero derivative). The second part is proved by applying inequality **(1)** to the differentiated equation for $k = d$ together with Sobolev embedding.

Proof of inequality **(4)**. It follows from **(3)** by duality. □

Corollary 3.13. *For any $a > a_0$, there exist $n \geq 1$ and a constant such that for any spaces E and $\mathcal{E}$ of the type $W^{\sigma,p}(m)$ as defined above, there holds*

$$\forall\, t \geq 0, \quad \|T_n(t) f\|_E \lesssim e^{at}\, \|f\|_{\mathcal{E}}. \tag{3.28}$$

Proof of Corollary 3.13. The proof follows from the application of Lemma 2.4 and Lemma 3.12 that implies that $\mathcal{A}\mathscr{S}_{\mathcal{B}}(t)\mathcal{A}$ maps any $W^{\sigma,p}(m)$ to $H^d(\mu^{-1/2})$ with some constant $Ct^{-\Theta}$ with some $\Theta > 0$. □

3.5. **End of the proof of Theorem 3.1.** In the cases $\sigma \geq 0$, $1 \leq p < \infty$ and $\sigma = -1$, $1 < p \leq \infty$ estimate (3.7) is an immediate consequence of Theorem 1.1 together with Lemma 3.6, Lemma 3.4, Lemma 3.8, Lemma 3.9, Lemma 3.10, Lemma 3.7 and Lemma 2.4.

In the case $\sigma = 0$, $p = \infty$ so that $L^\infty(m)$ is not dense in $L^2(\mu^{-1/2})$ (for any choice of the weight m), we remark that for any $\varepsilon > 0$ (small enough) there exists p_ε and m_ε so that $L^\infty(m) \subset L^p(m_\varepsilon)$ for any $p \geq p_\varepsilon$, so that estimate (3.7) holds in $L^p(m_\varepsilon)$, then in $L^\infty(m_\varepsilon)$ by passing to the limit $p \to \infty$ and finally in $L^\infty(m)$ by passing to the limit $\varepsilon \to 0$. We handle the two last cases in (3.7) in a similar way.

In order to prove (3.9), we first observe that combining Theorem 1.1 together with Lemma 3.6, Lemma 3.4, Lemma 3.11 and Lemma 2.4, we have established

$$\|\mathscr{S}_{\mathcal{L}}(t) f_0 - \mathscr{S}_{\mathcal{L}}(t) g_0\|_{(\mathcal{F}_\infty)'} \leq C_a\, e^{at}\, \|f_0 - g_0\|_{(\mathcal{F}_\infty)'}.$$

Next, for any two probability measures f, g with bounded first moment, we have

$$\begin{aligned} W_1(f,g) &= \sup_{\|\nabla\phi\|_{L^\infty} \leq 1} \int_{\mathbb{R}^d} (f-g)\,\phi\,\mathrm{d}v \\ &= \sup_{\|\nabla\phi\|_{L^\infty} \leq 1} \int_{\mathbb{R}^d} (f-g)\,(\phi - \phi(0))\,\mathrm{d}v \\ &= \sup_{\max\{\|\langle v\rangle^{-1}\,\psi\|_{L^\infty}, \|\nabla\psi\|_{L^\infty}\} \leq 1} \int_{\mathbb{R}^d} (f-g)\,\psi\,\mathrm{d}v, \end{aligned}$$

where we have used the Kantorovich-Rubinstein theorem (see for instance [19, Theorem 1.14]) in the first line, the mass condition in the second line and the change of test functions $\psi := \phi - \phi(0)$ on the last line. As a consequence the W_1 distance and the distance associated to the duality norm $\|\cdot\|_{(\mathcal{F}_\infty)'}$ are equivalent, which ends the proof.

4. The kinetic Fokker-Planck equation with potential confinement

4.1. **Main result.** Consider the kinetic Fokker-Planck equation in the whole space with a space confinement potential

$$\begin{cases} \partial_t f = \mathcal{L} f := \mathcal{C} f + \mathcal{T} f, \\ \mathcal{C} f := \nabla_v \cdot (\nabla_v f + v\, f)\,, \\ \mathcal{T} f := -v \cdot \nabla_x f + G \cdot \nabla_v f, \end{cases} \tag{4.1}$$

on the density $f = f(t,x,v)$, $t \ge 0$, $x \in \mathbb{R}^d$, $v \in \mathbb{R}^d$, where the (exterior) force field $G = G(x) \in \mathbb{R}^d$ is given by

$$G(x) = \nabla_x \Psi(x) := x \, \langle x \rangle^{\beta-2} \quad \text{with} \quad \forall \, |x| \ge R_1, \ \Psi(x) := \frac{1}{\beta} \langle x \rangle^\beta + \Psi_0 \tag{4.2}$$

for some constants $R_1 > 0$ and $\beta \ge 1$.

The unique stationary state of the kinetic Fokker-Planck equation (4.1) is

$$\mu(x,v) = \exp(-\Psi(x) - |v|^2/2),$$

with the choice of the constant $\Psi_0 \in \mathbb{R}$ so that μ is a probability measure.

We define the Hamiltonian function

$$H(x,v) := 1 + \Psi(x) + \frac{|v|^2}{2},$$

and we consider the following assumptions:

Assumptions on the functional spaces

Polynomial weights: For $\beta \ge 2$ and $p \in [1,+\infty]$, we introduce the weight functions $m := H^k$ with $k > k(d,p)$ for some explicit $k(d,p) > d/p'$ from the proofs.

Stretched exponential weights: For any $\beta \ge 1$ and $p \in [1,+\infty]$, we introduce the weight functions $m := e^{\kappa H^s}$ with $s \in (0,1)$ and $\kappa > 0$.

Definition of the spaces: We then define on $\mathbb{R}^d \times \mathbb{R}^d$ the associated weighted Lebesgue spaces $\mathcal{E} := W^{\sigma,p}(m)$, $\sigma \in \{-1,0,+1\}$, $p \in [1,+\infty]$, in the same way as in the previous section in the case of $\mathbb{T}^d \times \mathbb{R}^d$.

For any $f \in \mathcal{E}$, the terms $\langle f \rangle$, $\langle\!\langle f \rangle\!\rangle$ and $\Pi_1^\perp f$ are defined as before.

Theorem 4.1. *Consider one of the spaces $\mathcal{E}$ defined above. Then there exists $a = a(\mathcal{E}) < 0$ such that for any $f_0, g_0 \in \mathcal{E}$ with same mass, the associated solutions f_t, g_t of the kinetic Fokker-Planck equation (4.1) satisfy*

$$\|f_t - g_t\|_\mathcal{E} \le C_a \, e^{at} \, \|f_0 - g_0\|_\mathcal{E},$$

which implies the relaxation to equilibrium

$$\|f_t - \langle\!\langle f_0 \rangle\!\rangle \, \mu\|_\mathcal{E} \le C_a \, e^{at} \, \|f_0 - \langle\!\langle f_0 \rangle\!\rangle \mu\|_\mathcal{E},$$

for some constructive constant $C_a > 0$.

Remarks 4.2.
(1) Such a semigroup spectral gap result for the kinetic Fokker-Planck equation in the whole space with a confining potential (and the same harmonic potential for the friction force acting on velocities) has been proved in the Sobolev spaces $H^\sigma(\mu^{-1/2})$, $\sigma \in \mathbb{N}^*$ in [10, 12, 20] and in the Lebesgue space $L^2(\mu^{-1/2})$ in [7, 6] (inspiring from [10]). These last references provide also constructive estimates.
(2) We did not include it in the statement for the sake of clarity but our method of proof can recover the semigroup growth estimate in $L^2(\mu^{-1/2})$ as a consequence of the known growth estimate in $H^1(\mu^{-1/2})$ proved in [10, 12, 20], which provides an alternative argument to those in [7, 6].
(3) We believe that Theorem 4.1 can be extended to to Sobolev space $W^{\sigma,p}$ for $\sigma = \pm 1$ by combining the new estimates in this section with the strategy of the previous section.

The strategy of the proof follows the same structure as in the previous section, and we start from the following H^1 spectral gap estimate that has been established in [20] for potentials Ψ under our assumptions, with constructive proof. See also [5, 12, 9] for previous results in that direction.

Theorem 4.3. ([20, Theorem 35]) *The result in Theorem 4.1 is true in the Hilbert space $H^1(\mu^{-1/2})$, and satisfy quantitative hypodissipativity estimate for the equivalent norm*

$$\Big(\|f\|^2_{L^2(\mu^{-1/2})} + a\|\nabla_x f\|^2_{L^2(\mu^{-1/2})} + b\|\nabla_v f\|^2_{L^2(\mu^{-1/2})} + 2c\langle \nabla_x f, \nabla_v f\rangle_{L^2(\mu^{-1/2})}\Big)^{1/2}$$

for appropriate choice of $a, b, c > 0$ with $c < \sqrt{ab}$.

4.2. **Dissipativity property of $\mathcal{B}$.** We define $\mathcal{A}$ and $\mathcal{B}$ as follows:

$$\mathcal{A}f := M\chi_R f, \qquad \mathcal{B}f := \mathcal{C}f + \mathcal{T}f - M\chi_R f \tag{4.3}$$

where $M > 0$, $\chi_R(x,v) = \chi(H(x,v)/R)$, $R > 1$, and $0 \le \chi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ is such that $\chi(x,v) = 1$ for any $|x|^2 + |v|^2 \le 1$.

We start with Lebesgue spaces:

Lemma 4.4. *We have:*

- (Polynomial weights) *For any $\beta \ge 2$, $p \in [1,+\infty]$ and $k > k(d,p)$ for some $k(d,p) > d/p'$ from the proof, there is $a < 0$ such that the operator $\mathcal{B} - a$ is dissipative in the space $L^p(H^k)$.*
- (Exponential weights) *For any $\beta \ge 1$, $p \in [1,\infty]$ and $s \in (0,1]$ (with the extra condition $\kappa < 1$ in the case $s = 1$), there is $a < 0$ such that the operator $\mathcal{B} - a$ is dissipative in the space $L^p(e^{\kappa H^s})$.*

Proof of Lemma 4.4. The proofs in the two cases will be similar: we give full details for the first case and less for the second case.

Step 1: Polynomial weight. Let us first consider $\beta \ge 2$ and $m(x,v) = H^k$, and the following weight multiplier:

$$W(x,v) := m\,w, \quad w := \left(1 + \frac{1}{2}\frac{x\cdot v}{H_\alpha}\right), \quad H_\alpha := 1 + \alpha\frac{\langle x\rangle^\beta}{\beta} + \frac{1}{\alpha}\frac{|v|^2}{2}.$$

Observe that $(x \cdot v) \le H_\alpha$ by Young's inequality (for any $\alpha > 0$ and $\beta \ge 2$), which proves that $w \in [1/2, 3/2]$ and $(1/2)m \le W \le (3/2)m$. We then consider a solution to the equation $\partial_t f = \mathcal{B}f$ and compute

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\mathbb{R}^d} f^p W^p \,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbb{R}^d\times\mathbb{R}^d} f^{p-1}\mathcal{C}f W^p\,\mathrm{d}x\,\mathrm{d}v + \int_{\mathbb{R}^d\times\mathbb{R}^d} f^{p-1}\mathcal{T}f W^p\,\mathrm{d}x\,\mathrm{d}v - M\int_{\mathbb{R}^d\times\mathbb{R}^d} f^p W^p\,\chi_R\,\mathrm{d}x\,\mathrm{d}v.$$

On the one hand, we recall the following computation picked up from the proof of Lemma 3.8

$$\begin{aligned}&\int_{\mathbb{R}^d\times\mathbb{R}^d} f^{p-1}\mathcal{C}fW^p\,\mathrm{d}x\,\mathrm{d}v := \int_{\mathbb{R}^d\times\mathbb{R}^d} f\,|f|^{p-2}\,(\mathcal{C}f)\,W^p\,\mathrm{d}x\,\mathrm{d}v\\ &= -(p-1)\int_{\mathbb{R}^d\times\mathbb{R}^d}|\nabla_v(Wf)|^2\,|Wf|^{p-2}\,\mathrm{d}x\,\mathrm{d}v\\ &+ \int_{\mathbb{R}^d\times\mathbb{R}^d}|f|^p\,W^p\,\left[\frac{2}{p'}\frac{|\nabla_v W|^2}{W^2} + \left(\frac{2}{p}-1\right)\frac{\Delta_v W}{W} + \frac{d}{p'} - \frac{v\cdot\nabla_v W}{W}\right]\mathrm{d}x\,\mathrm{d}v\end{aligned}$$

where $p' = p/(p-1)$, and we compute

$$\begin{aligned}
&\frac{\nabla_v m}{m} = \frac{kv}{H}, \quad \frac{\Delta_v m}{m} = \frac{kd}{H} + \frac{k(k-1)|v|^2}{H^2},\\
&\frac{\nabla_v W}{W} = \frac{\nabla_v m}{m} + \frac{\nabla_v w}{w}, \quad \frac{v\cdot\nabla_v W}{W} = \frac{v\cdot\nabla_v m}{m} + \frac{v\cdot\nabla_v w}{w},\\
&\frac{|\nabla_v W|^2}{W^2} \le 2\frac{|\nabla_v m|^2}{m^2} + 2\frac{|\nabla_v w|^2}{w^2}.
\end{aligned}$$

We deduce

$$\begin{aligned}
&\left[\frac{2}{p'}\frac{|\nabla_v W|^2}{W^2} + \left(\frac{2}{p}-1\right)\frac{\Delta_v W}{W} + \frac{d}{p'} - \frac{v\cdot\nabla_v W}{W}\right]\\
&\le \left[\frac{4}{p'}\frac{|\nabla_v m|^2}{m^2} + \left(\frac{2}{p}-1\right)\frac{\Delta_v m}{m} + \frac{d}{p'} - \frac{v\cdot\nabla_v m}{m}\right]\\
&+ \left[\frac{4}{p'}\frac{|\nabla_v w|^2}{w^2} + \left(\frac{2}{p}-1\right)\frac{\Delta_v w}{w} - \frac{v\cdot\nabla_v w}{w}\right] + 2\left(\frac{2}{p}-1\right)\frac{\nabla_v m\cdot\nabla_v w}{mw}\\
&\le \frac{d}{p'} + \frac{C_1}{\sqrt{H}} - \frac{1}{2}\frac{(x\cdot v)}{H_\alpha} + \frac{1}{2\alpha}\frac{(x\cdot v)|v|^2}{H_\alpha^2} - k\frac{|v|^2}{H}
\end{aligned}$$

for some constant $C_1 > 0$. The RHS is not negative at infinity, where infinity means $H >> 1$. This explains the need for the additional correction term w in W. We compute

$$\begin{aligned}
\int_{\mathbb{R}^d\times\mathbb{R}^d} f^{p-1}\mathcal{T} f W^p \,\mathrm{d}x\,\mathrm{d}v &:= \frac{1}{p}\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathcal{T}(f^p) W^p \,\mathrm{d}x\,\mathrm{d}v\\
&:= -\int_{\mathbb{R}^d\times\mathbb{R}^d} f^p W^{p-1}\mathcal{T} W \,\mathrm{d}x\,\mathrm{d}v := -\int_{\mathbb{R}^d\times\mathbb{R}^d} f^p W^p \frac{\mathcal{T} w}{w}\,\mathrm{d}x\,\mathrm{d}v
\end{aligned}$$

where we have used $\mathcal{T} H = 0$. We have then

$$-\frac{\mathcal{T} w}{w} \le C_2 + \frac{C_3}{\alpha^2}\frac{|v|^2}{H} - \frac{1}{4}\frac{\langle x\rangle^\beta}{H_\alpha}$$

by differentiating and using Young inequality and the form of the potential $\Psi(x)$ at infinity, for some constants $C_2, C_3 > 0$. We deduce by taking R, M large enough that for any $\eta > 0$ as small as wanted

$$\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\mathbb{R}^d} f^p W^p \,\mathrm{d}x\,\mathrm{d}v\\
&\le \int_{\mathbb{R}^d\times\mathbb{R}^d} f^p W^p \left[C(d,p) + \left(\frac{C_3}{\alpha^2} - k\right)\frac{|v|^2}{H} - \frac{1}{4}\frac{\langle x\rangle^\beta}{H_\alpha} - M\chi_R\right]\mathrm{d}x\,\mathrm{d}v
\end{aligned}$$

where we have used $w \approx 1$. Finally we restrict to $H \ge R$ with R large enough, and observe that we have either $|v|^2/2 \ge H/3$ or $|v|^2/2 \le H/3$, together with $|v|^2/2 + \langle x\rangle^\beta/\beta \ge 2H/3$. In the first case

$$C(d,p) + \left(\frac{C_3}{\alpha^2} - k\right)\frac{|v|^2}{H} \le C(d,p) + \frac{2}{3}\left(\frac{C_3}{\alpha^2} - k\right) \le -\frac{k}{2}$$

for k large enough. In the second case we only have $(C_3\alpha^{-2} - k)|v|^2/H \le 0$ but we can use the second negative term since now $\langle x\rangle^\beta/\beta \ge H/3$ and $H_\alpha^{-1} \ge 1/(\alpha H)$:

$$C(d,p) - \frac{1}{4}\frac{\langle x\rangle^\beta}{H_\alpha} \le C(d,p) - \frac{\beta}{12\alpha} \le -\frac{\beta}{24\alpha}$$

for α small enough. All in all we deduce finally, for k large enough and α small enough (depending on p and d) and M large enough:

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\mathbb{R}^d} f^p W^p \,\mathrm{d}x\,\mathrm{d}v \le -K\int_{\mathbb{R}^d\times\mathbb{R}^d} f^p W^p\,\mathrm{d}x\,\mathrm{d}v$$

for some constant $K>0$, which concludes the proof.

Step 2: (Stretched) exponential weight. Let us now consider $\beta\ge 1$, $s\in(0,1]$, $\kappa>0$ (with $\kappa<1$ in the case $s=1$) and $m(x,v)=e^{\kappa H^s}$, and the corrected weight:

$$W(x,v) := m\,w, \quad w := \left(1+\frac{1}{2}\frac{x\cdot v}{H_\alpha}\right), \quad H_\alpha := 1+\alpha\frac{\langle x\rangle^\beta}{\beta}+\frac{1}{\alpha}\frac{|v|^2}{2}$$

which satisfies again $W\approx m$. We compute as before

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} f^{p-1}\mathcal{C}fW^p\,\mathrm{d}x\,\mathrm{d}v = -(p-1)\int_{\mathbb{R}^d\times\mathbb{R}^d}|\nabla_v(Wf)|^2\,|Wf|^{p-2}\,\mathrm{d}x\,\mathrm{d}v$$
$$+\int_{\mathbb{R}^d\times\mathbb{R}^d}|f|^p\,W^p\,\left[\frac{|\nabla_v W|^2}{W^2}+\frac{d}{p'}-v\cdot\frac{\nabla_v W}{m}\right]\,\mathrm{d}x\,\mathrm{d}v$$

where $p'=p/(p-1)$, and we compute again

$$\left[\frac{2}{p'}\frac{|\nabla_v W|^2}{W^2}+\left(\frac{2}{p}-1\right)\frac{\Delta_v W}{W}+\frac{d}{p'}-\frac{v\cdot\nabla_v W}{W}\right]$$
$$\le\left[\frac{4}{p'}\frac{|\nabla_v m|^2}{m^2}+\left(\frac{2}{p}-1\right)\frac{\Delta_v m}{m}+\frac{d}{p'}-\frac{v\cdot\nabla_v m}{m}\right]$$
$$+\left[\frac{4}{p'}\frac{|\nabla_v w|^2}{w^2}+\left(\frac{2}{p}-1\right)\frac{\Delta_v w}{w}-\frac{v\cdot\nabla_v w}{w}\right]+2\left(\frac{2}{p}-1\right)\frac{\nabla_v m\cdot\nabla_v w}{mw}$$

with

$$\left[\frac{4}{p'}\frac{|\nabla_v m|^2}{m^2}+\left(\frac{2}{p}-1\right)\frac{\Delta_v m}{m}+\frac{d}{p'}-\frac{v\cdot\nabla_v m}{m}\right]\le\frac{d}{p'}+\frac{1}{H}\left[-\kappa s|v|^2H^s+\kappa^2s^2|v|^2H^{2s-1}\right]$$
$$\left[\frac{4}{p'}\frac{|\nabla_v w|^2}{w^2}+\left(\frac{2}{p}-1\right)\frac{\Delta_v w}{w}-\frac{v\cdot\nabla_v w}{w}\right]+2\left(\frac{2}{p}-1\right)\frac{\nabla_v m\cdot\nabla_v w}{mw}$$
$$\le\frac{C_1}{\sqrt{H}}-\frac{1}{2}\frac{(x\cdot v)}{H_\alpha}+\frac{1}{2\alpha}\frac{(x\cdot v)|v|^2}{H_\alpha^2}$$

for some constant $C_1>0$. Since $s\in(0,1)$ we have $2s-1<s$ and the term $\kappa^2s^2|v|^2H^{2s-1}$ is dominated by the previous negative term for M large enough. Then we have

$$-\frac{\mathcal{T}w}{w}\le C_2-C_3\frac{\langle x\rangle^{\beta-2}|x|^2}{H_\alpha}$$

for two other constants $C_2,C_3>0$. We deduce that

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\mathbb{R}^d} f^pW^p\,\mathrm{d}x\,\mathrm{d}v$$
$$\le\int_{\mathbb{R}^d\times\mathbb{R}^d} f^pW^p\left[C(d,p)-\kappa s\frac{|v|^2H^s}{H}-C_3\frac{\langle x\rangle^{\beta-2}|x|^2}{H_\alpha}-M\chi_R\right]\,\mathrm{d}x\,\mathrm{d}v$$

for some constant $C(d,p)>0$. Finally we restrict to $H\ge R$ with R large enough, and observe that we have either $|v|^2/2\ge H/3$ or $|v|^2/2\le H/3$, together with

$|v|^2/2 + \langle x\rangle^\beta/\beta \geq 2H/3$. In the first case

$$C(d,p) - \kappa s \frac{|v|^2 H^s}{H} \leq C(d,p) - \kappa s R^s \frac{|v|^2}{H} \leq C(d,p) - \frac{\kappa s R^s}{3} \leq -\frac{\kappa s R^s}{6}$$

for R large enough. In the second case we use the second negative term since now $\langle x\rangle^\beta/\beta \geq H/3$ and $H_\alpha^{-1} \geq 1/(\alpha H)$ and $|x| \geq (H/3)^{1/\beta} \geq (R/3)^{1/\beta}$ is non-zero:

$$C(d,p) - C_3 \frac{\langle x\rangle^{\beta-2}|x|^2}{H_\alpha} \leq C(d,p) - \frac{C_3}{2}\frac{\langle x\rangle^\beta}{H_\alpha} \leq C(d,p) - \frac{C_3\beta}{6\alpha} \leq -\frac{C_3\beta}{12\alpha}$$

for α small enough. All in all we deduce finally, for R large enough and α small enough (depending on p and d) and M large enough:

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\mathbb{R}^d} f^p W^p \,\mathrm{d}x\,\mathrm{d}v \leq -K \int_{\mathbb{R}^d\times\mathbb{R}^d} f^p W^p \,\mathrm{d}x\,\mathrm{d}v$$

for some constant $K > 0$, which concludes the proof. □

Then, by arguing exactly similarly as in Section 3 and using the previous calculations for the differentiated equation and the adjoint operators, we obtain the following lemma. We omit the proof in order not the repeat closely related technical estimates.

Lemma 4.5. *We have:*

- (Polynomial weights) *For any $\beta \geq 2$, $p \in [1,+\infty]$ and $k > k(d,p)$ for some $k(d,p) > d/p'$ from the proof, there is $a < 0$ such that the operator $\mathcal{B} - a$ is dissipative in the spaces $W^{1,p}(H^k)$ and $W^{-1,p}(H^k)$.*
- (Exponential weights) *For any $\beta \geq 1$, $p \in [1,\infty]$ and $s \in (0,1]$ (with the extra condition $\kappa < 1$ in the case $s = 1$), there is $a < 0$ such that the operator $\mathcal{B} - a$ is dissipative in the spaces $W^{1,p}(e^{\kappa H^s})$ and $W^{-1,p}(e^{\kappa H^s})$.*

Remark 4.6. Observe that the previous lemma implies the hypodissipativity of B in $H^1(\mu^{-1/2})$ (as needed in the application of our abstract theorem to this Fokker-Planck equation with confinement). This result could also be obtained easily by slightly modifying the proof of [20, Theorem 35]. It is also possible to deduce this hypodissipativity from that of L together with estimates quantifying the gain of decay at infinity in x and v. Since we could prove the latter estimates using the ideas developped in this paper, and they seem of independent interest and not available in the litterature, we include them in a short appendix.

4.3. **Regularisation estimates.** We prove a regularization property of the kinetic Fokker-Planck equation with a confining potential. It is again related to the theory of hypoellipticity, but is slightly less well-known due to the use of weighted norms defined in the whole space. The argument follows the same method as before.

Lemma 4.7. *The semigroup $\mathscr{S}_\mathcal{B}$ satisfies similar inequalities as in Lemma 3.12, where now $\mu = e^{-H}$ is the (x,v)-dependent equilibrium, and ℓ is a number large enough (note the additional H^ℓ weight in L^2 and L^1 norms)*

$$(\mathbf{1})\quad \forall\, t \in [0,1],\ \forall\, k \in \mathbb{N}^* \quad \begin{cases} \|\mathscr{S}_\mathcal{B}(t) f\|_{H^k(\mu^{-1/2})} \lesssim \dfrac{1}{t^{3k/2}}\, \|f\|_{L^2(H^{\ell/2}\mu^{-1/2})}, \\[2ex] \|\mathscr{S}_\mathcal{B}(t) f\|_{L^2(H^{-\ell/2}\mu^{-1/2})} \lesssim \dfrac{1}{t^{3k/2}}\, \|f\|_{H^{-k}(\mu^{-1/2})}. \end{cases}$$

second (gain of integrability at order zero)

$$(\mathbf{2}) \quad \forall\, t \in [0,1], \quad \left\{ \begin{array}{l} \|\mathscr{S}_{\mathcal{B}}(t) f\|_{L^2(\mu^{-1/2})} \lesssim \dfrac{1}{t^{(5d+1)/2}} \, \|f\|_{L^1(H^\ell \mu^{-1/2})}, \\[2ex] \|\mathscr{S}_{\mathcal{B}}(t) f\|_{L^\infty(H^{-\ell} \mu^{-1/2})} \lesssim \dfrac{1}{t^{(5d+1)/2}} \, \|f\|_{L^2(\mu^{-1/2})} \end{array} \right.$$

third (gain of integrability at order one)

$$(\mathbf{3}) \quad \forall\, t \in [0,1], \quad \left\{ \begin{array}{l} \|\nabla \mathscr{S}_{\mathcal{B}}(t) f\|_{L^2(\mu^{-1/2})} \lesssim \dfrac{1}{t^{(5d+1)/2}} \, \|\nabla f\|_{L^1(H^\ell \mu^{-1/2})}, \\[2ex] \|\nabla \mathscr{S}_{\mathcal{B}}(t) f\|_{L^\infty(H^{-\ell} \mu^{-1/2})} \lesssim \dfrac{1}{t^{(5d+1)/2}} \, \|\nabla f\|_{L^2(\mu^{-1/2})} \end{array} \right.$$

fourth (gain of integrability at ordre minus one)

$$(\mathbf{4}) \quad \forall\, t \in [0,1], \quad \left\{ \begin{array}{l} \|\mathscr{S}_{\mathcal{B}}(t) f\|_{W^{-1,\infty}(H^{-\ell} \mu^{-1/2})} \lesssim \dfrac{1}{t^{(5d+1)/2}} \, \|f\|_{W^{-1,2}(\mu^{-1/2})}, \\[2ex] \|\mathscr{S}_{\mathcal{B}}(t) f\|_{W^{-1,2}(\mu^{-1/2})} \lesssim \dfrac{1}{t^{(5d+1)/2}} \, \|f\|_{W^{-1,1}(H^\ell \mu^{-1/2})}. \end{array} \right.$$

Proof of Lemma 4.7. We only sketch the proof which is similar that of Lemma 3.12.

We begin with the first part of inequality **(1)**, in the case $k = 1$. We write down energy estimates for the solution f, its first derivatives, and the product of the first derivatives. On the energy estimate for f we add up a certain (large enough) power

ℓ of the Hamiltonian to the usual weight μ^{-1}:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\mathbb{R}^d} f^2 H^\ell \mu^{-1}\,\mathrm{d}x\,\mathrm{d}v &\leq -\int_{\mathbb{R}^d\times\mathbb{R}^d} \left|\nabla_v (f/\mu)\right|^2 H^\ell \mu\,\mathrm{d}x\,\mathrm{d}v + C\int_{\mathbb{R}^d\times\mathbb{R}^d} f^2 H^\ell \mu^{-1}\,\mathrm{d}x\,\mathrm{d}v \\
&\leq -\int_{\mathbb{R}^d\times\mathbb{R}^d} \left|\nabla_v f\right|^2 H^\ell \mu^{-1}\,\mathrm{d}x\,\mathrm{d}v + C\int_{\mathbb{R}^d\times\mathbb{R}^d} f^2 H^\ell \mu^{-1}\,\mathrm{d}x\,\mathrm{d}v \\
\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_{x_i} f\|_{L^2(\mu^{-1/2})} &\leq -\int_{\mathbb{R}^d\times\mathbb{R}^d} \left|\nabla_v (\partial_{x_i} f/\mu)\right|^2 \mu\,\mathrm{d}x\,\mathrm{d}v \\
&\quad + \int_{\mathbb{R}^d\times\mathbb{R}^d} |\partial_{x_i} F||\partial_{x_i} f||\partial_{v_i} f|\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v \\
\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_{v_i} f\|_{L^2(\mu^{-1/2})} &\leq -\int_{\mathbb{R}^d\times\mathbb{R}^d} \left|\nabla_v (\partial_{v_i} f/\mu)\right|^2 \mu\,\mathrm{d}x\,\mathrm{d}v \\
&\quad - \int_{\mathbb{R}^d\times\mathbb{R}^d} \partial_{v_i} f \partial_{x_i} f \mu^{-1}\,\mathrm{d}x\,\mathrm{d}v + \int_{\mathbb{R}^d\times\mathbb{R}^d} \left|\partial_{v_i} f\right|^2 \mu^{-1}\,\mathrm{d}x\,\mathrm{d}v \\
&\quad + \frac{M}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} \left|\partial_{v_i}^2 \chi_R\right| |f|^2 \mu^{-1}\,\mathrm{d}x\,\mathrm{d}v \\
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d\times\mathbb{R}^d} \partial_{x_i} f \partial_{v_i} f \mu^{-1}\,\mathrm{d}x\,\mathrm{d}v &\leq -\int_{\mathbb{R}^d\times\mathbb{R}^d} \left|\nabla_x f\right|^2 \mu^{-1}\,\mathrm{d}x\,\mathrm{d}v \\
&\quad - 2\int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla_v\left(\partial_{v_i} f/\mu\right)\cdot\nabla_v\left(\partial_{x_i} f/\mu\right)\mu\,\mathrm{d}x\,\mathrm{d}v \\
&\quad + \int_{\mathbb{R}^d\times\mathbb{R}^d} |\partial_{x_i} F||\partial_{v_i} f|^2 \mu^{-1}\,\mathrm{d}x\,\mathrm{d}v \\
&\quad + 2M\int_{\mathbb{R}^d\times\mathbb{R}^d} \chi_R \partial_x f \partial_v f \mu^{-1}\,\mathrm{d}x\,\mathrm{d}v \\
&\quad + M\int_{\mathbb{R}^d\times\mathbb{R}^d} \left|\partial_{v_i}\chi_R\right| |f||\partial_x f|\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v.
\end{aligned}
$$

We then define the energy functional

$$
\begin{aligned}
\mathcal{F}(t,f_t) := A\|f_t\|^2_{L^2(H^{\ell/2}\mu^{-1/2})} &+ at\|\nabla_v f_t\|^2_{L^2(\mu^{-1/2})} \\
&+ 2ct^2\langle \nabla_v f_t, \nabla_x f_t\rangle_{L^2(\mu^{-1/2})} + bt^3\|\nabla_x f_t\|^2_{L^2(\mu^{-1/2})}
\end{aligned}
$$

with $a, b, c > 0$, $c < \sqrt{ab}$ (positive definite) and A large enough, and compute from above

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t, f_t) \leq -A \int_{\mathbb{R}^d\times\mathbb{R}^d} |\nabla_v f_t|^2 H^\ell \mu^{-1} \,\mathrm{d}x\,\mathrm{d}v + C \int_{\mathbb{R}^d\times\mathbb{R}^d} f_t^2 H^\ell \mu^{-1} \,\mathrm{d}x\,\mathrm{d}v$$
$$+ a\|\nabla_v f_t\|^2_{L^2(\mu^{-1/2})} + 4ct\langle \nabla_v f_t, \nabla_x f_t\rangle_{L^2(\mu^{-1/2})} + 3bt^2 \|\nabla_x f_t\|^2_{L^2(\mu^{-1/2})}$$
$$- bt^3 \sum_{i=1}^d \int_{\mathbb{R}^d\times\mathbb{R}^d} |\nabla_v(\partial_{x_i} f/\mu)|^2 \mu \,\mathrm{d}x\,\mathrm{d}v - bt^3 \sum_{i=1}^d \int_{\mathbb{R}^d\times\mathbb{R}^d} |\partial_{x_i} F||\partial_{x_i} f||\partial_{v_i} f|\mu^{-1} \,\mathrm{d}x\,\mathrm{d}v$$
$$- at \sum_{i=1}^d \int_{\mathbb{R}^d\times\mathbb{R}^d} |\nabla_v(\partial_{v_i} f/\mu)|^2 \mu \,\mathrm{d}x\,\mathrm{d}v - at \sum_{i=1}^d \int_{\mathbb{R}^d\times\mathbb{R}^d} \partial_{v_i} f \partial_{x_i} f \mu^{-1} \,\mathrm{d}x\,\mathrm{d}v$$
$$+ at \int_{\mathbb{R}^d\times\mathbb{R}^d} |\nabla_v f|^2 \mu^{-1} \,\mathrm{d}x\,\mathrm{d}v + \frac{atM}{2} \sum_{i=1}^d \int_{\mathbb{R}^d\times\mathbb{R}^d} \left|\partial^2_{v_i}\chi_R\right| |f|^2 \mu^{-1} \,\mathrm{d}x\,\mathrm{d}v$$
$$- 2ct^2 \int_{\mathbb{R}^d\times\mathbb{R}^d} |\nabla_x f|^2 \mu^{-1} \,\mathrm{d}x\,\mathrm{d}v - 4ct^2 \sum_{i=1}^d \int_{\mathbb{R}^d\times\mathbb{R}^d} \nabla_v \left(\partial_{v_i} f/\mu\right) \cdot \nabla_v \left(\partial_{x_i} f/\mu\right) \mu \,\mathrm{d}x\,\mathrm{d}v$$
$$+4cMt^2 \sum_{i=1}^d \int_{\mathbb{R}^d\times\mathbb{R}^d} \chi_R \partial_{x_i} f \partial_{v_i} f \mu^{-1} \,\mathrm{d}x\,\mathrm{d}v + 2cMt^2 \sum_{i=1}^d \int_{\mathbb{R}^d\times\mathbb{R}^d} |\partial_{v_i}\chi_R| |f||\partial_x f|\mu^{-1} \,\mathrm{d}x\,\mathrm{d}v$$
$$2ct^2 \int_{\mathbb{R}^d\times\mathbb{R}^d} |\partial_{x_i} F||\partial_{v_i} f|^2 \mu^{-1} \,\mathrm{d}x\,\mathrm{d}v.$$

The additional terms as compared to Lemma 3.12 are treated as before using that $|\nabla_x F|^2, |\nabla_x F|^4 \lesssim H^\ell$ for ℓ large enough, and it implies when the compatible conditions $c < \sqrt{ab}$, $2c > 3b$ and $A >> a, b, c, M$ are satisfied:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}(t, f) \leq -K \left(\|\nabla_v f_t\|^2_{L^2(H^{\ell/2}\mu^{-1/2})} + t^2 \|\nabla_x f_t\|^2_{L^2(\mu^{-1/2})}\right) + C \int_{\mathbb{R}^d\times\mathbb{R}^d} f^2 \mu^{-1} \,\mathrm{d}x\,\mathrm{d}v$$

for some constants $K, C > 0$. Since the $L^2(\mu^{-1/2})$ norm is decreasing over $t \in [0, 1]$ we deduce that

$$\forall\, t \in [0, 1], \quad \mathcal{F}(t, f_t) \leq \mathcal{F}(0, f_0) + C\|f_0\|_{L^2(\mu^{-1/2})} \lesssim \mathcal{F}(0, f_0)$$

which yields the first part of **(1)** by simple iteration of this gain. The rest of the proof of similar to that of Lemma 3.12. □

The proof of the growth estimate on $T_n(t)$ and the completion of the proof of Theorem 4.1 are then done as in Corollary 3.13 and Theorem 3.1.

Appendix A. Quantitative compactness estimates on the resolvent

In this appendix we amplify the ideas of this article in order to give quantitative estimates of compactness on the resolvent of the kinetic Fokker-Planck considered. More precisely: One way to understand the compactness of resolvent is to split it into a local gain of *regularity* and a gain of *decay at infinity*, and we focus here on the gain of decay at infinity. The gain of regularity can then be recovered by local hypoelliptic estimates along the theory of Hörmander. Note that another route for deriving estimates on the gain of decay at infinity is to use the *global* hypoellipticity estimates as in [11] and [20, Section A.21] with Gaussian weight and *deduce* the gain of decay at infinity by applying some forms of "strenghtened" Poincaré inequality;

however the fractional derivatives involved would likely create technical difficulties, whereas our estimates based on weight multiplicators is elementary. Our estimates also do not require regularity on the solution.

Let us first say a word on the case of the periodic confinement. In this case it is enough to use the strenghtened Poincaré inequality in velocity only:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\mathbb{T}^d\times\mathbb{R}^d} f^2\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v &\le -\int_{\mathbb{T}^d\times\mathbb{R}^d}\left|\nabla_v\left(\frac{f}{\mu}\right)\right|\mu\,\mathrm{d}x\,\mathrm{d}v\\ \le -K\int_{\mathbb{T}^d\times\mathbb{R}^d} f^2(1+|v|^2)\mu^{-1} &+ C\int_{\mathbb{T}^d\times\mathbb{R}^d} f^2\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v\end{aligned}$$

for some constants $C, K > 0$, and therefore we deduce

$$-\langle\mathcal{L}f, f\rangle_{L^2(\mu^{-1})} \ge K\int_{\mathbb{T}^d\times\mathbb{R}^d} f^2(1+|v|^2)\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v - C\int_{\mathbb{T}^d\times\mathbb{R}^d} f^2\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v$$

and finally

$$\int_{\mathbb{T}^d\times\mathbb{R}^d} f^2(1+|v|^2)\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v \lesssim \|\mathcal{L}f\|^2_{L^2(\mu^{-1})} + \|f\|^2_{L^2(\mu^{-1})}$$

which gives the gain of decay at infinity on the resolvent. Combined with hypocoercivity estimates that provide bounds $\|f\|_{L^2(\mu^{-1})} \lesssim \|\mathcal{L}f-\xi\|_{L^2(\mu^{-1})}$ for certain $\xi\in\mathbb{C}$, this allows to control $\int f^2(1+|v|^2)\mu^{-1}$.

Let us now turn to the more interesting case of the potential confinement. We now differentiate the following norm

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} f^2 W\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v, \quad W(x,v) := \left[a|x|^{\beta/3} + b|v|^2 + 2c|x|^{\beta/6-1}(x\cdot v)\right],$$

for some appropriate choice of $a, b, c > 0$ so that $c < \sqrt{ab}$. Then

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\mathbb{T}^d\times\mathbb{R}^d} f^2|x|^{\beta/3}\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v\\ &\qquad\le -\int_{\mathbb{T}^d\times\mathbb{R}^d}\left|\nabla_v\left(\frac{f}{\mu}\right)\right||x|^{\beta/3}\mu\,\mathrm{d}x\,\mathrm{d}v + C_0\int_{\mathbb{T}^d\times\mathbb{R}^d} f^2|x|^{-\beta/3}|v|\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v\\ &\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\mathbb{T}^d\times\mathbb{R}^d} f^2|v|^2\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v\\ &\qquad\le -\int_{\mathbb{T}^d\times\mathbb{R}^d}\left|\nabla_v\left(\frac{f}{\mu}\right)\right||v|^2\mu\,\mathrm{d}x\,\mathrm{d}v + C_0\int_{\mathbb{T}^d\times\mathbb{R}^d} f^2(1+|v|^2)\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v\\ &\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\mathbb{T}^d\times\mathbb{R}^d} f^2|x|^{\beta/6-1}(x\cdot v)\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v\\ &\qquad\le -\int_{\mathbb{T}^d\times\mathbb{R}^d} f^2|x|^{2\beta/3}\mu\,\mathrm{d}x\,\mathrm{d}v + C_0\int_{\mathbb{T}^d\times\mathbb{R}^d} f^2\left(1+|x|^{-\beta/3}|v|^2\right)\mu^{-1}\,\mathrm{d}x\,\mathrm{d}v\\ &\qquad+ C_0\int_{\mathbb{T}^d\times\mathbb{R}^d}\left|\nabla_v\left(\frac{f}{\mu}\right)\right||x|^{\beta/6}|v|\mu\,\mathrm{d}x\,\mathrm{d}v\end{aligned}$$

for some constant $C_0 > 0$, which implies by using Young's inequality and adjusting the constants $a, b, c > 0$ that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d \times \mathbb{R}^d} f^2 W \mu^{-1} \,\mathrm{d}x \,\mathrm{d}v = \int_{\mathbb{R}^d \times \mathbb{R}^d} (\mathcal{L} f) f W \mu^{-1} \,\mathrm{d}x \,\mathrm{d}v$$
$$\leq -K \int_{\mathbb{R}^d \times \mathbb{R}^d} f^2 W^2 \mu^{-1} \,\mathrm{d}x \,\mathrm{d}v + C \int_{\mathbb{R}^d \times \mathbb{R}^d} f^2 \mu^{-1} \,\mathrm{d}x \,\mathrm{d}v$$

for some constants $C, K > 0$, and finally

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} f^2 W^2 \mu^{-1} \,\mathrm{d}x \,\mathrm{d}v \lesssim \|\mathcal{L} f\|^2_{L^2(\mu^{-1})} + \|f\|^2_{L^2(\mu^{-1})},$$

which is again a quantitative estimate of gain of decay at infinity for the resolvent.

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Stéphane Mischler
Université Paris-Dauphine & IUF
CEREMADE, UMR CNRS 7534
Place du Maréchal de Lattre de Tassigny
75775 Paris Cedex 16 FRANCE
e-mail: mischler@ceremade.dauphine.fr

Clément Mouhot
University of Cambridge
DPMMS, Centre for Mathematical Sciences
Wilberforce road, Cambridge CB3 0WA, UK
e-mail: C.Mouhot@dpmms.cam.ac.uk