# Propagation of chaos for the 2D viscous Vortex model

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- Aim of the talk
  - quick introduction to mean field limit / propagation of chaos
  - statement of our propagation of chaos result for the **2D viscous Vortex model** (example of "singular" McKean-Vlasov model)
  - sketch of the proof

- the results are taken from
  - Hauray, M., "On Kac's chaos and related problems", HAL-2012
  - Fournier, Hauray, M., *"Propagation of chaos for the 2D viscous vortex model"*, to appear in JEMS

# Outlines of the talk

# Plan

# Micro to macro

• How to go **rigorously** from a microscopic description to a statistical description?

how to derive (justify) the equation at the macroscopic level ?

how to get something (simpler) from a microscopic description with a huge number of particles ?

• (Kac's) mean field limit ($\neq$ Boltzmann-Grad limit) in the sense that each particle interacts with all the other particles with an intensity of order $\mathcal{O}(1/N)$
$\Rightarrow$ statistical description $=$ *law of large numbers limit* of a $N$-particle system

• at the formal level the identification of the limit is quite clear when one assumes the molecular chaos for the limit model

• main difficulty : propagation of chaos

$\triangleright$ chaos for $\infty$ particles $=$ Boltzmann's molecular chaos (stochastic independence)
$\triangleright$ chaos for $N \to \infty$ particles $=$ Kac's chaos (asymptotic stochastic independence)
$\triangleright$ propagation of chaos: holds at time $t = 0$ implies holds for any $t > 0$
$\triangleright$ propagation of chaos is necessary in order to identify the limit as $N \to \infty$

Consider a system of $N$ indistinguishable (exchangeable) particles, each particle being described by its state (position, velocity) $\mathcal{Z}_1^N, ..., \mathcal{Z}_N^N \in E$, $E = \mathbb{R}^d$, which evolves according to

$$d\mathcal{Z}_i = \frac{1}{N} \sum_{j=1}^{N} a(\mathcal{Z}_i - \mathcal{Z}_j)\, dt \qquad \text{(ODE)}$$

$$d\mathcal{Z}_i = \frac{1}{N} \sum_{j=1}^{N} a(\mathcal{Z}_i - \mathcal{Z}_j)\, dt + \sqrt{2\nu}\, d\mathcal{B}_i \qquad \text{(Brownian SDE)}$$

$$d\mathcal{Z} = \frac{1}{N} \sum_{i,j=1}^{N} \int_{S^{d-1}} (\mathcal{Z}'_{ij} - \mathcal{Z})\, b\, d\mathcal{N}(d\sigma, i, j) \qquad \text{(Boltzmann-Kac)}$$

where $a$ is a pairwise interaction force field, $\mathcal{B}_i$ Brownian motions, $\mathcal{N}$ Poisson measure, $\mathcal{Z}'_{ij} = (\mathcal{Z}_1, ..., \mathcal{Z}'_i, ..., \mathcal{Z}'_j, ..., \mathcal{Z}_N)$ represents the system after collision of the pair $(\mathcal{Z}_i, \mathcal{Z}_j)$, $b$ cross-section

The law $G^N(t) := \mathcal{L}(\mathcal{Z}_t^N)$ satisfies the Master (Liouville or backward Kolmogorov) equation

$$\partial_t \langle G^N, \varphi \rangle = \langle G^N, \Lambda^N \varphi \rangle \qquad \forall\, \varphi \in C_b(E^N)$$

where the generator $\Lambda^N$ writes

$$(\Lambda^N \varphi)(Z) := \frac{1}{N} \sum_{i,j=1}^{N} a(z_i - z_j) \cdot \nabla_i \varphi \qquad \text{(ODE)}$$

$$(\Lambda^N \varphi)(Z) := \frac{1}{N} \sum_{i,j=1}^{N} a(z_i - z_j) \cdot \nabla_i \varphi + \nu \sum_{i=1}^{N} \Delta_i \varphi \qquad \text{(SDE)}$$

$$(\Lambda^N \varphi)(Z) = \frac{1}{N} \sum_{1 \le i < j \le N}^{N} \int_{\mathbb{S}^{d-1}} \left[ \varphi(Z'_{ij}) - \varphi(Z) \right] \tilde{b}_{ij} \, \mathrm{d}\sigma \qquad \text{(Boltzmann-Kac)}$$

with $\tilde{b}_{ij} := \tilde{b}(z_i - z_j, \sigma)$.

Is it possible to identify the limit of the law $\mathcal{L}(\mathcal{Z}_1^N)$ of one typical particle?

More precisely, we want to show that $\mathcal{L}(\mathcal{Z}_1^N) \to f = f(t,dz)$ and that $f \in C([0,\infty); P(E))$ is a solution to

$$\partial_t f = \mathrm{div}_z[(a * f)f] \qquad (Vlasov)$$

$$\partial_t f = \mathrm{div}_z[(a * f)f] + \nu \Delta f \qquad (McKean - Vlasov)$$

$$\partial_t f = \int_{\mathbb{R}^d \times S^{d-1}} [f(z')f(v') - f(z)f(v)]\, b\, dz d\sigma \qquad \text{(Boltzmann)},$$

depending of the $N$-particle dynamics

Assuming that

$$\mathcal{L}(\mathcal{Z}_1^N) \to f = f(t, dz), \quad \mathcal{L}(\mathcal{Z}_1^N, \mathcal{Z}_2^N) \to g = g(t, dz, dv),$$

we easily (formally) show by taking $\varphi(Z) = \varphi(z_1)$ in the Master equation

$$\partial_t f = \mathrm{div}_z \Big[ \int a(z-v) g(dz, dv) \Big]$$

$$\partial_t f = \mathrm{div}_z \Big[ \int a(z-v) g(dz, dv) \Big] + \nu \Delta f,$$

$$\partial_t f = \int_{\mathbb{R}^d \times S^{d-1}} [g(z', v') - g(z, v)] \, \tilde{b} \, dz d\sigma.$$

We obtain the Vlasov equation, the McKean-Vlasov equation and the Boltzmann equation if we make the additional

independence / molecular chaos assumption $g(v, z) = f(v) \, f(z)$.

# Difficulty

• The above picture is not that easy because for $N$ fixed particles the states $\mathcal{Z}_1(t)$, ..., $\mathcal{Z}_N(t)$ are **never independent** for positive time $t > 0$ even if the initial states $\mathcal{Z}_1(0), ..., \mathcal{Z}_N(0)$ are assumed to be independent : that is an inherent consequence of the fact that **particles do interact!**

• Equations are written in spaces with increasing dimension $N \to \infty$.
To overcome that difficulty we work in **fixed spaces** using:
empirical probability measure

$$X \in E^N \mapsto \mu_X^N := \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \in \mathbb{P}(E)$$

and/or marginal densities

$$F^N \in \mathbb{P}_{sym}(E^N) \;\mapsto\; F_j^N := \int_{E^{N-j}} F^N dz_{j+1} ... dz_N \in \mathbb{P}_{sym}(E^j)$$

• The nonlinear PDE can be obtained as a *"law of large numbers"* for a **not independent array of exchangeable random variables** in the mean-field limit.

• That is more demanding that the usual LLN. We need to **propagate** some asymptotic independence = Kac's stochatstic chaos.

# Even more difficult for singular models

- We need at least
  - ▷ a priori estimates on the $N$-particle system
  - ▷ uniqueness for the limit nonlinear PDE

- Most of the works has been done in a **probability measures framework**.

In order that everything make sense, it is then needed that coefficients are not singular (they must be smooth enough, say $C^0$).

There is some (few) works on **singular stochastic** dynamics:
- Osada, Proc. Japan Acad. 1986 & ... (vortex with diffusion)
- Caglioti, Lions, Marchioro, Pulvirenti, CMP 1995 & 1995 (stationary problem)
- Cépa, Lépingle, PTRF 1997 ($D = 1$)

For deterministic dynamics we refer to the talk by Maxime Hauray.

- Our goal: Understand the work by Osada. Recover and generalize his result.

# Plan

Consider a system of $N$ indistinguishable (exchangeable) particles, each particle being described by its position $\mathcal{X}_1^N, ..., \mathcal{X}_N^N \in \mathbb{R}^2$, which evolves according to

$$d\mathcal{X}_i = \frac{1}{N} \sum_{j=1}^N K(\mathcal{X}_i - \mathcal{X}_j)\, dt + \sqrt{2\nu} d\mathcal{B}_i \qquad \text{(Brownian SDE)}$$

where $\nu > 0$ is the viscosity and $K : \mathbb{R}^2 \to \mathbb{R}^2$ is the Biot-Savart kernel defined by

$$\forall\, x = (x_1, x_2) \in \mathbb{R}^2, \quad K(x) = \frac{x^\perp}{|x|^2} = \Big(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2}\Big) = \nabla^\perp \log |x|,$$

The associated mean field limit is the $2D$ Navier-Stokes equation written in vorticity formulation

$$\partial_t w_t(x) = (K \star w_t)(x) \cdot \nabla_x w_t(x) + \nu \Delta_x w_t(x), \qquad (1)$$

where $w : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}_+$ is the vorticity function

**All that can be done for vortices which turn in both (trigonometric and reverse) senses and thus** $w : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$

## Theorem (first version)

*(1) If $\mathcal{X}_0^N$ is $w_0$-Kac's chaotic and "appropriately bounded"*
*then $\mathcal{X}_t^N$ is $w_t$-Kac's chaotic for any time $t > 0$.*

*(2) If $\mathcal{X}_0^N$ is $w_0$-entropy chaotic and has bounded moment of order $k \in (0,1]$*
*then $\mathcal{X}_t^N$ is $w_t$–entropy chaotic for any time $t > 0$.*

- Definitions of chaos
- sketch of the proof

Chaos is the asymptotic independence as $N \to \infty$ for a sequence $(\mathcal{Z}^N)$ of exchangeable random variables with values in $E^N$

$$\begin{array}{ccc} \mathcal{Z}^N = (\mathcal{Z}_1^N, ..., \mathcal{Z}_N^N) \in E^N & \to & F^N := \mathcal{L}(\mathcal{Z}^N) \in \mathbb{P}_{sym}(E^N) \\ \updownarrow & & \updownarrow \\ \mu^N_{\mathcal{Z}^N} := \dfrac{1}{N} \displaystyle\sum_{i=1}^N \delta_{\mathcal{Z}_i^N} \in \mathbb{P}(E) & \to & \hat{F}^N := \mathcal{L}(\mu^N_{\mathcal{Z}^N}) \in \mathbb{P}(\mathbb{P}(E)) \end{array}$$

For $\mathcal{Y}$ r.v taking values in $E$ with law $\mathcal{L}(\mathcal{Y}) = f \in \mathbb{P}(E)$ we say that $(\mathcal{Z}^N)$ is $\mathcal{Y}$-Kac's chaotic if

- $\mathcal{L}(\mathcal{Z}_1^N, ..., \mathcal{Z}_j^N) \rightharpoonup f^{\otimes j}$ weakly in $\mathbb{P}(E^j)$ as $N \to \infty$;
- $\mu^N_{\mathcal{Z}^N} \Rightarrow f$ in law as $N \to \infty$,
  meaning $\mathcal{L}(\mu^N_{\mathcal{Z}^N}) \to \delta_f$ in $\mathbb{P}(\mathbb{P}(E))$ as $N \to \infty$;
- $\mathbb{E}(|\mathcal{X}^N - \mathcal{Y}^N|) \to 0$ as $N \to \infty$ for a sequence $\mathcal{Y}^N$ of i.i.d.r.v with law $f$

Exchangeable means: $\mathcal{L}(\mathcal{Z}^N_{\sigma(1)}, ..., \mathcal{Z}^N_{\sigma(N)}) = \mathcal{L}(\mathcal{Z}_1^N, ..., \mathcal{Z}_N^N)$ for any permutation $\sigma$ of the coordinates

For a given sequence $(F^N)$ in $\mathbb{P}_{sym}(E^N)$ we define

- the marginals $F_j^N \in \mathbb{P}_{sym}(E^j)$ by

$$F_j^N = \int_{E^{N-j}} F^N dz_{j+1}...dz_N$$

- the projection $\hat{F}^N \in \mathbb{P}(\mathbb{P}(E))$ by

$$\langle \hat{F}^N, \Phi \rangle = \int_{E^N} \Phi(\mu_X^N) \, F^N(dX) \quad \forall \, \Phi \in C_b(\mathbb{P}(E))$$

- the normalized MKW distance on $\mathbb{P}(E^j)$ by

$$W_1(F, G) := \inf_{\pi \in \Pi(F,G)} \int_{E^j \times E^j} \Big( \frac{1}{j} \sum_{i=1}^{j} |x_i - x_j| \wedge 1 \Big) \, \pi(dX, dY).$$

- the MKW distance $\mathcal{W}_1$ on $\mathbb{P}(\mathbb{P}(E))$ by

$$\mathcal{W}_1(\alpha, \beta) := \inf_{\pi \in \Pi(\alpha,\beta)} \int_{\mathbb{P}(E) \times \mathbb{P}(E)} W_1(\rho, \eta) \, \pi(d\rho, d\eta).$$

## Definition of chaos = not about random variables but their laws !

For a given sequence $(F^N)$ in $\mathbb{P}_{sym}(E^N)$ we define

− the marginals $F^N_j \in \mathbb{P}_{sym}(E^j)$,

− the projection $\hat F^N \in \mathbb{P}(\mathbb{P}(E))$,

− the normalized MKW distance $W_1$ on $\mathbb{P}(E^j)$,

− the MKW distance $\mathcal{W}_1$ on $\mathbb{P}(\mathbb{P}(E))$,

and for $f \in \mathbb{P}(E)$ we say that $(F^N)$ is $f$-Kac's chaotic if (equivalently)

- $\mathcal{D}_j(F^N; f) := W_1(F^N_j, f^{\otimes j}) = \mathbb{E}(|(\mathcal{X}^N_1, ..., \mathcal{X}^N_j) - (\mathcal{X}^N_1, ..., \mathcal{X}^N_j)|) \to 0$
- $\mathcal{D}_\infty(F^N; f) := \mathcal{W}_1(\hat F^N, \delta_f) = \mathbb{E}(W_1(\mu^N_{\mathcal{Z}^N}, f) \to 0$

More precisely, for $E = \mathbb{R}^d$

### Lemma (Hauray, M.)

*For given $M$ and $k > 1$ there exist some constants $\alpha_i, C > 0$ such that $\forall\, f \in \mathbb{P}(E)$, $\forall\, F^N \in \mathbb{P}_{sym}(E^N)$ with $M_k(F^N_1), M_k(f) \le M$*

$$\forall\, j, \ell \in \{1, ..., N, \infty\},\ \ell \neq 1 \quad \mathcal{D}_j(F^N; f) \le C \left(\mathcal{D}_\ell(F^N; f)^{\alpha_1} + \frac{1}{N^{\alpha_2}}\right).$$

## Stronger chaos: entropy and Fisher's chaos

For $F^N \in \mathbb{P}_{sym}(E^N)$, $E = \mathbb{R}^d$, we define the normalized functionals

$$H(F^N) := \frac{1}{N} \int_{E^N} F^N \log F^N, \quad I(F^N) := \frac{1}{N} \int_{E^N} \frac{|\nabla F^N|^2}{F^N}.$$

### Definition

Consider a sequence $F^N \in \mathbb{P}_{sym}(E^N)$ and $f \in \mathbb{P}(E)$

$(F^N)$ is $f$-entropy chaotic if $F_1^N \rightharpoonup f$ weakly in $\mathbb{P}(E)$ and $H(F^N) \to H(f)$

$(F^N)$ is $f$-Fisher's chaotic if $F_1^N \rightharpoonup f$ weakly in $\mathbb{P}(E)$ and $I(F^N) \to I(f)$

### Theorem (Hauray, M.)

*In the list below, each assertion implies the one which follows*

*(i) $(F^N)$ is Fisher's chaotic;*

*(ii) $(F^N)$ is Kac's chaotic and $I(F^N)$ is bounded;*

*(iii) $(F^N)$ is entropy chaotic;*

*(iv) $(F_j^N)$ converges in $L^1$ for any $j \geq 1$;*

*(v) $(F^N)$ is Kac's chaotic.*

We say that $\mathcal{X} = (\mathcal{X}_t)_{t>0}$ a continuous stochatsic process with values in $\mathbb{R}^2$ is a solution to the stochastic NS vortex equation if it satisfies the Brownian EDS

$$d\mathcal{X}_t = (K * w_t)(\mathcal{X}_t) + \sqrt{2\nu}\, d\mathcal{B}_t$$

for some given brownian motion $\mathcal{B}$ and where $w_t = \mathcal{L}(\mathcal{X}_t)$ is the law of $\mathcal{X}_t$.

It is important to point out that (thanks to Ito formula) the law $w_t$ of $\mathcal{X}_t$ then satisfies the NS vortex equation

$$\partial_t w_t = (K * w_t) \cdot \nabla_x w_t + \nu \Delta_x w_t.$$

## Theorem (second version)

*Consider $w_0 \geq 0$ a function such that*

$$\int_{\mathbb{R}^2} w_0 \, (1 + |x|^k + |\log w_0|) \, dx < \infty, \quad k \in (0, 1],$$

*the vortices trajectories $\mathcal{X}^N = (\mathcal{X}^N_t)_{t \geq 0}$ associated to an i.c. $\mathcal{X}^N_0 \sim w_0^{\otimes N}$ and $\mathcal{X}$ the solution to the stochastic NS vortex equation associated to an i.c. $\mathcal{X}_0 \sim w_0$. There holds*

$$\mu^N_{\mathcal{X}^N} \;\Rightarrow\; \mathcal{X} \quad \text{in law in } \mathbb{P}(C([0,\infty);\mathbb{R}^2)) \text{ as } N \to \infty$$
$$\mathcal{L}(\mathcal{X}^N_1(t)) \to w_t = \mathcal{L}(\mathcal{X}_t) \quad \text{strongly in } L^1(\mathbb{R}^2) \text{ as } N \to \infty$$

The first convergence means

$$\mathcal{L}(\mu^N_{\mathcal{X}^N}) \;\rightharpoonup\; \delta_{\mathcal{L}(\mathcal{X})} \quad \text{weakly in } \mathbb{P}(\mathbb{P}(C([0,\infty);\mathbb{R}^2)) \text{ as } N \to \infty$$

and the second can be improved into

$$\mathcal{L}(\mathcal{X}^N_1(t), ..., \mathcal{X}^N_j(t)) \to w_t^{\otimes j} \quad \text{strongly in } L^1(\mathbb{R}^2)^j \text{ as } N \to \infty$$

The proof follow the by-now well-known "weak stability on nonlinear martingales" approach, which goes back to Sznitmann 1984.
Everything is standard except the fact that we use the Fisher information bound in each step.

• A priori estimates (on entropy, moment and Fisher information)

• tightness of the law $Q^N$ of the empirical process $\mu^N_{\mathcal{X}^N}$ in $\mathbb{P}(\mathbb{P}(E))$

• pass to the limit and identify the set of limit points $\mathcal{S}$ as the probablity measures $q \in \mathbb{P}(E)$ associated to a process $\mathcal{X}$ which solves the (Martingale problem associated to the) stochastic NS vortex equation and has finite Fisher information.

• if $q \in \mathcal{S}$ and $q = \mathcal{L}(\mathcal{X})$ then $w_t := \mathcal{L}(\mathcal{X}_t)$ is the unique solution to the NS vortex PDE

• the Martingale problem has a unique solution $\bar{\mathcal{X}}$ and then $\mathcal{S} = \{\bar{q}\}$ where $\mathcal{L}(\bar{q}) = \bar{\mathcal{X}}$.

In conclusion, $Q^N \rightharpoonup \delta_{\bar{q}}$ in $\mathbb{P}(\mathbb{P}(E))$. (that Kac's chaos)

# Plan

Using $\mathrm{div} K = 0$, we get the entropy identity

$$H(F_t^N) + \nu \int_0^t I(F_s^N)\, ds = H(F_0^N).$$

As usually we need a control of a moment of $F_t^N$ in order to take advantage of the entropy bound (we need a lower bound on $H$).

We define the moment $M_k$ of order $k \in (0,1]$ by

$$M_k(F^N) = \int_{\mathbb{R}^{2N}} F^N \frac{1}{N} \sum_{j=1} \langle x_j \rangle^k = \int_{\mathbb{R}^2} F_1^N \langle x \rangle^k dx$$

We then compute

$$\begin{aligned}\frac{d}{dt} M_k(F_t^N) &= \nu \int_{\mathbb{R}^2} F_{1t}^N \,\Delta \langle x \rangle^k + \int_{\mathbb{R}^4} F_{2t}^N \, K(x_1 - x_2) \cdot \nabla_1 \langle x_1 \rangle^k \\ &\le C_1 \int_{\mathbb{R}^2} F_{1t}^N + C_2 \int_{\mathbb{R}^4} F_{2t}^N \, \frac{1}{|x_1 - x_2|}\end{aligned}$$

Defining $g^N := \mathcal{L}(X_2 - X_1)$ and using classical (Carlen 1991) results on Fisher information, we have

$$\frac{1}{2} I_1(g^N) \le I_2(F_2^N) \le I_N(F^N)$$

## Control given by the Fisher information

Defining $g^N := \mathcal{L}(X_2 - X_1)$ and using classical (Carlen 1991) results on Fisher information, we have

$$\frac{1}{2} I_1(g^N) \le I_2(F_2^N) \le I_N(F^N)$$

Next, one can prove some Gagliardo-Niremberg type inequalities in 2D (using Sobolev inequality plus Holder inequality)

$$\forall\, g \in \mathbb{P}(\mathbb{R}^2),\ \forall\, p \in [1,\infty) \quad \|g\|_{L^p} \le C_p\, I(g)^{1-1/p}$$

$$\forall\, g \in \mathbb{P}(\mathbb{R}^2),\ \forall\, q \in [1,2) \quad \|\nabla g\|_{L^q} \le C_q\, I(g)^{3/2-1/q}$$

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&\forall\, g \in \mathbb{P}(\mathbb{R}^2),\ \forall\, q \in [1,2) \quad \|\nabla g\|_{L^q} \le C_q\, I(g)^{3/2-1/q}
\end{aligned}$$

Coming back to the singular term in the moment equation, we compute

$$\begin{aligned}
\int_{\mathbb{R}^4} \frac{F_{2t}^N}{|x_1 - x_2|}\, dx_1 dx_2 &= \sqrt{2} \int_{B_1} \frac{g_t^N(x)}{|x|}\, dx + \sqrt{2} \int_{B_1^c} \frac{g_t^N(x)}{|x|}\, dx \\
&\le \sqrt{2}\, \| |\cdot|^{-1} \|_{L^{3/2}(B_1)} \|g_t^N\|_{L^3(B_1)} + \sqrt{2}\, \|g_t^N\|_{L^1(B_1^c)}
\end{aligned}$$

Defining $g^N := \mathcal{L}(X_2 - X_1)$

$$\frac{1}{2} I_1(g^N) \le I_2(F_2^N) \le I_N(F^N)$$

Gagliardo-Niremberg type inequality in 2D

$$\forall\, g \in \mathbb{P}(\mathbb{R}^2),\ \forall\, p \in [1,\infty) \quad \|g\|_{L^p} \le C_p\, I(g)^{1-1/p}$$

For the singular term in the moment equation, we compute

$$\begin{aligned}
\int_{\mathbb{R}^4} \frac{F_{2t}^N}{|x_1 - x_2|}\, dx_1 dx_2 &\le \sqrt{2}\, \||\cdot|^{-1}\|_{L^{3/2}(B_1)}\, \|g_t^N\|_{L^3(B_1)} + \sqrt{2}\, \|g_t^N\|_{L^1(B_1^c)} \\
&\le C_3\, I(g_t^N)^{2/3} + C_4 \\
&\le \frac{\nu}{4 C_2}\, I(g_t^N) + C_5 \\
&\le \frac{\nu}{2 C_2}\, I(F_t^N) + C_5
\end{aligned}$$

Summing up the two equations on the entropy and on the moment of order $k$, we find

**Lemma (a priori estimates)**

*Uniformly in $N$*

$$H(F_t^N) + M_k(F_t^N) + \frac{\nu}{2}\int_0^t I(F_s^N)\,ds \\ \leq H(F_0^N) + M_k(F_0^N) + (C_1 + C_2)t$$

# Plan

We denote

$$\mathcal{X}^N := (\mathcal{X}_1^N, ..., \mathcal{X}_N^N) \text{ the exchangeable r.v. with value in } E^N$$

where $\mathcal{X}_i^N = (\mathcal{X}_i^N(t))_{t\geq 0} \in E := C([0,\infty); \mathbb{R}^2)$ solution to the SDE

$$\mathcal{X}_i(t) = \mathcal{X}_i(0) + \int_0^t (K * \mu^N_{\mathcal{X}(s)})(\mathcal{X}_i(s))\, ds + \sqrt{2\nu}\, \mathcal{B}_i(t)$$

and we want to show that each particle behaves asymptotically like $N$ independent copies of the same process $\mathcal{X} = (\mathcal{X}(t))_{t\geq 0}$ defined as the solution to the nonlinear SDE

$$\mathcal{X}(t) = \mathcal{X}(0) + \int_0^t (K * w_s)(\mathcal{X}(s))\, ds + \sqrt{2\nu}\, \mathcal{B}(t),$$

where $w_s := \mathcal{L}(X(s))$ and then is a solution (Ito formula) to the NS vortex equation

$$\partial_t w = (K \star w) \cdot \nabla_x w + \nu \Delta_x w.$$

## Lemma

*the family of laws* $\mathcal{L}(\mu^N_{\mathcal{X}^N})_{N\geq 1}$ *is tight in* $\mathbb{P}(\mathbb{P}(E))$

From classical compactness criterium (Sznitmann 1984) it is enough to prove that the family of laws $\mathcal{L}(\mathcal{X}^N_1)_{N\geq 1}$ is tight in $\mathbb{P}(E)$. That is a consequence of

## Lemma

*For all* $T > 0$, $\theta \in (0, 1/2)$

$$\mathbb{E}\Big[\sup_{0<s<t<T} \frac{|\mathcal{X}^N_1(t) - \mathcal{X}^N_1(s)|}{(t-s)^\theta}\Big] \leq C\Big(1 + \int_0^T I(G^N_u)\, du\Big)$$

By Prokhorov, we get

## Lemma

*There exists* $Q \in \mathbb{P}(\mathbb{P}(E))$ *such that*

$$Q^N \rightharpoonup Q \quad in \quad \mathbb{P}(\mathbb{P}(E)).$$

Using the SDE equation we have

$$|\mathcal{X}_1^N(t) - \mathcal{X}_1^N(s)| \le (\mathcal{Z}_T + \mathcal{U}_T^N + 1)\,(t-s)^{1/3}$$

with

$$\mathcal{Z}_T := \sup_{0<s<t<T} |\mathcal{B}_1(t) - \mathcal{B}_2(s)|/(t-s)^{1/3}$$

and (using Holder inequality in the interaction term)

$$\mathcal{U}_T^N := \int_0^T \frac{1}{N} \sum_{j\neq 1} |\mathcal{X}_1(u) - \mathcal{X}_j(u)|^{-3/2}\,du$$

We conclude using that $\mathbb{E}(\mathcal{Z}_T) < \infty$ and

$$\begin{aligned}
\mathbb{E}(\mathcal{U}_T^N) &= \int_0^T \mathbb{E}(|\mathcal{X}_1(u) - \mathcal{X}_2(u)|^{-3/2}) \\
&\approx \int_0^T \int_{\mathbb{R}^4} \frac{G_2^N}{|x_1 - x_2|^{3/2}} \approx \int_0^T \int_{\mathbb{R}^2} \frac{g^N(du,x)}{|x|^{3/2}}\,dx \\
&\le C \int_0^T \|g_u\|_{L^{10}}\,du \le C\Big(1 + \int_0^T I(G_u^N)\,du\Big)
\end{aligned}$$

# Identification of the the limit thanks to "Sznitman" argument.

## Lemma

*Assume that $Q \in \mathbb{P}(\mathbb{P}(C([0,+\infty),\mathbb{R}^2)) = \mathbb{P}(\mathbb{P}(E))$ is a mixture measure obtained as a limit point of some subsequence of $Q^N$. Then supp $Q \subset \mathcal{S}$*

$$\mathcal{S} := \left\{ \begin{array}{l} q \text{ is the law of some } \mathcal{X} \text{ solution to stoch. NS vortex eq.} \\ \forall T > 0, \quad \int_0^T I(\mathcal{L}(X_s))\, ds < +\infty \end{array} \right\} = \mathcal{S}_0 \cap \mathcal{S}_1$$

- $q \approx \mathcal{X}$ solves the stoch. NS vortex eq. iif for all times $s, t$, $\psi$, $\varphi$...

$$\mathcal{F}(q) := \iint_{E^2} q(dx)q(dy)\psi(x(s \le t))\Big[\varphi(x(t)) - \varphi(x(s)) - \int_s^t K(x(u) - y(u)) \cdot \nabla\varphi(x(u))du - \nu \int_s^t \Delta\varphi(x(u))du\Big] = 0$$

- $Q$ concentrated on $\mathcal{S}_0 \iff \mathbb{E}_Q[|\mathcal{F}(\cdot)|^2] = 0$ for all $s, t, \psi, \varphi$.
- $\mathbb{E}_{Q^N}[|\mathcal{F}(\cdot)|^2] \to 0$ as $N \to +\infty$.
- Continuity $\mathbb{P}(\mathbb{P}(E)) \ni R \mapsto \mathbb{E}_R[|\mathcal{F}(\cdot)|^2]$ under the condition $\mathbb{E}_R\big[\int_0^t I(\cdot_s)\, ds\big] < +\infty$.
- $\mathbb{E}_P\big[\int_0^t I(\cdot_s)\, ds\big] < +\infty$, which is equivalently $P \in \mathcal{S}_1$.

# Plan

## Level 3 Fisher information (for a mixture of probability measures)

Consider $\pi \in \mathbb{P}(\mathbb{P}(E))$, $E = \mathbb{R}^2$, and define

$$\mathcal{I}(\pi) := \int_{\mathbb{P}(E)} I(\rho)\, \pi(d\rho), \quad \mathcal{I}'(\pi) := \sup_{j \ge 1} I(\pi_j) = \lim_{j \to \infty} I(\pi_j)$$

where $\pi_j$ is given by (the easy part of) Hewitt and Savage theorem

$$\pi_j := \int_{\mathbb{P}(E)} \rho^{\otimes j}\, \pi(d\rho) \in \mathbb{P}_{sym}(E^j).$$

### Theorem (representation formula, Hauray-M.)

$$\forall\, \pi \in \mathbb{P}(\mathbb{P}(E)) \quad \mathcal{I}(\pi) = \mathcal{I}'(\pi).$$

A similar formula is known for the entropy (Robinson-Ruelle, 1967)

Application: the Fisher information is Γ-lsc in the sense

$$\mathbb{P}_{sym}(E^N) \ni F^N \rightharpoonup \pi \in \mathbb{P}(\mathbb{P}(E)) \text{ implies } \mathcal{I}(\pi) \le \liminf I(F^N).$$

One line proof: for any $j \ge 1$ by lsc of $I_j$

$$I_j(\pi_j) \le \liminf I_j(F_j^N) \le \liminf I_N(F^N).$$

(ii) $I(f^{\otimes j}) = I(f)$ (good normalization)
(iii) $I$ is lsc, convex, proper and $\geq 0$ on $\mathbb{P}_{sym}(E^j)$, $\forall\, j \geq 1$
(iv) $\mathcal{I}'$ is linear on disjoint convex combination in the sense that

$$\mathcal{I}'(\pi) = \alpha_1 \mathcal{I}'(\gamma^1) + ...\alpha_M \mathcal{I}'(\gamma^M)$$

if

$$\pi = \alpha_1\, \gamma^1 + ...\alpha_M \gamma^M, \;\; \alpha_i \geq 0, \;\; \sum_i \alpha_i = 1, \;\; \mathrm{supp}\gamma^i \cap \mathrm{supp}\gamma^j = \emptyset$$

On the one hand, we have by (ii) and Jensen inequality

$$\mathcal{I}(\pi) = \int_{\mathbb{P}(E)} I(\rho^{\otimes j})\, \pi(d\rho) \geq I\Big( \int_{\mathbb{P}(E)} \rho^{\otimes j}\, \pi(d\rho)\Big) = I(\pi_j)$$

On the other hand, we write thanks to (iv) and the Jensen inequality

$$\begin{array}{rcll} \mathcal{I}'(\pi) & = & \alpha_1\, \mathcal{I}'(\gamma^1) + ... + \alpha_M\, \mathcal{I}'(\gamma^M), & \gamma^i := \alpha_i^{-1} \pi_{|\omega_i} \\ & \geq & \alpha_1\, I(f_1) + ... + \alpha_M\, I(f_M), & f_i := \gamma^i_1 \\ & = & \mathcal{I}(\pi^M), & \pi^M := \alpha_1^M\, \delta_{f_1^M} + ... + \alpha_M^M\, \delta_{f_M^M}. \end{array}$$

As $\pi^M \to \pi$ we get the inverse inequality $\mathcal{I}(\pi) \leq \liminf \mathcal{I}(\pi^M) \leq \mathcal{I}'(\pi)$.

For $G_j,\ F_j \in \mathbb{P}_{sym}(E^j)$, we write the identity

$$\theta I(F_j) + (1-\theta) I(G_j) - I(\theta F_j + (1-\theta) G_j) = \theta\,(1-\theta)\,J_j$$

with for $G_j = g^{\otimes j}$, $F_j = f^{\otimes j}$, $f \neq g$ so that $W_1(f,g) =: 2\delta > 0$,

$$\begin{aligned} J_j &:= \int_{E^j} \frac{G_j F_j}{\theta F_j + (1-\theta) G_j} \, |\nabla_1 \log \frac{G_j}{F_j}|^2 \\ &\le C \int_{W_1(\mu^N_X, f) \ge \delta} \frac{G_j F_j}{\theta F_j + (1-\theta) G_j} + \dots \\ &\le C \int_{W_1(\mu^N_X, f) \ge \delta} \frac{F_j}{1-\theta} + \dots \\ &\le \frac{C}{\delta(1-\theta)} \int_{E^j} W_1(\mu^N_X, f) f^{\otimes j}(dX) + \dots \to 0 \text{ as } j \to \infty, \end{aligned}$$

by the functional law of large number $\mu^N_{\mathcal{X}^N} \to f$ if $\mathcal{X}^N \sim f^{\otimes N}$.
We deduce (in the limit $j \to \infty$)

$$\theta \mathcal{I}'(\delta_f) + (1-\theta) \mathcal{I}'(\delta_g) - \mathcal{I}'(\theta \delta_f + (1-\theta)\delta_g) = 0$$

We know (from tightness) that

$$\mathcal{L}(\mu^N_{\mathcal{X}}) \rightharpoonup Q \quad \text{weakly in } \mathbb{P}(\mathbb{P}(E))$$

with here $E := C([0,\infty);\mathbb{R}^2)$. We define $Q_t :=$ projection on the section $\mathbb{P}(\mathbb{P}(\{t\}\times\mathbb{R}^2))$ so that

$$G^N_t = \mathcal{L}(\mathcal{X}^N_t), \mathcal{L}(\mu^N_{\mathcal{X}^N_t}) \rightharpoonup Q_t \quad \text{weakly in } \mathbb{P}(\mathbb{P}(\mathbb{R}^2))$$

As a consequence, by Fubini, Γ-lsc property of the Fisher information and Fatou

$$\begin{aligned}
\int_{\mathbb{P}(E)} \int_0^T I(q_t) dt \, Q(dq) &= \int_0^T \int_{\mathbb{P}(E))} I(q_t) \, Q(dq) \, dt \\
&= \int_0^T \mathcal{I}(Q_t) \, dt \\
&\le \int_0^T \liminf_N I(G^N_t) dt \le \liminf_N \int_0^T I(G^N_t) dt.
\end{aligned}$$

This last quantity is finite, so that $\int_0^T I(q_t) dt < \infty$ $Q$-a.s. and $\operatorname{supp} Q \subset \mathcal{S}_1$.

# Plan

We claim that

$$\forall\, q \in \mathcal{S},\ q = \mathcal{L}(\mathcal{X}), \quad w_t := \mathcal{L}(\mathcal{X}_t) = \bar{w}_t := \text{ unique solution of NS vortex.}$$

• First, for $q \in \mathcal{S}$, it is clear that $w_t := \mathcal{L}(\mathcal{X}_t)$ satisfies

$$w \in C([0,T); \mathbb{P}(R^2)), \quad I(w) \in L^1(0,T)$$

and $w$ is a weak solution to (take $\nu = 1$)

$$\partial_t w = \Delta w + (K * w) \cdot \nabla w.$$

• Second, the uniqueness is known (Ben-Artzi 1994, Brézis 1994, improved by Gallagher-Gallay 2005) in the class of function

$$t^{1/4} \, \|w(t,.)\|_{L^{4/3}} \to 0 \,\text{as}\ t \to 0.$$

• We have to prove by a "regularity argument" that $w$ satisfies the Ben-Artzi & Brézis criterium

# Our weak solution are in the Ben-Artzi & Brézis class

- A priori bound and renormalization. We recall the GN inequalities

$$\forall\, g \in \mathbb{P}(\mathbb{R}^2),\ \forall\, p \in [1,\infty) \quad \|g\|_{L^p} \le C_p\, I(g)^{1-1/p}$$
$$\forall\, g \in \mathbb{P}(\mathbb{R}^2),\ \forall\, q \in [1,2) \quad \|\nabla g\|_{L^q} \le C_q\, I(g)^{3/2-1/q}$$

which in turn imply

$$g \in L^6_t(L^{6/5}_x) \text{ and } \nabla g \in L^{6/5}_t(L^{3/2}_x) \quad \text{take } p = 6/5,\ q = 3/2.$$

Together with the Hardy-Littlewood-Sobolev inequality we get

$$\nabla_x(K * g) \in L^{6/5}_t(L^6_x),$$

and then the commutator appearing in the DiPerna-Lions renomalizing theory converges to 0 in $L^1_{loc}$. As a consequence, we may renormalize the equation

$$\partial_t \beta(w) + \beta''(w)|\nabla w|^2 = \Delta\beta(w) + (K * w)\nabla\beta(w).$$

- Renormalization and better bounds. Thanks, to the renormalization equation

$$\partial_t \beta(w) + \beta''(w)|\nabla w|^2 = \Delta \beta(w) + (K * w)\nabla \beta(w)$$

we get (smoothing effect)

$$w \in C((0,T); L^1 \cap L^\infty) \quad \text{and} \quad w \in L^\infty_t(0,T; L\log L \cap L^1_k).$$

- Thanks to Nash inequality we have

$$\frac{d}{dt}\|f\|^2_{L^2} = -\|\nabla f\|^2_{L^2} \leq C\,\|f\|^4_{L^2}$$

and

$$t^{1/2}\,\|f\|_{L^2} \leq C \quad \text{(just like for the heat equation)}$$

- Together with the entropy uniform bound we get

$$t^{1/4}\,\|w(t,.)\|_{L^{4/3}} \to 0 \,\text{as}\; t \to 0.$$

If $q \in \mathcal{S}$ we consider the associated linear SDE

$$\mathcal{X}_t = \mathcal{X}_0 + \int_0^t u_s(\mathcal{X}_s)\, ds + \nu B_t, \qquad u_s = K * \bar{w}_s,$$

## Lemma

*Strong uniqueness for the previous linear SDE holds (and thus weak uniqueness by Yamada-Watanabe theorem). In other words,* $\mathcal{S} = \{\bar{q}\}$.

### Sketch of the proof

- Use argument used by Crippa-De Lellis for uniqueness in ODE with low regularity.
- Two solutions $\mathcal{X}$ and $\mathcal{Y}$ satisfies

$$\forall \delta > 0, \;\; \mathbb{E}\left[\ln\Big(1 + \frac{1}{\delta}\sup_{s\le t}|\mathcal{X}_s - \mathcal{Y}_s|\Big)\right] \le \mathbb{E}\left[\int_0^t [M\nabla u_s(\mathcal{X}_s) + M\nabla u_s(\mathcal{Y}_s)]\, ds\right]$$

  where $M$ stands for maximal function.
- Standard estimates + bounds on Fischer information helps to bound the r.h.s.
- A variant of Chebichev ineq. allows to conclude.

# Plan

## Chaos entropic

From

$$H(F_t^N) + \int_0^t I(F_s^N)\, ds = H(F_0^N)$$

and

$$H(w_t) + \int_0^t I(w_s)\, ds = H(w_0),$$

as well as the $\Gamma$-lsc of $H$ and $I$ we get if

$$H(F_0^N) \to H(w_0),$$

the conclusion

$$\begin{aligned} H(w_t) + \int_0^t I(w_s)\, ds &\le \liminf_{N\to\infty} \left\{ H(F_t^N) + \int_0^t I(F_s^N)\, ds \right\} \\ &\le \limsup_{N\to\infty} \left\{ H(F_t^N) + \int_0^t I(F_s^N)\, ds \right\} \\ &= \limsup_{N\to\infty} H(F_0^N) = H(w_0) \end{aligned}$$

and then

$$H(F_0^N) \to H(w_0) \quad \forall\, t > 0$$

# A word of conclusion:

We use arguments coming from several areas of mathematics:

• "true" probability (non linear martingale problem)

• functional analysis in finite, increasing and infinite dimension (level-3 Fisher information)

• PDE (renormalization argument for a singular parabolic equation and sharp uniqueness result)

Open problems: Is-it possible to adapt the method to other singular models?

• Kac-Landau model (for soft potential)?

• Keller-Segel model?

▷ propagation of chaos for subcritical Keller-Segel model by D. Godinho, C. Quininao