Spectral analysis of semigroups in Banach spaces and applications to PDEs

S. Mischler

(Paris-Dauphine & IUF)

in collaboration with K. Carrapatoso,
M. Gualdani, C. Mouhot and J. Scher

Boltzmann, Vlasov and related equations: last results and open problems
Cartagena, Colombia, June 23-27, 2014

Outline of the talk

Outline of the talk

1. Introduction

2. Spectral theory in an abstract setting
   - Spectral mapping, Weyl, Krein-Rutmann, small perturbation and extension theorems
   - sketch of the proof for the enlargement theorem

3. Increasing the rate of convergence for the Boltzmann equation
   - Increasing the rate of convergence
   - sketch of the proof of the stability result in a large space

4. Parabolic-parabolic Keller-Segel equation in chemotaxis
   - An asymptotic self-similar result
   - By a non standard perturbation argument

Revisit the spectral theory in an abstract setting

Spectral theory for general operator and its semigroup in general (large) Banach space, without regularity ($\neq$ eventually norm continuous), without symmetry ($\neq$ Hilbert space and self-adjoint op) and without (or with) positivity (Banach lattice)

- *Spectral map Theorem* $\hookrightarrow$ $\Sigma(e^{t\Lambda}) \simeq e^{t\Sigma(\Lambda)}$ and $\omega(\Lambda) = s(\Lambda)$
- *Weyl's Theorem* $\hookrightarrow$ (quantified) compact perturbation $\Sigma_{ess}(\mathcal{A}+\mathcal{B}) \simeq \Sigma_{ess}(\mathcal{B})$
- *Small perturbation* $\hookrightarrow$ $\Sigma(\Lambda_\varepsilon) \simeq \Sigma(\Lambda)$ if $\Lambda_\varepsilon \to \Lambda$
- *Krein-Rutmann Theorem* $\hookrightarrow$ $s(\Lambda) = \sup \Re e \Sigma(\Lambda) \in \Sigma_d(\Lambda)$ when $S_\Lambda \geq 0$
- *functional space extension (enlargement and shrinkage)*
  $\hookrightarrow$ $\Sigma(L) \simeq \Sigma(\mathcal{L})$ when $L = \mathcal{L}_{|E}$
  $\hookrightarrow$ tide of spectrum phenomenon

Structure: operator which splits as

$$\Lambda = \mathcal{A} + \mathcal{B}, \quad \mathcal{A} \prec \mathcal{B}, \quad \mathcal{B} \,\text{dissipative}$$

Examples: Boltzmann, Fokker-Planck, Growth-Fragmentation operators and $W^{\sigma,p}(m)$ weighted Sobolev spaces

• (1) Convergence rate in large Banach space for linear dissipative and hypodissipative PDEs (ex: Fokker-Planck, growth-fragmentation)

• (2) Long time asymptotic for nonlinear PDEs via the spectral analysis of linearized PDEs (ex: Boltzmann, Landau, Keller-Segel) in natural φ space

• (3) Existence, uniqueness and stability of equilibrium in "small perturbation regime" in large space (ex: inelastic Boltzmann, Wigner-Fokker-Planck, parabolic-parabolic Keller-Segel, neural network)

Is it new?

• Simple and quantified versions, unified theory (sectorial, KR, general) which holds for the "principal" part of the spectrum

• first enlargement result in an abstract framework by Mouhot (CMP 2006)

• Unusual splitting

$$\Lambda = \underbrace{\mathcal{A}_0}_{compact} + \underbrace{\mathcal{B}_0}_{dissipative} = \underbrace{\mathcal{A}_\varepsilon}_{smooth} + \underbrace{\mathcal{A}^c_\varepsilon + \mathcal{B}_0}_{dissipative}$$

• The applications to these linear(ized) "kinetic" equations and to these nonlinear problems are clearly new

• Fredholm, Hilbert, Weyl, Stone (Funct Analysis & sG Hilbert framework) $\leq$ 1932

• Hyle, Yosida, Phillips, Lumer, Dyson (sG Banach framework & dissipative operators) 1940-1960 and also Dunford, Schwartz

• Kato, Pazy, Voigt (analytic op., positive op.) 1960-1975

• Engel, Nagel, Gearhart, Metz, Diekmann, Prüss, Arendt, Greiner, Blake, Mokhtar-Kharoubi, Yao, ... 1975-

Spectral analysis of the linearized (in)homogeneous Boltzmann equation and convergence to the equilibrium

• Hilbert, Carleman, Grad, Ukai, Arkeryd, Esposito, Pulvirenti, Wennberg, Guo, Mouhot, Strain, ...

Spectral tide/spectral analysis in large space

• Bobylev (for linearized Boltzmann with Maxwell molecules, 1975), Gallay-Wayne (for harmonic Fokker-Planck, 2002)

Still active research field

• Semigroup school (≥ 0, bio): Arendt, Blake, Diekmann, Engel, Gearhart, Greiner, Metz, Mokhtar-Kharoubi, Nagel, Prüss, Webb, Yao, ...

• Schrodinger school / hypocoercivity and fluid mechanic: Batty, Burq, Duyckaerts, Gallay, Helffer, Hérau, Lebeau, Nier, Sjöstrand, Wayne, ...

• Probability school (as in Toulouse): Bakry, Barthe, Bobkov, Cattiaux, Douc, Gozlan, Guillin, Fort, Ledoux, Roberto, Röckner, Wang, ...

• Kinetic school ($\sim$ Boltzmann):

▷ Carlen, Carvalho, Toscani, Otto, Villani, ... (log-Sobolev inequality)

▷ Desvillettes, Villani, Mouhot, Baranger, Neuman, Strain, Dolbeault, Schmeiser, ... (Poincaré inequality & hypocoercivity)

▷ Guo school related to Ukai, Arkeryd, Esposito, Pulvirenti, Wennberg, ... (existence in "small spaces" and "large spaces")

A list of related papers

- Mouhot, *Rate of convergence to equilibrium for the spatially homogeneous Boltzmann equation with hard potentials*, CMP 2006
- M., Mouhot, *Stability, convergence to self-similarity and elastic limit for the Boltzmann equation for inelastic hard spheres*, CMP 2009
- Gualdani, M., Mouhot, *Factorization for non-symmetric operators and exponential H-Theorem*, arXiv 2010
- Arnold, Gamba, Gualdani, M., Mouhot, Sparber, *The Wigner-Fokker-Planck equation: Stationary states and large time behavior*, M3AS 2012
- Cañizo, Caceres, M., *Rate of convergence to the remarkable state for fragmentation and growth-fragmentation equations*, JMPA 2011 & CAIM 2011
- Egaña, M., *Uniqueness and long time asymptotic for the Keller-Segel equation - Part I. The parabolic-elliptic case*, arXiv 2013
- M., Mouhot, *Exponential stability of slowing decaying solutions to the kinetic Fokker-Planck equation*, in progress
- M., Scher, *Spectral analysis of semigroups and growth-fragmentation eqs*, arXiv 2013
- Carrapatoso, *Exponential convergence ... homogeneous Landau equation*, arXiv 2013
- Tristani, *Boltzmann equation for granular media with thermal force in a weakly inhomogeneous setting*, arXiv 2013
- Carrapatoso, M., *Uniqueness and long time asymptotic for the parabolic-parabolic Keller-Segel equation*, put today on arXiv

Outline of the talk

For a given operator Λ in a Banach space X, we want to prove

$$(1) \quad \Sigma(\Lambda) \cap \Delta_a = \{\xi_1\} \text{ (or } = \emptyset), \quad \xi_1 = 0$$

with $\Sigma(\Lambda) = $ spectrum, $\Delta_\alpha := \{z \in \mathbb{C},\ \Re e\, z > \alpha\}$

$$(2) \quad \Pi_{\Lambda,\xi_1} = \text{finite rank projection}, \quad \text{i.e. } \xi_1 \in \Sigma_d(\Lambda)$$

$$(3) \quad \|S_\Lambda(I - \Pi_{\Lambda,\xi_1})\|_{X \to X} \le C_a\, e^{at}, \quad a < \Re e \xi_1$$

Definition:
We say that $L - a$ is hypodissipative iff $\|e^{tL}\|_{X \to X} \le C\, e^{at}$
$s(\Lambda) := \sup \Re e \Sigma(\Lambda)$
$\omega(\Lambda) := \inf\{a \in \mathbb{R}, \text{ s.t. } L - a \text{ is hypodissipative }\}$

Th 1. (M., Scher)
(0) $\Lambda = \mathcal{A} + \mathcal{B}$, where $\mathcal{A}$ is $\mathcal{B}^{\zeta'}$-bounded with $0 \le \zeta' < 1$,
(1) $\|S_\mathcal{B} * (\mathcal{A} S_\mathcal{B})^{(*\ell)}\|_{X \to X} \le C_\ell \, e^{at}$, $\forall\, a > a^*$, $\forall\, \ell \ge 0$,
(2) $\|S_\mathcal{B} * (\mathcal{A} S_\mathcal{B})^{(*n)}\|_{X \to D(\Lambda^\zeta)} \le C_n \, e^{at}$, $\forall\, a > a^*$, with $\zeta > \zeta'$,
(3) $\Sigma(\Lambda) \cap (\Delta_{a^{**}} \backslash \Delta_{a^*}) = \emptyset$, $a^* < a^{**}$,

is equivalent to

(4) there exists a projector Π which commutes with Λ such that $\Lambda_1 := \Lambda_{|X_1} \in \mathcal{B}(X_1)$, $X_1 := R\Pi$, $\Sigma(\Lambda_1) \subset \bar{\Delta}_{a^{**}}$

$$\|S_\Lambda(t)\,(I - \Pi)\|_{X \to X} \le C_a \, e^{at}, \quad \forall\, a > a^*$$

In particular

$$\Sigma(e^{t\Lambda}) \cap \Delta_{e^{at}} = e^{t\Sigma(\Lambda) \cap \Delta_a} \quad \forall\, t \ge 0,\ \ a > a^*$$

and

$$\max(s(\Lambda), a^*) = \max(\omega(\Lambda), a^*)$$

Th 2. (M., Scher)
(0) $\Lambda = \mathcal{A} + \mathcal{B}$, where $\mathcal{A}$ is $\mathcal{B}^{\zeta'}$-bounded with $0 \le \zeta' < 1$,
(1) $\|S_\mathcal{B} * (\mathcal{A} S_\mathcal{B})^{(*\ell)}\|_{X \to X} \le C_\ell \, e^{at}$, $\forall \, a > a^*$, $\forall \, \ell \ge 0$,
(2) $\|S_\mathcal{B} * (\mathcal{A} S_\mathcal{B})^{(*n)}\|_{X \to X_\zeta} \le C_n \, e^{at}$, $\forall \, a > a^*$, with $\zeta > \zeta'$,
(3′) $\int_0^\infty \|(\mathcal{A} S_\mathcal{B})^{(*n+1)}\|_{X \to Y} \, e^{-at} \, dt < \infty$, $\forall \, a > a^*$, with $Y \subset\subset X$,
is equivalent to

(4′) there exist $\xi_1, ..., \xi_J \in \bar\Delta_a$, there exist $\Pi_1, ..., \Pi_J$ some finite rank projectors, there exists $T_j \in \mathcal{B}(R\Pi_j)$ such that $\Lambda \Pi_j = \Pi_j \Lambda = T_j \Pi_j$, $\Sigma(T_j) = \{\xi_j\}$, in particular

$$\Sigma(\Lambda) \cap \bar\Delta_a = \{\xi_1, ..., \xi_J\} \subset \Sigma_d(\Sigma)$$

and there exists a constant C_a such that

$$\|S_\Lambda(t) - \sum_{j=1}^{J} e^{t T_j} \Pi_j\|_{X \to X} \le C_a \, e^{at}, \quad \forall \, a > a^*$$

Th 3. (M. & Mouhot; Tristani)
Assume
(0) $\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ in X_i, $X_{-1} \subset\subset X_0 = X \subset\subset X_1$, $\mathcal{A}_\varepsilon \prec \mathcal{B}_\varepsilon$,
(1) $\|S_{\mathcal{B}_\varepsilon} * (\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{(*\ell)}\|_{X_i \to X_i} \le C_\ell \, e^{at}$, $\forall\, a > a^*$, $\forall\, \ell \ge 0$, $i = 0, \pm 1$,
(2) $\|S_{\mathcal{B}_\varepsilon} * (\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{(*n)}\|_{X_i \to X_{i+1}} \le C_n \, e^{at}$, $\forall\, a > a^*$, $i = 0, -1$,
(3) $X_{i+1} \subset D(\mathcal{B}_{\varepsilon|X_i}), D(\mathcal{A}_{\varepsilon|X_i})$ for $i = -1, 0$ and

$$\|\mathcal{A}_\varepsilon - \mathcal{A}_0\|_{X_i \to X_{i-1}} + \|\mathcal{B}_\varepsilon - \mathcal{B}_0\|_{X_i \to X_{i-1}} \le \eta_1(\varepsilon) \to 0, \quad i = 0, 1,$$

(4) the limit operator satisfies (in both spaces X_0 and X_1)

$$\Sigma(\Lambda_0) \cap \Delta_a = \{\xi_1, ..., \xi_k\} \subset \Sigma_d(\Lambda_0).$$

Then

$$\Sigma(\Lambda_\varepsilon) \cap \Delta_a = \{\xi^\varepsilon_{1,1}, ..., \xi^\varepsilon_{1,d^\varepsilon_1}, ..., \xi^\varepsilon_{k,1}, ..., \xi^\varepsilon_{k,d^\varepsilon_k}\} \subset \Sigma_d(\Lambda_\varepsilon),$$
$$|\xi_j - \xi^\varepsilon_{j,j'}| \le \eta(\varepsilon) \to 0 \quad \forall\, 1 \le j \le k, \ \forall\, 1 \le j' \le d_j;$$
$$\mathrm{dim} R(\Pi_{\Lambda_\varepsilon, \xi^\varepsilon_{j,1}} + ... + \Pi_{\Lambda_\varepsilon, \xi^\varepsilon_{j,d_j}}) = \mathrm{dim} R(\Pi_{\Lambda_0, \xi_j});$$

Th 4. (M. & Scher) Consider a semigroup generator Λ on a "Banach lattice of functions" X,
(1) Λ such as in Weyl's Theorem for some $a^* \in \mathbb{R}$;
(2) $\exists b > a^*$ and $\psi \in D(\Lambda^*) \cap X'_+ \backslash \{0\}$ such that $\Lambda^* \psi \geq b\, \psi$;
(3) S_Λ is positive (and Λ satisfies Kato's inequalities);
(4) $-\Lambda$ satisfies a strong maximum principle.
Defining $\lambda := s(\Lambda)$, there holds

$$a^* < \lambda = \omega(\Lambda) \quad \text{and} \quad \lambda \in \Sigma_d(\Lambda),$$

and there exists $0 < f_\infty \in D(\Lambda)$ and $0 < \phi \in D(\Lambda^*)$ such that

$$\Lambda f_\infty = \lambda\, f_\infty, \quad \Lambda^* \phi = \lambda\, \phi, \quad R\Pi_{\Lambda,\lambda} = \mathrm{Vect}(f_\infty),$$

and then

$$\Pi_{\Lambda,\lambda} f = \langle f, \phi \rangle\, f_\infty \quad \forall\, f \in X.$$

Moreover, there exist $\alpha \in (a^*, \lambda)$ and $C > 0$ such that for any $f_0 \in X$

$$\| S_\Lambda(t) f_0 - e^{\lambda t}\, \Pi_{\Lambda,\lambda} f_0 \|_X \leq C\, e^{\alpha t}\, \| f_0 - \Pi_{\Lambda,\lambda} f_0 \|_X \qquad \forall\, t \geq 0.$$

Change (enlargement and shrinkage) of the functional space of the spectral analysis and semigroup decay

Th 5. (Mouhot 06, Gualdani, M. & Mouhot) Assume

$$\mathcal{L} = \mathcal{A} + \mathcal{B}, \ \ L = A + B, \ \ A = \mathcal{A}_{|E}, \ \ B = \mathcal{B}_{|E}, \ \ E \subset \mathcal{E}$$

(i) $(B - a)$ is hypodissipative on E, $(\mathcal{B} - a)$ is hypodissipative on $\mathcal{E}$;
(ii) $A \in \mathcal{B}(E)$, $\mathcal{A} \in \mathcal{B}(\mathcal{E})$;
(iii) there is $n \ge 1$ and $C_a > 0$ such that

$$\left\| (\mathcal{A} S_{\mathcal{B}})^{(*n)}(t) \right\|_{\mathcal{E} \to E} \le C_a \, e^{at}.$$

Then the following for $(X, \Lambda) = (E, L)$, $(\mathcal{E}, \mathcal{L})$ are equivalent:
$\exists \xi_j \in \Delta_a$ and finite rank projector $\Pi_{j,\Lambda} \in \mathcal{B}(X)$, $1 \le j \le k$, which commute with Λ and satisfy $\Sigma(\Lambda_{|\Pi_{j,\Lambda}}) = \{\xi_j\}$, so that

$$\forall \, t \ge 0, \quad \Big\| S_\Lambda(t) - \sum_{j=1}^{k} S(t) \, \Pi_{j,\Lambda} \Big\|_{X \to X} \le C_{\Lambda,a} \, e^{a\,t}$$

• In Theorems 1, 2, 3, 4, 5 one can take $n = 1$ in the simplest situations (most of space homogeneous equations in dimension $d \leq 3$), but one need to take $n \geq 2$ for the space inhomogeneous Boltzmann equation

• Open problem: Beyond the "dissipative case"?
$\triangleright$ example of the Fokker-Planck equation for "soft confinement potential" and relation with "weak Poincaré inequality" by Röckner-Wang
$\triangleright$ Links with semi-uniform stability by Lebeau & co-authors, Burq, Liu-R, Bátkal-E-P-S, Batty-D, ...
$\triangleright$ applications to the Boltzmann and Landau equations associated with "soft potential"

Proof of the enlargement theorem

We split the semigroup into invariant linear sub-manifolds (eigenspaces)

$$S_{\mathcal{L}} = \Pi\, S_{\mathcal{L}} + (I - \Pi)\, S_{\mathcal{L}}\, (I - \Pi)$$

and write the (iterated) Duhamel formula or "stopped" Dyson-Phillips series (the Dyson-Phillips series corresponds to the choice $n = \infty$)

$$S_{\mathcal{L}} = \sum_{\ell=0}^{n-1} S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{B}})^{(*\ell)} \quad + \quad S_{\mathcal{L}} * (\mathcal{A} S_{\mathcal{B}})^{(*n)}$$

$$\text{or} \quad + \quad (\mathcal{A} S_{\mathcal{B}})^{(*n)} * S_{\mathcal{L}}.$$

These two identities together

$$\begin{aligned} S_{\mathcal{L}} \quad &= \quad \Pi\, S_{\mathcal{L}} + (I - \Pi)\, \{ \sum_{\ell=0}^{n-1} S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{B}})^{(*\ell)} \}\, (I - \Pi) \\ &+ \quad \{(I - \Pi)\, S_{\mathcal{L}}\} * (\mathcal{A} S_{\mathcal{B}})^{(*n)} (I - \Pi) \\ \text{or} \quad &+ \quad (I - \Pi)(\mathcal{A} S_{\mathcal{B}})^{(*n)} * \{S_{\mathcal{L}} (I - \Pi)\} \end{aligned}$$

Outline of the talk

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f,f), \quad f(0) = f_0$$

- $f = f(t,x,v) \geq 0$ time-dependent probability density of particles (in L^1),
- position $x \in \Omega = \mathbb{T} \subset \mathbb{R}^d$ the torus, velocity $v \in \mathbb{R}^d$,
- $v \cdot \nabla_x$ free flow transport term,
- Q collision term, modelling elastic binary collisions

$$\{v\} + \{v_*\} \xrightarrow{B} \{v'\} + \{v'_*\} \quad \text{with} \quad \left\{ \begin{array}{l} v' + v'_* = v + v_* \\ |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2 \end{array} \right.$$

$B = |v - v_*|$ hard spheres collision kernel (dictates outcome velocities v' et v'_*)

One possible parametrisation is $\left\{ \begin{array}{l} v' = \dfrac{v+v_*}{2} + \sigma \dfrac{|v_* - v|}{2} \\[2ex] v'_* = \dfrac{v+v_*}{2} - \sigma \dfrac{|v_* - v|}{2} \end{array} \right\} \quad \sigma \in S^{d-1}$

The Boltzmann kernel

$$Q(f,g) = \int_{\mathbb{R}^d \times S^{d-1}} B\,[f'\,g_* - f\,g_*]\,d\sigma\,dv_*$$

- Collision invariants are mass, momentum and energy:

$$\int_{\mathbb{R}^d} Q(f,f) \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = 0$$

- the "entropy dissipation" has a sign

$$D(f) := -\int_{\mathbb{R}^d} Q(f,f)\,\log f\,dv \geq 0$$

and fulfills

$$(f \in \mathcal{H}_v \text{ and } D(f) = 0) \quad \Leftrightarrow \quad f = M$$

with $M = (2\pi)^{-d/2}\exp(-|v|^2/2)$ is the normalized Maxwellian and

$$\mathcal{H}_v := \left\{ f \in L^1_2(\mathbb{R}^d) \Big/ \int_{\mathbb{R}^d} f \begin{pmatrix} 1 \\ v \\ |v|^2/2 \end{pmatrix} dv = \begin{pmatrix} 1 \\ 0 \\ d \end{pmatrix},\ H(f) := \int_{\mathbb{R}^d} f \log f\,dv < \infty \right\}$$

where we define the weighted Lebesgue spaces by the norms

$$\|f\|_{L^p(m)} := \|f\,m\|_{L^p}, \quad L^p_k = L^p(\langle v\rangle^k), \quad \langle v\rangle^2 = 1 + |v|^2.$$

Main properties of the equation

Conservation of mass, momentum and energy

$$\frac{d}{dt}\int_{\mathbb{T}}\int_{\mathbb{R}^d} f \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = 0$$

Boltzmann's H-Theorem

$$\frac{d}{dt}\int_{\mathbb{T}}\int_{\mathbb{R}^d} f \log f \, dxdx = -\int_{\mathbb{T}} D(f)\, dx \le 0$$

As a consequence, one expects that

$$f_0 \in \mathcal{H}_{v,x} \quad \text{implies} \quad f(t) \to M \text{ as } t \to \infty.$$

Th. DiPerna-Lions, Ann. Math. 1989, Existence & H-Theorem

For any $f_0 \in \mathcal{H}_{v,x}$ there exists a global renormalized solution $f \in C([0,\infty); L^1(\mathbb{T} \times \mathbb{R}^d))$ and

$$f(t) \to M_e \text{ as } t \to \infty$$

in a weak sense and with possible loss of energy $\langle M_e, |v|^2/2\rangle = e \le d$.

▷ We have $M_e = M$ under the (unverified) tightness of the energy distribution.

Conditionally (up to time uniform strong estimate) exponential H-Theorem

Th. Desvillettes, Villani, Invent. Math. 2005

Assuming that for some $s \geq s_0$, $k \geq k_0$

$$(*) \qquad \sup_{t \geq 0} \left(\|f_t\|_{H^s} + \|f_t\|_{L^1(\langle v \rangle^k)} \right) \leq C_{s,k} < \infty.$$

there exist $C_{s,k}, \tau_{s,k} < \infty$ such that

$$\forall\, t \geq 0 \qquad \int_{\mathbb{T} \times \mathbb{R}^d} f_t \log \frac{f_t}{M} \, dvdx \leq C_{s,k} \, (1+t)^{-\tau_{s,k}}$$

Th 6. Gualdani-M.-Mouhot, arXiv 2011

$\exists\, s_1, k_1$: if f_t satisfies $(*)$ then for any $a > \lambda_2$ exists C_a

$$\forall\, t \geq 0 \qquad \int_{\mathbb{T} \times \mathbb{R}^d} f_t \log \frac{f_t}{M} \, dvdx \leq C_a \, e^{\frac{a}{2}\, t},$$

with $\lambda_2 < 0$ (2^{nd} eigenvalue of the linearized Boltzmann eq. in $L^2(M^{-1/2})$).

Global existence, uniqueness and exponential stability for weakly inhomogeneous initial data for the elastic inhomogeneous Boltz eq for hard spheres interactions in the torus

Th 7. Gualdani-M.-Mouhot

For any $F_0 \in L^1_3(\mathbb{R}^d)$ there exists $\varepsilon_0 > 0$ such that if $f_0 \in L^1_3(L^\infty_x)$ satisfies $\|f_0 - F_0\|_{L^1_3(L^\infty_x)} \le \varepsilon_0$ then

- there exists a unique global mild solution f starting from f_0;
- $f(t) \to M$ when $t \to \infty$ (with exponential rate).

• Extend to a larger class of initial data similar results due to Ukai, Arkeryd-Esposito-Pulvirenti, Wennberg, Guo, Strain and collaborators

• It can be adapted to other situations

▷ homogeneous Boltzmann eq for hard spheres (Mouhot 2006)
▷ homogeneous weakly inelastic Boltzmann eq for HS (M-Mouhot 2009)
▷ homogeneous Landau eq for hard potential (Carrapatoso 2013)
▷ inhomogeneous weakly inelastic Boltzmann eq for HS (Tristani 2013)
▷ parabolic-elliptic Keller-Segel eq (Egaña-M 2013)
▷ homogeneous Boltz eq for hard potential without cut-off (Tristani 2014)

• Open problems: inhomogeneous Boltzmann and Landau equation for soft potential, Navier-Stokes limit

We write $f = M + h$. The function h satisfies

$$\partial_t h = \mathcal{L} h + Q(h,h)$$

with

$$\mathcal{L} h := -v \cdot \nabla_x + Q(h,M) + Q(M,h)$$

• $L := \mathcal{L}_{|E_v}$ is self-adjoint and dissipative (Hilbert, Carleman, Grad, Ukai, Baranger-Mouhot) in the small space homogeneous functional space

$$E_v := \left\{ h;\ h \in L^2_v(M^{-1/2});\ \int_{\mathbb{R}^d} h \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dv = 0 \right\}$$

because

$$\begin{aligned} -\langle Lh, h\rangle &:= \frac{d^2}{\varepsilon^2} D(M + \varepsilon h)_{|\varepsilon=0} \\ &= \int_{\mathbb{R}^d\times\mathbb{R}^d\times S^{d-1}} (g' + g'_* - g - g_*)^2 \, B \, M \, M_*, \quad g := h/M \\ &\geq \lambda \|g M^{1/2}\|^2_{L^2} = \lambda \, \|h\|^2_E \end{aligned}$$

• $L := \mathcal{L}_{|E}$ is hypo-dissipative (Ukai, Guo, Mouhot-Strain, ...) in the small space inhomogeneous functional space

$$E := \left\{ h;\ h \in H^1_{xv}(M^{-1/2});\ \int_{\mathbb{T}\times\mathbb{R}^d} h \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} dvdx = 0 \right\}$$

because one can choose $\varepsilon > 0$ small enough such for the Hilbert norm

$$|||h|||^2 := \|h\|^2 + \|\nabla_x h\|^2 + \varepsilon(\nabla_x h, \nabla h) + \varepsilon^{3/2}\|\nabla_v h\|^2$$

where $\|\cdot\| := \|\cdot\|_{L^2(M^{-1/2})}$, there holds

$$-(\!(Lh, h)\!) \geq \lambda_2\, |||h|||^2$$

• In the large space inhomogeneous functional space $\mathcal{E}_k := L^1_k(L^\infty)$ we introduce the splitting

$$\begin{aligned} \mathcal{L}h &= Q(h,M) + Q(M,h) - v\cdot\nabla_x h \\ &= Q^+(h,M) + Q^+(M,h) - L(h)\,M - L(M)h - v\cdot\nabla_x h \\ &= \underbrace{\mathcal{Q}^{+*}_\delta[h]}_{=:\mathcal{A}_\delta h} + \underbrace{\mathcal{Q}^{+*,c}_\delta[h] - L(M)h - v\cdot\nabla_x h}_{=:\mathcal{B}^2_\delta h + \mathcal{B}^1 h = \mathcal{B}h} \end{aligned}$$

▷ $\mathcal{A}_\delta : \mathcal{E}_k \to H^s_{comp}(L^\infty)\ \forall\, s \ge 0$,
▷ $\mathcal{B}^2_\delta : \mathcal{E}_{k+1} \to \mathcal{E}_k$ small with bound $\frac{4}{k+2} + \mathcal{O}(\delta)$
▷ $\mathcal{B}_\delta - a$ is "strongly" dissipative in $\mathcal{E}_k$ for any $a \ge a^* = a^*_{k,\delta}$ with $a^* < 0$ for any $k > 2$, $\delta > 0$ small and $a^* \to -\lambda_0$ when $k \to \infty$, $\delta \to 0$

$$\frac{d}{dt}\|S_\mathcal{B}(t)h\|_{\mathcal{E}_k} \le a\,\|S_\mathcal{B}(t)h\|_{\mathcal{E}_{k+1}}$$

• As a consequence, the following dissipativity estimate hods

$$\forall\,\ell \ge 0,\ \forall\, a > a^* \quad (\mathcal{A}S_\mathcal{B})^{(*\ell)} : \mathcal{E}_k \to \mathcal{E}_k \text{ is } \mathcal{O}(e^{at})$$

$\triangleright$ Iterated averaging lemma

$$(\mathcal{A}S_{\mathcal{B}})^{(*2)} : W^{s,1}_k \to W^{s+1/2,1}_{comp} \quad \forall\, s \geq 0.$$

which in turn implies the

$$\exists\, n \geq 2,\ \forall\, a > a^* \quad (\mathcal{A}S_{\mathcal{B}})^{(*n)} : \mathcal{E}_k \to H^1(M^{-1/2}) \text{ is } \mathcal{O}(e^{at})$$

• Thanks to the "extension theorem" we obtain that $\mathcal{L} - a$ is hypodissipative in

$$\mathcal{E}_{k,0} := \Big\{ h \in \mathcal{E}_k;\ \langle h, (1, v, |v|^2) \rangle = 0 \Big\}.$$

and better, for the equivalent norm

$$|||f|||_k := \eta\, \|h\|_{\mathcal{E}_k} + \int_0^\infty \|e^{s\mathcal{L}} h\|_{\mathcal{E}_k}\, ds,$$

there holds

$$\frac{d}{dt} |||S_{\mathcal{L}}(t)|||_k \leq a |||S_{\mathcal{L}}(t)|||_{k+1}.$$

For the nonlinear term, the following estimate holds

$$\forall\, h \in \mathcal{E}_{k+1} \quad \|Q(h,h)\|_{\mathcal{E}_k} \le C\, \|h\|_{\mathcal{E}_k}\, \|h\|_{\mathcal{E}_{k+1}}$$

All together, the solution to the NL Boltzmann equation

$$\partial_t h = \mathcal{L} h + Q(h,h)$$

satisfies (formally) the differential inequality

$$\frac{d}{dt} |||h|||_k \le a |||h|||_{k+1} + C |||h|||_k |||h|||_{k+1}$$

which provides invariant regions $\{|||h|||_k \le \delta |a|/C\}$, $0 < \delta \le 1$ and then exponential convergence to 0.

▷ Analysis of the linearized equation is a bit long but it is straightforward for the nonlinear equation

Outline of the talk

Parabolic-parabolic Keller-Segel (ppKS) system of equations

$$\begin{cases} \partial_t f = \Delta f - \nabla(f \nabla u) \\ \varepsilon \partial_t u = \Delta u + f - \alpha u \end{cases}$$

- $f = f(t,x) \geq 0$ time-dependent density of cells (in L^1),
- $u = u(t,x) \geq 0$ time-dependent chemo-attractant concentration (in L^2),
- $x \in \mathbb{R}^2$, $t \geq 0$,
- $\varepsilon > 0$, $\alpha \geq 0$ parameters

The (first) equation being in divergence form the mass is (formally) conserved:

$$\int_{\mathbb{R}^2} f(t,x)dx = \int_{\mathbb{R}^2} f_0 dx =: M.$$

• The case $\varepsilon = 0$ corresponds to the parabolic-elliptic Keller-Segel equation for which $M = 8\pi$ is a threshold (Blanchet-Dolbeault-Perthame):

$$\begin{array}{lcl} M \leq 8\pi & \Rightarrow & \text{solutions are global in time} \\ M > 8\pi & \Rightarrow & \text{solutions blows up in finite time} \end{array}$$

About global existence an uniqueness

- The case $\varepsilon > 0$ is more involved since

$$M \leq 8\pi \text{ or } \varepsilon >> M \quad \Rightarrow \quad \text{solutions are global in time}$$

- In the case $M < 8\pi$ global "free energy" solutions are known to exist (Calvez-Corrias) when at initial time $\mathcal{F}(f_0, u_0) < \infty$, with

$$\mathcal{F}(f,u) := \int f \log f + \int f \log\langle x\rangle - \int fu + \frac{1}{2}\int |\nabla u|^2 + \alpha \int u^2.$$

Also the existence of solutions "à la Kato" has been established recently by Mizoguchi, Corrias-Escobedo-Matos, Biler-Guerra-Karch. These solutions are global in time when ε large enough ($\Rightarrow$ small nonlinearity).

Th 8. Carrapatoso-M.

For any (f_0, u_0) such that $\mathcal{F}(f_0, u_0) < \infty$ there exists a unique "free energy" solution on a maximal time interval $(0, T^*)$ with the alternative

$$T_* = +\infty \quad \text{or} \quad (T_* < \infty,\ \mathcal{F}(f(t), u(t)) \to \infty \text{ as } t \to T^*).$$

Improve uniqueness result (in L^∞ framework) by Carrillo-Lisini-Mainini

- We restrict ourself to the case $\alpha = 0$ and $M < 8\pi$.
- We introduce the rescaled functions g and v defined by

$$f(t,x) := \tau^{-2} g(\log \tau, \tau^{-1} x), \quad u(t,x) := v(\log \tau, \tau^{-1} x),$$

with $\tau := (1+t)^{1/2}$. The rescaled ppKS system reads

$$\partial_t g = \Delta g + \nabla (\frac{1}{2}\, x\, g - g\, \nabla v) \quad \text{in} \quad (0,\infty) \times \mathbb{R}^2,$$

$$\varepsilon \partial_t v = \Delta v + g + \frac{\varepsilon}{2}\, x \cdot \nabla v \quad \text{in} \quad (0,\infty) \times \mathbb{R}^2.$$

A stationary solution (G, V) to the rescaled ppKS system is called a self-similar profile and the functions

$$F(t,x) = \frac{1}{\tau^2} G_\varepsilon(\frac{x}{\tau}), \quad U(t,x) = V_\varepsilon(\frac{x}{\tau})$$

is a self-similar solution for the non-rescaled ppKS system.

- It is known (Naito-Suzuki-Yoshida, Biler-Corrias-Dolbeault, Corrias-Escobedo-Matos) that for any $\varepsilon > 0$ and $M \in (0, 8\pi)$ there exists a unique self-similar profile such that the mass of G is equal to M. The functions G and V are radially symmetric and smooth.

Asymptotic behaviour = self-similarity (in the radially symmetric case)

- We want to prove

$$(f,u) \underset{t\to\infty}{\sim} (F,U) \quad \text{or equivalently} \quad (g,v) \underset{t\to\infty}{\to} (G,V)$$

- Difficulty: we do not have uniform in time estimates (except the mass!)
- We assume g, v are radially symmetric and we define

$$|||(g,v)||| := \|g\|_{H^1_k} + \|v\|_{H^2}, \quad k > 7,$$

Th 9. Carrapatoso-M.

$\forall\, M \in (0, 8\pi)\ \exists\, \varepsilon^* > 0$, $\exists\, \delta^* > 0$ such that $\forall\, \varepsilon \in (0, \varepsilon^*)$, $\forall\, (g_0, v_0)$ satisfying

$$|||(g_0,v_0) - (G,V)||| \le \delta^*, \quad \int_{\mathbb{R}^2} g_0\, dx = \int_{\mathbb{R}^2} G\, dx = M,$$

the solution (g,v) to the ppKS system satisfies

$$|||(g(t),v(t)) - (G,V)||| \le C_a\, e^{at} \quad \forall\, a \in (-1/3, \infty),\ \forall\, t \ge 0$$

We want to take advantage of the fact that $(G_\varepsilon, V_\varepsilon) \to (G_0, V_0)$ as $\varepsilon \to 0$, where (G_0, V_0) is the self-similar profile to the parabolic-elliptic KS equation which is known to be linearly exponential stable.

More precisely, the linearized equation of the ppKS system on the variation $f = g - G$, $u = v - V$ writes

$$\begin{aligned} \partial_t f &= Af + Bu \quad = \overbrace{\Delta f + \nabla(\frac{1}{2}\, x\, f - f\, \nabla V_\varepsilon)}^{Af} \overbrace{-\nabla(G_\varepsilon\, \nabla u)}^{Bu} \\ \varepsilon \partial_t u &= f + Cu + \varepsilon Du \quad = f + \overbrace{\Delta u}^{Cu} + \overbrace{\frac{\varepsilon}{2}\, x \cdot \nabla u}^{\varepsilon Du}. \end{aligned}$$

In the limit case $\varepsilon = 0$ the second equation writes as the time independent equation

$$0 = f + \Delta u = f + Cu.$$

In the limit case $\varepsilon = 0$ the system then reduces in a single equation

$$\partial_t f = \Omega f := A_0 f + B_0 (-C)^{-1} f$$

That equation is known (Campos-Dolbeault, Egaña-M.) to be dissipative

$$\| S_\Omega(t) \|_{\mathcal{B}(L^2_{k,0})} \le C\, e^{-t}, \quad L^2_{k,0} = \{ h \in L^2_k, \, \langle h \rangle = 0 \}$$

Matrix form and splitting - (proof 2/5)

We write the system in matrix form

$$\frac{d}{dt}\begin{pmatrix} f \\ u \end{pmatrix} = \mathcal{L}^\varepsilon \begin{pmatrix} f \\ u \end{pmatrix} = \begin{pmatrix} A & B \\ \varepsilon^{-1} & \varepsilon^{-1} C + D \end{pmatrix} \begin{pmatrix} f \\ u \end{pmatrix}$$

with

$$Af := \Delta f + \nabla(\frac{1}{2}\, x\, f - f\, \nabla V_\varepsilon), \quad Bu := -\nabla(G_\varepsilon\, \nabla u)$$
$$Cu := \Delta u, \quad Du := \frac{1}{2}\, x \cdot \nabla u,$$

We split

$$\mathcal{L}^\varepsilon = \mathcal{A} + \mathcal{B}^\varepsilon$$

with

$$\mathcal{A}\begin{pmatrix} f \\ u \end{pmatrix} = \begin{pmatrix} N\chi_R[f] \\ 0 \end{pmatrix}, \quad \chi_R[f] = \chi_R\, f - \chi_1 \langle \chi_R f \rangle,$$

χ_R being the truncation function $\chi_R(x) := \chi(x/R)$, $\chi \in \mathcal{P}(\mathbb{R}^2) \times \mathcal{D}(\mathbb{R}^2)$.

We define

$$X := X_1 \times X_2, \quad X_1 := L^2_{rad} \cap L^2_{k,0}, \quad X_2 := L^2_{rad}$$

$$Y := Y_1 \times Y_2, \quad Y_1 := X_1 \cap H^1_k, \quad Y_2 := X_2 \cap H^1$$

- $\mathcal{A} \in \mathcal{B}(X, X \cap L^2_{k+1})$, $\mathcal{A} \in \mathcal{B}(Y)$
- $\mathcal{B}^\varepsilon - a$ is dissipative for the equivalent norm

$$\|(f,u)\|^2_{X_*} := \|f\|^2_{L^2_k} + \eta \|u - \kappa * f\|^2_{L^2}, \quad \kappa = \text{ Poisson kernel},$$

for any $a \in (-1/2, 0)$ by choosing η, ε small and R, N large. We then deduce

$$\|S_{\mathcal{B}^\varepsilon}(t)\|_{\mathcal{B}(X)} \le C\, e^{at}, \quad \|S_{\mathcal{B}^\varepsilon}(t)\|_{\mathcal{B}(Y)} \le C\, e^{at}, \quad \|S_{\mathcal{B}^\varepsilon}(t)\|_{\mathcal{B}(X,Y)} \le C\, t^{-1}\, e^{at}.$$

The Weyl's theorem implies

Proposition

$$\Sigma(\mathcal{L}^\varepsilon) \cap \Delta_a \subset \Sigma_d(\mathcal{L}^\varepsilon) \cap B(0, r^*)$$

For $z \in \mathbb{C}$, we denote

$$\mathcal{L}^\varepsilon(z) = \mathcal{L}^\varepsilon - z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with

$$a = A(z) = A - z, \quad b = B, \quad c := \varepsilon^{-1} I, \quad d := \varepsilon^{-1} C + D(z), \quad D(z) = D - z.$$

If $d = d(z)$ is invertible as well as its Schur's complement

$$s_\varepsilon = s_\varepsilon(z) := a - b d^{-1} c = A(z) - B(C + \varepsilon D(z))^{-1}$$

is invertible, the resolvent of $\mathcal{L}^\varepsilon$ is given by

$$\mathcal{R}_{\mathcal{L}^\varepsilon}(z) = \mathcal{L}^\varepsilon(z)^{-1} = \begin{pmatrix} s_\varepsilon^{-1} & -s_\varepsilon^{-1} b d^{-1} \\ -d^{-1} c s_\varepsilon^{-1} & d^{-1} + d^{-1} c s_\varepsilon^{-1} b d^{-1} \end{pmatrix} =: \begin{pmatrix} \mathcal{R}_{11}^{\mathcal{L}^\varepsilon} & \mathcal{R}_{12}^{\mathcal{L}^\varepsilon} \\ \mathcal{R}_{21}^{\mathcal{L}^\varepsilon} & \mathcal{R}_{22}^{\mathcal{L}^\varepsilon} \end{pmatrix}.$$

Then at least formally, we see that

$$\mathcal{R}_{\mathcal{L}^\varepsilon}(z) \underset{\varepsilon \to 0}{\longrightarrow} \begin{pmatrix} \mathcal{R}_\Omega(z) & 0 \\ -C^{-1}\mathcal{R}_\Omega(z) & 0 \end{pmatrix} =: U(z),$$

with $U \in \mathcal{H}(\Delta_{-1}; \mathcal{B}(X))$.

We are not able to prove the above convergence but

Proposition

$\forall \rho > 0 \quad \mathcal{R}_{\mathcal{L}^\varepsilon}(z) \in \mathcal{H}(\Delta_{-1/3} \cap B(0,\rho); \mathcal{B}(X)) \quad$ for ε small enough

For the hardest term, we have

$$\mathcal{R}_{11}^{\mathcal{L}^\varepsilon}(z) = s_\varepsilon^{-1}$$

with

$$s_\varepsilon = \Omega(z) + r_\varepsilon, \quad \|r_\varepsilon\|_{Y_1 \to X_1} \to 0$$

We may apply the perturbation argument and we get

$$s_\varepsilon^{-1} \in \mathcal{H}(\Delta_{-1/3} \cap B(0,\rho); \mathcal{B}(X_1)).$$

- The two propositions and the spectral mapping theorem imply

$$\|S_{\mathcal{L}^\varepsilon}(t)\|_{\mathcal{B}(X)} \le C\, e^{at}, \quad \forall\, a \in (-1/3, 0).$$