Relaxation in time elapsed neuron network models in the weak connectivity regime

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Abstract

In order to describe the firing activity of a homogenous assembly of neurons, we consider time elapsed models, which give mathematical descriptions of the probability density of neurons structured by the distribution of times elapsed since the last discharge. Under general assumption on the firing rate and the delay distribution, we prove the uniqueness of the steady state and its nonlinear exponential stability in the weak connectivity regime. The result generalizes some similar results obtained in [10] in the case without delay. Our approach uses the spectral analysis theory for semigroups in Banach spaces developed recently by the first author and collaborators.

Keywords. Neuron networks, time elapsed dynamics, semigroup spectral analysis, weak connectivity, long time asymptotic.

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1 Introduction

In nervous systems, neuronal circuits carry out tasks of information transmission and processing. Many neurons generate trains of stereotyped electrical pulses in response to incoming stimulations. Following each discharge, the neuron undergoes a period of refractoriness during which it is less responsive to inputs, before recovering its excitability [11]. The main carrier of information is the discharge times or some statistics of the discharge times. In this work, we consider a simple neuronal model which neglects the mechanisms underlying spike generation and focusses on describing the neuronal dynamics in terms of discharge times. More precisely, we consider a model which has been introduced and studied in [3, 10, 11] and which describes the post-discharge recovery of the neuronal membranes through an instantaneous firing rate that depends on the time elapsed since the last discharge and the inputs by neurons. We refer to these papers for biologic motivation and discussions. We also refer to [1, 2, 13, 12] where these models (or similar ones) are obtained as a mean field limit of finite number of neuron network models.

The neuronal network is described here by the density number of neurons $f = f(t,x) \ge 0$ which at time $t \ge 0$ is in the state $x \ge 0$. The state of a neuron is a local time (or internal clock) which corresponds to the elapsed time since the last discharge. The dynamic of the neuron network is given by the following nonlinear time elapsed (or of age structured type) evolution equation

$$\partial_t f = -\partial_x f - a(x, \varepsilon\, m(t)) f =: \mathcal{L}_{\varepsilon m(t)} f, \tag{1.1a}$$

$$f(t,0) = p(t), \quad f(0,x) = f_0(x). \tag{1.1b}$$

Here $a(x,\varepsilon\,\mu) \ge 0$ represents the firing rate of a neuron in the state x for a network activity $\mu \ge 0$ and a network connectivity parameter $\varepsilon \ge 0$.

The function $p(t)$ represents the total density of neurons which undergo a discharge at time t and is defined through

$$p(t) := \mathcal{P}[f(t); m(t)],$$

where

$$\mathcal{P}[g, \mu] = \mathcal{P}_\varepsilon[g, \mu] := \int_0^\infty a(x, \varepsilon\mu) g(x) \mathrm{d}x.$$

The function $m(t)$ represents the network activity at time $t \geq 0$ resulting from earlier discharges and is defined by

$$m(t) := \int_0^\infty p(t-y) b(\mathrm{d}y),$$

where the delay distribution b is a probability measure which takes into account the persistence of the electric activity in the network resulting from discharges (synaptic integration). In the sequel, we will consider the two following situations :

- The *case without delay*, when $b = \delta_0$ and then $m(t) = p(t)$.
- The *case with delay*, when b is a smooth function.

We observe that in both cases, the solution f of the time elapsed equation (1.1) satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^\infty f(t,x) \mathrm{d}x = f(t,0) - \int_0^\infty a(x, \varepsilon m(t)) f(t,x) \mathrm{d}x = 0.$$

As a consequence, the total density number of neurons (also called *mass* in the sequel) is conserved and we can normalize that mass to be 1. In other words, we may always assume

$$\langle f(t,.) \rangle = \langle f_0 \rangle = 1, \quad \forall t \geq 0, \quad \langle g \rangle := \int_0^\infty g(x) \mathrm{d}x.$$

A (normalized) steady state for the time elapsed evolution equation (1.1) is a couple $(F_\varepsilon, M_\varepsilon)$ of a density number of neurons $F_\varepsilon = F_\varepsilon(x) \geq 0$ and a network activity $M_\varepsilon \geq 0$ such that

$$0 = -\partial_x F_\varepsilon - a(x, \varepsilon\, M_\varepsilon) F_\varepsilon = \mathcal{L}_{\varepsilon M_\varepsilon} F_\varepsilon, \tag{1.2a}$$

$$F_\varepsilon(0) = M_\varepsilon, \quad \langle F_\varepsilon \rangle = 1. \tag{1.2b}$$

It is worth emphasizing that for a steady state the associated network activity and discharge activity are two equal constants because of the normalization of the delay distribution, i.e. $\langle b \rangle = 1$.

Our main purpose in this paper is to prove that solutions to the time elapsed evolution equation (1.1) converge to a stationary state under a weak connectivity assumption. Before stating that result, let us present the precise mathematical assumptions we will need on the firing rate a and on the delay distribution b.

We make the physically reasonable assumption

$$\partial_x a \geq 0, \quad a' = \partial_\mu a \geq 0, \tag{1.3}$$

$$0 < a_0 := \lim_{x \to \infty} a(x, 0) \leq \lim_{x,\mu \to \infty} a(x, \mu) =: a_1 < \infty, \tag{1.4}$$

as well as the smoothness assumption

$$a \in W^{2,\infty}(\mathbb{R}^2_+). \tag{1.5}$$

In the delay case, we assume that $b(dy) = b(y)\, dy$ satisfies the smoothness and lost of memory conditions

$$\exists \delta > 0, \quad \int_0^\infty e^{\delta y} \left(b(y) + |b'(y)|\right) \mathrm{d}y < \infty. \tag{1.6}$$

Under the same assumption as above, the proof of existence (and uniqueness in the weak connectivity regime) are presented in the companion paper [16]. The main result we establish in the paper is the following long-time asymptotic result on the solutions.

Theorem 1.1. *We assume that the firing rate a satisfies* (1.3), (1.4) *and* (1.5). *We also assume that the delay distribution b satisfies $b = \delta_0$ or* (1.6). *There exists $\varepsilon_0 > 0$, small enough, such that for any $\varepsilon \in (0, \varepsilon_0)$ the steady state $(F_\varepsilon, M_\varepsilon)$ is unique. There also exists some constants $\alpha < 0$, $C \geq 1$ and $\eta > 0$ such that for any connectivity parameter $\varepsilon \in (0, \varepsilon_0)$ and any initial datum $0 \leq f_0 \in L^1$ with mass 1 and such that $\|f_0 - F_\varepsilon\|_{L^1} \leq \eta/\varepsilon$, the (unique, positive and mass conserving) solution f to the evolution equation* (1.1) *exhibited in [16] satisfies*

$$\|f(t,.) - F_\varepsilon\|_{L^1} \leq C e^{\alpha t}, \qquad \forall\, t \geq 0.$$

Theorem 1.1 extends some similar results obtained in [10, 11] in the case without delay and for a firing rate given by

$$a(x,\mu) = \mathbf{1}_{x>\sigma(\mu)}, \quad \sigma, \sigma^{-1} \in W^{1,\infty}(\mathbb{R}_+), \quad \sigma' \le 0.$$

It is worth mentioning that the above firing rate does not fall in the class of rate considered in the present paper because condition (1.5) is not met. On the other hand, we are able to tackle the case with delay, what it was not the case in [10, 11].

Our proof follows a strategy of "perturbation of semigroup spectral gap" initiated in [7] for studying long time convergence to the equilibrium for the homogeneous inelastic Boltzmann equation and used recently in [8] for a neuron network equation. More precisely, we introduce the linearized equation for the variation functions $(g, n, q) = (f, m, p) - (F_\varepsilon, M_\varepsilon, M_\varepsilon)$ around a stationary state $(F_\varepsilon, M_\varepsilon, M_\varepsilon)$, which writes

$$\partial_t g = -\partial_x g - a(x, \varepsilon\, M_\varepsilon) g - n(t)\, \varepsilon (\partial_\mu a)(x, \varepsilon M_\varepsilon) F_\varepsilon, \tag{1.7a}$$

$$g(t,0) = q(t), \quad g(0,x) = g_0(x), \tag{1.7b}$$

with

$$q(t) = \int_0^\infty a(x, \varepsilon\, M_\varepsilon) g \,\mathrm{d}x + n(t)\, \varepsilon \int_0^\infty (\partial_\mu a)(x, \varepsilon M_\varepsilon) F_\varepsilon \,\mathrm{d}x \tag{1.8}$$

and

$$n(t) := \int_0^\infty q(t-y) b(\mathrm{d}y). \tag{1.9}$$

We associate to that linear evolution equation a generator Λ_ε (which acts on an appropriate space to be specified in the two cases without and with delay) and its semigroup S_{Λ_ε}. It turns out that we may split the operator Λ_ε as

$$\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon,$$

for some α-hypodissipative operator $\mathcal{B}_\varepsilon$, $\alpha < 0$, and some bounded and $\mathcal{B}_\varepsilon$-power regular operator $\mathcal{A}_\varepsilon$ as defined in [15, 4, 9, 6]. In particular, the version of the Spectral Mapping Theorem of [9, 6] and the version of the Weyl's Theorem of [15, 4, 9, 6] imply that the semigroup S_{Λ_ε} as a finite dimensional dominant part. Moreover, the semigroup S_0 being positive, we may use the Krein-Rutman Theorem established in [9, 6] in order to get that the stationary state (F_0, M_0, M_0) is unique and exponentially stable. Using next a perturbative argument developed in [7, 14, 6], we get that the unique stationary state $(F_\varepsilon, M_\varepsilon, M_\varepsilon)$ is also exponentially stable in the weak connectivity regime. We conclude the proof of Theorem 1.1 by a somewhat classical nonlinear exponential stability argument.

Let us end the introduction by describing the plan of the paper. In Section 2, we introduce the strategy, we prove the stationary state result and we establish Theorem 1.1 in the case without delay. In section 3, we establish Theorem 1.1 in the case with delay.

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2 Case without delay

The present section is devoted to the proof of our main result Theorem 1.1 in the case of without delay.

2.1 The stationary problem

Theorem 2.1. *Assume* (1.3)-(1.4)-(1.5). *For any $\varepsilon \geq 0$, there exists at least one solution $(F_\varepsilon(x), M_\varepsilon) \in BV(\mathbb{R}_+) \times \mathbb{R}_+$ to the stationary problem* (1.2). *Moreover, there exists $\varepsilon_0 > 0$, small enough, such that the above solution is unique for any $\varepsilon \in [0, \varepsilon_0)$.*

Proof. Step 1. We prove the existence of a solution. We set

$$A(x,m) := \int_0^x a(y,m)\mathrm{d}y, \quad \forall, x, m \geq 0.$$

For any $m \geq 0$, we can solve the equation (1.2a), by writing

$$F_{\varepsilon,m}(x) := T_m e^{-A(x,\varepsilon m)},$$

where, $T_m \geq 0$ is chosen in order that $F_{\varepsilon,m}$ satisfies the mass normalized condition, namely

$$T_m^{-1} = \int_0^\infty e^{-A(x,\varepsilon m)}\mathrm{d}x.$$

In order to conclude the existence of a solution, we just have to find a real number $m = M_\varepsilon$ such that $m = F_{\varepsilon,m}(0) = T_m$. Equivalently, we need to find $M_\varepsilon \geq 0$ such that

$$\Phi(\varepsilon, M_\varepsilon) = 1, \tag{2.1}$$

where

$$\Phi(\varepsilon, m) = mT_m^{-1} := m\int_0^\infty e^{-A(x,\varepsilon m)}\mathrm{d}x.$$

From the assumption (1.4) of a, there exists $x_0 \in [0,\infty)$ such that $a(x,\mu) \geq \frac{a_0}{2}$, for any $x \geq 0$, $\mu \geq 0$, and therefore

$$\frac{a_0}{2}(x-x_0)_+ \leq A(x,\mu) \leq a_1 x, \quad \forall\, x \geq 0,\ \forall\, \mu \geq 0. \tag{2.2}$$

We deduce that $\Phi(\varepsilon, .)$ is a continuous function (from the Lebesgue dominated convergence theorem) and that $\Phi(0) = 0$, $\Phi(\infty) = \infty$. From the intermediate values theorem, we immediately conclude.

Step 2. We prove the uniqueness of the solution in the weak connectivity regime. Obviously, there exists a unique $M_0 := (\int_0^\infty e^{-A(x,0)}\mathrm{d}x)^{-1} \in (0,\infty)$ such that $\Phi(0, M_0) = 1$. Moreover, we compute

$$\frac{\partial}{\partial m}\Phi(\varepsilon,m) = \int_0^\infty e^{-A(x,\varepsilon m)}(1 - m\varepsilon\frac{\partial A}{\partial m}(x,\varepsilon m))\mathrm{d}x,$$

which is continuous as a function of the two variables because of (1.5). We then easily obtain that $\Phi \in C^1$. Since moreover

$$\frac{\partial}{\partial m}\Phi(\varepsilon,m)|_{\varepsilon=0} = \int_0^\infty e^{-A(x,0)}\mathrm{d}x > 0,$$

the implicit function theorem implies that there exists $\varepsilon_0 > 0$, small enough, such that the equation (2.1) has a unique solution for any $\varepsilon \in [0,\varepsilon_0)$. $\square$

Remark 2.2. *In the above proof, we do not need* (1.5) *but only the weaker smoothness assumption that A and $\partial_m A$ are continuous.*

2.2 Linearized equation and structure of the spectrum

To go one step further, we introduce the linearized equation around the stationary solution $(F_\varepsilon, M_\varepsilon)$. On the variation (g,n), the linearized equation writes

$$\partial_t g + \partial_x g + a_\varepsilon g + a'_\varepsilon F_\varepsilon n(t) = 0,$$
$$g(t,0) = n(t) = \int_0^\infty (a_\varepsilon g + a'_\varepsilon F_\varepsilon n(t))\,\mathrm{d}x, \quad g(0,x) = g_0(x),$$

with $a_\varepsilon := a(x, \varepsilon M_\varepsilon)$, $a'_\varepsilon := \varepsilon\,(\partial_\mu a)(x, \varepsilon M_\varepsilon)$. Since there exists $\varepsilon_0 > 0$, small enough, such that

$$\forall\, \varepsilon \in (0, \varepsilon_0) \qquad \kappa := \int_0^\infty a'_\varepsilon F_\varepsilon \mathrm{d}x < 1,$$

we may define

$$\mathcal{M}_\varepsilon[g] := (1-\kappa)^{-1} \int_0^\infty a_\varepsilon g\,\mathrm{d}x, \tag{2.3}$$

and the linearized equation is then equivalent to

$$\partial_t g + \partial_x g + a_\varepsilon g + a'_\varepsilon F_\varepsilon\, \mathcal{M}_\varepsilon[g(t,.)] = 0, \tag{2.4}$$
$$g(t,0) = \mathcal{M}_\varepsilon[g(t,.)], \quad g(0,x) = g_0(x). \tag{2.5}$$

To the above linear evolution equation, one can probably classically associates a semigroup acting on $L^1(\mathbb{R}_+)$. Here we use another approach by considering the boundary term as a source term, and then rewriting the equation as

$$\partial_t g = \Lambda_\varepsilon g := -\partial_x g - a_\varepsilon g - a'_\varepsilon F_\varepsilon \mathcal{M}_\varepsilon[g] + \delta_{x=0} \mathcal{M}_\varepsilon[g], \tag{2.6}$$

acting on the space of bounded Radon measures

$$X := M^1(\mathbb{R}_+) = \{g \in (C_0(\mathbb{R}))';\ \operatorname{supp} g \subset \mathbb{R}_+\},$$

endowed with the weak $*$ topology $\sigma(M^1, C_0)$. We also denoted by $BV(\mathbb{R}_+)$ the space of bounded variation measures.

Theorem 2.3. *Assume* (1.3)-(1.4)-(1.5) *and define* $\alpha := -a_0/2 < 0$. *The operator* Λ_ε *is the generator of a weakly $*$ continuous semigroup* S_{Λ_ε} *acting on* X *endowed with the weak $*$ topology* $\sigma(M^1, C_0)$. *Moreover, there exists a finite rank projector* $\Pi_{\Lambda_\varepsilon, \alpha}$ *which commutes with* S_{Λ_ε}, *an integer* $j \geq 0$ *and some complex numbers*

$$\xi_1, ..., \xi_j \in \Delta_\alpha := \{z \in \mathbb{C},\ \Re e\, z > \alpha\},$$

such that on $E_1 := \Pi_{\Lambda_\varepsilon,\alpha} X$ the restricted operator satisfies

$$\Sigma(\Lambda_{\varepsilon|E_1}) \cap \Delta_\alpha = \{\xi_1, ..., \xi_j\}$$

(with the convention $\Sigma(\Lambda_{\varepsilon|E_1}) \cap \Delta_\alpha = \emptyset$ when $j = 0$) and for any $a > \alpha$ there exists a constant C_a such that the remainder semigroup satisfies

$$\|S_{\Lambda_\varepsilon}(I - \Pi_{\Lambda_\varepsilon,\alpha})\|_{\mathscr{B}(X)} \le C_a e^{at}, \quad \forall\, t \ge 0.$$

The proof of the result is a direct consequence of the fact that the operator Λ_ε splits as $\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ where $\mathcal{A}_\varepsilon$ and $\mathcal{B}_\varepsilon$ are defined on X by

$$\mathcal{A}_\varepsilon g := \mu_\varepsilon \mathcal{M}_\varepsilon[g], \quad \mu_\varepsilon := \delta_0 - a'_\varepsilon F_\varepsilon, \tag{2.7}$$

$$\mathcal{B}_\varepsilon g := -\partial_x g - a_\varepsilon g, \tag{2.8}$$

for which can apply the Spectral Mapping Theorem of [9, 6] and the Weyl's Theorem of [15, 4, 9, 6]. The picture is not that simple, because Λ_ε does not generates a strongly continuous semigroup on X and then does not lie in the framework developed in [9]. However, we may apply the theory to an operator and its semigroup acting on $C_0(\mathbb{R})$ and then deduce the result by duality or simply observe that the above results extend straightforwardly to a weakly $*$ continuous framework. We refer for full details to the companion paper [16] as well as to [6] where the weak $*$ continuous framework is discussed.

We recall the definition of hypodissipativity introduced in [4]. We say that the generator L of a semigroup of bounded operators on a Banach space X is α-hypodissipative if there exists an equivalent norm $|\!|\!| \cdot |\!|\!|$ on X such that

$$\forall\, f \in D(\Lambda),\ \exists\, \varphi \in F_{|\!|\!|\cdot|\!|\!|}(f) \qquad \Re e\, \langle \varphi, (L - \alpha)\, f \rangle \le 0,$$

where, for any $f \in X$, the associated dual set $F_{|\!|\!|\cdot|\!|\!|}(f) \subset X'$ is defined by

$$F_{|\!|\!|\cdot|\!|\!|}(f) := \{\varphi \in X';\ \langle \varphi, f \rangle = |\!|\!| f |\!|\!|_X^2 = |\!|\!| \varphi |\!|\!|_{X'}^2\}.$$

We also recall that L is α-hypodissipative if, and only if, there exists a constant $M \ge 1$ such that the associated semigroup S_L satisfies the growth estimate

$$\|S_L(t)\|_{\mathscr{B}(X)} \le M\, e^{\alpha\, t}, \quad \forall\, t \ge 0,$$

where $\mathscr{B}(X)$ denotes the space of linear and bounded operators on X. We will sometime abuse by saying that S_L is α-hypodissipative when it satisfies the above growth estimate. We refer to [4, 6] for details.

We start with the properties of the two auxiliary operators.

Lemma 2.4. *Assume that a satisfies conditions (1.3)-(1.4). The operators $\mathcal{A}_\varepsilon$ and $\mathcal{B}_\varepsilon$ satisfy the following properties.*

(i) $\mathcal{A}_\varepsilon \in \mathscr{B}(X, Y)$, where $Y = \mathbb{C}\mu_\varepsilon \subset X$ with compact embedding.

(ii) $S_{\mathcal{B}_\varepsilon}$ is α-hypodissipative in X.

(iii) The family of operators $S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}$ satisfies

$$\|(S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})(t)\|_{X \to BV} \leq Ce^{\alpha t}, \quad \forall t \geq 0.$$

Proof. In order to shorten notation, we write $a(x) = a(x, \varepsilon M_\varepsilon)$ and $A(x) = A(x, \varepsilon M_\varepsilon)$.

(i) We obtain $\mathcal{A}_\varepsilon \in \mathscr{B}(X, Y)$ from the fact that $\mathcal{N}_\varepsilon[\cdot] \in \mathscr{B}(X, \mathbb{R})$ because $\|a\|_\infty \leq a_1$ from (1.4).

(ii) We write $S_{\mathcal{B}_\varepsilon}$ with the explicit formula

$$S_{\mathcal{B}_\varepsilon}(t)g(x) = e^{-A(x)}g(x-t)\mathbf{1}_{x-t\geq 0} =: S(t). \tag{2.9}$$

From the inequality (2.2) on A, we have

$$\begin{aligned} \|S_{\mathcal{B}_\varepsilon}(t)g\|_X &= \|e^{-A(x+t)}g(x)\|_X \\ &\leq \|e^{-\frac{a_0}{2}(t-x_0)_+}g(x)\|_X \\ &\leq Ce^{\alpha t}\|g(x)\|_X, \end{aligned}$$

with $C = e^{\frac{a_0 x_0}{2}} > 0$ and $\alpha := -\frac{a_0}{2} < 0$.

(iii) We have

$$\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(t)g = \mu_\varepsilon\, N(t),$$

with

$$N(t) := \mathcal{M}_\varepsilon[S(t)g] = (1-\kappa)^{-1}\int_0^\infty a(x)e^{-A(x)}g(x-t)\mathbf{1}_{x-t\geq 0}\,\mathrm{d}x \, \in C_b(\mathbb{R}_+),$$

because $a\, e^{-A} \in C_b(\mathbb{R}_+)$. Moreover, we have

$$|N(t)| \leq C\, a_1 \int_0^\infty e^{\alpha x}\, |g(x-t)|\mathbf{1}_{x-t\geq 0}\,\mathrm{d}x \leq C\, e^{\alpha t}\|g\|_X,$$

for any $t \geq 0$. We deduce

$$\begin{aligned}
(S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})(t)g(x) &= \int_0^t (S_{\mathcal{B}_\varepsilon}(s)\mu_\varepsilon)(x) N(t-s)\,\mathrm{d}s \\
&= \int_0^t e^{-A(x)}\,\mu_\varepsilon(x-s)N(t-s)\mathbf{1}_{x-s\geq 0}\,\mathrm{d}s \\
&= e^{-A(x)}\,(\mu_\varepsilon * \check{N}_t)(x),
\end{aligned}$$

with the classical notation $\check{N}_t(s) = N(t-s)$. Next, differentiating the above function, we get

$$\begin{aligned}
\partial_x[(S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})(t)g](x) &= -\int_0^t a(x)e^{-A(x)}\,\mu_\varepsilon(x-s)N(t-s)\mathbf{1}_{x-s\geq 0}\,\mathrm{d}s \\
&\quad -e^{-A(x)}\,\mu_\varepsilon(x-t)N(0)\mathbf{1}_{x-t\geq 0} \\
&\quad -\int_0^t e^{-A(x)}\,\mu_\varepsilon(x-s)N'(t-s)\mathbf{1}_{x-s\geq 0}\,\mathrm{d}s \\
&= -a(x)e^{-A(x)}(\mu_\varepsilon * \check{N}_t)(x) - e^{-A(x)}(\mu_\varepsilon * \check{N}'_t)(x) \\
&\quad -e^{-A(x)}\,\mu_\varepsilon(x-t)N(0)\mathbf{1}_{x-t\geq 0}
\end{aligned}$$

with $\check{N}'_t(s) = N'(t-s)$. We also compute

$$N'(t) = (1-\kappa)^{-1}\int_0^\infty \partial_x[a(x)e^{-A(x)}]g(x-t)\mathbf{1}_{x-t\geq 0}\,\mathrm{d}x = (1-\kappa)^{-1}[(a\,e^{-A})' * \check{g}(t)$$

with $\check{g}(x) = g(-x)$, so that $N' \in M^1(\mathbb{R}+)$, as a convolution of two bounded measures (remind that $\partial_x a \in \mathcal{D}'(\mathbb{R})$ and it is positive) and more precisely

$$\int_0^\infty |N'|e^{-\alpha t} \leq (1-\kappa)^{-1}\int_0^\infty |(ae^{-A})'|e^{-\alpha x}\int_{-\infty}^0 |\check{g}|e^{-\alpha x} < \infty.$$

From (1.5) and $(\mu_\varepsilon * \check{N}_t)(x) \in M^1(\mathbb{R}_+)$, we get $\partial_x[(S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})(t)g] \in M^1(\mathbb{R}_+)$ for any $t \geq 0$. We deduce that

$$\|\partial_x(S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})(t)g\|_X \leq Ce^{\alpha t}\|g\|_X,$$

and the similar estimate for $\|(S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})(t)g\|_X$. □

Proof of Theorem 2.3. In order to apply the Spectral Mapping Theorem [9, Theorem 2.1] and the Weyl's Theorem [9, Theorem 3.1] (see also the variant results in [6]), we only need to check that the following assumptions are satisfied.

(A1) Given some $\alpha \in \mathbb{R}$, for any $a > \alpha$ and $\ell \in \mathbb{N}$, there exists a positive constant $C_{a,\ell}$ such that the following growth estimate holds

$$\|S_{\mathcal{B}_\varepsilon} * (\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{(*\ell)}(t)\|_{\mathscr{B}(X)} \leq C_{a,\ell}\, e^{at}, \quad \forall t \geq 0. \tag{2.10}$$

It is obvious that (**A1**) is true for $\ell = 0$ from Lemma 2.4$-(ii)$. Since $(\mathcal{B}_\varepsilon - a)$ is hypodissipative in X for any $a > \alpha$ and $\mathcal{A}_\varepsilon \in \mathscr{B}(X)$, we get that (**A1**) holds for all $\ell \in \mathbb{N}$.

(A2) The operator $\mathcal{A}_\varepsilon$ is bounded and satisfies the estimate

$$\|S_{\mathcal{B}_\varepsilon} * (\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})(t)\|_{\mathscr{B}(X,D(\Lambda))} \leq C'_{a,1} e^{at}, \quad \forall t \geq 0, \tag{2.11}$$

holds for any $a > \alpha$ and a positive constant $C'_{a,1}$. That is nothing but Lemma 2.4$-(iii)$ observing that $D(\Lambda) = BV(\mathbb{R}_+)$.

(A3) The family of operators $(\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{(*2)}(t)$ satisfies the growth and compactness estimate

$$\int_0^\infty \|\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(t)\|_{X \to Y}\, e^{-at} \mathrm{d}t \leq C''_{2,a}, \quad \forall a > \alpha, \tag{2.12}$$

for some positive constant $C''_{2,a} \geq 0$ and some (separable) Banach space Y with compact embedding $Y \subset X$, what clearly holds true from Lemma 2.4$-(i)$&(iii) and $a > \alpha$. $\square$

2.3 The vanishing connectivity regime

When the network connectivity parameter vanishes, $\varepsilon = 0$, the linearized time elapsed operator simplifies

$$\Lambda_0 g = -\partial_x g - a(x,0)g + \delta_{x=0}\mathcal{M}_0[g], \tag{2.13}$$

where $\mathcal{M}_0[g] = \int_0^\infty a(x,0)g(x)\mathrm{d}x$.

Theorem 2.5. *There exist some constants $\alpha < 0$ and $C > 0$ such that $\Sigma(\Lambda_0) \cap \Delta_\alpha = \{0\}$ and for any $g_0 \in X$, $\langle g_0 \rangle = 0$, there holds*

$$\|S_{\Lambda_0}(t) g_0\|_X \leq C e^{\alpha t}\, \|g_0\|_X, \quad \forall\, t \geq 0. \tag{2.14}$$

We denote

$$X_+ := \{g \in M^1(\mathbb{R}_+);\ g \geq 0\}$$

the space of bounded and nonnegative Radon measures.

We start with two elementary auxiliary results.

Lemma 2.6. *S_{Λ_0} is positive: $S_{\Lambda_0}(t)g \in X_+$ for any $g \in X_+$ and any $t \geq 0$.*

Proof. We introduce a dual problem of (2.13) defined on the space $C_0(\mathbb{R})$ by

$$\partial_t \varphi = \tilde{\Lambda}\varphi = \tilde{\mathcal{B}}\varphi + \tilde{\mathcal{A}}\varphi \tag{2.15}$$

with

$$\tilde{\mathcal{B}}\varphi = \partial_x \varphi - a(x,0)\varphi, \quad \tilde{\mathcal{A}}\varphi = a(x,0)\varphi(0).$$

A solution φ to (2.15) then satisfies

$$\varphi(t) = S_{\tilde{\mathcal{B}}}(t)\varphi_0 + (S_{\tilde{\mathcal{B}}} * \tilde{\mathcal{A}}\varphi)(t).$$

Let us fix $\varphi_0 \in C_0(\mathbb{R})$ such that $\varphi_0 \leq 0$ and let us prove that $\varphi(t) \leq 0$ for any $t \geq 0$. We obviously have that $S_{\tilde{\mathcal{B}}}$ is a positive operator and it is a contraction in $C_0(\mathbb{R})$. Taking the positive part in (2.15) we get

$$\begin{aligned} \varphi_+(t) &\leq S_{\tilde{\mathcal{B}}}(t)\varphi_{0+} + (S_{\tilde{\mathcal{B}}} * \tilde{\mathcal{A}}\varphi_+)(t) \\ &\leq a_1 \int_0^t S_{\tilde{\mathcal{B}}}(t-s)\varphi_+(0)\,\mathrm{d}s, \end{aligned}$$

so that

$$\|\varphi_+(t)\|_{L^\infty} \leq C \int_0^t \|\varphi_+(s)\|_{L^\infty}\,\mathrm{d}s.$$

From the Grönwall lemma, we deduce that $\varphi_+(t) = 0$ for any $t \geq 0$ and then $\varphi \leq 0$. We conclude by observing that S_{Λ_0} is the dual of $S_{\tilde{\Lambda}}$. □

Lemma 2.7. *$-\Lambda_0$ satisfies the following version of the strong maximum principle: for any given $g \in X_+$ and $\mu \in \mathbb{R}$, there holds*

$$g \in D(\Lambda_0) \setminus \{0\} \text{ and } (-\Lambda_0 + \mu)g \geq 0 \text{ imply } g > 0.$$

Proof. Suppose that there holds $(-\Lambda_0 + \mu)g \geq 0$ for g satisfying the above conditions, it is only necessary to prove that g does not vanish in $\mathbb{R}_+$. Since $g \not\equiv 0$, there exists $x^* \in \mathbb{R}_+$ such that $g(x^*) > 0$. Rewrite the assumption as

$$\partial_x g + \big(a(x,0) + \mu\big)g \geq \delta_{x=0} \int_0^\infty a(x,0) g \mathrm{d}x,$$

and we observe that

$$\partial_x(e^{A(x,0)+\mu x} g) = e^{A(x,0)+\mu x}\Big(\partial_x g + \big(a(x,0)+\mu\big)g\Big) \geq 0. \tag{2.16}$$

(i) For $x \in (x^*, \infty)$, we have

$$e^{A(x,0)+\mu x} g \geq e^{A(x^*,0)+\mu x^*} g(x^*) > 0.$$

(ii) For $x \in (0, x^*)$, by integrate the same equation on $(0,x)$, we obtain

$$\begin{aligned} e^{A(x,0)+\mu x} g &\geq \int_0^x \delta_{x=0} e^{A(x,0)+\mu x} \int_0^\infty a(y,0)g(y)\mathrm{d}y\mathrm{d}x + g(0) \\ &\geq \int_0^\infty a(y,0)g(y)\mathrm{d}y. \end{aligned}$$

From the positivity assumption (1.4) on a and step (i), we have

$$\int_0^\infty a(y,0)g(y)\mathrm{d}y > \frac{a_0}{2}\int_{\max\{x_0,x^*\}}^\infty g(y)\mathrm{d}y > 0.$$

Therefore, g does not vanish on $(0,\infty)$. □

Proof of Theorem 2.5. First, we know from Theorem 2.1 that there exists at least one nonnegative and non-vanishing solution F_0 to the eigenvalue problem $\Lambda_0 F_0 = 0$ and the associated dual eigenvector is $\psi = 1$. Next, we observe that, defining the sign$f$ operator for $f \in D(\Lambda_0^2)$ by

$$[(\mathrm{sign} f)^*\,\psi](x) := \frac{1}{2|f(x)|}\,[\bar{f}(x)\psi(x) + f(x)\bar{\psi}(x)], \quad \forall\,\psi \in C_0(\mathbb{R}),$$

we have, for any $\psi \in C_0(\mathbb{R})_+$,

$$\begin{aligned} \Re e\langle(\mathrm{sign} f)\mathcal{M}_0[f]\,,\psi\rangle &= \Re e\langle\mathcal{M}_0[f]\,,(\mathrm{sign} f)^*\psi\rangle \\ &= \Re e\Big[\int a_0 f\,dx\Big]\frac{\Re e f(0)}{|f(0)|}\,\psi(0) \\ &\leq \int a_0|f|dx\,\psi(0) = \langle\mathcal{M}_0[|f|]\,,\psi\rangle, \end{aligned}$$

which is nothing but the complex Kato's inequality

$$\forall\, f \in D(\Lambda_0^2), \quad \Re e(\mathrm{sign} f)\,\Lambda_0 f \leq \Lambda_0|f|. \tag{2.17}$$

We also observe that $D(\Lambda_0^2) \subset C_b(0,\infty)$, and, as a consequence, $g \in D(\Lambda_0^2)$ and $|g| > 0$ implies $g > 0$ or $g < 0$. We then may use exactly the same argument as in [9, Proof of Theorem 5.3] (see also [6]):

- Kato's inequality (2.17) and the strong maximum principle imply that the eigenvalue $\lambda = 0$ is simple and the associated eigenspace is $\mathrm{Vect}(F_0)$;

- together with the fact that S_{Λ_0} is a positive semigroup, one deduces that $\lambda = 0$ is the only eigenvalue with nonnegative real part.

We conclude to the spectral gap estimate (2.14) with the help of Theorem 2.3. □

2.4 Weak connectivity regime - exponential stability of the linearized equation

We extend the exponential stability property which holds for a vanishing connectivity to the weak connectivity regime.

Theorem 2.8. *There exist some constants $\varepsilon_0 > 0$, $\alpha < 0$ and $C > 0$ such that for any $\varepsilon \in [0, \varepsilon_0]$ there hold $\Sigma(\Lambda_\varepsilon) \cap \Delta_\alpha = \{0\}$ and*

$$\|S_{\Lambda_\varepsilon}(t) g_0\|_X \le C e^{\alpha t} \|g_0\|_X, \quad \forall\, t \ge 0, \tag{2.18}$$

for any $g_0 \in X$, $\langle g_0 \rangle = 0$.

The proof uses the stability theory for semigroups developed in Kato's book [5] and revisited in [7, 14, 6]. Now, we present several results needed in the proof of Theorem 2.8.

Proof of Theorem 2.8. With the definitions (2.3), (2.7) and (2.8) of $\mathcal{M}_\varepsilon$, $\mathcal{A}_\varepsilon$ and $\mathcal{B}_\varepsilon$, we have

$$(\mathcal{B}_\varepsilon - \mathcal{B}_0) g = (a(x,0) - a(x, \varepsilon M_\varepsilon)) g$$

and

$$(\mathcal{A}_\varepsilon - \mathcal{A}_0) g = (\mathcal{M}_\varepsilon[g] - \mathcal{M}_0[g])\, \delta_0 - \varepsilon (\partial_\mu a)(x, \varepsilon M_\varepsilon)\, F_\varepsilon\, \mathcal{M}_\varepsilon[g].$$

Together with the smoothness assumption (1.5), we deduce

$$\|\mathcal{B}_\varepsilon - \mathcal{B}_0\|_{\mathscr{B}(X)} + \|\mathcal{A}_\varepsilon - \mathcal{A}_0\|_{\mathscr{B}(X)} \le C\, \varepsilon, \quad \forall\, \varepsilon \ge 0. \tag{2.19}$$

We then argue similarly as in the proof of [14, Theorem 2.15] (see also [5, 7, 6]) and therefore just sketch the proof. For the generator L of a semigroup S_L we define the resolvent set $\rho(L)$ and the spectrum set $\Sigma(L)$ by

$$\rho(L) := \{z \in \mathbb{C};\ L - z \text{ is a bijection}\}, \quad \Sigma(L) := \mathbb{C}\backslash\rho(L),$$

as well as the resolvent (operator) $R_L(z) := (L - z)^{-1}$ for any $z \in \rho(L)$. We now define

$$K_\varepsilon(z) := (\Lambda_\varepsilon - \Lambda_0)\, R_{\Lambda_0}(z)\, \mathcal{A}_\varepsilon R_{\mathcal{B}_\varepsilon}(z),$$

and we deduce from (2.19) and the estimates (i) and (ii) in Lemma 2.4 that for $\varepsilon_0 > 0$, small enough, and $C > 0$, we have $\|K_\varepsilon(z)\|_{\mathscr{B}(X)} \le C\varepsilon < 1$ for any $z \in \Delta_\alpha\backslash B(0,\eta)$, $\eta < |\alpha|$, and $\varepsilon \in (0, \varepsilon_0)$. That implies

$$R_{\Lambda_\varepsilon} = (R_{\mathcal{B}_\varepsilon} - R_{\Lambda_0}\mathcal{A}_\varepsilon R_{\mathcal{B}_\varepsilon})(I + K_\varepsilon)^{-1}$$

on $\Delta_\alpha\backslash B(0,\eta)$ and for any $\varepsilon \in (0, \varepsilon_0)$. In particular, $\Sigma(\Lambda_\varepsilon) \cap \Delta_\alpha \subset B(0,\eta)$. Using the definition of the eigenprojector Π_ε on the eigenspace associated to the spectral values of Λ_ε lying in $B(0,\eta)$ by mean of Dunford integral (see [5, Section III.6.4] or [4, 6]), namely

$$\Pi_\varepsilon := \frac{i}{2\pi}\int_{|z|=\eta} R_{\Lambda_\varepsilon}(z)\, dz,$$

we get

$$\|\Pi_\varepsilon - \Pi_0\|_{\mathscr{B}(X)} < 1.$$

From the classical result [5, Section I.4.6] (or more explicitly [14, Lemma 2.18]), we deduce that there exists $\xi_\varepsilon \in \Delta_\alpha$ such that

$$\Sigma(\Lambda_\varepsilon) \cap \Delta_\alpha = \{\xi_\varepsilon\}, \quad \xi_\varepsilon \text{ is a simple eigenvalue},$$

for any $\varepsilon \in [0, \varepsilon_0]$ (up to take a smaller real number $\varepsilon_0 > 0$). We conclude by observing that $\xi_\varepsilon = 0$ because $1 \in X'$ and $\Lambda_\varepsilon^* 1 = 0$ (which is nothing but the mass conservation). □

2.5 Weak connectivity regime - nonlinear exponential stability

Now, we focus on the nonlinear exponential stability of the solution to the evolution equation (1.1) in the case without delay. We start with an auxiliary result. We define the function $\Phi : L^1(\mathbb{R}_+) \times \mathbb{R} \to \mathbb{R}$ by

$$\Phi[g, \mu] := \int_0^\infty a(x, \varepsilon\mu) g(x)\, \mathrm{d}x - \mu.$$

We denote by W_1 the optimal transportation Monge-Kantorovich-Wasserstein distance on the probability measures set $\mathbf{P}(\mathbb{R}_+)$ associated to the distance $d(x, y) = |x - y| \wedge 1$, or equivalently defined by

$$\forall\, f, g \in \mathbf{P}(\mathbb{R}_+), \quad W_1(f, g) := \sup_{\varphi, \|\varphi\|_{W^{1,\infty}} \le 1} \int_0^\infty (f - g)\, \varphi.$$

Lemma 2.9. *Assume* (1.5). *There exists* $\varepsilon_0 > 0$ *and for any* $\varepsilon \in (0, \varepsilon_0)$ *there exists a function* $\varphi_\varepsilon : \mathbf{P}(\mathbb{R}) \to \mathbb{R}$ *which is Lipschitz continuous for the weak topology of probability measures and such that* $\mu = \varphi_\varepsilon[g]$ *is the unique solution to the equation*

$$\mu \in \mathbb{R}_+, \quad \Phi(g, \mu) = 0.$$

Proof of Lemma 2.9. *Step 1. Existence.* For any $g \in \mathbf{P}(\mathbb{R})$ we have $\Phi(g, 0) > 0$ and for any $g \in \mathbf{P}(\mathbb{R})$ and $\mu \ge 0$, we have

$$\Phi(g, \mu) \le \|a\|_{L^\infty} - \mu,$$

so that $\Phi(g, \mu) < 0$ for $\mu > \|a\|_{L^\infty}$. By the intermediate values theorem and the continuity property of Φ, for any fixed $g \in \mathbf{P}(\mathbb{R}_+)$ and $\varepsilon \ge 0$, there exists at least one solution $\mu \in (0, \|a\|_{L^\infty}]$ to the equation $\Phi(g, \mu) = 0$.
Step 2. Uniqueness and Lipschitz continuity. Fix $f, g \in \mathbf{P}(\mathbb{R}_+)$ and consider $\mu, \nu \in \mathbb{R}_+$ such that

$$\Phi(f, \mu) = \Phi(g, \nu) = 0.$$

We have

$$\nu - \mu = \int_0^\infty a(x, \varepsilon\nu)(g - f) + \int_0^\infty (a(x, \varepsilon\nu) - a(x, \varepsilon\mu)) f,$$

with

$$\Big| \int_0^\infty a(x, \varepsilon\nu)(g - f) \Big| \le \|a(\cdot, \varepsilon\nu)\|_{W^{1,\infty}}\, W_1(g, f),$$

and

$$\Big| \int_0^\infty \big(a(x, \varepsilon\nu) - a(x, \varepsilon\mu)\big) f \Big| \le \|a(\cdot, \varepsilon\nu) - a(\cdot, \varepsilon\mu)\|_{L^\infty} \le \varepsilon\, \|\partial_\mu a\|_{L^\infty} |\mu - \nu|.$$

We then obtain

$$|\mu - \nu| \, (1 - \varepsilon \|\partial_\mu a\|_{L^\infty}) \le \|a(\cdot, \varepsilon\nu)\|_{W^{1,\infty}} \, W_1(g, f), \tag{2.20}$$

and we may fix $\varepsilon_0 > 0$ such that $1 - \varepsilon_0 \|\partial_\mu a\|_{L^\infty} \in (0,1)$, $\varepsilon \in [0, \varepsilon_0]$. On the one hand, for $f = g$, we deduce that $\mu = \nu$ and that uniquely defines the mapping $\varphi_\varepsilon[g] := \mu$. On the other hand, the function is Lipschitz continuous because of (2.20). $\square$

We also recall the following classical Grönwall's type lemma.

Lemma 2.10. *Assume that $u \in C([0,\infty); \mathbb{R}_+)$ satisfies the integral inequality*

$$u(t) \le C_1 e^{at} \, u_0 + C_2 \int_0^t e^{a(t-s)} u(s)^2 \, \mathrm{d}s, \quad \forall \, t > 0,$$

for some constants $C_1 \ge 1$, $C_2, u_0 \ge 0$ and $a < 0$. Under the smallness assumption

$$a + 2C_2 u_0 < 0,$$

there holds

$$u(t) \le \Big(1 + \frac{C_1 u_0 C_2}{|a + 2C_2 u_0|}\Big) C_1 \, e^{at} \, u_0, \quad \forall \, t \ge 0.$$

Proof of Lemma 2.10. We fix $A \in (C_1 u_0, 2C_1 u_0)$, so that $C_1 u(t) \le A$ at least on a small interval, that is for any $t \in [0, \tau]$, $\tau > 0$ small enough, and then the integral inequality implies on the same interval

$$u(t) \le C_1 e^{at} \, u_0 + C_2 C_1^{-1} A \int_0^t e^{a(t-s)} u(s) \, \mathrm{d}s.$$

The classical Grönwall's lemma (for linear integral inequality) and the smallness assumption $a + C_2 C_1^{-1} A \le 0$ imply

$$u(t) \le C_1 \, u_0 \, e^{(a + C_2 C_1^{-1} A)t} \le C_1 \, u_0 < A$$

on that interval. By a continuity argument, the first above inequality holds on $\mathbb{R}_+$ and then with $A := C_1 u_0$. Next, replacing that first estimate in the integral inequality we started with, we get

$$u(t) \le C_1 e^{at} \, u_0 + C_2 C_1^2 u_0^2 e^{at} \int_0^t e^{(a + 2C_2 u_0)s} \, \mathrm{d}s, \quad \forall \, t > 0,$$

from which we immediately conclude. $\square$

We come to the proof of our main result Theorem 1.1 in the case without delay.

Proof of Theorem 1.1 in the case without delay. We split the proof into two steps.
Step 1. New formulation. We start giving a new formulation of the solutions to the evolution and stationary equations in the weak connectivity regime $\varepsilon \in (0, \varepsilon_0]$, where ε_0 is defined in Lemma 2.9. For a given initial datum $0 \le f_0 \in L^1(\mathbb{R}_+)$ with unit mass the solution $f \in C([0,\infty); L^1(\mathbb{R}_+))$ to the evolution equation (1.1) and the solution F_ε to the stationary equation (1.2) clearly satisfy

$$\begin{aligned} \partial_t f + \partial_x f + a(\varepsilon\varphi[f]) f = 0, &\qquad f(t,0) = \varphi[f(t,\cdot)], \\ \partial_x F + a(\varepsilon M) F = 0, &\qquad F(0) = M = \varphi[F], \end{aligned}$$

where here and below the ε and x dependency is often removed without risk of misleading.

We introduce the variation function $g := f - F$ which satisfies the PDE

$$\partial_t g = -\partial_x g - a(\varepsilon M) g - \varepsilon a'(\varepsilon M) F\, \mathcal{M}[g] - Q[g] \tag{2.21}$$

with

$$Q[g] := a(\varepsilon\varphi[f]) f - a(\varepsilon\varphi[F]) F - a(\varepsilon\varphi[F]) g - \varepsilon a'(\varepsilon\varphi[F]) F\, \mathcal{M}[g],$$

where $\mathcal{M} = \mathcal{M}_\varepsilon$ is defined in (2.3). The above PDE is complemented with the boundary condition

$$g(t,0) \ = \ \varphi[f(t,\cdot)] - \varphi[F],$$

and we may write again

$$\begin{aligned} \varphi[f] - \varphi[F] &= \int_0^\infty a(\varepsilon\varphi[f]) f - \int_0^\infty a(\varepsilon\varphi[F]) F \\ &= \int_0^\infty \big(a(\varepsilon M) g + \varepsilon a'(\varepsilon M) F\, \mathcal{M}[g] \big) + \int_0^\infty Q[g]\, \mathrm{d}x \\ &= \mathcal{M}[g] + \mathcal{Q}[g], \qquad \mathcal{Q}[g] := \langle Q[g] \rangle. \end{aligned}$$

As a consequence, we have proved that the variation function g satisfies the equation

$$\partial_t g = \Lambda_\varepsilon g + Z[g], \quad Z[g] := -Q[g] + \delta_0 \mathcal{Q}[g]. \tag{2.22}$$

Step 2. The nonlinear term. On the one hand, we obviously have

$$\langle Z[g] \rangle = 0, \quad \forall\, g \in M^1(\mathbb{R}_+). \tag{2.23}$$

On the other hand, in order to get an estimate on the nonlinear term $Z[g]$, we introduce the notation

$$\psi(u) = a(x, \varepsilon m_u) f_u,$$

where, for some fixed $g \in \mathbf{P}(\mathbb{R}_+)$, $\langle g \rangle = 0$, we have set

$$f := F + g, \quad f_u := uf + (1-u)F, \quad m_u := \varphi[f_u].$$

We first notice that $\psi(0) = a(\varepsilon\varphi[F])F$ and $\psi(1) = a(\varepsilon\varphi[f])f$. Second, we have

$$\psi'(u) = a'_\varepsilon(m_u) f_u \, m'_u + a_\varepsilon(m_u) g. \tag{2.24}$$

In order to compute m'_u, we differentiate with respect to u the identity

$$m_u = \int_0^\infty a_\varepsilon(m_u) f_u \mathrm{d}x,$$

and we have

$$m'_u = \int_0^\infty a'_\varepsilon(m_u) f_u \mathrm{d}x \, m'_u + \int_0^\infty a_\varepsilon(m_u) \, g \mathrm{d}x,$$

which implies

$$m'_u = \big(1 - \int_0^\infty a'_\varepsilon(m_u) f_u \mathrm{d}x\big)^{-1} \int_0^\infty a_\varepsilon(m_u) \, g \mathrm{d}x. \tag{2.25}$$

We may thus observe that $m'_0 = \mathcal{M}[g]$, so that $\psi'(0) = a'_\varepsilon(M) F \mathcal{M}_\varepsilon[g] + a_\varepsilon(M) g$, and therefore

$$Q[g] = \psi(1) - \psi(0) - \psi'(0).$$

Third, from (2.24), we have

$$\psi''(u) = a''_\varepsilon(m_u) f_u \, (m'_u)^2 + 2 a'_\varepsilon(m_u) g m'_u + a'_\varepsilon(m_u) f_u \, m''_u,$$

and from (2.25), we have

$$\begin{aligned} m''(u) \;&=\; 2\Big(1 - \int_0^\infty a'_\varepsilon f_u\Big)^{-2} \int_0^\infty a_\varepsilon g \int_0^\infty a'_\varepsilon g \\ &\quad + 2\Big(1 - \int_0^\infty a'_\varepsilon f_u\Big)^{-3} \int_0^\infty a''_\varepsilon f \Big(\int_0^\infty a_\varepsilon g\Big)^2. \end{aligned}$$

In the small connectivity regime $\varepsilon \in (0, \varepsilon_0]$, $\varepsilon_0 \|a'\|_\infty < 1$, we get the bound

$$\begin{aligned}
\|\psi''(u)\|_X &\le \|a''_\varepsilon\|_\infty \, |m'_u|^2 + 2\|a'_\varepsilon\|_\infty \|g\|_X |m'_u| + \|a'_\varepsilon\|_\infty \, |m''_u| \\
&\le \varepsilon^2 \frac{\|a''\|_\infty \|a\|_\infty^2}{(1-\varepsilon\|a'\|_\infty)^2} \, \|g\|_X^2 + 2\varepsilon \frac{\|a'\|_\infty \|a\|_\infty}{1-\varepsilon\|a'\|_\infty} \, \|g\|_X^2 \\
&\quad + 2\varepsilon^2 \frac{\|a'\|_\infty^2 \|a\|_\infty}{(1-\varepsilon\|a'\|_\infty)^2} \, \|g\|_X^2 + 2\varepsilon^3 \frac{\|a''\|_\infty \|a'\|_\infty \|a\|_\infty}{(1-\varepsilon\|a'\|_\infty)^3} \, \|g\|_X^2 \\
&\le \varepsilon \, K \, \|g\|_X^2,
\end{aligned}$$

for some constant $K \in (0, \infty)$. Using the Taylor expansion

$$Q[g] = \psi(1) - \psi(0) - \psi'(0) = \int_0^1 (1-u)\psi''(u) \mathrm{d}u,$$

we then obtain

$$\|Z[g]\|_X \le 2\|Q[g]\|_X \le \int_0^1 (1-u)\|\psi''(u)\|_X \mathrm{d}u \le C \, \|g\|^2.$$

Step 3. Decay estimate. Thanks to the Duhamel formula, the solution g to the evolution equation (2.22) satisfies

$$g(t) = S_{\Lambda_\varepsilon}(t)(f_0 - F) + \int_0^t S_{\Lambda_\varepsilon}(t-s) Z[g(s)] \, \mathrm{d}s.$$

Using Theorem 2.8 and the second step, we deduce

$$\begin{aligned}
\|g(t)\|_X &\le C \, e^{\alpha t} \, \|g_0\|_X + \int_0^t C \, e^{\alpha(t-s)} \, \|Z[g(s)]\|_X \, \mathrm{d}s \\
&\le C \, e^{\alpha t} \, \|g_0\|_X + C \, \varepsilon \, K \int_0^t e^{\alpha(t-s)} \, \|g(s)\|_X^2 \, \mathrm{d}s,
\end{aligned}$$

for any $t \ge 0$ and for some constant $C \ge 1$, $\alpha < 0$, independent of $\varepsilon \in (0, \varepsilon_0]$. Observing that $\|g(t)\|_X = \|g(t)\|_{L^1} \in C([0,\infty)$, we conclude thanks to Lemma 2.10. □

3 Case with delay

This section is devoted to the proof of our main result, Theorem 1.1, in the case with delay by following the same strategy as in the case without delay but adaptation the functional framework. We have already proved in Theorem 2.1 the existence of a unique stationary solution $(F_\varepsilon, M_\varepsilon)$ in the weak connectivity regime and we may then focus on the evolution equation.

3.1 Linearized equation and structure of the spectrum

In order to write as a time autonomous equation the linearized equation (1.7)-(1.8)-(1.9), we introduce the following intermediate evolution equation on a function $v = v(t, y)$

$$\partial_t v + \partial_y v = 0, \quad v(t,0) = q(t), \quad v(0,y) = 0, \tag{3.1}$$

where $y \geq 0$ represent the local time for the network activity. That last equation can be solved with the characteristics method

$$v(t,y) = q(t-y)\mathbf{1}_{0\leq y\leq t}.$$

Therefore, equation (1.9) on the variation $n(t)$ of network activity writes

$$n(t) = \mathcal{D}[v(t)], \quad \mathcal{D}[v] := \int_0^\infty v(y) b(\mathrm{d}y),$$

and then equation (1.8) on the variation $q(t)$ of discharging neurons writes

$$q(t) = \mathcal{O}_\varepsilon[g(t), v(t)],$$

with

$$\mathcal{O}_\varepsilon[g,v] := \mathcal{N}_\varepsilon[g] + \kappa_\varepsilon \, \mathcal{D}[v],$$

$$\mathcal{N}_\varepsilon[g] := \int_0^\infty a_\varepsilon(M_\varepsilon) g \, \mathrm{d}x, \quad \kappa_\varepsilon := \int_0^\infty a'_\varepsilon(M_\varepsilon) F_\varepsilon \, \mathrm{d}x.$$

As a consequence, we may rewrite the linear system (1.7)-(1.8)-(1.9), as the autonomous system

$$\partial_t(g,v) = \mathscr{L}_\varepsilon(g,v), \tag{3.2}$$

where the operator $\mathscr{L}_\varepsilon = (\mathscr{L}_\varepsilon^1, \mathscr{L}_\varepsilon^2)$ is defined by

$$\begin{aligned} \mathscr{L}_\varepsilon^1(g,v) &:= -\partial_x g - a_\varepsilon g - a'_\varepsilon F_\varepsilon \mathcal{D}[v] + \delta_{x=0} \mathcal{O}_\varepsilon[g,v], \\ \mathscr{L}_\varepsilon^2(g,v) &:= -\partial_y v + \delta_{y=0} \mathcal{O}_\varepsilon[g,v], \end{aligned}$$

in the space

$$X = X_1 \times X_2 := M^1(\mathbb{R}_+) \times M^1(\mathbb{R}_+, \mu)$$

with $\mu(x) = e^{-\delta x}$ and $\delta > 0$ is the same as in the condition (1.6).

Theorem 3.1. *Assume* (1.3)-(1.4)-(1.5) *and* (1.6). *The conclusions of Theorem 2.3 holds true with* $\alpha := \max\{-a_0/2, -\delta\} < 0$.

The result follows from the Spectral Mapping theorem and the Weyl's Theorem established in [9, 6] by introducing a convenient splitting of the operator $\mathscr{L}_\varepsilon$. More precisely, we write $\mathscr{L}_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ with

$$\mathcal{B}_\varepsilon(g,v) = \begin{pmatrix} \mathcal{B}^1_\varepsilon(g,v) \\ \mathcal{B}^2_\varepsilon(g,v) \end{pmatrix} = \begin{pmatrix} -\partial_x g - a_\varepsilon g \\ -\partial_y v \end{pmatrix}$$

and

$$\mathcal{A}_\varepsilon(g,v) = \begin{pmatrix} \mathcal{A}^1_\varepsilon(g,v) \\ \mathcal{A}^2_\varepsilon(g,v) \end{pmatrix} = \begin{pmatrix} -a'_\varepsilon F_\varepsilon \mathcal{D}[v] + \delta_{x=0}\mathcal{O}_\varepsilon[g,v] \\ \delta_{y=0}\mathcal{O}_\varepsilon[g,v] \end{pmatrix},$$

and we just check the properties enjoyed by the operators $\mathcal{A}_\varepsilon$ and $\mathcal{B}_\varepsilon$. We skip the rest of the proof and we refer to the proof of Theorem 2.3 for more details.

Lemma 3.2. *(i) $\mathcal{A}_\varepsilon \in \mathscr{B}(X,Y)$, where $Y = (\mathbb{C}\delta_0 + BV(\mathbb{R}_+)) \times \mathbb{C}\delta_0 \subset X$ with compact embedding;*

(ii) $S_{\mathcal{B}_\varepsilon}(t)$ is α-hypodissipative in X;

(iii) the family of operators $S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}$ satisfies

$$\|(S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})(t)\|_{X\to D(\Lambda_\varepsilon)} \le C e^{\alpha t}, \quad \forall t \ge 0.$$

Proof. (i) It is an immediate consequence of the fact that $\mathcal{D} \in \mathcal{B}(X_2;\mathbb{R})$ (because of (1.6)) and $\mathcal{N}_\varepsilon \in \mathscr{B}(X_1;\mathbb{R})$ (because of (1.5)).

(ii) Since $S_{\mathcal{B}^1_\varepsilon}$ is nothing but the semigroup $S_{\mathcal{B}_\varepsilon}$ defined in (2.9) which is $-a_0/2$-dissipative thanks to Lemma 2.4-(ii), we just have to prove the dissipativity of the translation semigroup $S_{\mathcal{B}^2_\varepsilon}$ which is given by the explicit formula $[S_{\mathcal{B}^2_\varepsilon}(t)v](y) = v(y-t)\mathbf{1}_{y-t\ge 0}$. That follows from

$$\|S_{\mathcal{B}^2_\varepsilon}(t)v\|_{X_2} = \int_0^\infty |v(y-t)|\mathbf{1}_{y-t\ge 0}\, e^{-\delta y}\, dy = e^{-\delta t}\|v\|_{X_2},$$

for any $v \in X_2$ and any $t \ge 0$.

(iii) On the one hand, we have

$$\mathcal{A}^1_\varepsilon S_{\mathcal{B}_\varepsilon}(t)(g,v)(x) = \mu(x)D(t) + \delta_{x=0}N(t),$$

with

$$\begin{aligned} \mu(x) &:= \kappa_\varepsilon \delta_{x=0} - a'_\varepsilon(x) F_\varepsilon(x), \\ N(t) &:= \mathcal{N}_\varepsilon[S^1_{\mathcal{B}_\varepsilon}(t) g] = \int_0^\infty a(x) g(x-t) \mathbf{1}_{x-t\geq 0} \mathrm{d}x, \\ D(t) &:= \mathcal{D}[S^2_{\mathcal{B}_\varepsilon}(t) v] = \int_0^\infty v(y-t) \mathbf{1}_{y-t\geq 0} b(\mathrm{d}y), \end{aligned}$$

and we observe that $|D(t)| \leq e^{\alpha t} \|v\|_{X_2}$ from (1.6). We also have

$$|D'(t)| \leq \Big| \int_0^\infty |v(y-t)| \mathbf{1}_{y-t\geq 0} \, |b'| \, \mathrm{d}y \leq C \, e^{\alpha \, t} \, \|v\|_{X_2}, \quad \forall \, t \geq 0.$$

Next, we denote

$$T_1(t)(g,v)(x) := (S_{\mathcal{B}^1_\varepsilon} * \mathcal{A}^1_\varepsilon S_{\mathcal{B}_\varepsilon})(t)(g,v)(x).$$

We compute

$$\mathcal{A}^1_\varepsilon S_{\mathcal{B}_\varepsilon}(t)(g,v) = -a'_\varepsilon(x) F_\varepsilon(x) \mathcal{N}_\varepsilon[S_1(t)] + \delta_{x=0} \mathcal{O}_\varepsilon[S_1(t), S_2(t)],$$

and then

$$\begin{aligned} T_1(t)(g,v)(x) &= (S_{\mathcal{B}^1_\varepsilon} * \mathcal{A}^1_\varepsilon S_{\mathcal{B}_\varepsilon})(t)(g,v)(x) \\ &= \int_0^t S_{\mathcal{B}^1_\varepsilon}(s) \mathcal{A}^1_\varepsilon S_{\mathcal{B}_\varepsilon}(t-s)(g,v)(x) \mathrm{d}s \\ &= \int_0^t e^{-A_\varepsilon(x)} \mathcal{A}^1_\varepsilon S_{\mathcal{B}^1_\varepsilon}(t-s)(g,v)(x-s) \mathbf{1}_{x-s\geq 0} \mathrm{d}s \\ &= \int_0^t e^{-A_\varepsilon(x)} \{ -a'_\varepsilon(x-s) F_\varepsilon(x-s) \mathcal{N}_\varepsilon[S_2(t-s)] \\ &\quad + \delta_{x-s=0} \{ \mathcal{N}_\varepsilon[S_2(t-s)] \int_0^\infty a'_\varepsilon F_\varepsilon \mathrm{d}x + \int_0^\infty a_\varepsilon S_1(t-s) \mathrm{d}x \} \mathbf{1}_{x-s\geq 0} \mathrm{d}s \\ &= -\int_0^{t\wedge x} e^{-A_\varepsilon(x)} a'_\varepsilon(x-s) F_\varepsilon(x-s) \mathcal{N}_\varepsilon[S_2(t-s)] \mathrm{d}s \\ &\quad + \mathbf{1}_{t\geq x} e^{-A_\varepsilon(x)} \{ \mathcal{N}_\varepsilon[S_2(t-x)] \int_0^\infty a'_\varepsilon F_\varepsilon \mathrm{d}x + \int_0^\infty a_\varepsilon S_1(t-x) \mathrm{d}x \}. \end{aligned}$$

Next, differentiating the above identity, we get

$$\begin{aligned}\partial_x T_1(t)(g,v)(x) &= -\mathbf{1}_{t\geq x} e^{-A_\varepsilon(x)} a_\varepsilon'(0) F_\varepsilon(0) \mathcal{N}_\varepsilon[S_2(t-x)] \\ &\quad + \int_0^{t\wedge x} a_\varepsilon(x) e^{-A_\varepsilon(x)} a_\varepsilon'(x-s) F_\varepsilon(x-s) \mathcal{N}_\varepsilon[S_2(t-s)]\mathrm{d}s \\ &\quad - \int_0^{t\wedge x} e^{-A_\varepsilon(x)} a_\varepsilon''(x-s) F_\varepsilon(x-s) \mathcal{N}_\varepsilon[S_2(t-s)]\mathrm{d}s \\ &\quad - \int_0^{t\wedge x} e^{-A_\varepsilon(x)} a_\varepsilon'(x-s) F_\varepsilon'(x-s) \mathcal{N}_\varepsilon[S_2(t-s)]\mathrm{d}s \\ &\quad -\mathbf{1}_{t\geq x} a_\varepsilon(x) e^{-A_\varepsilon(x)} \{\mathcal{N}_\varepsilon[S_2(t-x)] \int a_\varepsilon' F_\varepsilon + \int_0^\infty a_\varepsilon S_1(t-x)\mathrm{d}x\}. \\ &\quad -\mathbf{1}_{t\geq x} e^{-A_\varepsilon(x)} \{\mathcal{N}_\varepsilon[S_2'(t-x)] \int a_\varepsilon' F_\varepsilon + \int_0^\infty a_\varepsilon S_1'(t-x)\mathrm{d}x\}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\partial_y T_2(t)(g,v)(y) &= \partial_y S_{\mathcal{B}_\varepsilon^2} * \mathcal{A}_\varepsilon^2 S_{\mathcal{B}_\varepsilon}(t)(g,v)(y) \\ &= -\mathcal{N}_\varepsilon[S_2'(t-y)] \int a_\varepsilon' F_\varepsilon - \int_0^\infty a_\varepsilon S_1'(t-y)\mathrm{d}x.\end{aligned}$$

We easily deduce that

$$\begin{aligned}\|\partial_x T_1(t)(g,v)(x)\|_{X_1} &\leq Ce^{\alpha t}\|(g,v)\|_X \\ \|\partial_y T_2(t)(g,v)(y)\|_{X_2} &\leq Ce^{\alpha t}\|(g,v)\|_X,\end{aligned}$$

and the similar estimate for $\|(S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})(t)(g,v)\|_X$. Thus, the announced estimate holds for the family of operators $S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}$. □

3.2 The vanishing connectivity regime

When the network connectivity parameter vanishes, $\varepsilon = 0$, the linearized operator simplifies into

$$\mathscr{L}_0 \begin{pmatrix} g \\ v \end{pmatrix} = \begin{pmatrix} -\partial_x g - a(x,0)g + \delta_{x=0}\mathcal{O}_0[g,v] \\ -\partial_y v + \delta_{y=0}\mathcal{O}_0[g,v] \end{pmatrix}, \qquad (3.3)$$

where $\mathcal{O}_0[g,v] = \mathcal{N}_0[g] = \int_0^\infty a(x,0)g(x)\mathrm{d}x$.

Theorem 3.3. *There exist some constants $\alpha < 0$ and $C > 0$ such that $\Sigma(\mathscr{L}_0) \cap \Delta_\alpha = \{0\}$ and for any $(g_0, v_0) \in X$, $\langle g_0 \rangle = 0$, there holds*

$$\|S_{\mathscr{L}_0}(t)(g_0,\, v_0)\|_X \leq C e^{\alpha t}\, \|(g_0,\, v_0)\|_X, \quad \forall\, t \geq 0. \tag{3.4}$$

Proof of Theorem 3.3. Since $\mathscr{L}_0^1 = \Lambda_0$, we have already proved that there exists some $\beta < 0$ such that $g(t) := S_{\mathscr{L}_0^1}(t) g_0$ satisfies $\|g(t)\| \leq C e^{\beta t}\, \|g_0\|_{X_1}$ for any $t \geq 0$ from Theorem 3.3. We then just focus on $\mathscr{L}_0^2$. The Duhamel formula associated to the equation $\partial_t v = \mathscr{L}_0^2(g, v)$ writes

$$v(t) = S_{\mathcal{B}_0^2}(t) v_0 + \int_0^t S_{\mathcal{B}_0^2}(t-s) \mathcal{A}_0^2\big(g(s), v(s)\big)\, \mathrm{d}s.$$

Using the already known estimate on $g(t)$, we deduce

$$\begin{aligned} \|S_{\mathscr{L}_0^2} v_0(t)\|_{X_2} &= \|v(t)\|_{X_2} \leq \|S_{\mathcal{B}_0^2}(t) v_0\|_{X_2} + \int_0^t \|S_{\mathcal{B}_0^2}(t-s)\delta_0 \mathcal{N}_0[g(s)]\|_{X_2}\, \mathrm{d}s \\ &\leq e^{-\delta t}\|v_0\|_{X_2} + \int_0^t e^{-\delta(t-s)} C\, e^{\beta s}\|g_0\|_{X_1}\, \mathrm{d}s \\ &\leq C\, e^{\alpha t}\|(g_0,\, v_0)\|_X \end{aligned}$$

for some $0 > \alpha > \max\{\beta, -\delta\}$, which yields our conclusion. □

3.3 Weak connectivity regime - exponential stability of the linearized equation

In this part, we shall discuss the geometry structure of the spectrum of the linearized time elapsed equation in weak connectivity regime taking delay into account.

Theorem 3.4. *There exists some constants $\varepsilon_0 > 0$, $C \geq 1$ and $\alpha < 0$ such that for any $\varepsilon \in [0, \varepsilon_0]$ there holds $\Sigma(\mathscr{L}_\varepsilon) \cap \Delta_\alpha = \{0\}$ and*

$$\|S_{\mathscr{L}_\varepsilon}(t)(g_0, v_0)\|_X \leq C e^{\alpha t}\|(g_0, v_0)\|_X, \tag{3.5}$$

for any $(g_0, v_0) \in X$ such that $\langle g_0 \rangle = 0$.

We present a technical result needed in the proof of Theorem 3.4.

Lemma 3.5. *The operator $\mathscr{L}_\varepsilon$ is continuous with respect to ε, and more precisely*

$$\|\mathscr{L}_\varepsilon - \mathscr{L}_0\|_{\mathscr{B}(X)} \leq O(\varepsilon). \tag{3.6}$$

Proof. For all $(g, v) \in X$, we have

$$\mathscr{L}_\varepsilon \begin{pmatrix} g \\ v \end{pmatrix} = \begin{pmatrix} -\partial_x g - a_\varepsilon g - a'_\varepsilon F_\varepsilon \mathcal{D}_\varepsilon[v] + \delta_{x=0}\mathcal{O}_\varepsilon[g,v] \\ -\partial_y v \qquad\qquad\qquad\qquad + \delta_{y=0}\mathcal{O}_\varepsilon[g,v] \end{pmatrix}, \tag{3.7a}$$

$$\mathscr{L}_0 \begin{pmatrix} g \\ v \end{pmatrix} = \begin{pmatrix} -\partial_x g - a(x,0) g \qquad + \delta_{x=0}\mathcal{O}_0[g,v] \\ -\partial_y v \qquad\qquad\qquad + \delta_{y=0}\mathcal{O}_0[g,v] \end{pmatrix}. \tag{3.7b}$$

By (3.3)-(3.7b), we deduce

$$(\mathscr{L}_\varepsilon - \mathscr{L}_0) \begin{pmatrix} g \\ v \end{pmatrix} = \begin{pmatrix} (a(x,0) - a_\varepsilon) g - a'_\varepsilon F_\varepsilon \mathcal{D}_\varepsilon[v] + \delta_{x=0}(\mathcal{O}_\varepsilon[g,v] - \mathcal{O}_0[g,v]) \\ \delta_{y=0}(\mathcal{O}_\varepsilon[g,v] - \mathcal{O}_0[g,v]) \end{pmatrix}.$$

Then we compute

$$\begin{aligned} \|(\mathscr{L}_\varepsilon - \mathscr{L}_0)(g,v)\|_X &= \|(a(x,0) - a_\varepsilon) g\|_{X_1} + \|a'_\varepsilon F_\varepsilon \mathcal{D}_\varepsilon[v]\|_{X_1} + 2|\mathcal{O}_\varepsilon[g,v] - \mathcal{O}_0[g,v]| \\ &\leq 3\|(a_\varepsilon - a_0) g\|_{X_1} + 2\|a'_\varepsilon F_\varepsilon \mathcal{D}_\varepsilon[v]\|_{X_2} \\ &\leq 3\varepsilon \|a'\|_\infty \|g\|_{X_1} + 2\varepsilon a_1 \|a'\|_\infty (1 - \varepsilon\|a'\|_\infty) \|F_\varepsilon\|_{X_1} \|v\|_{X_2} \\ &= C\varepsilon \|(g,v)\|_X, \end{aligned}$$

which is nothing but (3.6). □

Proof of Theorem 3.4. With the help of Lemma 3.5, we may proceed exactly as in the proof of Theorem 2.8 (see also again [14]) and we conclude that

$$\Sigma(\mathscr{L}_\varepsilon) \cap \Delta_\alpha = \{\xi_\varepsilon\},$$

with $|\xi_\varepsilon| \leq O(\varepsilon)$ and ξ_ε is algebraically simple. We observe that

$$\mathscr{L}^*_\varepsilon \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} \partial_x \varphi - a_\varepsilon \varphi + a_\varepsilon(\varphi(0) + \psi(0)) \\ \partial_y \psi + \kappa_\varepsilon b\, \psi(0) + \kappa_\varepsilon b\, \varphi(0) - b \int a'_\varepsilon F_\varepsilon\, \varphi \,\mathrm{d}x \end{pmatrix},$$

from which we deduce that $\mathscr{L}^*(1,0) = 0$. Then $0 \in \Sigma(\mathscr{L}^*_\varepsilon)$ and $\xi_\varepsilon = 0$. Moreover, the orthogonality condition $\langle g_0 \rangle = \langle (g_0, v_0), (1,0) \rangle_{X,X'} = 0$ implies that the exponential estimate (3.5) holds. □

3.4 Weak connectivity regime - nonlinear exponential stability

We finally come back on the nonlinear problem and we present the proof of the second part of our main result for the case with delay.

Proof of Theorem 1.1 in case with delay. We write the system as

$$\begin{array}{rcl}\partial_t f &=& -\partial_x f - a_\varepsilon(\mathcal{D}[u])f + \delta_0 \mathscr{P}[f,\mathcal{D}[u]] \\ \partial_t u &=& -\partial_y u + \delta_0 \mathscr{P}[f,\mathcal{D}[u]],\end{array}$$

with

$$\mathscr{P}[f,m] = \int a(m) f, \quad \mathcal{D}[u] = \int bu.$$

We recall that the steady state (F,U), $U := M\mathbf{1}_{y\geq 0}$, satisfies

$$\begin{array}{rcl}0 &=& -\partial_x F - a_\varepsilon(M)F + \delta_0 M \\ 0 &=& -\partial_y U + \delta_0 M, \quad M = \mathcal{D}[U] = \mathscr{P}[F,\mathcal{D}[U]].\end{array}$$

We introduce the variation $g := f - F$ and $v = u - U$. The equation on g is

$$\begin{array}{rcl}\partial_t g &=& -\partial_x g - a_\varepsilon(\mathcal{D}[u])f + a_\varepsilon(M)F + \delta_0(\mathscr{P}[f,\mathcal{D}[u]] - \mathscr{P}[F,\mathcal{D}[U]]) \\ &=& -\partial_x g - a_\varepsilon(M)f - a'_\varepsilon F\mathcal{D}[v] - Q[g,v] + \delta_0\mathcal{O}[g,v] + \delta_0\mathcal{Q}[g,v] \\ &=& \mathscr{L}^1_\varepsilon(g,v) + \mathcal{Z}^1[g,v],\end{array}$$

with

$$\begin{array}{rcl}Q[g,v] &:=& a_\varepsilon(M)F - a_\varepsilon(\mathcal{D}[u])f + a_\varepsilon(M)f + a'_\varepsilon F\mathcal{D}[v] \\ &=& \Phi(0) - \Phi(1) + \Phi'(0),\end{array}$$

where $\Phi(k) = a_\varepsilon(\mathcal{D}[k\,u + (1-k)U])(k\,f + (1-k)F)$ and $\mathcal{Q}[g,v] = \langle Q[g,v]\rangle$, $\mathcal{Z}^1[g,v] := -Q[g,v] + \delta_0\mathcal{Q}[g,v]$. The equation on v is

$$\begin{array}{rcl}\partial_t v &=& -\partial_y v + \delta_0(\mathscr{P}[f,\mathcal{D}[u]] - \mathscr{P}[F,\mathcal{D}[U]]) \\ &=& -\partial_y v + \delta_0\mathcal{O}[g,v] + \delta_0\mathcal{Q}[g,v] \\ &=& \mathscr{L}^2_\varepsilon(g,v) + \mathcal{Z}^2[g,v], \quad \mathcal{Z}^2[g,v] := \delta_0\mathcal{Q}[g,v].\end{array}$$

We then write the associated Duhamel formula

$$(g(t),v(t)) = S_{\mathscr{L}_\varepsilon}(t)(g_0,v_0) + \int_0^t S_{\mathscr{L}_\varepsilon}(t-s)\mathcal{Z}[g(s),v(s)]\,\mathrm{d}s.$$

Because $\|\mathcal{Z}[g,v]\|_X \leq C\,\|(g,\,v)\|_X^2$ we may conclude as in the proof of Theorem 1.1. $\square$

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