THE FOKKER-PLANCK EQUATION WITH SUBCRITICAL CONFINEMENT FORCE

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ABSTRACT. We consider the Fokker-Planck equation with subcritical confinement force field which may not derive from a potential function. We prove the existence of an equilibrium (in the case of a general force) and we establish some (polynomial and stretch exponential) rate of convergence to the equilibrium (depending on the space to which belongs the initial datum). Our results improve similar results established by Toscani, Villani [29] and Röchner, Wang [27]: the force field is more general, the spaces are more general, the rates are sharper.

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CONTENTS

1. INTRODUCTION

In the present work, we consider the Fokker-Planck equation

$$\partial_t f = \mathcal{L} f = \Delta f + \mathrm{div}(f\,\mathbf{F}) \tag{1.1}$$

on the density function $f = f(t,x)$, $t > 0$, $x \in \mathbb{R}^d$, $d \geq 1$, which is complemented with an initial condition

$$f(0,x) = f_0(x), \quad \forall\, x \in \mathbb{R}^d. \tag{1.2}$$

We will always assume that the force field $\mathbf{F} \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ satisfies

$$x \cdot \mathbf{F}(x) \geq |x| \, \langle x \rangle^{\gamma-1}, \quad \operatorname{div}(\mathbf{F}(x)) \leq C_F \, |x|^{\gamma-2}, \quad \forall \, x \in B_{R_0}^c \, , \tag{1.3}$$

as well as

$$|D\mathbf{F}(x)| \leq C'_F \, \langle x \rangle^{\gamma-2}, \quad \forall \, x \in \mathbb{R}^d, \tag{1.4}$$

for some constants $C_F \geq d$, $R_0 > 0$, $C'_F > 0$ and an exponent

$$\gamma \in (0,1). \tag{1.5}$$

Here and below, we denote $\langle x \rangle := (1 + |x|^2)^{1/2}$ for any $x \in \mathbb{R}^d$.

It is worth mentioning that we have made two normalization hypotheses by taking a diffusion coefficient equal to 1 in (1.1) as well as a lower bound constant equal to 1 in the first condition in (1.3). Of course, these two normalization hypotheses can be removed by standard scaling arguments (in the time and position variables) and thus do not restrict the generality of our analysis.

A typical example of a force field is the one associated to a confinement potential

$$\mathbf{F}(x) := \nabla V(x), \quad V(x) := \frac{\langle x \rangle^\gamma}{\gamma} + V_0, \quad V_0 \in \mathbb{R}.$$

In this case, we may observe that

$$G(x) := \mathrm{e}^{-V(x)} \in L^1(\mathbb{R}^d) \cap C^2(\mathbb{R}^d), \tag{1.6}$$

is a stationary solution of (1.1), and even an equilibrium state. We may assume that G is a probability measure, by choosing the constant V_0 adequately. We also observe that contrary to the case $\gamma \geq 1$, a Poincaré inequality of the type

$$\exists \, c > 0, \qquad \int_{\mathbb{R}^d} |f(x)|^2 \exp(-V(x)) dx \leq c \int_{\mathbb{R}^d} |\nabla f(x)|^2 \exp(-V(x)),$$

for f such that $\int_{\mathbb{R}^d} f(x) \exp(-V(x)) dx = 0$, does not hold but only a weak version of this inequality remains true (see [27], below (1.11) and section 4). In particular, there is no spectral gap for the associated operator $\mathcal{L}$, nor is there an exponential trend to the equilibrium for the associated semigroup. Similarly, the classical *logarithmic Sobolev inequality* does not hold but only a *modified* version of it, see the discussion in [29, Section 2].

In the general case of a force field which is not the gradient of a potential, one may see easily that the above Fokker-Planck equation preserves positivity, that is

$$f(t,.) \geq 0, \quad \forall \, t \geq 0, \quad \text{if} \quad f_0 \geq 0,$$

and that it conserves mass, that is

$$M(f(t,\cdot)) = M(f_0), \quad \forall \, t \geq 0, \quad \text{with} \quad M(g) := \int_{\mathbb{R}^d} g(x) \, dx.$$

Moreover, the Fokker-Planck operator $\mathcal{L}$ generates a positive semigroup in many Lebesgue spaces. However, due to the lack of compactness of this associated semigroup, the standard Krein-Rutman theory does not apply directly in the case $\gamma \in (0,1)$, and the existence of a stationary solution is not straightforward. We refer to the recent work [26] and the references therein where the Fokker-Planck equation with general force field (1.1)–(1.5) in the case $\gamma \geq 1$ is considered.

Before stating our existence result, let us introduce some notations. For any exponent $p \in [1,\infty]$, we define the polynomial and stretch exponential weight functions $m : \mathbb{R}^d \to \mathbb{R}_+$, by

$$m(x) := \langle x \rangle^k, \qquad \text{for some } \ k > \max(d, k^*), \text{ with } \ k^* := C_F/p', \tag{1.7}$$

where $p' := p/(p-1)$ is the conjugated exponent associated to p,

$$m(x) := \exp(\kappa \langle x \rangle^s), \qquad \text{for some } \ 0 < s < \gamma \ \text{ and } \ \kappa > 0, \tag{1.8}$$

or

$$m(x) := \exp(\kappa \langle x \rangle^\gamma), \qquad \text{for some } \ \kappa \in (0, 1/\gamma), \tag{1.9}$$

as well as the associated Lebesgue spaces

$$L^p(m) = \{f \in L^1_{\rm loc}(\mathbb{R}^d);\ \|f\|_{L^p(m)} := \|fm\|_{L^p} < \infty\}.$$

We also use the shorthands $L^p_k = L^p(m)$ when $m(x) = \langle x \rangle^k$. It is noteworthy that, for such a choice of weights $m(x)$ and when the parameter k satisfies $k(p-1)/p > d$, we have $L^p(m) \subset L^1(\mathbb{R}^d)$.

As a first step, we have the following existence and uniqueness result.

Theorem 1.1. *For any exponent $p \in [1,\infty]$, any weight function m satisfying either of definitions (1.7), (1.8) or (1.9), and an initial datum $f_0 \in L^p(m)$, there exists a unique global solution f to the Fokker-Planck equation (1.1)–(1.2), such that for any $T > 0$,*

$$f \in C([0,T]; L^1(\mathbb{R}^d)) \cap L^\infty(0,T; L^p(m)).$$

Moreover the associated flow preserves positivity and conserves mass. Also, the operator $\mathcal{L}$ generates a strongly continuous semigroup $S_{\mathcal{L}}(t)$ in $L^p(m)$ when $p \in [1,\infty)$.

On the other hand, there exists a unique positive, unit mass, stationary solution $G > 0$ such that for all $\kappa \in (0, 1/\gamma)$

$$G \in L^\infty(\exp(\kappa \langle x \rangle^\gamma)) \qquad \textit{and} \qquad \Delta G + \mathrm{div}(G\,\mathbf{F}) = 0.$$

Next, we are interested in the long time behaviour of the solution $f(t,\cdot)$. We consider separately the following two cases:

Case 1. Following [27], we consider the case when furthermore the above steady state G fulfills a *weak Poincaré inequality.* More precisely, we assume that there exist some constants $R_0, c_1, c_2 > 0$ such that the function $V := -\log G \in C^1(\mathbb{R}^d)$ satisfies

$$\forall\, x \in B^c_{R_0}, \qquad c_1\,|x|^\gamma \le V(x) \le c_2\,|x|^\gamma, \tag{1.10}$$

and also that there exists $\mu > 0$ such that for any $f \in \mathcal{D}(\mathbb{R}^d)$ with $M(f) = 0$, we have

$$\int |\nabla(f/G)|^2\, G\, dx \ge \mu \int f^2\, \langle x \rangle^{2\gamma-2}\, G^{-1}\, dx. \tag{1.11}$$

Such a weak Poincaré inequality is known to hold when $c_1 = c_2$ in (1.10), see [27, Example 1.4(c)].

The weak Poincaré inequality is also a consequence of a *"local Poincaré inequality"* together with the fact that the following Lyapunov condition (see for instance [4, 3])

$$\Delta w - \nabla V \cdot \nabla w \le w\,(-\zeta(x) + M\chi_R), \quad \forall\, x \in \mathbb{R}^d, \tag{1.12}$$

holds for some well chosen function $w : \mathbb{R}^d \to [1,\infty)$. Here it is assumed that M and R are two positive constants, $\chi_R(x) := \chi(x/R)$ is a truncation function defined through a certain $\chi \in \mathcal{D}(\mathbb{R}^d)$ such that $\mathbf{1}_{|x|\le 1} \le \chi \le \mathbf{1}_{|x|\le 2}$ and $\zeta(x) := \zeta_0 \langle x\rangle^{2(1-\gamma)}$ for a constant $\zeta_0 > 0$. It is worth emphasizing that, in this case, the force field $\mathbf{F}$ can be written as

$$\mathbf{F} = \nabla V + \mathbf{F}_0, \qquad \mathrm{div}(\mathrm{e}^{-V}\,\mathbf{F}_0) = 0, \tag{1.13}$$

with no other specific condition on $\mathbf{F}_0$ except that $\mathbf{F}$ still satisfies conditions (1.3)–(1.5). Under these circumstances, we can give a sharper rate of decay to the equilibrium and provide a simpler proof than in the general case. For future use, we define the critical decay exponent by setting

$$\sigma^*_{\mathcal{L}} = \sigma^*_{\mathcal{B}} := \frac{\gamma}{2-\gamma}. \tag{1.14}$$

Case 2. This corresponds to the general case when $\mathbf{F}$ satisfies only conditions (1.3)–(1.5), without any further assumption on the stationary state G, which in general cannot be determined explicitly. Using the above notations for M, R, χ_R and ζ used in the inequality (1.12), the assumptions (1.3)–(1.5) made on $\mathbf{F}$ imply in particular the following inequality

$$\mathcal{L}^* m^p := \Delta m^p - \mathbf{F}\cdot\nabla m^p \le m^p\,(-\zeta(x) + M\chi_R), \quad \forall\, x \in \mathbb{R}^d, \tag{1.15}$$

which is a generalization of (1.12). In this case, we define the critical decay exponents $\sigma^*_{\mathcal{L}}$ and $\sigma^*_{\mathcal{B}}$ by setting

$$\sigma^*_{\mathcal{L}} := \frac{1}{\lfloor 2/\gamma\rfloor}\,, \qquad\qquad \sigma^*_{\mathcal{B}} := \frac{\gamma}{2-\gamma}\,, \tag{1.16}$$

where $\lfloor s\rfloor$ stands for the integer part of the real number s.

The main and fundamental difference between these two cases is that the first one involves the equilibrium state $G = \mathrm{e}^{-V}$, and a certain type of behavior on it, while the second one only involves the force field $\mathbf{F}$.

When $a(t) \ge 0$ and $b(t) \ge 0$ are two functions of time $t > 0$, we write $a(t) \lesssim b(t)$ to mean that there exists a positive constant c_0 independent of t such that one has $a(t) \le c_0\, b(t)$ for all $t > 0$.

Our main result is as follows:

Theorem 1.2. *Let $\mathbf{F}$ satisfy (1.3)–(1.5), and let the exponent $\sigma^*_{\mathcal{L}}$ be defined by (1.16) corresponding to Case 2, or by (1.14) when $\mathbf{F}$ satisfies the conditions (1.10) and (1.11), which correspond to Case 1 above. Then for any θ, with $0 \le \theta < 1$, any $p \in [1,\infty]$, any weight function m satisfying either of definitions (1.7), (1.8) or (1.9), and any initial datum $f_0 \in L^p(m)$, the associated solution $f = f(t,x)$ to the Fokker-Planck equation (1.1) satisfies*

$$\|f(t,.) - M(f_0)\,G\|_{L^p} \lesssim \Theta_m(t)\,\|f_0 - M(f_0)\,G\|_{L^p(m)}, \tag{1.17}$$

with the function Θ_m being defined as follows. When $m(x) = \langle x\rangle^k$, we take $\beta \in (0, (k-k^*)/(2-\gamma))$ arbitrary and we take

$$\Theta_m(t) := (1+t)^{-\beta}. \tag{1.18}$$

When $m(x) = \exp(\kappa\,\langle x\rangle^s)$, with notation (1.16) and $\sigma \in (0, \min(\sigma^*_{\mathcal{L}}, s/(2-\gamma)]$, for some $\lambda \in (0,\infty)$, we take

$$\Theta_m(t) := \exp(-\lambda t^\sigma). \tag{1.19}$$

Remark 1.3. When the force field $\mathbf{F}$ and the equilibrium G are as described in Case 1, some previous results are known. A less accurate result than the one given by (1.17)-(1.18), actually a decay rate of order $\mathcal{O}(t^{-(k-2)/(2(2-\gamma))})$ in L^1-norm, has been proved in [29, Theorem 3] under the additional assumptions that the initial datum f_0 is nonnegative, it has finite energy and finite Boltzmann entropy, and that $\gamma \in (0,2)$. Estimate (1.17)-(1.19) has been proved in [27] for a subclass of functional spaces, which corresponds essentially, with the settings presented here, to the case $p > 2$, $s = \gamma$ and $\kappa = 1/2$.

Remark 1.4. Estimate (1.17)-(1.18) has been proved in [14, Corollary 3.5] in **Case 1** when $\gamma \in [1,2)$.

Remark 1.5. The same kind of decay estimates remain true when, on the left hand side of (1.17), the norm L^p is replaced by a weighted Lebesgue norm $L^p(m^\theta)$ with $0 \le \theta < 1$. More precisely, when $m(x) = \langle x\rangle^k$ and θ is such that $k^*/k < \theta < 1$, we choose $\beta := k(1-\theta)/(2-\gamma)$, and if $0 \le \theta \le k^*/k$, we choose $\beta \in (0,(k-k^*)/(2-\gamma))$ arbitrary. In both cases, we define Θ_m through (1.18). When $m(x) = \exp(\kappa\,\langle x\rangle^s)$ the definition of the decay rate Θ_m is unchanged.

Remark 1.6. When the weight function is given by $m(x) := \exp(\kappa\langle x\rangle^\gamma)$, one could consider a field force $\mathbf{F}$ satisfying the first condition of (1.3) and $\operatorname{div}(\mathbf{F}) \le C'_F\,\langle x\rangle^{\gamma'-2}$, with $\gamma' < 2\gamma$, or $\gamma' = 2\gamma$ but with C'_F small enough, and obtain similar results. However we do not push our investigations in that direction, since the general ideas of the proof are essentially the same.

Remark 1.7. When the weight function $m(x) = \langle x\rangle^k$, the decay rate in (1.17) is given by (1.18), which is better than the decay rate one might obtain by a mere interpolation argument between $L^2(\exp(\kappa\langle x\rangle^\gamma))$ and $L^1(\mathbb{R}^d)$. More precisely, assume that when the weight function $m(x) = \exp(\kappa\langle x\rangle^\gamma)$, the function Θ_m being given by (1.19) with $s := \gamma/(2-\gamma)$, we have (1.17), as well as the estimate $\|f(t)\|_{L^1} \lesssim \|f_0\|_{L^1}$ for any $f_0 \in L^1(\exp(\kappa\langle x\rangle^\gamma))$ such that $M(f_0) = 0$. Then for any $R > 0$ we have $M(f_0 1_{B_R}) = -M(f_0 1_{B_R^c})$, and thus we may write $f_0 = f_{01} + f_{02} + f_{03}$ where

$$f_{01} := (f_0 - M(f_0 1_{B_R}))\,1_{B_R}, \quad f_{02} := f_0 1_{B_R^c}, \quad f_{03} := M(f_0 1_{B_R^c}) 1_{B_R}.$$

Therefore for $t > 0$, denoting by $f_j(t)$ the solution of (1.1) with initial datum f_{0j}, one has

$$\begin{aligned}
\|f(t)\|_{L^1} &\le \|f_1(t)\|_{L^1} + \|f_2(t)\|_{L^1} + \|f_3(t)\|_{L^1} \\
&\lesssim \exp(-\lambda t^{\gamma/(2-\gamma)})\,\|f_{01}\|_{L^1(\exp(\kappa\langle x\rangle^\gamma))} + \|f_{02}\|_{L^1} + \|f_{03}\|_{L^1} \\
&\lesssim \exp(-\lambda t^{\gamma/(2-\gamma)})\,\|(f_0 - M(f_0\,\mathbf{1}_{B_R}))\mathbf{1}_{B_R}\|_{L^1(\exp(\kappa\langle x\rangle^\gamma))} \\
&\qquad + \|M(f_0\,\mathbf{1}_{B_R^c})\mathbf{1}_{B_R}\|_{L^1} + \|f_0\,\mathbf{1}_{B_R^c}\|_{L^1} \\
&\lesssim \exp(-\lambda t^{\gamma/(2-\gamma)})\,\mathrm{e}^{\kappa\,\langle R\rangle^\gamma}\,\|f_0\|_{L^1} + (R^{d-k} + R^{-k})\|f\|_{L^1_k} \\
&\lesssim \Big(\exp(-\lambda t^{\gamma/(2-\gamma)} + \kappa\langle R\rangle^\gamma) + R^{d-k}\Big)\,\|f\|_{L^1_k}.
\end{aligned}$$

Assuming that $t > (2\kappa/\lambda)^{(2-\gamma)/\gamma}$, we may choose R so that $\kappa\langle R\rangle^\gamma = \lambda t^{\gamma/(2-\gamma)}/2$, we find that when $k > d$, for any $t > (2\kappa/\lambda)^{(2-\gamma)/\gamma}$ we have

$$\|f(t)\|_{L^1} \lesssim t^{\frac{-(k-d)}{2-\gamma}}\,\|f_0\|_{L^1_k}.$$

This decay estimate is not as sharp as the one given by Theorem 1.2 when the weight function is $m := \langle x\rangle^k$, since according to the definition (1.18) in this case we have actually a decay rate of $\mathcal{O}(t^{-K})$ for any $K \in (0, k/(2-\gamma))$. □

Remark 1.8. In a few previous papers, due in particular to Caflisch [7, 8], Liggett [19], Toscani and Villani [29], Guo [15] or Aoki and Golse [1], a certain number of models, arising from statistical physics, has been considered for which only polynomial or stretch exponential (but not exponential) rate of decay to the equilibrium can be established. As it is the present case, one can associated to each of these models a linear(ized) operator which does not enjoy any spectral gap in its spectrum set and that is the reason why exponential rate of convergence fails.

We also refer to Röckner and Wang [27] where the same problem for the Fokker-Planck equation with subcritical confinement force corresponding to case 1 is considered and where a cornerstone step of the strategy we follow here has been devised. This work has been subsequently carried on by Guillin and collaborators in [10] and [11, Theorem 3.2 & Section 5.1]. These last works are based on the Lyapunov condition method, that we discuss in the presentation of case 1 and which we believe is related to the method we use here. The Lyapunov condition method has been extensively studied during the last decade, and we refer for instance to [4, 3] and the references therein for more details.

An abstract theory for non-uniformly exponentially stable semigroups (with non exponential decay rate) has also been recently developed and we refer the interested reader to [5, 6] and the references therein. We finally refer to [9] where similar semigroup analysis as here is developed and applied in order to establish the well-posedness of the Landau equation in large spaces.

Let us briefly explain the main ideas behind our method of proof. In Case 1, and as a first step, we may use the argument introduced in [27] (see also [16, Lemma 1.3]) which we briefly recall now. We consider three Banach spaces E_2, E_1 and E_0, such that $E_2 \subset E_1 \subset E_0 \subset L^1$, and more precisely E_1 is an interpolation space of order $1-1/\alpha$ between E_0 and E_2 for some $\alpha \in (1,\infty)$, that is

$$\|f\|_{E_1} \le C_\alpha \, \|f\|_{E_0}^{1/\alpha} \, \|f\|_{E_2}^{1-1/\alpha}, \quad \forall\, f \in E_2, \tag{1.20}$$

and such that the semigroup $S_{\mathcal{L}}(t)$ associated to the Fokker-Planck equation can be solved in each of these spaces. Moreover, assume that for any $f_0 \in E_2$, the solution $S_{\mathcal{L}}(t)f_0 = f(t)$ to the Fokker-Planck equation (1.1)–(1.2) satisfies the following two differential inequalities

$$\frac{d}{dt}\|f(t)\|_{E_1} \le -\lambda\,\|f(t)\|_{E_0}, \qquad \frac{d}{dt}\|f(t)\|_{E_2} \le 0, \tag{1.21}$$

for some constant $\lambda > 0$. Using the fact that $\|f(t)\|_{E_2} \le \|f_0\|_{E_2}$, as a consequence of the above second differential inequality, together with (1.20), we obtain the closed differential inequality

$$\frac{d}{dt}\|f(t)\|_{E_1} \le -\lambda\, C_\alpha^{-\alpha}\,\|f_0\|_{E_2}^{-(\alpha-1)}\|f(t)\|_{E_1}^{\alpha}.$$

We may readily integrate it and we obtain the estimate

$$\|f(t)\|_{E_1} \lesssim t^{-1/(\alpha-1)}\|f_0\|_{E_2}. \tag{1.22}$$

Now, choosing $E_1 = L^2(G^{-1/2})$, $E_0 := L^2(G^{-1/2}\langle x\rangle^{\gamma-1})$ and $E_2 = L^\infty(G^{-1})$, one may see that the first differential inequality in (1.21) is an immediate consequence of the weak Poincaré inequality (1.11). The second differential inequality is a kind of generalized relative entropy principle (see [27, 20]). The above estimate (1.22) is a somewhat rough variant of estimate (1.19). It is noteworthy that for $\alpha \in (1,2)$, we get that the associated semigroup $S_{\mathcal{L}}$ defined by $S_{\mathcal{L}}(t)f_0 = f(t)$ satisfies in particular $\|S_{\mathcal{L}}\|_{E_2 \to E_1} \in L^1(0,\infty)$.

We then generalize the decay estimate to a wider class of Banach spaces by adapting the extension theory introduced in [25] and developed in [14, 22]. In order to do so, we consider the two Banach spaces $\mathcal{E}_2 := L^p(m)$ and $\mathcal{E}_1 := L^p$, and we introduce a splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$, where $\mathcal{A}$ is an appropriately defined bounded operator so that $\mathcal{B}$ becomes a dissipative operator. Then, we show that for $i = 1$ and $i = 2$

$$\|S_{\mathcal{B}}\mathcal{A}\|_{\mathcal{E}_i \to \mathcal{E}_i} \in L^1(\mathbb{R}_+), \qquad \|S_{\mathcal{B}}\|_{\mathcal{E}_2 \to \mathcal{E}_1} \in L^1(\mathbb{R}_+), \qquad \|\mathcal{A}S_{\mathcal{B}}\|_{\mathcal{E}_1 \to \mathcal{E}_1} \in L^1(\mathbb{R}_+).$$

If $\mathcal{T}_i$, with $i = 1, 2$, are two given operator valued measurable functions defined on $(0,\infty)$, we denote by

$$(\mathcal{T}_1 * \mathcal{T}_2)(t) := \int_0^t \mathcal{T}_1(\tau)\mathcal{T}_2(t-\tau)\,d\tau$$

their convolution on $\mathbb{R}_+$. We then set $\mathcal{T}^{(*0)} := I$, $\mathcal{T}^{(*1)} := \mathcal{T}$ and, for any $k \geq 2$, $\mathcal{T}^{(*k)} := \mathcal{T}^{*(k-1)} * \mathcal{T}$. We may show that for $n \in \mathbb{N}$ sufficiently large (actually $n \geq 1 + (d/2)$ is enough), we have

$$\|(S_{\mathcal{B}}\mathcal{A})^{(*n)}\|_{E_1 \to \mathcal{E}_1} \in L^1(\mathbb{R}_+), \quad \|(\mathcal{A}S_{\mathcal{B}})^{(*n)}\|_{\mathcal{E}_1 \to E_1} \in L^1(\mathbb{R}_+). \tag{1.23}$$

Next, from the usual Duhamel formula, the solution of (1.1) can be written as $f(t) = S_{\mathcal{B}}(t)f_0 + \int_0^t S_{\mathcal{B}}(t-\tau)\mathcal{A}S_{\mathcal{L}}(\tau)f_0\,d\tau$. Thus, using the above notations for the convolution of operators valued functions, we have $S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{L}})$, and interchanging the role played by $\mathcal{L}$ and $\mathcal{B}$ in this expression, we get the following operators versions of Duhamel formulas

$$S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{L}}) = S_{\mathcal{B}} + (S_{\mathcal{B}}\mathcal{A}) * S_{\mathcal{L}} \tag{1.24}$$

$$= S_{\mathcal{B}} + S_{\mathcal{L}} * (\mathcal{A}S_{\mathcal{B}}) = S_{\mathcal{B}} + (S_{\mathcal{L}}\mathcal{A}) * S_{\mathcal{B}}. \tag{1.25}$$

Upon replacing recursively $S_{\mathcal{L}}$ in either of the expressions on the right hand side by either of the Duhamel's formula, we get, for instance:

$$\begin{aligned} S_{\mathcal{L}} &= S_{\mathcal{B}} + S_{\mathcal{B}} * \mathcal{A}\{S_{\mathcal{B}} + (S_{\mathcal{B}}\mathcal{A}) * S_{\mathcal{L}}\} \\ &= S_{\mathcal{B}} + S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}}) + (S_{\mathcal{B}}\mathcal{A})^{(*2)} * S_{\mathcal{L}}. \end{aligned}$$

By induction on the integers $n_1 \geq 0$ and $n_2 \geq 0$, we thus obtain

$$S_{\mathcal{L}} = \sum_{k=0}^{n_1+n_2-1} S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*k)} + (S_{\mathcal{B}}\mathcal{A})^{(*n_1)} * S_{\mathcal{L}} * (\mathcal{A}S_{\mathcal{B}})^{(*n_2)}. \tag{1.26}$$

Using the above formulas (1.26) and estimates (1.23), as well as the decay estimate (1.22) for initial data in the space E_2, we conclude that $\|S_{\mathcal{L}}\|_{\mathcal{E}_2 \to \mathcal{E}_1} \in L^1(\mathbb{R}_+)$ which is nothing but a rough version of the estimates presented in Theorem 1.2. While the method leading to (1.22) in E_i can be performed only in very specific (Hilbert) spaces, the last extension method is very general and can be used in a large class of Banach spaces $\mathcal{E}_i$ (once we already know the decay in one pair of spaces (E_1, E_2)).

Finally, in Case 2, we start proving an equivalent to estimate (1.22) in one appropriate pair of (small) spaces. In order to do so, we adapt the Krein-Rutman theory to the present context. On the one hand, it is a simple version of the Krein-Rutman theory because the equation is mass conserving, a property which implies that the largest eigenvalue of $\mathcal{L}$ is $\lambda_1 = 0$. On the other hand, it is not a classical version because the operator $\mathcal{L}$ does not have a compact resolvent (however it has power-compact resolvent in the sense of Voigt [30]) and more importantly 0 is not necessarily an isolated point in the spectrum. First adapting (from [13, 24] for instance) some more or less standard arguments, we prove that there exists G, a unique stationary solution of (1.1) which is positive, has unit mass and is such that $G \in L^\infty(\exp(\kappa\langle x\rangle^\gamma))$, for all $\kappa \in (0, 1/\gamma)$. Next, we prove an estimate similar to (1.22) by establishing a set of accurate estimates on the resolvent operators $\mathcal{R}_\mathcal{B}(z)$, $R_\mathcal{L}(z)$ and by using the iterated Duhamel formula

$$S_\mathcal{L} = \sum_{k=0}^{5} S_\mathcal{B} * (\mathcal{A}S_\mathcal{B})^{(*k)} + S_\mathcal{L} * (\mathcal{A}S_\mathcal{B})^{(*6)},$$

together with the inverse Laplace formula

$$S_\mathcal{L} * (\mathcal{A}S_\mathcal{B})^{(*6)}(t) = \frac{\mathrm{i}}{2\pi}\,\frac{1}{t^n}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{e}^{zt}\,\frac{d^n}{dz^n}\left[R_\mathcal{L}(z)(\mathcal{A}R_\mathcal{B}(z))^6\right] dz,$$

which holds true for any time $t > 0$ and any integer n.

To finish this introduction, let us describe the plan of the paper. In Section 2, we introduce an appropriate splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and present the main estimates on the semigroup $S_\mathcal{B}$. In Section 3, we deduce that the semigroup $S_\mathcal{L}$ is bounded in the spaces $L^p(m)$. In Section 4, the proof of Theorem 1.2 is carried out in the case when a weak Poincaré inequality is satisfied (Case 1). Finally, Section 5 is devoted to the proof of Theorem 1.2 in the general case (Case 2).

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2. The splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and growth estimates on $S_\mathcal{B}$

We introduce the splitting of the operator $\mathcal{L}$ defined by

$$\mathcal{A}f := M\chi_R f, \qquad \mathcal{B}f := \mathcal{L}f - M\chi_R f \tag{2.1}$$

where M is positive constant, and for a fixed truncation function $\chi \in \mathcal{D}(\mathbb{R}^d)$ such that $1_{B(0,1)} \le \chi \le 1_{B(0,2)}$, and for $R > 1$ which will be chosen appropriately as well as M, we set $\chi_R(x) := \chi(x/R)$.

2.1. **Basic growth estimates.**

Lemma 2.1. *For any exponent $p \in [1, \infty]$ and any stretch exponential or polynomial weight function m given by* (1.7), (1.8) *or* (1.9), *we can choose R, M large enough in the definition* (2.1) *of $\mathcal{B}$ such that the operator $\mathcal{B}$ is dissipative in $L^p(m)$, namely*

$$\|S_\mathcal{B}(t)\|_{L^p(m)\to L^p(m)} \le 1, \qquad \forall\, t \ge 0. \tag{2.2}$$

Moreover, if $m(x) = \langle x\rangle^k$, set $\beta := k(1-\theta)/(2-\gamma)$ for $k^*/k < \theta < 1$, and $\beta \in (0,(k-k^*)/(2-\gamma))$ arbitrary when $\theta \le k^*/k$. Then the function Θ_m being defined by (1.18)*, we have*

$$\|S_{\mathcal{B}}(t)\|_{L^p(m)\to L^p(m^\theta)} \lesssim \Theta_m(t). \tag{2.3}$$

If $m(x) = \exp(\kappa\langle x\rangle^s)$ satisfies (1.8) *or* (1.9)*, the above inequality holds, provided the function Θ_m is defined by*

$$\Theta_m(t) := \exp(-\lambda t^{s/(2-\gamma)}),$$

where $\lambda > 0$ can be chosen arbitrarily when $s < \gamma$, and $\lambda < \lambda_*$, with

$$\lambda_* := (\kappa(1-\theta))^{(2-2\gamma)/(2-\gamma)}(\kappa\gamma(1-\kappa\gamma))^{\gamma/(2-\gamma)},$$

when $s = \gamma$.

Proof of Lemma 2.1. The proof is similar to the proof of [14, Lemma 3.8] and [22, Lemma 3.8].

Step 1. We first fix $p \in [1,\infty)$, assuming m is as in the statement of the Lemma. We start recalling an identity satisfied by the operator $\mathcal{B}$ (see the proof of [22, Lemma 3.8]). For any smooth, rapidly decaying and positive function f, we have

$$\begin{aligned}\int_{\mathbb{R}^d} (\mathcal{B} f)\, f^{p-1}\, m^p dx = \\ = -(p-1)\int_{\mathbb{R}^d} |\nabla(mf)|^2\,(mf)^{p-2}dx + \int_{\mathbb{R}^d} f^p\, m^p\, \psi^0_{m,p} dx,\end{aligned}$$

with

$$\psi^0_{m,p} := \frac{(2-p)}{p}\frac{\Delta m}{m} + \frac{2}{p'}\frac{|\nabla m|^2}{m^2} + \frac{1}{p'}\operatorname{div}(\mathbf{F}) - \mathbf{F}\cdot\frac{\nabla m}{m} - M\,\chi_R. \tag{2.4}$$

Observe that

$$\begin{aligned}\frac{\nabla m}{m} &= k\kappa x\langle x\rangle^{s-2} \\ \frac{\Delta m}{m} &= k\kappa d\langle x\rangle^{s-2} + s(s-2)\kappa|x|^2\langle x\rangle^{s-4} + \nu|x|^2\langle x\rangle^{2s-4},\end{aligned}$$

where we have set

$$\begin{aligned}&s := 0, \quad \kappa := 1, \quad \nu := k(k-2), \quad &&\text{when } m(x) = \langle x\rangle^k, \\ &k := s, \quad \nu := (s\kappa)^2, &&\text{when } m(x) = \exp(\kappa\,\langle x\rangle^s).\end{aligned}$$

In this latter case, for $s \in (0,\gamma]$, the third term in the definition of $\psi^0_{m,p}$ is negligible with respect to the first and second terms, and thus

$$\begin{aligned}\psi^0_{m,p}\,|x|^{2-\gamma-s} &\underset{|x|\to\infty}{\longrightarrow} -a^* := (\kappa\gamma)^2 - \kappa\gamma < 0 \quad \text{if } s = \gamma, \\ \psi^0_{m,p}\,|x|^{2-\gamma-s} &\underset{|x|\to\infty}{\longrightarrow} -a^* := -\infty \quad \text{if } 0 < s < \gamma.\end{aligned}$$

When $m = \langle x\rangle^k$, and $k > k^*(p) := C_F/p'$, the first and second terms are negligible with respect to the third term, and then

$$\limsup_{|x|\to\infty} \psi^0_{m,p}\,|x|^{2-\gamma} \le -a^* := (1-\frac{1}{p})\,C_F - k < 0.$$

We deduce that for any $a \in (0, a^*(m,p))$, we can choose $R > 1$ and M large enough in such a way that $\psi^0_{m,p}(x) \le -a\langle x\rangle^{\gamma+s-2}$ for all $x \in \mathbb{R}^d$, and then

$$\int (\mathcal{B}f)\, f^{p-1}\, m^p \le -a \int |f|^p\, m^p\, \langle x\rangle^{\gamma+s-2} - (p-1)\int |\nabla(fm)|^2 (fm)^{p-1}. \tag{2.5}$$

In particular, using only the fact that the RHS term is negative, we conclude that the operator $\mathcal{B}$ is dissipative and we classically deduce that the semigroup $S_\mathcal{B}$ is well-defined on $L^p(m)$ for $p \in [1,\infty)$ and that it is a strongly continuous contraction semigroup, in other words, (2.2) holds for any $p \in [1,\infty)$. Since we may choose R, M such that the above inequality holds true for any $p \in [1,\infty)$ when $a \in (0, a^*(m,\infty))$, we may pass to the limit as $p \to \infty$ in (2.2) and we conclude that $S_\mathcal{B}$ is a contraction semigroup in $L^p(m)$, for any $p \in [1,\infty]$.

Step 2. Take $p \in [1,\infty)$ and $k > k^*(p) = C_F/p'$, and finally, assuming first that $\theta > k^*/k$, set $\ell := \theta k \in (k^*, k)$. If $f_0 \in L^p(m)$ with $m := \langle x\rangle^k$, denote $f(t) := S_\mathcal{B}(t) f_0$. Dropping the last term in (2.5), we have for $a \in (0, a^*(m,p))$

$$\frac{d}{dt}\int |f|^p\, \langle x\rangle^{p\ell} \le -ap \int |f|^p\, \langle x\rangle^{p\ell+\gamma-2}.$$

Using Hölder's inequality

$$\int f^p\, \langle x\rangle^{p\ell} \le \Big(\int f^p\, \langle x\rangle^{p\ell+\gamma-2}\Big)^\eta \Big(\int f^p\, \langle x\rangle^{pk}\Big)^{1-\eta}$$

with $\eta := (k-\ell)/[k-\ell+(2-\gamma)/p] \in (0,1)$, and the fact that the semigroup $S_\mathcal{L}$ is a contraction semigroup in L^p_k by (2.2), upon denoting $\alpha := \eta/(1-\eta) = p\,(k-\ell)/(2-\gamma)$, we get

$$\frac{d}{dt} Y_\theta(t) \le -a\,p\, Y_\theta(t)^{(\alpha+1)/\alpha}\, Y_1(0)^{-1/\alpha}, \qquad \text{where } Y_\tau(t) := \int f^p \langle x\rangle^{p\tau}.$$

Integrating the above differential inequality yields

$$Y_\theta(t) \le \big(\frac{\alpha}{apt}\big)^\alpha\, Y_1(0),$$

which in turn implies (2.3) with $\Theta_m(t)$ replaced with $\Big(\frac{(k-\ell)/(2-\gamma)}{at}\Big)^{\frac{k-\ell}{2-\gamma}}$. Since for $0 \le t \le 1$ we have clearly $Y_\theta(t) \lesssim Y_1(0)$ the proof of (2.3) is complete when $p < \infty$ and $m(x) = \langle x\rangle^k$ and $\ell := k\theta > k^*$.

In the case where $\ell = k\theta \le k^*$, it is enough to pick $\theta_0 > \theta$ so that $k\theta_0 > k^*$ and observe that we have $Y_\theta \le Y_{\theta_0}$: in this way one is convinced that (2.3) holds for all $p < \infty$ and $0 \le \theta < 1$.

We deduce the same estimate for $p = \infty$ by letting $p \to \infty$ in (2.3).

Step 3. Similarly, when the weight function m is an stretch exponential as defined in (1.8) or (1.9), take $p \in [1,\infty)$. Given an initial datum $f_0 \in L^p(m)$, denote $f(t) := S_\mathcal{B}(t) f_0$, and set $Y_\theta(t) := \|f(t)\|^p_{L^p(m^\theta)}$. Thanks to the above Step 1 we have for all $t \ge 0$ and $0 < \theta \le 1$

$$Y_\theta(t) \le Y_\theta(0).$$

For $\rho > 0$ denote by B_ρ the ball of $\mathbb{R}^d$ centered at the origin with radius ρ. Using the estimate (2.5) with the weight function m^θ, neglecting the last term of that inequality, we have successively

$$\begin{aligned}
\frac{d}{dt} Y_\theta(t) &= p \int (\mathcal{B} f)\, f^{p-1}\, m^{p\theta} \\
&\le -a\,p \int_{B_\rho} |f|^p\, m^{p\theta}\, \langle x\rangle^{\gamma+s-2} \\
&\le -a\,p\, \langle\rho\rangle^{\gamma+s-2} \int_{B_\rho} |f|^p\, m^{p\theta} \\
&\le -a\,p\, \langle\rho\rangle^{\gamma+s-2} Y_\theta + a\,p\, \langle\rho\rangle^{\gamma+s-2} \int_{B_\rho^c} |f|^p\, m^{p\theta} \\
&\le -a\,p\, \langle\rho\rangle^{\gamma+s-2} Y_\theta + a\,p\, \langle\rho\rangle^{\gamma+s-2} m(\rho)^{-p(1-\theta)} \int_{B_\rho^c} |f_0|^p\, m^p \\
&\le -a\,p\, \langle\rho\rangle^{\gamma+s-2} Y_\theta + a\,p\, \langle\rho\rangle^{\gamma+s-2}\, m(\rho)^{-p(1-\theta)} \int |f_0|^p\, m^p.
\end{aligned}$$

Integrating this differential inequality we deduce

$$\begin{aligned}
Y_\theta(t) &\le \exp(-ap\,t\,\langle\rho\rangle^{\gamma+s-2})\, Y_\theta(0) + m(\rho)^{-p(1-\theta)}\, Y_\theta(0). \\
&\le \big(\exp(-ap\,t\,\langle\rho\rangle^{\gamma+s-2}) + \exp(-p(1-\theta)\,\rho^s)\big)\; Y_\theta(0).
\end{aligned}$$

We may choose ρ such that $a\,\langle\rho\rangle^{\gamma+s-2} t = (1-\theta)\,\rho^s$, that is we may take ρ of order $t^{1/(2-\gamma)}$, which allows us to conclude that (2.3) also holds in the exponential case. As indicated above, the estimate (2.3) for $p=\infty$ is obtained by letting $p \to \infty$. $\square$

The following two lemmas state that when the weight function is exponential, that is $m(x) := \exp(\kappa \langle x\rangle^\gamma)$, the semigroup $S_\mathcal{B}$ is ultracontractive in the spaces $L^p(m)$, that is it maps $L^1(m)$ into $L^\infty(m)$ for $t > 0$. As it is pointed out in [17] (see Remark 2.2 of this reference for a proof based on Probability arguments, and Remark 5.2 for a simple proof based on comparison theorems for parabolic equations), when one considers an operator of the type $Lf := \Delta f + \nabla V \cdot \nabla f$ with V satisfying, for some constants $R > 0$ and $c_0 \ge 0$,

$$\forall x \in B_R^c, \qquad \frac{\Delta V^{1/2}}{V^{1/2}} + c_0 \ge 0,$$

and if there exists a positive constant $c_1 > 0$ such that $V(x) \ge c_1$ for all $x \in B_R^c$, then the semigroups $S_L(t)$ and $S_{L^*}(t)$ are ultracontractive in the spaces $L^p(\exp(V))$ (the above condition on $\Delta V^{1/2}/V^{1/2}$ is a sort of convexity condition at infinity). Here the operator $\mathcal{B}$ is not exactly of the same type as L, but nevertheless the ultracontractivity of the semigroup $S_\mathcal{B}(t)$, as well as that of $(S_\mathcal{B})^*$, holds. (Recall also that the ultracontractivity of the semigroup $S_\mathcal{B}$ is equivalent to an appropriate form of Nash inequality for the operator $\mathcal{B}$).

Lemma 2.2. *Consider the weight function $m_0 := \exp(\kappa \langle x\rangle^\gamma)$, for $0 < \kappa\gamma < 1$. Then there exists $R_0, M_0 > 0$ such that for $M \ge M_0$ and $R \ge R_0$ we have*

$$\forall\, t > 0, \qquad \|S_\mathcal{B}(t)\|_{L^1(m_0) \to L^2(m_0)} \lesssim t^{-d/4}. \tag{2.6}$$

Proof. The proof is similar to the proof of [14, Lemma 3.9], see also [22, Section 3], [23, Section 2] and [16]. For the sake of completeness we sketch it below.

Consider $f_0 \in L^2(m_0)$ and denote $f(t) = S_{\mathcal{B}}(t) f_0$. From (2.4) with $p = 2$ and throwing out the last term in that inequality, we find

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}^d} f(t)^2\, m_0^2 dx \le -\int_{\mathbb{R}^d} |\nabla (f(t) m_0)|^2 dx.$$

Using Nash's inequality for $g := f(t) m_0$ ([18, Chapter 8]) stating that for some constant $c > 0$

$$\int_{\mathbb{R}^d} g^2 dx \le c \left(\int_{\mathbb{R}^d} |\nabla g|^2 dx\right)^{\frac{d}{d+2}} \left(\int_{\mathbb{R}^d} |g| dx\right)^{\frac{4}{d+2}} \tag{2.7}$$

we get (for another constant $c > 0$)

$$X'(t) \le -2\, c\, Y(t)^{-4/d}\, X(t)^{1+\frac{2}{d}}, \tag{2.8}$$

where for brevity of notations we have set

$$X(t) := \|f(t)\|^2_{L^2(m_0)}, \qquad Y(t) := \|f(t)\|_{L^1(m_0)}.$$

Since according to (2.2) we have $Y(t) \le Y_0$ for $t > 0$, we may integrate the differential inequality (2.8) and obtain (2.6).

□

The next result states that the adjoint of $\mathcal{B}$ generates also an ultracontractive semigroup in the spaces $L^p(m_0)$.

Lemma 2.3. *Consider the weight function $m_0 := \exp(\kappa \langle x\rangle^\gamma)$, for $0 < \kappa\gamma < 1$. Then there exists $R_1 \ge R_0$ and $M_1 \ge M_0$ (where M_0 and R_0 are defined in the previous lemma) such that for $M \ge M_1$ and $R \ge R_1$, the semigroup generated by $\mathcal{B}_*$, the formal adjoint of $\mathcal{B}$, satisfies*

$$\forall\, t > 0, \qquad \|S_{\mathcal{B}_*}(t)\|_{L^1(m_0)\to L^2(m_0)} \lesssim t^{-d/4}. \tag{2.9}$$

Consequently, for $M \ge M_1$ and $R \ge R_1$, we have

$$\forall\, t > 0, \qquad \|S_{\mathcal{B}}(t)\|_{L^2(m_0)\to L^\infty(m_0)} \lesssim t^{-d/4}. \tag{2.10}$$

Proof. We first observe that if the operator B is of the form

$$Bf = \Delta f + \mathbf{b}(x)\cdot\nabla f + a(x)\, f,$$

and we make the transform $h := f\, m$, then the corresponding operator $B_m h := m\, B(m^{-1} h)$ is of the same type and is given by

$$B_m h = \Delta h + \left[\mathbf{b}(x) - 2\,\frac{\nabla m}{m}\right]\cdot\nabla h + \left[-\frac{\Delta m}{m} + 2\,\frac{|\nabla m|^2}{m^2} + a - \mathbf{b}(x)\cdot\frac{\nabla m}{m}\right] h.$$

Observe also that the formal adjoint of B, denoted by B_* to avoid any misunderstanding, is given by

$$B_* g = \Delta g - \mathbf{b}(x)\cdot\nabla g + (a(x) - \mathrm{div}(\mathbf{b}(x)))\, g.$$

Applying these observations to $\mathcal{B} f = \Delta f + \mathbf{F}\cdot\nabla f + (\mathrm{div}(\mathbf{F}) - M\chi_R)\, f$, we get that for $h := g\, m_0$ the operator $\mathcal{B}_{*,m}$, associated to the formal adjoint $\mathcal{B}_*$, is given by

$$\mathcal{B}_{*,m} h = \Delta h - \left[\mathbf{F} - 2\,\frac{\nabla m}{m}\right]\cdot\nabla h + \left[\frac{\Delta m}{m} - M\chi_R - \mathbf{F}\cdot\frac{\nabla m}{m}\right] h. \tag{2.11}$$

Thus, if $g_0 \ge 0$ is a smooth initial datum, then the solution g of

$$\partial_t g = \mathcal{B}_* g, \qquad g(0,x) = g_0(x),$$

yields the function $h := g\, m_0$ which satisfies the evolution equation

$$\partial_t h = \mathcal{B}_{*,m_0} h, \qquad h(0,x) = h_0(x). \tag{2.12}$$

Now, one can verify easily that for $h_0 \in C_c^\infty(\mathbb{R}^d)$ and $h_0 \geq 0$, the solution h to the equation

$$\partial_t h = \Delta h + \mathbf{b}(x) \cdot \nabla h + a(x)\, h, \qquad h(0,x) = h_0(x)$$

satisfies $h(t,x) \geq 0$ and for $1 \leq p < \infty$ we have the identity

$$\frac{d}{dt}\frac{1}{p}\int h(t,x)^p\, dx = -(p-1)\int |\nabla h|^2\, h^{p-2}\, dx + \frac{1}{p}\int (p\, a(x) - \mathrm{div}(\mathbf{b}(x)))\, h^p\, dx.$$

As a consequence, applying this to the operator $\mathcal{B}_{*,m_0}$, we have that the solution h of equation (2.12) verifies

$$\frac{d}{dt}\frac{1}{p}\int h(t,x)^p\, dx \leq -(p-1)\int |\nabla h|^2 h^{p-2}\, dx + \int h^p\, \psi_{*,m_0,p}\, dx,$$

where, for convenience, we have set

$$\psi_{*,p,m_0} := \frac{(p-2)}{p})\frac{\Delta m_0}{m_0} + \frac{2}{p}\frac{|\nabla m_0|^2}{m_0^2} + \frac{1}{p}\,\mathrm{div}(\mathbf{F}) - \mathbf{F} \cdot \frac{\nabla m_0}{m_0} - M\chi_R. \tag{2.13}$$

Proceeding as we did above in the study of the function defined in (2.4), we may choose, if necessary, M and R large enough (in particular larger than M_0, R_0 given by Lemma 2.2), so that for all $x \in \mathbb{R}^d$ we have $\psi_{*,m_0,} \leq 0$. Therefore we conclude that

$$\frac{d}{dt}\frac{1}{p}\|h(t)\|_{L^p}^p \leq -(p-1)\int |\nabla h|^2 h^{p-2}\, dx. \tag{2.14}$$

On the one hand, taking $p := 1$, we deduce that the semigroup generated by $\mathcal{B}_{*,m_0}$ is a contraction semigroup in $L^1(\mathbb{R}^d)$, that is $\|h(t)\|_{L^1} \leq \|h_0\|_{L^1}$ for all $t \geq 0$.

On the other hand, taking $p := 2$ and using Nash's inequality (2.7), together with the fact that $\|h(t)\|_{L^1}$ is non increasing, we deduce that if we set $X(t) := \|h(t)\|_{L^2}^2$, then for some constant $c > 0$, the function $X(t)$ satisfies the differential inequality

$$\frac{d}{dt}X(t) \leq -c\, \|h_0\|_{L^1}^{-4/d}\, X(t)^{(2+d)/d}.$$

Integrating this, we get that for all $t > 0$

$$\|h(t)\|_{L^2} \lesssim t^{-d/4}\, \|h_0\|_{L^1}, \quad \text{that is} \quad \|g(t)\|_{L^2(m_0)} \lesssim t^{-d/4}\, \|g_0\|_{L^1(m_0)}.$$

From this, by a density argument and the splitting of any initial datum as the difference of two nonnegative functions, we conclude that for any $g_0 \in L^1(m_0)$ the associated solution to $\partial_t g = \mathcal{B}_* g$ satisfies

$$\|S_{\mathcal{B}_*}(t) g_0\|_{L^2(m_0)} = \|g(t)\|_{L^2(m_0)} \lesssim t^{-d/4}\, \|g_0\|_{L^1(m_0)}, \quad \forall t > 0, \tag{2.15}$$

which is precisely (2.9). To conclude the proof of the Lemma, observe that for $f, g \in C_c^\infty(\mathbb{R}^d)$ we have

$$\begin{aligned}
(\mathcal{B} f | g)_{L^2(m_0)} &= \int \mathcal{B} f\, g\, m_0^2\, dx = \int \mathcal{B}_{m_0}(f\, m_0)\, (g\, m_0)\, dx \\
&= \int (f\, m_0)\, \mathcal{B}_{*,m_0}(g\, m_0)\, dx = \int (f\, m_0)\, m_0^{-1} \mathcal{B}_{*,m_0}(g\, m_0)\, m_0(x)^2\, dx \\
&= (f | \mathcal{B}_* g)_{L^2(m_0)}.
\end{aligned}$$

This allows one to verify that $(S_{\mathcal{B}}(t))^* = S_{\mathcal{B}_*}(t)$, the adjoint being taken in the sense of the Hilbert space $L^2(m_0)$, where we assume that this space is identified with its dual. Therefore, since with these conventions we have that $(L^1(m_0))' = L^\infty(m_0)$, thanks to (2.15), we conclude that (2.10) is established. $\square$

Putting together the previous estimates, we get the following ultracontractivity result on the semigroup $S_{\mathcal{B}}$ and on the iterated convolution family of operators $(\mathcal{A}S_{\mathcal{B}})^{(*n)}$.

Lemma 2.4. *Consider the weight function $m_0 := \exp(\kappa\langle x\rangle^\gamma)$, for $0 < \kappa\gamma < 1$. Then, M, R being large enough as in Lemma 2.3, there exists $\lambda_* \in (0,\infty)$ such that for any $p, q \in [1,\infty]$, $p \le q$, and for any $0 \le \theta_2 < \theta_1 \le 1$, the semigroup $S_{\mathcal{B}}$ satisfies*

$$\|S_{\mathcal{B}}(t)\|_{L^p(m_0^{\theta_2}) \to L^q(m_0^{\theta_1})} \lesssim t^{-(d/2)(1/p-1/q)}\, \mathrm{e}^{-\lambda_*\, t^{\gamma/(2-\gamma)}}, \quad \forall\, t > 0. \tag{2.16}$$

Moreover, if $n \ge d/2$ is an integer, for all $\lambda < \lambda_*$ and all $t > 0$, we have

$$\|(\mathcal{A}S_{\mathcal{B}})^{(*n)}(t)\|_{L^1(m_0) \to L^\infty(m_0^2)} \lesssim \mathrm{e}^{-\lambda\, t^{\gamma/(2-\gamma)}}. \tag{2.17}$$

Proof. Writing $S_{\mathcal{B}}(t) = S_{\mathcal{B}}(t/2)S_{\mathcal{B}}(t/2)$, and using (2.6) together (2.10) we deduce that for any $t > 0$ we have

$$\begin{aligned}\|S_{\mathcal{B}}(t)\|_{L^1(m_0)\to L^\infty(m_0)} &\le \|S_{\mathcal{B}}(t/2)\|_{L^2(m_0)\to L^\infty(m_0)}\, \|S_{\mathcal{B}}(t/2)\|_{L^1(m_0)\to L^2(m_0)} \\ &\lesssim t^{-d/2}.\end{aligned} \tag{2.18}$$

Since on the other hand we have also $\|S_{\mathcal{B}}(t) f_0\|_{L^p(m_0)} \le \|f_0\|_{L^p(m_0)}$, a classical interpolation argument yields that for $1 \le p \le q \le \infty$ we have

$$\|S_{\mathcal{B}}(t)\|_{L^p(m_0)\to L^q(m_0)} \lesssim t^{-(d/2)(1/p-1/q)}. \tag{2.19}$$

Using Lemma 2.1, since

$$\|S_{\mathcal{B}}(t)\|_{L^q(m_0)\to L^q(m_0^\theta)} \lesssim \exp(-\lambda t^{\gamma/(2-\gamma)}),$$

one sees that the proof of (2.16) with $\theta_1 = 1$ is complete. To see that (2.16) holds with $\theta_1 < 1$, it is enough to observe that $m_0^{\theta_1}(x)$ is of the same type as $m_0(x) = \exp(\kappa\langle x\rangle^\gamma)$ provided κ is replaced with $\theta_1\kappa$.

In order to show (2.17), first note that the operator $\mathcal{A}$ consisting simply in a multiplication by a smooth compactly supported function, thanks to the above Lemmas we clearly have, for all $t > 0$,

$$\|\mathcal{A}S_{\mathcal{B}}(t)\|_{L^{p_1}(m_0)\to L^{p_2}(m_0^2)} \lesssim t^{-\alpha}\, e^{-\lambda\, t^{\sigma^*_{\mathcal{B}}}}, \tag{2.20}$$

where it is understood that $\alpha := d/2$ if $(p_1, p_2) := (1, \infty)$, and $\alpha := 0$ when $p_1 = p_2$. We claim that the three estimates for three choices $(p_1, p_2) = (1, \infty)$, and $p_1 = p_2 = 1$, as well as $p_1 = p_2 = \infty$, imply that for all integers $n \ge 1$ we have

$$\|(\mathcal{A}S_{\mathcal{B}})^{(*n)}(t)\|_{L^{p_1}(m_0)\to L^{p_2}(m_0^2)} \le C_n\, t^{n-1-\alpha}\, e^{-\lambda\, t^{\sigma^*_{\mathcal{B}}}}, \tag{2.21}$$

from which one readily deduces (2.17). We prove (2.21) by induction. Estimates (2.21) are clearly true for $n = 1$. Let us assume that $p_1 = 1$ and $p_2 = \infty$, for which

(2.21) is true for a certain $n \geq 1$. Introducing the shorthand notation $u := \mathcal{A} S_{\mathcal{B}}$ and $\| \cdot \|_{p_1 \to p_2} = \| \cdot \|_{L^{p_1}(m_0) \to L^{p_2}(m_0^2)}$, we have

$$\begin{aligned}
\|u^{(*(n+1))}(t)\|_{1\to\infty} &\leq \int_0^{t/2} \|u^{(n)}(t-s)\|_{1\to\infty} \, \|u(s)\|_{1\to 1} \, ds \\
&\quad + \int_{t/2}^{t} \|u^{(n)}(t-s)\|_{\infty\to\infty} \, \|u(s)\|_{1\to\infty} \, ds \\
&\leq C_n \, C_1 \, e^{-\lambda \, t^{\sigma^*_{\mathcal{B}}}} \int_0^{t/2} (t-s)^{-\alpha+n-1} \, ds \\
&\quad + C_n \, C_1 \, e^{-\lambda \, t^{\sigma^*_{\mathcal{B}}}} \int_{t/2}^{t} (t-s)^{n-1} \, s^{-\Theta} \, ds \\
&\leq \quad C_n \, C_1 \, e^{-\lambda \, t^{\sigma^*_{\mathcal{B}}}} t^{-\alpha+n} \left\{ \int_0^{1/2} (1-\tau)^{-\alpha+n-1} \, d\tau \right. \\
&\qquad \left. + \int_{1/2}^{1} (1-\tau)^{n-1} \, \tau^{-\alpha} \, d\tau \right\},
\end{aligned}$$

where we have used that $t^\sigma \leq (t-s)^\sigma + s^\sigma$ for any $0 \leq s \leq t$ and $\sigma \in (0,1)$. This proves estimate (2.21) at rank $n+1$ and $(p_1, p_2) = (1, \infty)$.

The proof of the other cases $(p_1, p_2) = (1, 1)$ and $(p_1, p_2) = (\infty, \infty)$ is similar, if not much simpler, and can be left to the reader. $\square$

2.2. **Additional growth estimates.** In order to deal with the general case in Section 5, we will need a more accurate version of the previous estimates.

Lemma 2.5. *Consider $m_0 := \mathrm{e}^{2\kappa \langle x \rangle^\gamma}$ with $\kappa \in (0, 1/(4\gamma))$ and define the sequence of spaces*

$$X_k := L^2(m_k), \quad m_k := \frac{m_0}{\nu_k}, \quad \nu_k(x) := \sum_{\ell=0}^{k} \frac{(\kappa \, \langle x \rangle^\gamma)^\ell}{\ell!}, \tag{2.22}$$

for any $k \in \mathbb{N}$. There exist some constants R and M in the definition of $\mathcal{B}$ and some constant $\beta > 0$ such that for any $k, j \in \mathbb{N}$, $k \geq j$ and any $\alpha \in (0, \alpha^*)$, $\alpha^* := 1/2(1-\gamma)$, the semigroup $S_{\mathcal{B}}$ satisfies the growth estimate

$$\|S_{\mathcal{B}}(t)\|_{X_{k-j} \to X_k} \lesssim \mathrm{e}^{-\lambda \, \langle t^\alpha \rangle^{2(\gamma-1)} t} + \Big(1 \wedge \frac{k^j}{\kappa^j \, \langle t^\alpha \rangle^{\gamma j}}\Big) \quad \forall \, t \geq 0. \tag{2.23}$$

Proof. We easily compute

$$\frac{\nabla m_k}{m_k} = \gamma \, \kappa \, x \, \langle x \rangle^{\gamma-2} \Big[2 - \frac{1 + ... + (\kappa \, \langle x \rangle^\gamma)^{k-1}/(k-1)!}{1 + ... + (\kappa \, \langle x \rangle^\gamma)^k/k!} \Big],$$

from what we deduce

$$\Big| \frac{\nabla m_k}{m_k} \Big| \leq 2 \, \gamma \, \kappa \, |x| \, \langle x \rangle^{\gamma-2},$$

and

$$-\mathbf{F} \cdot \frac{\nabla m_k}{m_k} \leq -\gamma \, \kappa \, |x|^\gamma \, \langle x \rangle^{\gamma-2} \quad \forall \, x \in B^c_{R_0}.$$

As a consequence, we have

$$\frac{d}{dt} \int f^2 \, m_k^2 \leq - \int |\nabla f|^2 \, m_k^2 + \int f^2 \, m_k^2 \, \psi_k,$$

with

$$\begin{aligned} \psi_k &= \frac{|\nabla m_k|^2}{m_k^2} + \frac{1}{2}\,\mathrm{div}(\mathbf{F}) - \mathbf{F}\cdot\frac{\nabla m_k}{m_k} - M\,\chi_R \\ &\le 4\,\gamma^2\,\kappa^2\,|x|^2\,\langle x\rangle^{2\gamma-4} + \frac{1}{2}\,C'_F\,\langle x\rangle^{\gamma-2} \\ &\qquad -\gamma\,\kappa\,|x|^\gamma\,\langle x\rangle^{\gamma-2}\,\mathbf{1}_{B^c_{R_0}} - M\,\chi_R \\ &\le -2\lambda\,\langle x\rangle^{2(\gamma-1)} \end{aligned}$$

for any $x \in \mathbb{R}^d$ and $k \in \mathbb{N}$, by fixing $\kappa > 0$ small enough (as we did) and then R and M large enough. We deduce

$$\frac{d}{dt}\int f^2\,m_k^2 \le -\int |\nabla f|^2\,m_k^2 - 2\lambda\int f^2\,m_k^2\,\langle x\rangle^{2(\gamma-1)}, \tag{2.24}$$

and in particular

$$Y_k(t) := \int f^2\,m_k^2 \le Y_k(0) \quad \text{for any} \quad k \ge 0.$$

We now observe that for any $j \in \mathbb{N}$, $0 \le j \le k$, there hold $m_k \le m_{k-j}$ as well as

$$m_k(x) \le \frac{m_0(x)}{\nu_{k-j}(x)\,(\kappa\,\langle x\rangle^\gamma/k)^j} = \frac{k^j}{\kappa^j\,\langle x\rangle^{\gamma j}}\,m_{k-j}(x) \quad \forall\,x \in \mathbb{R}^d.$$

The two inequalities together, we have proved

$$\forall\,j \le k,\ \forall\,x \in \mathbb{R}^d \quad m_k(x) \le \Big(1 \wedge \frac{k^j}{\kappa^j\,\langle x\rangle^{\gamma j}}\Big)\,m_{k-j}(x). \tag{2.25}$$

As a consequence, for any $\rho > 0$, we have

$$\begin{aligned} Y_k &= \int_{B_\rho} f^2\,m_k^2 + \int_{B^c_\rho} f^2\,m_k^2 \\ &\le \langle\rho\rangle^{2(1-\gamma)}\int f^2\,m_k^2\,\langle x\rangle^{2(\gamma-1)} + \Big(1 \wedge \frac{k^j}{\kappa^j\,\langle\rho\rangle^{\gamma j}}\Big)^2\,Y_{k-j}. \end{aligned}$$

Coming back to (2.24), we deduce

$$\frac{d}{dt}Y_k \le -2\lambda\,\langle\rho\rangle^{2(\gamma-1)}\,Y_k + 2\lambda\,\langle\rho\rangle^{2(\gamma-1)}\Big(1 \wedge \frac{k^j}{\kappa^j\,\langle\rho\rangle^{\gamma j}}\Big)^2\,Y_{k-j}(0),$$

which in turn implies

$$Y_k(t) \le \Big\{\mathrm{e}^{-2\lambda\,\langle\rho\rangle^{2(\gamma-1)}\,t} + \Big(1 \wedge \frac{k^j}{\kappa^j\,\langle\rho\rangle^{\gamma j}}\Big)^2\Big\}Y_{k-j}(0) \quad \forall\,t \ge 0,\ \rho > 0.$$

We conclude by making the choice $\rho = t^\alpha$ for any $\alpha \in (0, \alpha^*)$. □

Lemma 2.6. *Consider $m_0 := \mathrm{e}^{\kappa\langle x\rangle^\gamma}$ with $\kappa \in (0, 1/(2\gamma))$. There exist constants $C, \lambda \in (0,\infty)$ such that $S_\mathcal{B}$ satisfies*

$$\|S_\mathcal{B}(t)\|_{L^2(m_0)\to H^1} \le \frac{C}{t^{1/2}}\,\mathrm{e}^{-\lambda\,t^{\sigma^*_\mathcal{B}}} \quad \forall\,t > 0. \tag{2.26}$$

Proof. The function $f = S_\mathcal{B}(t)f_0$ satisfies

$$\partial_t\partial_i f = \Delta\partial_i f + (\partial_i\mathrm{div}(\mathbf{F}))f + \mathrm{div}(\mathbf{F})\,\partial_i f + \partial_i\mathbf{F}_j\partial_j f + \mathbf{F}\cdot\nabla\partial_i f - M\partial_i\chi_R f - M\chi_R\partial_i f,$$

so that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int|\nabla f|^2 m_0^2 &= -\int|D^2 f|^2 m_0^2 + \frac{1}{2}\int|\nabla f|^2\Delta m_0^2 - \int(\mathrm{div}\mathbf{F})\, f(\Delta f)\, m_0^2\\ &-\int(\mathrm{div}(\mathbf{F}))\, f\,\nabla f\cdot\nabla m_0^2 - \int\partial_i\mathbf{F}_j\partial_i f\partial_j f\, m_0^2 - \frac{1}{2}\int(\mathrm{div}(\mathbf{F}))\,|\nabla f|^2 m_0^2\\ &-\frac{1}{2}\int|\nabla f|^2\mathbf{F}\cdot\nabla m_0^2 + M\int\mathrm{div}(\nabla\chi_R m_0^2) f^2 - \int M\chi_R|\nabla f|^2\\ &\le -\frac{1}{2}\int|D^2 f|^2 m_0^2 + \int|\nabla f|^2 m_0^2\psi^1 + \int f^2 m_0^2\psi^2,\end{aligned}$$

with

$$\psi^1 := \frac{1}{2}\frac{\Delta m_0^2}{m_0^2} + \frac{1}{2}|\mathrm{div}(\mathbf{F})|\frac{|\nabla m_0^2|}{m_0^2} + \frac{3}{2}|D\mathbf{F}| - \mathbf{F}\cdot\frac{\nabla m_0}{m_0} - M\chi_R$$

and

$$\psi^2 := (\mathrm{div}(\mathbf{F}))^2 + \frac{1}{2}\frac{\Delta m_0^2}{m_0^2} + \frac{1}{2}|\mathrm{div}\mathbf{F}|\frac{|\nabla m_0^2|}{m_0^2} + M|\Delta\chi_R| + M|\nabla\chi_R|\frac{|\nabla m_0^2|}{m_0^2}.$$

Choosing M and R large enough, we have

$$\psi^1 \le -a\,\langle x\rangle^{2(\gamma-1)},\quad a>0,\quad \psi^2\le C\,\langle x\rangle^{2(\gamma-1)},\quad C\in\mathbb{R},$$

and, choosing then $\eta>0$ small enough, we get

$$\begin{aligned}\frac{d}{dt}\int(f^2+\eta|\nabla f|^2)m_0^2 \quad &\le\quad -\frac{1}{2}\int(|\nabla f|^2+\eta|D^2 f|^2)m_0^2\\ &\qquad -a\int(f^2+\eta|\nabla f|^2)\, m_0^2\,\langle x\rangle^{2(\gamma-1)}.\end{aligned}$$

On the one hand, keeping only the second term and arguing as in the proof of Lemma 2.1, we get

$$\|S_{\mathcal{B}}(t)\|_{\mathscr{B}(H^1(m_0),H^1)} \le \Theta_m(t),\quad \forall\, t\ge 0. \tag{2.27}$$

On the other hand, using the elementary inequality

$$\int|\nabla f|^2 m_0^2 = -\int f D^2 f m_0^2 + \frac{1}{2}\int f^2\Delta m_0^2 \le \|f\|_{L^2(m_0)}\,\|f\|_{H^2(m_0)} + C\,\|f\|^2_{L^2(m_0)},$$

the differential inequality

$$\frac{d}{dt}\|\nabla f\|^2_{L^2(m_0)} \le -\|D^2 f\|^2_{L^2(m_0)} + \|\psi^2\|_{L^\infty}\|f\|^2_{L^2(m_0)}$$

and the contraction in $L^2(m_0)$, we then obtain

$$\frac{d}{dt}\|\nabla f\|^2_{L^2(m_0)} \le -\|f_0\|^{-2}_{L^2(m_0)}\,\|\nabla f\|^4_{L^2(m_0)} + \|\psi^2\|_{L^\infty}\|f_0\|^2_{L^2(m_0)}.$$

As in the proof of Lemma 2.1, we deduce

$$\|\nabla f\|^2_{L^2(m_0)} \le \frac{C}{t}\|f_0\|^2_{L^2(m_0)}\quad \forall\, t\in(0,1). \tag{2.28}$$

We easily deduce (2.26) by gathering (2.27) and (2.28). $\square$

3. Boundedness of $S_{\mathcal{L}}$

In this section, we establish some estimates on $S_{\mathcal{L}}$ which are simple results yielded by the iterated Duhamel formula (1.26) and the previous estimates on $S_{\mathcal{B}}$.

Lemma 3.1. *For any exponent $p \in [1,\infty]$ and any weight function m given by (1.7), (1.8) or (1.9), there exists $C(m,p)$ such that*

$$\|S_{\mathcal{L}}(t)\|_{\mathscr{B}(L^p(m))} \le C(m,p), \quad \forall\, t \ge 0.$$

Proof. First the estimate

$$\|S_{\mathcal{L}}(t)\|_{\mathscr{B}(L^1)} \le 1, \quad \forall\, t \ge 0, \tag{3.1}$$

is clear and is a consequence of the fact that $S_{\mathcal{L}}$ is a positive and mass conserving semigroup.

Step 1. We consider first the case $p=1$ and we write the Duhamel formula

$$S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{B}} * \mathcal{A} S_{\mathcal{L}},$$

which allows one to deduce that

$$\|S_{\mathcal{L}}\|_{L^1_k \to L^1_k} \le \|S_{\mathcal{B}}\|_{L^1_k \to L^1_k} + \|S_{\mathcal{B}}\|_{L^1_{k+\ell} \to L^1_k} * \|\mathcal{A} S_{\mathcal{L}}\|_{L^1 \to L^1_{k+\ell}} \le C,$$

because the second term on the right hand side of the first inequality is a bounded function of time thanks to (3.1) and, by choosing $\ell := 2(2-\gamma)$ and applying (2.1), we have $\|S_{\mathcal{B}}\|_{L^1_{k+\ell} \to L^1_k} \lesssim \langle t \rangle^{-2}$, which is clearly in $L^1(0,\infty)$.

Step 2. In order to study the case $1 < p \le \infty$, we write the iterated Duhamel formula (1.26) with $n_1 := \ell + 1$ and $n_2 := 0$ and we get

$$S_{\mathcal{L}} = S_{\mathcal{B}} + \dots + S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{B}})^{(*(\ell-1))} + S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{B}})^{(*\ell)} * (\mathcal{A} S_{\mathcal{L}})$$

with $\ell := d/2 + 1$ and we get

$$\begin{aligned}\|S_{\mathcal{L}}\|_{\mathscr{B}(L^p(m))} \le & \sum_{\ell'=0}^{\ell-1} \|S_{\mathcal{B}}\|_{\mathscr{B}(L^p(m))} * \|\mathcal{A} S_{\mathcal{B}}\|^{(*\ell)}_{\mathscr{B}(L^p(m))} \\ & + \|S_{\mathcal{B}}\|_{L^p(\omega) \to L^p(m)} * \|(\mathcal{A} S_{\mathcal{B}})^{(*\ell)}\|_{L^1(\omega) \to L^\infty(\omega^2)} * \|\mathcal{A} S_{\mathcal{L}}\|_{L^p(m) \to L^1(\omega)},\end{aligned}$$

where $\omega := \mathrm{e}^{\nu \langle x \rangle^\gamma}$ with $\nu \in (0, 1/\gamma)$ large enough. Using (2.20) and (2.17), we see that the RHS term is uniformly bounded as the sum of $\ell + 1$ functions, each one being the convolution of one L^∞ function with less that ℓ integrable functions. $\square$

Remark 3.2. When $p = 1$, $m = \langle x \rangle^k$ and $f(t) \ge 0$, the above estimate means exactly that for $k > 2 - \gamma > 1$, there exists a constant C_k such that

$$M_k(S_{\mathcal{L}}(t) f_0) \le C_k\, M_k(f_0), \quad M_k(f) := \int_{\mathbb{R}^d} f\, \langle x \rangle^k\, dx.$$

We then recover (with a simpler proof) and generalize (to a wider class of confinement force fields) a result obtained by Toscani and Villani in [29, section 2].

4. The case where a Poincaré inequality holds

In this section we restrict our analysis to the case when furthermore the steady state G to the Fokker-Planck equation (1.1) is such that the "weak Poincaré inequality" (1.11) holds true and satisfies the large x behavior (1.10).

4.1. **Decay estimate in a small space.**

We first recall the following decay estimate in a small space. We define

$$E_1 := L^2(G^{-1/2}), \quad E_2 := L^\infty(G^{-1}).$$

Theorem 4.1 (Röckner-Wang [27])**.** *Under assumptions* (1.11)–(1.10) *on the steady state G, set $\Theta_G(t) := \exp(-\lambda t^{\gamma/(2-\gamma)})$. Then there exists λ^* such that for $\lambda < \lambda_*$ the semigroup $S_\mathcal{L}$ satisfies for all $t > 0$*

$$\|S_\mathcal{L}(t) f_0 - M(f_0)\, G\|_{E_1} \lesssim \Theta_G(t)\, \|f_0 - M(f_0)\, G\|_{E_2}. \tag{4.1}$$

We briefly sketch the proof of Theorem 4.1 and we refer to [27] for more details. We define $V := -\log G$, we set $\mathbf{F}_0 := \mathbf{F} + \nabla V$ and we observe that $\mathrm{div}(\mathbf{F}_0 G) = 0$ and any solution f to the Fokker-Planck equation (1.1) satisfies

$$\partial_t f = \mathrm{div}[G \nabla (f/G) + f \mathbf{F}_0].$$

As a consequence, on the one hand, we compute

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int f^2 G^{-1} &= -\int G\, |\nabla (f/G)|^2 + \int \mathrm{div}[f\mathbf{F}_0] f/G \\ &\le -\mu \int f^2 \, \langle x\rangle^{2\gamma-2}\, G^{-1}, \end{aligned} \tag{4.2}$$

thanks to the weak Poincaré inequality (1.11) and because

$$\int \mathrm{div}[f\mathbf{F}_0] f/G = \int f/G\, (G\mathbf{F}_0)\cdot \nabla (f/G) = -\frac{1}{2}\int (f/G)^2 \mathrm{div}(G\mathbf{F}_0) = 0.$$

On the other hand, for any convex function $j : \mathbb{R} \to \mathbb{R}$, the following generalized relative entropy inequality holds (see [20], and also [27] and the references therein),

$$\begin{aligned} \frac{d}{dt}\int j\Big(\frac{f}{G}\Big)\, G &= \int j'\Big(\frac{f}{G}\Big) \mathrm{div}[G\nabla (f/G)] + \int j'\Big(\frac{f}{G}\Big)\mathrm{div}[f\mathbf{F}_0] \\ &= -\int j''\Big(\frac{f}{G}\Big) |\nabla (f/G)|^2 \le 0. \end{aligned}$$

In particular, taking $j(s) = |s|^p$, we obtain

$$\|f(t)/G\|_{L^p} \le \|f_0/\|_{L^p},$$

and passing to the limit $p \to \infty$, we get

$$\big\|f(t)/G\big\|_{L^\infty} \le \big\|f_0/G\big\|_{L^\infty}, \quad \forall\, t \ge 0. \tag{4.3}$$

We obtain (4.1) by gathering the two estimates (4.2) and (4.3). More precisely, we write, with $k := 2 - 2\gamma$,

$$\begin{aligned} \int f^2 G^{-1} &\le \int_{B_R} f^2\, G^{-1} + \int_{B_R^c} f^2\, G^{-1} \\ &\le R^k \int_{B_R} f^2 \; G^{-1}\, \langle x\rangle^{-k} + \|f/G\|^2_{L^\infty} \int_{B_R^c} G. \end{aligned}$$

Together with (4.2) and (4.3), we get for $R \ge R_0$

$$\frac{d}{dt}\int f^2 G^{-1} \le -R^{-k}\int f^2 G^{-1} + \|f_0/G\|^2_{L^\infty} \int_{B_R^c} \mathrm{e}^{-c_1 |x|^\gamma}.$$

Integrating in time, for any $\theta \in (0,1)$, we find a constant $C_\theta > 0$ such that

$$\begin{aligned} \int f^2 G^{-1} &\le e^{-R^{-k} t} \int f_0^2 \, G^{-1} + C_\theta \, e^{-\theta c_1 R^\gamma} \, \|f_0/G\|^2_{L^\infty} \\ &\le e^{-\lambda \, t^{\gamma/(2-\gamma)}} \, \|f_0/G\|^2_{L^\infty}, \end{aligned}$$

by choosing $R := (t/(\theta c_1))^{1/(2-\gamma)}$ for t large enough and defining $\lambda := (\theta c_1)^{-1/(2-\gamma)}$, which is nothing but (4.1).

4.2. **Rate of decay in a large space.** In this section we present the proof of our main Theorem 1.2 in the case when furthermore G satisfies the weak Poincaré inequality (1.11) and the asymptotic estimates (1.10). We recall that the projection operator Π is defined by

$$\Pi f := M(f) \, G.$$

Step 1. Here we consider the case $p \in [1,2]$. We begin by fixing m a polynomial or stretch exponential weight function, and we introduce a stronger confinement exponential weight $m_0(x) := \exp(\kappa_0 \, \langle x \rangle^\gamma)$, with $\kappa_0 \in (0, 1/\gamma)$. We denote by $\Theta_m(t)$, $\Theta_{m_0}(t)$ and $\Theta_G(t)$ the associated rate of decay defined in (1.18), (1.19) and in the statement of Theorem 4.1, and we may assume that $\Theta_{m_0}/\Theta_m, \Theta_G/\Theta_m \in L^1(0,\infty)$. We split the semigroup on invariant spaces

$$S_{\mathcal{L}} = \Pi \, S_{\mathcal{L}} + (I - \Pi) \, S_{\mathcal{L}},$$

and together with the iterated Duhamel formula (1.26) with $n_1 = n \ge d/2 + 1$ and $n_2 = 0$, we have

$$\begin{aligned} S_{\mathcal{L}} - \Pi \quad &= \quad (I - \Pi) \left\{ S_{\mathcal{B}} + \sum_{\ell=1}^{n-1} (S_{\mathcal{B}} \mathcal{A})^{(*\ell)} * S_{\mathcal{B}} \right\} \qquad (=: T_1) \\ &\quad + \{(I - \Pi) S_{\mathcal{L}}\} * (\mathcal{A} S_{\mathcal{B}})^{(*n)} \qquad (=: T_2). \end{aligned}$$

In order to estimate the first term T_1, we recall that

$$\|S_{\mathcal{B}}(t)\|_{L^p(m) \to L^p} \le \Theta_m(t), \quad \|S_{\mathcal{B}}(t) \mathcal{A}\|_{L^p \to L^p} \le \Theta_{m_0}(t),$$

thanks to Lemma 2.1. We write

$$\|T_1\|_{\mathscr{B}(L^p(m), L^p)} \le u_0 + ... + u_{n-1}$$

with, for $\ell \in \{1, ..., n-1\}$,

$$u_\ell := C \, \|S_{\mathcal{B}} \mathcal{A}\|^{(*\ell)}_{\mathscr{B}(L^p)} * \|S_{\mathcal{B}}\|_{\mathscr{B}(L^p(m), L^p)}.$$

Since clearly $\Theta_m^{-1}(t) \le \Theta_m^{-1}(s) \, \Theta_m^{-1}(t-s)$ for all $0 \le s \le t$, we deduce

$$\|\Theta_m^{-1} u_\ell\|_{L^\infty_t} \le \|\Theta_m^{-1} \, S_{\mathcal{B}} \mathcal{A}\|^\ell_{L^1_t(\mathscr{B}(L^p))} \, \|\Theta_m^{-1} \, S_{\mathcal{B}}\|_{L^\infty_t(\mathscr{B}(L^p(m), L^p))},$$

and then

$$\|T_1(t)\|_{\mathscr{B}(L^p(m), L^p)} \le C \, \Theta_m(t) \quad \forall \, t \ge 0.$$

In order to estimate the second term T_2, we recall that from Lemma 2.1, Lemma 2.4 and Theorem 4.1, we have

$$\begin{aligned} &\|\mathcal{A} S_{\mathcal{B}}(t)\|_{\mathscr{B}(L^p(m), L^1(m_0))} \le \Theta_m(t), \\ &\|(\mathcal{A} S_{\mathcal{B}})^{(*(n-1))}(t)\|_{\mathscr{B}(L^1(m_0), L^\infty(m_1))} \le \Theta_{m_0}(t), \\ &\|S_{\mathcal{L}}(t)(I - \Pi)\|_{\mathscr{B}(L^\infty(G^{-1}), L^2(G^{-1/2}))} \le \Theta_G(t), \end{aligned}$$

where $m_1 := m_0 + G^{-1}$. We thus obtain that T_2 satisfies the same estimate as T_1. We conclude to (1.17) by gathering the two estimates obtained on T_1 and T_2.

Step 2. The case $p \in [2, \infty]$. Thanks to the iterated Duhamel formula (1.26), we also have

$$\begin{aligned} S_{\mathcal{L}} - \Pi &= (I - \Pi) \{ \sum_{\ell=0}^{n} S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{B}})^{(*\ell)} \} \qquad (=: T_1) \\ &\quad + S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{B}})^{(*(n-1))} * \mathcal{A} \{ S_L (I - \Pi) \} * (\mathcal{A} S_{\mathcal{B}}) \qquad (=: T_3), \end{aligned}$$

and we conclude in a similar way as in Step 1. □

5. The general case

In this section we present the proof of our main Theorem 1.2 in the general case, that is assuming that the force field satisfies conditions (1.3)–(1.5), without any further assumption on the existence and particular properties of a positive stationary state. In order to do so, we define the "small" Hilbert space

$$X = X_0 := L^2(m_0), \quad m_0 := \exp(2\kappa \langle x \rangle^\gamma), \quad \kappa \in (0, 1/(8\gamma)),$$

and we perform a spectral analysis of $\mathcal{L}$ acting in X_0 and establish a semigroup decay in that space. The proof of the decay of the semigroup in a general Banach space $L^p(m)$, with a weight function m which is a polynomial or a stretch exponential function, then follows the same arguments as the ones which will be developed in section 4.2 and may be skipped here. The only difference comes from the fact that the rate of decay in the small space X_0 is worse than in the small space $L^2(G^{-1/2})$ (see section 4.2 for the notations), and then the rate of decay in the general Lebesgue space $L^p(m)$ has to be modified accordingly.

5.1. **Eigenvalue problem on the imaginary axis.**

We start recalling some standard notions and notations. For an operator Λ acting in a Banach space X, we denote $N(\Lambda) := \Lambda^{-1}(\{0\})$ its null space, $\Sigma(\Lambda)$ its spectrum, $\Sigma_P(\Lambda)$ its point spectrum set, that is the set of the eigenvalues of Λ, while $\rho(\Lambda) := \mathbb{C} \backslash \Sigma(\Lambda)$ will denote the resolvent set and $R_\Lambda(z)$, for $z \in \rho(\Lambda)$, the resolvent operator defined by

$$R_\Lambda(z) := (\Lambda - z)^{-1}.$$

We denote X_+ the positive cone of X. In this section we shall prove

Theorem 5.1. *There exists a unique steady state $G \in X$, positive and normalized (with total mass $M(G) = 1$) such that*

$$N(\mathcal{L}^n) = \mathrm{span}(G), \quad \forall\, n \geq 1. \tag{5.1}$$

Moreover, there holds

$$\Sigma_P(\mathcal{L}) \cap \{ z \in \mathbb{C} \,;\, \Re e\, z \geq 0 \} = \{0\}. \tag{5.2}$$

We start by recalling some elementary properties regarding the positivity of the semigroup $S_{\mathcal{L}}(t)$.

Proposition 5.2. *The operator $\mathcal{L}$ and its semigroup $S_{\mathcal{L}}$ satisfy the following properties:*

(a) *The semigroup $S_{\mathcal{L}}$ is positive, namely $S_{\mathcal{L}}(t) f \geq 0$ for any $f \geq 0$ and $t \geq 0$.*

(b) *The operators $\mathcal{L}$ and $\mathcal{L}^*$ satisfy Kato's inequality, that is if $f \in D(\mathcal{L})$ is complex valued, then we have*

$$\mathcal{L}|f| \ge \frac{1}{|f|}\Re e(\overline{f}\,\mathcal{L}f), \qquad \mathcal{L}^*|g| \ge \frac{1}{|g|}\Re e(\overline{g}\,\mathcal{L}^*g) \qquad \text{in } \mathcal{D}'(\mathbb{R}^d). \tag{5.3}$$

The above inequality means in particular that for all $\psi \in D(\mathcal{L}^*) \cap X_+$ and all $\varphi \in D(\mathcal{L}) \cap X_+$ we have

$$\langle |f|, \mathcal{L}^*\psi\rangle \ge \Re e\langle |f|^{-1}\,\overline{f}\,\mathcal{L}f, \psi\rangle, \qquad \langle \mathcal{L}\varphi, |g|\rangle \ge \Re e\langle \varphi, |g|^{-1}\,\overline{g}\,\mathcal{L}g\rangle.$$

(c) *The operator $-\mathcal{L}$ satisfies a "weak maximum principle", namely for any $a > 0$ and $g \in X_+$ there holds*

$$f \in D(\mathcal{L}) \text{ and } (-\mathcal{L} + a)f = g \quad \text{imply} \quad f \ge 0. \tag{5.4}$$

(d) *The opposite of the resolvent operator is a positive operator, namely for any $a > 0$ and $g \in X_+$ there holds $-R_{\mathcal{L}}(a)g \in X_+$.*

Proof. The argument to establish (5.3) is a classical one, but we outline it briefly. For a smooth complex valued function f if we set $f_\varepsilon := (\varepsilon^2 + |f|^2)^{1/2} - \varepsilon$, then for $1 \le j \le d$ we have

$$\partial_j f_\varepsilon = \frac{\Re e(\overline{f}\,\partial_j f)}{(\varepsilon^2 + |f|^2)^{1/2}}, \tag{5.5}$$

$$\partial_{jj} f_\varepsilon = \frac{\Re e(\overline{f}\,\partial_{jj} f)}{(\varepsilon^2 + |f|^2)^{1/2}} + \frac{|\partial_j f|^2}{(\varepsilon^2 + |f|^2)^{1/2}} - \frac{\big(\Re e(\overline{f}\,\partial_j f)\big)^2}{(\varepsilon^2 + |f|^2)^{3/2}}. \tag{5.6}$$

Observe that $f_\varepsilon \to |f|$ in $H^1_{\rm loc}(\mathbb{R}^d)$ as $\varepsilon \to 0$, we have $\partial_j|f| = |f|^{-1}\Re e(\overline{f}\,\partial_j f)$ in $H^1_{\rm loc}(\mathbb{R}^d)$. Also since

$$\frac{|\partial_j f|^2}{(\varepsilon^2 + |f|^2)^{1/2}} - \frac{\big(\Re e(\overline{f}\,\partial_j f)\big)^2}{(\varepsilon^2 + |f|^2)^{3/2}} \ge 0,$$

we conclude that $\partial_{jj} f_\varepsilon \ge f_\varepsilon^{-1}\,\Re e(\overline{f}\,\partial_{jj} f)$. Passing to the limit in the sense of the distributions we obtain $\partial_{jj}|f| \ge |f|^{-1}\,\Re e(\overline{f}\,\partial_{jj} f)$, and using the expressions of the operators $\mathcal{L}$ and $\mathcal{L}^*$ we obtain (5.3).

The properties (a), (c) and (d) are classical consequences of (5.3), applied to real valued functions, see for instance [2] or [21]. □

Remark 5.3. For later use, we point out that if $f \in H^2_{\rm loc}(\mathbb{R}^d)$ is complex valued and moreover is such that $|f| > 0$ on $\mathbb{R}^d$, then as $\varepsilon \to 0$ one may pass to the limit in (5.5) and (5.6) and obtain the following equalities

$$\partial_j |f| = \frac{\Re e(\overline{f}\,\partial_j f)}{|f|}, \tag{5.7}$$

$$\partial_{jj} |f| = \frac{\Re e(\overline{f}\,\partial_{jj} f)}{|f|} + \frac{|\partial_j f|^2}{|f|} - \frac{\big(\Re e(\overline{f}\,\partial_j f)\big)^2}{|f|}. \tag{5.8}$$

Proposition 5.4. *The operator $-\mathcal{L}$ satisfies a "strong maximum principle", namely for any given real valued function $f \in X\backslash\{0\}$, there holds*

$$f, |f| \in D(\mathcal{L}), \quad \mathcal{L}f = 0 \text{ and } \mathcal{L}|f| = 0 \quad \text{imply} \quad f > 0 \text{ or } f < 0.$$

Proof. Consider $f \in X\backslash\{0\}$ such that $f \in D(\mathcal{L})$ and $\mathcal{L}f = 0$. By a bootstrap regularization argument, we classically have $f \in C(\mathbb{R}^d)$. By assumption there exist then $x_0 \in \mathbb{R}^d$, and two constants $c, r > 0$, such that $|f| \geq c$ on $B(x_0, r)$. From Lemma 2.1, we also have that the operator $\mathcal{L} - aI$ is dissipative for a large enough, in the sense that

$$\forall\, f \in D(\mathcal{L}) \quad ((\mathcal{L} - a)f|f)_X \leq -\|f\|_X^2. \tag{5.9}$$

For instance one can take $a := M + 1$, where M is the constant entering in the definition of $\mathcal{B}$, using the fact that $\mathcal{B}$ is then dissipative and that the same holds for $\mathcal{L} - MI$ because $0 \leq \mathcal{A} \leq M$.

We next observe that for $\sigma > 0$ large enough, the function

$$g(x) := c\exp(\sigma r^2 - \sigma|x - x_0|^2)$$

satisfies $g = c$ on $\partial B(x_0, r)$ and

$$(-\mathcal{L} + a)g = (a + d\sigma + \sigma\mathbf{F}\cdot x - \operatorname{div}(\mathbf{F}) - \sigma^2|x|^2)\, g \leq 0 \quad \text{on} \quad B(x_0, r)^c.$$

We define $h := (g - |f|)^+$ and $\Omega := \mathbb{R}^d\backslash\overline{B}(x_0, r)$. We have $h \in H_0^1(\Omega, mdx)$ and

$$\begin{array}{rcl} (\mathcal{L} - a)h & \geq & \theta'(g - |f|)\, \mathcal{L}(g - |f|) - a\, h \\ & = & \theta'(g - |f|)\, [(\mathcal{L} - a)g + a|f|] \geq 0, \end{array}$$

where we have used the notation $\theta(s) = s^+$. Thanks to a straightforward generalization of (5.9) to $H_0^1(\Omega, m)$, we deduce

$$0 \leq ((\mathcal{L} - a)h|h)_{L^2(\Omega,m)} \leq -\|h\|^2_{L^2(\Omega,m)},$$

and then $h = 0$. This implies that we have $|f| \geq g$ on $\Omega = \overline{B}(x_0, r)^c$, and thus $|f| > 0$ on $\mathbb{R}^d$. Since $f \in C(\mathbb{R}^d)$, we must have either $f \geq g > 0$ or $f \leq -g < 0$. $\square$

Proof of Theorem 5.1. We split the proof into four steps.

Step 1. We prove that there exists $G \in N(\mathcal{L})$ such that $G > 0$, which yields in particular that $0 \in \Sigma_P(\mathcal{L})$. In other words, we prove that there exists a positive and normalized (with mass 1) steady state $G \in X$. For the equivalent norm $|||\cdot|||$ defined on X by

$$|||f||| := \sup_{t>0} \|S_{\mathcal{L}}(t)f\|_X,$$

we have $|||S_{\mathcal{L}}(t)f||| \leq |||f|||$ for all $t \geq 0$, that is the semigroup $S_{\mathcal{L}}$ is a contraction semigroup on $(X, |||\cdot|||)$. There exists $R > 0$ large enough such that the intersection of the closed hyperplane $H := \{f\ ;\ M(f) = 1\}$ and the closed ball of radius R in $(X, |||\cdot|||)$ is a convex, non empty subset. Then consider the closed, weakly compact convex set

$$\mathbb{K} := \{f \in X_+\ ;\ |||f||| \leq R, \quad M(f) = 1\}.$$

Since $S_{\mathcal{L}}(t)$ is a linear, weakly continuous, contraction in $(X, |||\cdot|||)$ and $M(S_{\mathcal{L}}(t)f) = M(f)$ for all $t \geq 0$, we see that $\mathbb{K}$ is stable under the action of the semigroup. Therefor we apply the Markov–Kakutani fixed point theorem (see for instance [12, Theorem 6, chapter V, § 10.5, page 456] or its (possibly nonlinear) variant [13, Theorem 1.2]) and we conclude that there exists $G \in \mathbb{K}$ such that $S_{\mathcal{L}}(t)G = G$. Therefore we have in particular $G \in D(\mathcal{L})$ and $\mathcal{L}G = 0$. Moreover since $G \geq 0$, $G \not\equiv 0$ and $G \in C(\mathbb{R}^d)$, we may conclude that $G > 0$ on $\mathbb{R}^d$.

Step 2. In this step we prove that $N(\mathcal{L}^2) = N(\mathcal{L})$, which implies that $N(\mathcal{L}^n) = N(\mathcal{L})$ for all $n \geq 1$. Otherwise, there would exist $g_1 \in D(\mathcal{L})$ with $\|g_1\| = 1$ and

$g_2 \in D(\mathcal{L})$ such that $\mathcal{L}g_1 = 0$ and $\mathcal{L}g_2 = g_1$. Then, since $S_\mathcal{L}(t)g_1 = g_1$ for all $t \geq 0$, one sees easily that one must have $S_\mathcal{L}(t)g_2 = g_2 + tg_1$. However, since $g_1 \neq 0$, this is in contradiction with Lemma 3.1, that is the fact that the semigroup $S_\mathcal{L}$ is bounded on X with $\|S_\mathcal{L}(t)\|_{X\to X} \leq C(m, 2)$.

Step 3. We prove here that $N(\mathcal{L}) = \mathrm{span}(G)$. Since $\mathcal{L}$ is a real operator, we may restrict ourselves to real valued eigenfunctions. Consider a real valued eigenfunction $f \in N(\mathcal{L})$ with $f \not\equiv 0$. First we observe that thanks to Kato's inequality

$$0 = (\mathcal{L}f)\,\mathrm{sgn}(f) \leq \mathcal{L}|f|.$$

Actually this inequality must be an equality, since otherwise we would have

$$0 \neq \langle \mathcal{L}|f|, 1\rangle = \langle |f|, \mathcal{L}^* 1\rangle = 0,$$

which is a contradiction. As a consequence, we have also $|f| \in N(\mathcal{L})$, so that the strong maximum principle, Proposition 5.4, implies that either $f > 0$ or $f < 0$, and without loss of generality we may assume that $f > 0$, and up to a multiplication by a normalization factor, we may also assume that $M(f) = 1$. Now, using once more Kato's inequality we have

$$0 = \mathcal{L}(f - G)\,\mathbf{1}_{[f-G>0]} \leq \mathcal{L}(f - G)^+,$$

and due to the same reasons as above, we may conclude that this last inequality is in fact an equality, that is $(f-G)^+ \in N(\mathcal{L})$. Since $(f-G)^+ = |(f-G)^+|$, the strong maximum principle implies that either $(f - G)^+ \equiv 0$ or $(f - G)^+ > 0$ on $\mathbb{R}^d$. This means that either we have $f \leq G$ or $f > G$ on $\mathbb{R}^d$. Thanks to the normalization hypothesis $M(f) = M(G) = 1$, the second possibility must be excluded and thus we have $f \leq G$ on $\mathbb{R}^d$. Repeating the same argument with $(G - f)^+$ we deduce that $G \leq f$ and we conclude that $f = G$.

Step 4. We prove here that $\mathrm{i}\mathbb{R}\cap\Sigma_P(\mathcal{L}) = \{0\}$: the only eigenvalue with vanishing real part is zero, or in other words, (5.2) holds. We consider a couple (f, μ) of eigenfunction and eigenvalue, with $\mu := \mathrm{i}\omega \in \mathrm{i}\mathbb{R}$, and normalized so that $M(|f|) = 1$. Using the complex version of Kato's inequality (5.3), we have

$$\mathcal{L}|f| \geq \frac{1}{|f|}\Re e(\overline{f}\,\mathcal{L}f) = 0. \tag{5.10}$$

Computing $\langle \mathcal{L}|f|, 1\rangle$, thanks to the above inequality, we get

$$0 \leq \langle \mathcal{L}|f|, 1\rangle = \langle |f|, \mathcal{L}^* 1\rangle = 0,$$

which implies that the inequality is in fact an equality, that is $\mathcal{L}|f| = 0$, and since $M(G) = M(|f|) = 1$, we conclude that $|f| = G$.

Next, using Remark 5.3 and (5.7)–(5.8), one sees that for any complex valued function f such that $|f| > 0$ on $\mathbb{R}^d$, we have

$$\begin{aligned}|f|\,\mathcal{L}|f| &= \Re e(\overline{f}\,\Delta f) + \Re e(\overline{f}\,\mathbf{F}\cdot\nabla f) + \mathrm{div}(\mathbf{F})|f|^2 + |\nabla f|^2 - \frac{1}{|f|^2}\,\left|\Re e(\overline{f}\,\nabla f)\right|^2 \\ &= \Re e(\overline{f}\,\mathcal{L}f) + |\nabla f|^2 - \frac{1}{|f|^2}\,\left|\Re e\left(\overline{f}\,\nabla f\right)\right|^2.\end{aligned}$$

Because here $|f| = G > 0$, the above identity applies. Together with $\mathcal{L}G = 0$ and thus $\Re e(\overline{f}\,\mathcal{L}f) = 0$ from (5.10), we get

$$|\nabla f|^2 - \frac{1}{|f|^2}\,\left|\Re e(\overline{f}\,\nabla f)\right|^2 = 0. \tag{5.11}$$

This implies that $f = \exp(\mathrm{i}\,\theta)\,G$ for some constant $\theta \in [0, 2\pi]$, and thus $\mathcal{L}f = \exp(\mathrm{i}\,\theta)\,\mathcal{L}G = 0$ and $\omega = 0$.

Indeed, in order to see that $f = \exp(\mathrm{i}\,\theta)\,G$, for some $\theta \in [0, 2\pi]$, let us write $f = u + \mathrm{i}\,v$ for two real valued functions u and v. Then, since $\Re e(\overline{f}\,\nabla f) = u\nabla u + v\nabla v$, relation (5.11) means that

$$(u^2+v^2)\left(|\nabla u|^2 + |\nabla v|^2\right) = |u\nabla u + v\nabla v|^2\,,$$

which yields $v\nabla u - u\nabla v = 0$. Since u and v are not both identically equal to zero, assuming for instance that $u \not\equiv 0$, this means that $\nabla(v/u) = 0$, that is $v = \alpha\,u$ for some $\alpha \in \mathbb{R}$ and thus, setting $a := 1 + \mathrm{i}\alpha$ we have $f = a\,u$. Finally one sees that $|u| = G/|a| > 0$, and so u has a constant sign, that is $f = \pm a\,G/|a| = \exp(\mathrm{i}\,\theta)G$ for some $\theta \in [0, 2\pi]$. □

5.2. **Decay estimate.** We mainly adapt an argument used in the proof of [24, Theorem 2.1]. We consider the sequence of spaces $(X_k)_{k\in\mathbb{N}}$, as defined in (2.22), and $X_\infty := L^2(m_0^{1/2})$. We observe that $X_k \subset X_{k+1} \subset X_\infty$ for any $k \in \mathbb{N}$. For $0 \le \eta \le 1$ we also denote $X_{k,\eta}$ the space defined by Hilbertian interpolation between $X_{k,0} = X_k$ and $X_{k,1} := \{f \in X_k;\ \mathcal{L}f \in X_k\}$, that is, with the notations of L. Tartar [28, Chapter 22, page 109],

$$X_{k,\eta} := (X_k, X_{k,1})_{\eta,2}\,.$$

Lemma 5.5. *Let us fix an integer $j > 2(1-\gamma)/\gamma > 0$. There exists a constant C such that for any $\ell_1, \ell_2, k \in \mathbb{N}$, $k \ge j$, $\ell_i \ge 1$, we have for all $z \in \mathbb{C}$ with $\Re e\, z > 0$*

$$\|R_{\mathcal{B}}(z)\|_{X_{k-j}\to X_k} \le \mathcal{C}_k := C\,k^j, \tag{5.12}$$

$$\|\mathcal{A}R_{\mathcal{B}}(z)^{\ell_1}\mathcal{A}R_{\mathcal{B}}(z)^{\ell_2}\|_{X_0\to X_0} \le (\ell_1!\ell_2!)^j C^{\ell_1+\ell_2}/\langle y\rangle^{1/2}, \tag{5.13}$$

where $z = x + \mathrm{i}y$, $x, y \in \mathbb{R}$.

Proof. We use the representation formula

$$R_{\mathcal{B}}(z) = -\int_0^\infty \mathrm{e}^{-zt}\,S_{\mathcal{B}}(t)\,dt$$

together with the estimates established in Lemmas 2.1, 2.4 and 2.5 in the following way. In order to simplify the presentation, we only consider in the sequel the boundary case $\Re e\, z = x = 0$.

On the one hand, we define, as in Lemma 2.5, $\alpha^* := 1/2(1-\gamma)$, so that $\alpha^*\gamma j > 1$, and we observe that the LHS term of (2.23) belongs to $L^1(\mathbb{R}_+)$. Therefore, with the notations of Lemma 2.5, we have for any $y \in \mathbb{R}$

$$\|R_{\mathcal{B}}(\mathrm{i}y)\|_{X_{k-j}\to X_k} \le C_1 + \left(\frac{k}{\kappa}\right)^j \frac{1}{\alpha\gamma j - 1} \le C\,k^j.$$

On the other hand, from (2.26), we have

$$\sup_{y\in\mathbb{R}} \|R_{\mathcal{B}}(\mathrm{i}y)\|_{X_\infty\to H^1} \le \int_0^\infty \|S_{\mathcal{B}}(t)\|_{X_\infty\to H^1}\,dt \le C. \tag{5.14}$$

The latter estimate together with (5.12) yield that, for any $y \in \mathbb{R}$ and $\ell_2 \in \mathbb{N}^*$, we have

$$\begin{aligned} \|\mathcal{A} R_{\mathcal{B}}(\mathrm{i}y)^{\ell_2}\|_{X_0 \to X_{0,1/2}} &\le \\ &\le \|\mathcal{A} R_{\mathcal{B}}(\mathrm{i}y)\|_{X_\infty \to X_{0,1/2}} \|R_{\mathcal{B}}(\mathrm{i}y)\|_{X_{(\ell_2-2)j} \to X_{(\ell_2-1)j}} \cdots \|R_{\mathcal{B}}(\mathrm{i}y)\|_{X_0 \to X_j} \\ &\le (\ell_2!)^j C^{\ell_2}. \end{aligned} \tag{5.15}$$

On the other hand, from the identity

$$R_{\mathcal{B}}(z) = z^{-1}(\mathcal{R}_{\mathcal{B}}(z)\mathcal{B} - I)$$

and an interpolation argument, we deduce that

$$\|R_{\mathcal{B}}(\mathrm{i}y)\|_{X_{0,1/2} \to X_j} \le C/\langle y\rangle^{1/2},$$

and therefore for any $\ell_1 \in \mathbb{N}$

$$\|\mathcal{A} R_{\mathcal{B}}(\mathrm{i}y)^{\ell_1}\|_{X_{0,1/2} \to X_0} \le (\ell_1!)^j C^{\ell_1}/\langle y\rangle^{1/2}.$$

It is now clear that the above estimate together with (5.15) completes the proof of estimate (5.13). □

Lemma 5.6. *Let us fix again an integer $j > 2(1-\gamma)/\gamma > 0$. There exists a constant C such that for any $\ell \in \mathbb{N}^*$, denoting by Π the projection on G, that is $\Pi(f) := M(f)G$, we have*

$$\sup_{\Re e\, z > 0} \|(I - \Pi) R_{\mathcal{L}}(z)^\ell\|_{X_0 \to X_\infty} \le C^\ell\, (\ell!)^j. \tag{5.16}$$

Proof. Since the operator $\mathcal{L} - aI$ is dissipative for any $a > 0$, we clearly have

$$C_{\mathcal{L},a} := \sup_{\Re e\, z \ge a} \|(I-\Pi)R_{\mathcal{L}}(z)\|_{X_0 \to X_\infty} \lesssim \sup_{\Re e\, z \ge a} \|R_{\mathcal{L}}(z)\|_{X_0 \to X_\infty} < \infty,$$

and thus we have only to prove that the constant $C_{\mathcal{L},a}$ does not blow up when $a \to 0^+$.

Step 1. We claim that for any fixed M, there holds

$$\sup_{z \in \Omega_M} \|(I-\Pi)R_{\mathcal{L}}(z)\|_{X_{k-j} \to X_k} \le C_M\, \mathcal{C}_k, \tag{5.17}$$

where we define $\Omega_M := \{z = x + \mathrm{i}y \in \mathbb{C},\ 0 < x \le 1,\ |y| \le M\}$ and we recall that $\mathcal{C}_k$ is defined in (5.12). We argue by contradiction, assuming that there exists $y \in [-M, M]$ and a sequence (z_n) in Ω_M such that

$$z_n \to z := \mathrm{i}y \quad \text{and} \quad \mathcal{C}_n^{-1}\|R_{\mathcal{L}_1}(z_n)\|_{\mathscr{B}(X_{n-j}, X_n)} \to \infty,$$

with $\mathcal{L}_1 := \Pi^\perp \mathcal{L}$, where for brevity we note $\Pi^\perp := I - \Pi$, despite the fact that Π is not an orthogonal projection. The last family of blowing up estimates means that there exist sequences $(\widetilde{f}_n)_n$ and $(\widetilde{g}_n)_n$ such that

$$\mathcal{C}_n^{-1}\|\widetilde{f}_n\|_{X_n} \to \infty, \quad \|\widetilde{g}_n\|_{X_{n-j}} = 1, \quad \widetilde{f}_n = R_{\mathcal{L}_1}(z_n)\, \widetilde{g}_n,$$

or, equivalently, that there exist $(f_n)_n$ in X_n and $(g_n)_n$ in X_{n-j} satisfying

$$\|f_n\|_{X_n} = 1, \quad \mathcal{C}_n\|g_n\|_{X_{n-j}} \to 0, \quad g_n = (\mathcal{L}_1 - z_n)\, f_n.$$

This in turn would imply that

$$R_{\mathcal{B}}(z_n)\mathcal{A}\Pi^\perp f_n + \Pi^\perp f_n - z_n R_{\mathcal{B}}(z_n)\Pi f_n = R_{\mathcal{B}}(z_n) g_n, \tag{5.18}$$

with

$$\|R_{\mathcal{B}}(z_n) g_n\|_{X_n} \le \mathcal{C}_n \|g_n\|_{X_{n-j}} \to 0,$$

by using (5.12). Since $(f_n)_n$ is bounded in $X_n \subset X_\infty = L^2(e^{\kappa \langle x\rangle^\gamma})$, by weak compactness of this sequence in X_∞, we find $f \in X_\infty$ and a subsequence denoted again by $(f_n)_n$ such that $f_n \rightharpoonup f$ weakly in X_∞, and then $\mathcal{A}\Pi^\perp f_n \rightharpoonup \mathcal{A}\Pi^\perp$ weakly in X_0. Together with (5.14), we deduce that

$$R_\mathcal{B}(z_n)\mathcal{A}\Pi^\perp f_n \to R_\mathcal{B}(z)\mathcal{A}\Pi^\perp f \quad \text{strongly in } X_0. \tag{5.19}$$

Now, passing (weakly) to the limit in (5.18), we have

$$R_\mathcal{B}(z)\mathcal{A}\Pi^\perp f + \Pi^\perp f - zR_\mathcal{B}(z)\Pi f = 0.$$

We claim that $f \neq 0$. If not, we get from (5.19) that

$$R_\mathcal{B}(z_n)\mathcal{A}\Pi^\perp f_n \to 0 \qquad \text{and} \qquad \Pi f_n \to 0 \quad \text{strongly in } X_0$$

and then together with (5.18) that $\|\Pi^\perp f_n\|_{X_n} \to 0$. Thus we would have

$$1 = \|f_n\|_{X_n} \le \|\Pi f_n\|_{X_0} + \|\Pi^\perp f_n\|_{X_n} \to 0,$$

which is a contradiction. As a consequence, we have exhibited an $f \in X_\infty\backslash\{0\}$ such that $(\mathcal{L}_1 - zI)f = 0$. This means that f is an eigenvector for $\mathcal{L}_1 = \Pi^\perp\mathcal{L}$ associated to an eigenvalue $z \in i\mathbb{R}$; however this is in contradiction with the fact that $\Sigma_P(\Pi^\perp\mathcal{L}) \cap i\mathbb{R} = \emptyset$. Thus the proof of (5.17) is complete.

Step 2. In this step we complete the proof of the Lemma. We begin by recalling that $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and then we write

$$R_\mathcal{L}(z) = R_\mathcal{B}(z) - R_\mathcal{L}(z)\mathcal{A}R_\mathcal{B}(z)$$

and

$$R_\mathcal{L}(z)(1 - \mathcal{V}(z)) = R_\mathcal{B}(z) - R_\mathcal{B}(z)\,\mathcal{A}\,R_\mathcal{B}(z), \qquad \text{where } \mathcal{V}(z) := (\mathcal{A}\,R_\mathcal{B}(z))^2\,.$$

Let $\widetilde{X}_k$ be defined as X_k but with a coefficient $\widetilde{\kappa} > \kappa$ so that $\widetilde{X}_k \subset \widetilde{X}_\infty \subset X_0 \subset X_k$ with embedding constants uniformly bounded with respect to k. First we may fix M large enough such that $\|\mathcal{V}(z)\|_{\mathscr{B}(X_0)} \le 1/2$ and $\|\mathcal{V}(z)\|_{\mathscr{B}(\widetilde{X}_0)} \le 1/2$, for any $z = x + iy$, with $|y| \ge M$: this is indeed possible thanks to (5.13) by choosing $\ell_1 = \ell_2 = 1$. Next, we write the expansion

$$R_\mathcal{L}(z) = R_\mathcal{B}(z) - \big(R_\mathcal{B}(z) - R_\mathcal{B}(z)\,\mathcal{A}\,R_\mathcal{B}(z)\big)\Big(\sum_{j=0}^\infty \mathcal{V}(z)^j\Big)\mathcal{A}R_\mathcal{B}(z),$$

where the series converges normally in $\mathscr{B}(X_0)$ and in $\mathscr{B}(\widetilde{X}_0)$. More precisely, for $z = x + iy$ and $|y| \ge M$ we have

$$\begin{aligned}
\|R_\mathcal{L}(z)\|_{X_{k-j}\to X_k} &\le \|R_\mathcal{B}(z)\|_{X_{k-j}\to X_k} \\
&\quad + \|(R_\mathcal{B} - R_\mathcal{B}\,\mathcal{A}\,R_\mathcal{B})(z)\|_{\widetilde{X}_0\to X_0}\Big(\sum_{j=0}^\infty \|\mathcal{V}(z)\|^j_{\mathscr{B}(\widetilde{X}_0)}\Big)\,\|\mathcal{A}R_\mathcal{B}\|_{X_{k-j}\to \widetilde{X}_0}, \\
&\le \|R_\mathcal{B}(z)\|_{X_{k-j}\to X_k} + 2\,\|(R_\mathcal{B} - R_\mathcal{B}\,\mathcal{A}\,R_\mathcal{B})(z)\|_{\widetilde{X}_0\to X_0}\,\|\mathcal{A}R_\mathcal{B}\|_{X_{k-j}\to \widetilde{X}_0}. \qquad (5.20)
\end{aligned}$$

The right hand side of the above inequality being bounded by a constant $C\,\mathcal{C}_k$, we conclude that

$$\sup_{|\Re e\, z| \ge M} \|(I - \Pi)R_\mathcal{L}(z)\|_{X_{k-j}\to X_k} \le C\,\mathcal{C}_k.$$

Together with (5.17), we conclude the proof of (5.16) in the case when $\ell = 1$. For the general case $\ell \ge 1$, we argue similarly as we did in the proof of (5.13). $\square$

Theorem 5.7. *Let $\sigma^*_{\mathcal{L}} := 1/\lfloor 2/\gamma \rfloor$ be defined by (1.16), and let $m_0(x) := \exp(\kappa \langle x\rangle^\gamma)$ and $0 < \kappa\gamma < 1/8$. Then for any $\sigma \in (0, \sigma^*_{\mathcal{L}}]$ and $\theta < 1$ there exist $\lambda > 0$ such that for all $t > 0$*

$$\|f(t) - M(f_0)\, G\|_{L^2(m_0^\theta)} \lesssim \exp(-\lambda t^\sigma)\, \|f_0 - M(f_0)\, G\|_{L^2(m_0)}.$$

Proof. We write the representation formulas (taken from [24])

$$S_{\mathcal{L}}(t) f = \Pi f + \sum_{\ell=0}^{5} (I - \Pi) S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{B}})^{(*\ell)}(t) f + \mathcal{T}(t) f, \tag{5.21}$$

$$\text{where } \mathcal{T}(t) := \lim_{M\to\infty} \frac{\mathrm{i}}{2\pi} \int_{a-\mathrm{i}M}^{a+\mathrm{i}M} \mathrm{e}^{zt}\, (I-\Pi)\mathcal{R}_{\mathcal{L}}(z)\, (\mathcal{A}\mathcal{R}_{\mathcal{B}}(z))^6\, dz,$$

for any $f \in D(\mathcal{L})$, $t \ge 0$ and $a > 0$.

Thanks to Lemma 2.4, we know that each term $\|(I - \Pi) S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{B}})^{(*\ell)}(t) f\|$ in the above expression of $S_{\mathcal{L}}(t)$ is bounded by $\mathcal{O}(\Theta_{m_0}(t))$. In order to conclude, we have to estimate the last term.

We introduce the shorthands $\Phi_1 := R_{\mathcal{L}}(I - \Pi)$, $\Phi_\ell = \mathcal{A} R_{\mathcal{B}}$, for $2 \le \ell \le 7$, and we perform n integration by part in the formula giving $\mathcal{T}(t)$ to get

$$\mathcal{T}(t) = \frac{\mathrm{i}}{2\pi} \frac{1}{t^n} \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} \mathrm{e}^{zt}\, \frac{d^n}{dz^n} \big(\prod_{i=1}^{7} \Phi_i(z)\big)\, dz. \tag{5.22}$$

Using the fact that all the functions appearing in the integral are bounded on the imaginary axis, together with the resolvent identity

$$R^n_\Lambda(z) := \frac{d^n}{dz^n} R_\Lambda(z) = n! R_\Lambda(z)^n,$$

we find in $\mathscr{B}(X_0)$, thanks to Leibniz formula and for any $z = x + \mathrm{i}y \in \mathbb{C}$ with $0 \le y \le 1$,

$$\begin{aligned}
\|\frac{d^n}{dz^n}\big(\prod_{\ell=1}^{7} \Phi_\ell(z)\big)\| &\le 7^n \sup_{\alpha\in\mathbb{N}^n, |\alpha|=n} \|\prod_{\ell=1}^{7} \frac{d^{\alpha_\ell}}{dz^{\alpha_\ell}} \Phi_\ell(z)\| \\
&\le 7^n \sup_{\alpha\in\mathbb{N}^7, |\alpha|=n} \prod_{i=1}^{7} \|\alpha_1!(I-\Pi)\, R_{\mathcal{L}}^{1+\alpha_1} \alpha_2! \mathcal{A} R_{\mathcal{B}}^{1+\alpha_2} \ldots \alpha_7! \mathcal{A} R_{\mathcal{B}}^{1+\alpha_7}(z)\| \\
&\le C^n\, (n!)^{1+j}\, \langle y\rangle^{-3/2},
\end{aligned}$$

where in the last step we have used Lemma 5.6 for some integer j which will be fixed later. Next, using the bound $n! \le (C\, n)^n$, we get

$$\|\mathcal{T}(t)\| \le C^n\, n^{(1+j)n}\, t^{-n} \qquad \forall\, t > 0,\ \forall\, k \ge 0.$$

For any $t \ge t^*$, where t^* is large enough and depends on j, we choose $n \in \mathbb{N}$ such that

$$t \ge 2Cn^{1+j} \ge t/2,$$

and we obtain

$$\|\mathcal{T}(t)\| \le (Cn^{1+j} t^{-1})^n \le (1/2)^n \le (1/2)^{(t/4C)^{\frac{1}{j+1}}} \qquad \forall\, t > 0.$$

As a consequence, with the choice $j := \lfloor [2(1-\gamma)/\gamma \rfloor + 1$, we have proved that for all $t \ge 0$ we have

$$\|\mathcal{T}(t)\| \le \exp(-\lambda\, t^{\frac{1}{1+j}}),$$

which clearly ends the proof. □

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