Spectral analysis of semigroups in Banach spaces and applications to PDEs

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works by K. Carrapatoso, I. Tristani

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Outline of the talk

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Revisit the spectral theory in an abstract setting

Spectral theory for general operator and its semigroup in general (large) Banach space, without regularity ($\neq$ eventually norm continuous), without symmetry ($\neq$ Hilbert space and self-adjoint op) and without (or with) positivity (Banach lattice)

- *Spectral map Theorem* $\hookrightarrow$ $\Sigma(e^{t\Lambda}) \simeq e^{t\Sigma(\Lambda)}$ and $\omega(\Lambda) = s(\Lambda)$
- *Weyl's Theorem* $\hookrightarrow$ (quantified) compact perturbation $\Sigma_{ess}(\mathcal{A}+\mathcal{B}) \simeq \Sigma_{ess}(\mathcal{B})$
- *Small perturbation* $\hookrightarrow$ $\Sigma(\Lambda_\varepsilon) \simeq \Sigma(\Lambda)$ if $\Lambda_\varepsilon \to \Lambda$
- *Krein-Rutmann Theorem* $\hookrightarrow$ $s(\Lambda) = \sup \Re e \Sigma(\Lambda) \in \Sigma_d(\Lambda)$ when $S_\Lambda \geq 0$
- *functional space extension (enlargement and shrinkage)*
  $\hookrightarrow$ $\Sigma(L) \simeq \Sigma(\mathcal{L})$ when $L = \mathcal{L}_{|E}$
  $\hookrightarrow$ tide of spectrum phenomenon

Structure: operator which splits as

$$\Lambda = \mathcal{A} + \mathcal{B}, \quad \mathcal{A} \prec \mathcal{B}, \quad \mathcal{B}\,\text{dissipative}$$

Examples: Boltzmann, Fokker-Planck, Growth-Fragmentation operators and $W^{\sigma,p}(m)$ weighted Sobolev spaces

• (1) Convergence rate in large Banach space for linear dissipative and hypodisipative PDEs (ex: Fokker-Planck, growth-fragmentation)

• (2) Long time asymptotic for nonlinear PDEs via the spectral analysis of linearized PDEs (ex: Boltzmann, Landau, Keller-Segel) in natural φ space

• (3) Existence, uniqueness and stability of equilibrium in "small perturbation regime" in large space (ex: inelastic Boltzmann, Wigner-Fokker-Planck, parabolic-parabolic Keller-Segel, neural network)

Is it new?

• Simple and quantified versions, unified theory (sectorial, KR, general) which holds for the "principal" part of the specrtrum

• first enlargement result in an abstract framework by C. Mouhot (CMP06)

• Unusual splitting

$$\Lambda = \underbrace{\mathcal{A}_0}_{compact} + \underbrace{\mathcal{B}_0}_{dissipative} = \underbrace{\mathcal{A}_\varepsilon}_{smooth} + \underbrace{\mathcal{A}^c_\varepsilon + \mathcal{B}_0}_{dissipative}$$

• The applications to these nonlinear problems are clearly new

• Fredholm, Hilbert, Weyl, Stone (Funct Analysis & sG Hilbert framewrok) $\leq$ 1932

• Hyle, Yosida, Phillips, Lumer, Dyson (sG Banach framework & dissipative operators) 1940-1960 and also Dunford, Schwartz

• Kato, Pazy, Voigt (analytic op., positive op.) 1960-1975

• Engel, Nagel, Gearhart, Metz, Diekmann, Prüss, Arendt, Greiner, Blake, Mokhtar-Kharoubi, Yao, ... 1975-

Spectral analysis of the linearized (in)homogeneous Boltzmann equation and convergence to the equilibrium

• Hilbert, Carleman, Grad, Ukai, Arkeryd, Esposito, Pulvirenti, Wennberg, Guo, Strain, ...

Spectral tide/spectral analysis in large space

• Bobylev (for Boltzmann), Gallay-Wayne (for harmonic Fokker-Planck)

• Semigroup school (≥ 0, bio): Arendt, Blake, Diekmann, Engel, Gearhart, Greiner, Metz, Mokhtar-Kharoubi, Nagel, Prüss, Webb, Yao, ...

• Schrodinger school / hypocoercivity and fluid mechanic: Batty, Burq, Duyckaerts, Gallay, Helffer, Hérau, Lebeau, Nier, Sjöstrand, Wayne, ...

• Probability school (as in Toulouse): Bakry, Barthe, Bobkov, Cattiaux, Douc, Gozlan, Guillin, Fort, Ledoux, Roberto, Röckner, Wang, ...

• Kinetic school ($\sim$ Boltzmann):

▷ Carlen, Carvalho, Toscani, Otto, Villani, ... (log-Sobolev inequality)

▷ Desvillettes, Villani, Mouhot, Baranger, Neuman, Strain, Dolbeault, Schmeiser, ... (Poincaré inequality & hypocoercivity)

▷ Guo school related to Ukai, Arkeryd, Esposito, Pulvirenti, Wennberg, ... (existence in "small spaces" and "large spaces")

A list of related papers

- M., Mouhot, *Stability, convergence to self-similarity and elastic limit for the Boltzmann equation for inelastic hard spheres*, CMP 2009
- Gualdani, M., Mouhot, *Factorization for non-symmetric operators and exponential H-Theorem*, ArXiv 2010
- Arnold, Gamba, Gualdani, M., Mouhot, Sparber, *The Wigner-Fokker-Planck equation: Stationary states and large time behavior*, M3AS 2012
- Cañizo, Caceres, M., *Rate of convergence to the remarkable state for fragmentation and growth-fragmentation equations*, JMPA 2011 & CAIM 2011
- Egaña, M. *Uniqueness and long time asymptotic for the Keller-Segel equation - Part I. The parabolic-elliptic case*, arXiv 2013
- M., Mouhot *Semigroup factorisation in Banach spaces and kinetic hypoelliptic equations*, in progress
- M., Scher *Spectral analysis of semigroups and growth-fragmentation eqs*, arXiv 2013
- Carrapatoso, *Exponential convergence ... homogeneous Landau equation*, arXiv 2013
- Tristani, *Boltzmann equation for granular media with thermal force in a weakly inhomogeneous setting*, arXiv 2013

Outline of the talk

Examples of operators - I

1) - Linear Boltzmann, e.g. $k(v,v_*) = \sigma(v,v_*)\, M(v_*)$, $\sigma(v_*,v) = \sigma(v,v_*)$,

$$\Lambda f = \underbrace{\int k(v,v_*)\, f(v_*)\, dv_*}_{=:\mathcal{A}f} \underbrace{- \int k(v_*,v)\, dv_*\, f(v)}_{=:\mathcal{B}f}$$

2) - Fokker-Planck, with $E(v) \approx v\,|v|^{\gamma-2}$, $\gamma \ge 1$,

$$\Lambda = \underbrace{\Delta_v + \mathrm{div}_v(E(v)\,\cdot) - M\,\chi_R}_{=:\mathcal{B}} + \underbrace{M\,\chi_R(v)}_{=:\mathcal{A}}$$

3) - Inhomogeneous/kinetic Fokker-Planck

$$\Lambda = \underbrace{\mathcal{T} + \mathcal{C} - M\,\chi_R}_{=:\mathcal{B}} + \underbrace{M\,\chi_R(x,v)}_{=:\mathcal{A}}$$

with

$$\mathcal{T} := -v\cdot\nabla_x + F\cdot\nabla_x, \quad \mathcal{C}f := \Delta_v f + \mathrm{div}_v(E(v)\,f)$$

4) - Growth fragmentation

$$\Lambda = \mathcal{F}^+ - \mathcal{F}^- + \mathcal{D} = \underbrace{\mathcal{F}^+_\delta}_{=:\mathcal{A}} + \underbrace{\mathcal{F}^{+,c}_\delta - \mathcal{F}^- + \mathcal{D}}_{=:\mathcal{B}}$$

with

$$\mathcal{D} f = -\tau(x)\partial_x f - \nu f, \quad (\tau,\nu) = (1,0) \text{ or } (x,2)$$

$$\mathcal{F}^+(f) := \int_x^\infty k(y,x)\, f(y)\, dy, \quad \mathcal{F}^- f := K(x)\, f$$

Mass conservation of $\mathcal{F}^+ - \mathcal{F}^-$ implies

$$K(x) = \int_0^x \frac{y}{x}\, k(x,y)\, dy$$

Self-similarity in y/x

$$k(x,y) = K(x)\, x^{-1}\, \theta(y/x), \quad \int_0^1 z\theta(z)\, dz = 1,$$

with

$$\theta \in \mathcal{D}(0,1) \quad \text{or} \quad \theta(z) = 2\,\delta_{z=1/2} \quad \text{or} \quad \theta(z) = \delta_{z=0} + \delta_{z=1}$$

5) - Linearized Boltzmann

$$\begin{aligned} \Lambda h &= Q(h,M) + Q(M,h) \\ &= Q^+(h,M) + Q^+(M,h) - L(h)\,M - L(M)h \\ &= \underbrace{\mathcal{Q}_\delta^{+,*}[h]}_{=:\mathcal{A}h} + \underbrace{\mathcal{Q}_\delta^{+,*,c}[h] - L(M)h}_{=:\mathcal{B}h} \end{aligned}$$

6) - Inhomogeneous linearized Boltzmann (in the torus)

$$\Lambda h = \underbrace{\mathcal{Q}_\delta^{+,*}[h]}_{=:\mathcal{A}h} + \underbrace{\mathcal{Q}_\delta^{+,*,c}[h] - L(M)h + \mathcal{T}h}_{=:\mathcal{B}h}, \quad \mathcal{T} := -v\cdot\nabla_x$$

7) - other operators: homogeneous/inhomogeneous linearized inelastic Boltzmann, homogeneous linearized Landau, Fokker-Planck with fractional diffusion, linearized Keller-Segel (parabolic-elliptic), homogeneous Boltzmann for hard potential without angular cut-off

Operators and their decomposition

General rule 1 for FP/Boltzmann type operator

$$L := \textit{order } \leq 1 + \textit{order } 2$$

$$L := \textit{compact} + \textit{explicit}$$

$$\mathcal{L} := \underbrace{\textit{smooth}/\textit{order} \leq 1}_{\mathcal{A}} + \underbrace{\textit{small} + \textit{explicit}/\textit{order } 2}_{\mathcal{B}}$$

General rule 2 for non space homogeneous operator

$$\mathcal{L}_v := \mathcal{A}_v + \mathcal{B}_v$$

$$\mathcal{L}_{x,v} := \underbrace{\mathcal{A}_v}_{\mathcal{A}_{x,v}} + \underbrace{\mathcal{B}_v + T_x}_{\mathcal{B}_{x,v}}, \qquad T_x = -v \cdot \nabla_x + \nabla_x \Psi \cdot \nabla_v$$

Th 1. (M., Scher)

Assume that for $\gamma \geq 0$, $x_0 \geq 0$, $0 < K_0 \leq K_1 < \infty$:

$$K_0\, x^\gamma\, \mathbf{1}_{x \geq x_0} \leq K(x) \leq K_1\, x^\gamma.$$

There exists a (unique) (λ, f_∞) with $\lambda \in \mathbb{R}$ and f_∞ is the unique solution to

$$\mathcal{F} f_\infty + \mathcal{D} f_\infty = \lambda\, f_\infty, \quad f_\infty \geq 0, \quad \langle f_\infty, 1 \rangle = 1.$$

There exists $a < \lambda$, $C > 0$ such that $\forall\, f_0 \in L^1_\alpha$, $\alpha > 1$

$$\| fe^{\Lambda t}\, f_0 - e^{\lambda t}\, \Pi_0 f_0 \|_{L^1_\alpha} \leq C\, e^{at}\, \| f_0 - e^{\lambda t}\, \Pi_0 f_0 \|_{L^1_\alpha},$$

where Π_0 is the projector on the eigenspace $\mathrm{Vect}(f_\infty)$.

Improve and unify : Metz-Diekmann (1983), Escobedo-M-Rodriguez (2005), Michel-M-Perthame (2005), Perthame-Ryzhik (2005), Laurençot-Perthame (2009), Caceres-Cañizo-M (2010) &t (2011)

Consider

$$\partial_t f = \Lambda f = \Delta_v f + \mathrm{div}_v(F\, f)$$

with a (friction) force field F such that

$$F \cdot x \geq |x|^\gamma, \quad \mathrm{div} F \leq C_F\, |x|^{\gamma-2}, \quad \forall\, x \in B_R^c$$

There exists then a (unique) function $f_\infty \in \mathbf{P}(\mathbb{R}^d)$ which is a stationary solution, with $f_\infty = \exp(-\Phi + \Phi_0)$ when $F = \nabla\Phi$, $\Phi(v) = \frac{1}{\gamma}\, \langle v \rangle^\gamma$.

For an integrability exponent $p \in [1, \infty]$, a regularity exponent $\sigma \in \{-1, 0, 1\}$ and a polynomial weight

$$m = \langle v \rangle^k, \quad k > k^*(p, \sigma, \gamma), \quad \text{if } \gamma \geq 2,$$

or an exponential weight

$$m = e^{\kappa \langle v \rangle^s}, \quad s \in [2-\gamma, \gamma],\ \gamma \geq 1,\ \kappa < 1/\gamma \, \text{if}\, s = \gamma,$$

we define the abscissa

$$a_\sigma(p, m) = \text{ finite } < 0 \text{ in the limit cases, } = -\infty \text{ otherwise}$$

Th 2. Gualdani-M.-Mouhot; M.-Mouhot; Ndao

There exists

$$0 > a \ge a_\sigma(p,m)$$

there exists $C = C(a,p,\sigma,\gamma,m)$ such that for any $f_0 \in W^{\sigma,p}(m)$

$$\|e^{t\Lambda} f_0 - \langle f_0\rangle\, f_\infty\|_{W^{\sigma,p}(m)} \le C\, e^{at}\, \|f_0 - \langle f_0\rangle\, f_\infty\|_{W^{\sigma,p}(m)}.$$

If moreover, $\gamma \in [2, 2+1/(d-1)]$,

$$W_1(e^{t\Lambda} f_0), \langle f_0\rangle\, f_\infty) \le C\, e^{at}\, W_1(f_0, \langle f_0\rangle\, f_\infty)$$

• Generalize similar result known in $L^2(f_\infty^{-1/2})$

• The same result holds for the kinetic Fokker-Planck in the torus and may be extended to the kinetic Fokker-Planck in $\mathbb{R}^d_x$ with confinement potential

• In the case $E(v) = \nabla\Phi$ one can take $-a = \lambda :=$ the best constant in the Poincaré inequality if m is "increasing enough"

• A rate of decay in Wasserstein distance W_2 have been obtained by Bolley-Gentil-Guillin (2012)

Outline of the talk

- $(f_t)_{t\geq 0}$ solution to the inhomogeneous Boltzmann equation for hard spheres interactions in the torus with strong estimate

$$\sup_{t\geq 0}\left(\|f_t\|_{H^k}+\|f_t\|_{L^1(1+|v|^s)}\right)\leq C_{s,k}<\infty.$$

- Desvillettes, Villani proved [Invent. Math. 2005]: for any $s\geq s_0$, $k\geq k_0$

$$\forall\, t\geq 0 \qquad \int_{\mathbb{T}\times\mathbb{R}^d} f_t \log\frac{f_t}{G_1(v)}\, dvdx \leq C_{s,k}\,(1+t)^{-\tau_{s,k}}$$

with $C_{s,k}<\infty$, $\tau_{s,k}\to\infty$ when $s,k\to\infty$, $G_1 :=$ Maxwell function

Th 3. Gualdani-M.-Mouhot

$\exists\, s_1, k_1$ s.t. for any $a>\lambda_2$ exists C_a

$$\forall\, t\geq 0 \qquad \int_{\mathbb{T}\times\mathbb{R}^d} f_t \log\frac{f_t}{G_1(v)}\, dvdx \leq C_a\, e^{\frac{a}{2}\, t},$$

with $\lambda_2<0$ (2^{nd} eigenvalue of the linearized Boltzmann eq. in $L^2(G_1^{-1})$).

Global existence and uniqueness for weakly inhomogeneous initial data for the elastic and inelastic inhomogeneous Boltzmann equation for hard spheres interactions in the torus

Th 4. Gualdani-M.-Mouhot; Tristani

For any $F_0 \in L^1_3(\mathbb{R}^d)$ there exists $e_0 \in (0,1)$ and $\varepsilon_0 > 0$ such that if $f_0 \in W^{k,1}_x(\mathbb{T}^d; L^1_3(\mathbb{R}^d))$ satisfies $\|f_0 - F_0\| \le \varepsilon_0$ and if $e \in [e_0, 1]$ then

- there exists a unique global mild solution $f(t,x,v)$ starting from f_0;
- $f(t) \to G_1$ when $t \to \infty$ (with rate) when $e = 1$;
- $f(t) \to \bar{G}_e$ when $t \to \infty$ (with rate) when $e < 1$ (diffuse forcing).

• The case $e \sim 1$ is proved thanks to a small perturbation argument in a large space because $\bar{G}_e(v) \ge e^{-|v|^{3/2}} \notin L^2(G_1^{-1/2})$.

• The case $e = 1$ has been treated by non constructive arguments by Arkeryd-Esposito-Pulvirenti (CMP 1987), Wennberg (Nonlinear Anal. 1993) and for the space homogeneous analogous by Arkeryd (ARMA 1988), Wennberg (Adv. MAS 1992)

• Extend to a larger class of initial data similar results due to Ukai, Guo, Strain and collaborators

For

- homogeneous Boltzmann eq for hard spheres (Mouhot 2006)
- homogeneous weakly inelastic Boltzmann eq for hard spheres (M-Mouhot 2009)
- homogeneous Landau eq for hard potential (Carrapatoso 2013)
- parabolic-elliptic Keller-Segel eq (Egaña-M 2013)
- homogeneous Boltzmann eq for hard potential (Tristani, soon on arXiv)

In all these cases, we prove that under minimal assumptions on the initial datum f_0 (bounded mass, energy, entropy, ...) the associated solution $f(t)$ satisfies

$$f(t) \to G \text{ when } t \to \infty \text{ (with exponential rate)}$$

where G is the unique associated equilibrium/self-similar profile

We know (except for the inelastic Boltzmann eq) that the associated linearized operator $\mathcal{L}$ is self-adjoint and has a spectral gap in the very small space $L^2(G_1^{-1/2})$ in which a general solution does not belong (even for large time).
▷ we start by "enlarge" the space in which $\mathcal{L}$ has a spectral gap and then we (classically) prove a nonlinear stability result
▷ for the weakly inelastic Boltzmann eq we additionally use perturbation argument

Outline of the talk

For a given operator Λ in a Banach space X, we want to prove

$$\Sigma(\Lambda) \cap \Delta_a = \{\xi_1\}, \quad \xi_1 = 0$$

with $\Sigma(\Lambda)$ = spectrum, $\Delta_\alpha := \{z \in \mathbb{C},\ \Re e\, z > \alpha\}$

$$\Pi_{\Lambda,\xi_1} = \text{finite rank projection}, \quad \text{i.e. } \xi_1 \in \Sigma_d(\Lambda)$$

$$\|S_\Lambda(I - \Pi_{\Lambda,\xi_1})\|_{X \to X} \le C_a\, e^{at}, \quad a < \Re e \xi_1$$

Th 1. (M., Scher)
(0) $\Lambda = \mathcal{A} + \mathcal{B}$, where $\mathcal{A}$ is $\mathcal{B}^{\zeta'}$ with $0 \le \zeta' < 1$,
(1) $\|S_\mathcal{B} * (\mathcal{A} S_\mathcal{B})^{(*\ell)}\|_{X \to X} \le C_\ell \, e^{at}$, $\forall\, a > a^*$, $\forall\, \ell \ge 0$,
(2) $\|S_\mathcal{B} * (\mathcal{A} S_\mathcal{B})^{(*n)}\|_{X \to D(\Lambda^\zeta)} \le C_n \, e^{at}$, $\forall\, a > a^*$, with $\zeta > \zeta'$,
(3) $\Sigma(\Lambda) \cap (\Delta_{a^{**}} \backslash \Delta_{a^*}) = \emptyset$, $a^* < a^{**}$,

is equivalent to

(4) there exists a projector Π which commutes with Λ such that $\Lambda_1 := \Lambda_{|X_1} \in \mathcal{B}(X_1)$, $X_1 := R\Pi$, $\Sigma(\Lambda_1) \subset \Delta_{a^*}$

$$\|S_\Lambda(t)\,(I - \Pi)\|_{X \to X} \le C_a \, e^{at}, \quad \forall\, a > a^*$$

In particular

$$\Sigma(e^{t\Lambda}) \cap \Delta_{e^{at}} = e^{t\Sigma(\Lambda) \cap \Delta_a} \quad \forall\, t \ge 0, \;\; a > a^*$$

and

$$\max(s(\Lambda), a^*) = \max(\omega(\Lambda), a^*)$$

Th 2. (M., Scher)
(0) $\Lambda = \mathcal{A} + \mathcal{B}$, where $\mathcal{A}$ is $\mathcal{B}^{\zeta'}$ with $0 \le \zeta' < 1$,
(1) $\|S_\mathcal{B} * (\mathcal{A} S_\mathcal{B})^{(*\ell)}\|_{X \to X} \le C_\ell \, e^{at}$, $\forall\, a > a^*$, $\forall\, \ell \ge 0$,
(2) $\|S_\mathcal{B} * (\mathcal{A} S_\mathcal{B})^{(*n)}\|_{X \to X_\zeta} \le C_n \, e^{at}$, $\forall\, a > a^*$, with $\zeta > \zeta'$,
(3) $\int_0^\infty \|(\mathcal{A} S_\mathcal{B})^{(*n+1)}\|_{X \to Y} \, e^{-at} \, dt < \infty$, $\forall\, a > a^*$, with $Y \subset\subset X$,
is equivalent to

(4) there exist $\xi_1, ..., \xi_J \in \bar\Delta_a$, there exist $\Pi_1, ..., \Pi_J$ some finite rank projectors, there exists $T_j \in \mathcal{B}(R\Pi_j)$ such that $\Lambda \Pi_j = \Pi_j \Lambda = T_j \Pi_j$, $\Sigma(T_j) = \{\xi_j\}$, in particular

$$\Sigma(\Lambda) \cap \bar\Delta_a = \{\xi_1, ..., \xi_J\} \subset \Sigma_d(\Sigma)$$

and there exists a constant C_a such that

$$\|S_\Lambda(t) - \sum_{j=1}^{J} e^{t T_j} \Pi_j\|_{X \to X} \le C_a \, e^{at}, \quad \forall\, a > a^*$$

Th 3. (M. & Mouhot; Tristani)
Assume
(0) $\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ in X_i, $X_{-1} \subset\subset X_0 = X \subset\subset X_1$, $\mathcal{A}_\varepsilon \prec \mathcal{B}_\varepsilon$,
(1) $\|S_{\mathcal{B}_\varepsilon} * (\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{(*\ell)}\|_{X_i \to X_i} \le C_\ell \, e^{at}$, $\forall\, a > a^*$, $\forall\, \ell \ge 0$, $i = 0, \pm 1$,
(2) $\|S_{\mathcal{B}_\varepsilon} * (\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{(*n)}\|_{X_i \to X_{i+1}} \le C_n \, e^{at}$, $\forall\, a > a^*$, $i = 0, -1$,
(3) $X_{i+1} \subset D(\mathcal{B}_{\varepsilon|X_i}), D(\mathcal{A}_{\varepsilon|X_i})$ for $i = -1, 0$ and

$$\|\mathcal{A}_\varepsilon - \mathcal{A}_0\|_{X_i \to X_{i-1}} + \|\mathcal{B}_\varepsilon - \mathcal{B}_0\|_{X_i \to X_{i-1}} \le \eta_1(\varepsilon) \to 0, \quad i = 0, 1,$$

(4) the limit operator satisfies (in both spaces X_0 and X_1)

$$\Sigma(\Lambda_0) \cap \Delta_a = \{\xi_1, ..., \xi_k\} \subset \Sigma_d(\Lambda_0).$$

Then

$$\Sigma(\Lambda_\varepsilon) \cap \Delta_a = \{\xi^\varepsilon_{1,1}, ..., \xi^\varepsilon_{1,d_1^\varepsilon}, ..., \xi^\varepsilon_{k,1}, ..., \xi^\varepsilon_{k,d_k^\varepsilon}\} \subset \Sigma_d(\Lambda_\varepsilon),$$
$$|\xi_j - \xi^\varepsilon_{j,j'}| \le \eta(\varepsilon) \to 0 \quad \forall\, 1 \le j \le k, \ \forall\, 1 \le j' \le d_j;$$
$$\mathrm{dim} R(\Pi_{\Lambda_\varepsilon, \xi^\varepsilon_{j,1}} + ... + \Pi_{\Lambda_\varepsilon, \xi^\varepsilon_{j,d_j}}) = \mathrm{dim} R(\Pi_{\Lambda_0, \xi_j});$$

Th 4. (M. & Scher) Consider a semigroup generator Λ on a "Banach lattice of functions" X,
(1) Λ such as in Weyl's Theorem holds for some $a^* \in \mathbb{R}$;
(2) $\exists b > a^*$ and $\psi \in D(\Lambda^*) \cap X'_+ \backslash \{0\}$ such that $\Lambda^* \psi \ge b\, \psi$;
(3) S_Λ is positive (and Λ satisfies Kato's inequalities);
(4) $-\Lambda$ satisfies a strong maximum principle.
Defining $\lambda := s(\Lambda)$, there holds

$$a^* < \lambda = \omega(\Lambda) \quad \text{and} \quad \lambda \in \Sigma_d(\Lambda),$$

and there exists $0 < f_\infty \in D(\Lambda)$ and $0 < \phi \in D(\Lambda^*)$ such that

$$\Lambda f_\infty = \lambda\, f_\infty, \quad \Lambda^* \phi = \lambda\, \phi, \quad R\Pi_{\Lambda,\lambda} = \text{Vect}(f_\infty),$$

and then

$$\Pi_{\Lambda,\lambda} f = \langle f, \phi \rangle\, f_\infty \quad \forall\, f \in X.$$

Moreover, there exist $\alpha \in (a^*, \lambda)$ and $C > 0$ such that for any $f_0 \in X$

$$\| S_\Lambda(t) f_0 - e^{\lambda t}\, \Pi_{\Lambda,\lambda} f_0 \|_X \le C\, e^{\alpha t}\, \| f_0 - \Pi_{\Lambda,\lambda} f_0 \|_X \qquad \forall\, t \ge 0.$$

Change (enlargement and shrinkage) of the functional space of the spectral analysis and semigroup decay

Th 5. (M. & Mouhot) Assume

$$\mathcal{L} = \mathcal{A} + \mathcal{B}, \ \ L = A + B, \ \ A = \mathcal{A}_{|E}, \ \ B = \mathcal{B}_{|E}, \ \ E \subset \mathcal{E}$$

(i) $(B - a)$ is hypodissipative on E, $(\mathcal{B} - a)$ is hypodissipative on $\mathcal{E}$;
(ii) $A \in \mathcal{B}(E)$, $\mathcal{A} \in \mathcal{B}(\mathcal{E})$;
(iii) there is $n \geq 1$ and $C_a > 0$ such that

$$\left\| (\mathcal{A} S_{\mathcal{B}})^{(*n)}(t) \right\|_{\mathcal{E} \to E} \leq C_a \, e^{at}.$$

Then the following for $(X, \Lambda) = (E, L)$, $(\mathcal{E}, \mathcal{L})$ are equivalent:
$\exists \xi_j \in \Delta_a$ and finite rank projector $\Pi_{j,\Lambda} \in \mathcal{B}(X)$, $1 \leq j \leq k$, which commute with Λ and satisfy $\Sigma(\Lambda_{|\Pi_{j,\Lambda}}) = \{\xi_j\}$, so that

$$\forall \, t \geq 0, \quad \left\| S_\Lambda(t) - \sum_{j=1}^{k} S(t) \, \Pi_{j,\Lambda} \right\|_{X \to X} \leq C_{\Lambda,a} \, e^{a\,t}$$

• In Theorem 1, 2, 3, 4, one can take $n = 1$ in the simplest situations (most of space homogeneous equations), but one need to take $n = 2$ for the equal mitosis equation or for the space inhomogeneous Boltzmann equation

• In Theorem 5, one need to take $n > d/4$ for the space homogeneous Fokker-Planck equation in order to extend the spectral analysis from L^2 (well-known) to L^1

• Beyond the "dissipative case"?

▷ example of the Fokker-Planck equation when $\gamma \in (0, 1)$ and relation with "weak Poincaré inequality" by Röckner-Wang

▷ Links with semi-uniform stability by Lebeau & co-authors, Burq, Liu-R, Bátkal-E-P-S, Batty-D, ...

▷ applications to Boltzmann and Landau equation associated to "soft potential"

• inhomogeneous linearized Landau, linearized Keller-Segel (parabolic-parabolic), neural network, Fokker-Planck in the subcritical case $\gamma \in (0, 1)$

Outline of the talk

We split the semigroup into invariant linear sub-manifolds (eigenspaces)

$$S_{\mathcal{L}} = \Pi\, S_{\mathcal{L}} + (I - \Pi)\, S_{\mathcal{L}}\, (I - \Pi)$$

and write the (iterated) Duhamel formula or "stopped" Dyson-Phillips series (the Dyson-Phillips series corresponds to the choice $n = \infty$)

$$S_{\mathcal{L}} = \sum_{\ell=0}^{n-1} S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{B}})^{(*\ell)} \quad + \quad S_{\mathcal{L}} * (\mathcal{A} S_{\mathcal{B}})^{(*n)}$$

$$\text{or} \quad + \quad (\mathcal{A} S_{\mathcal{B}})^{(*n)} * S_{\mathcal{L}}.$$

These two identities together

$$\begin{aligned} S_{\mathcal{L}} &= \Pi\, S_{\mathcal{L}} + (I - \Pi)\, \{ \sum_{\ell=0}^{n-1} S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{B}})^{(*\ell)} \}\, (I - \Pi) \\ &+ \{ (I - \Pi)\, S_{\mathcal{L}} \} * (\mathcal{A} S_{\mathcal{B}})^{(*n)} (I - \Pi) \\ \text{or} &+ (I - \Pi)(\mathcal{A} S_{\mathcal{B}})^{(*n)} * \{ S_{\mathcal{L}} (I - \Pi) \} \end{aligned}$$

We introduce the resolvent

$$R_\Lambda(z) = (\Lambda - z)^{-1} = -\int_0^\infty S_\Lambda(t)\, e^{-zt}\, dt.$$

Using the inverse Laplace formula for $b > \omega(\Lambda) \geq s(\Lambda) = \sup \Re e \Sigma(\Lambda)$ and the fact that $\Pi^\perp R_\Lambda(z)$ is analytic in Δ_{a^*}, $\Pi^\perp := I - \Pi$, we get

$$\begin{aligned} S_\Lambda(t)\Pi^\perp &= \frac{i}{2\pi} \int_{b-i\infty}^{b+i\infty} e^{zt}\, \Pi^\perp\, R_\Lambda(z)\, dz \\ &= \lim_{M\to\infty} \frac{i}{2\pi} \int_{a-iM}^{a+iM} e^{zt}\, \Pi^\perp R_\Lambda(z)\, dz \end{aligned}$$

Similarly as for the (iterated) Duhamel formula, we have

$$R_\Lambda = \sum_{\ell=0}^{N-1} (-1)^\ell\, R_{\mathcal{B}} (\mathcal{A} R_{\mathcal{B}})^\ell + (-1)^N\, R_\Lambda (\mathcal{A} R_{\mathcal{B}})^N$$

These two identities together

$$\begin{aligned} S_{\mathcal{L}}(t)\Pi^{\perp} &= \Pi^{\perp}\sum_{\ell=0}^{N-1}(-1)^{\ell}\frac{i}{2\pi}\int_{a-i\infty}^{a+i\infty} e^{zt}\, R_{\mathcal{B}}(z)(\mathcal{A}R_{\mathcal{B}}(z))^{\ell}\, dz \\ &\quad +(-1)^{N}\,\Pi^{\perp}\,\frac{i}{2\pi}\int_{a-i\infty}^{a+i\infty} e^{zt}\, R_{\Lambda}(z)(\mathcal{A}R_{\mathcal{B}}(z))^{N}\, dz \\ &= \sum_{\ell=0}^{N-1}\Pi^{\perp}\, S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)} \\ &\quad +(-1)^{N}\,\frac{i}{2\pi}\int_{a-i\infty}^{a+i\infty} e^{zt}\,\Pi^{\perp}R_{\Lambda}(z)(\mathcal{A}R_{\mathcal{B}}(z))^{N}\, dz \end{aligned}$$

and we have to explain why the last term is of order $\mathcal{O}(e^{at})$. We clearly have

$$\sup_{z=a+iy,\, y\in[-M,M]} \|\Pi^{\perp}R_{\Lambda}(z)(\mathcal{A}R_{\mathcal{B}}(z))^{N}\|$$

and it is then enough to get the bound

$$\|R_{\Lambda}(z)(\mathcal{A}R_{\mathcal{B}}(z))^{N}\| \le C/|y|^2, \quad \forall\, z = a+iy,\ |y|\ge M,\ a > a_*$$

We assume (in order to make the proof simpler) that $\zeta = 1$, namely

$$\|(\mathcal{A}S_{\mathcal{B}})^{(*n)}\|_{X\to X_1} = \mathcal{O}(e^{at}) \quad \forall\, t \geq 0,$$

with $X_1 := D(\Lambda) = D(\mathcal{B})$, which implies

$$\|(\mathcal{A}R_{\mathcal{B}}(z))^n\|_{X\to X_1} \leq C_a \quad \forall\, z = a + iy,\ a > a_*.$$

We also assume (for the same reason) that $\zeta' = 0$, so that

$$\mathcal{A} \in \mathcal{L}(X) \ \text{ and } \ R_{\mathcal{B}}(z) = \frac{1}{z}(R_{\mathcal{B}}(z)\mathcal{B} - I) \in \mathcal{L}(X_1, X)$$

imply

$$\|\mathcal{A}R_{\mathcal{B}}(z)\|_{X_1\to X} \leq C_a/|z| \quad \forall\, z = a + iy,\ a > a_*.$$

The two estimates together imply

$$(*) \qquad \|(\mathcal{A}R_{\mathcal{B}}(z))^{n+1}\|_{X\to X} \leq C_a/|z| \quad \forall\, z = a + iy,\ a > a_*.$$

- In order to deal with the general case $0 \leq \zeta' < \zeta \leq 1$ one has to use some additional interpolation arguments

We write

$$R_\Lambda(1-\mathcal{V}) = \mathcal{U}$$

with

$$\mathcal{U} := \sum_{\ell=0}^{n} (-1)^\ell \, R_\mathcal{B}(\mathcal{A}R_\mathcal{B})^\ell, \quad \mathcal{V} := (-1)^{n+1} \, (\mathcal{A}R_\mathcal{B})^{n+1}$$

For M large enough

$$(**) \qquad \|\mathcal{V}(z)\| \le 1/2 \quad \forall\, z = a + iy,\ |y| \ge M,$$

and we may write the Neuman series

$$R_\Lambda(z) = \underbrace{\mathcal{U}(z)}_{\text{bounded}} \underbrace{\sum_{j=0}^{\infty} \mathcal{V}(z)^j}_{\text{bounded}}$$

For $N = 2(n+1)$, we finally get from $(*)$ and $(**)$

$$\|R_\Lambda(z)(\mathcal{A}R_\mathcal{B}(z))^N\| \le C/\langle y\rangle^2, \quad \forall\, z = a + iy,\, |y| \ge M$$

Perturbation argument

Uniqueness and linearized/nonlinear stability of the steady state for problems without "detailed balance condition" or "trivial stationary solution"

My personal favorite example: the inelastic Boltzmann equation

- steady state: $\exists\, G_e \in P(\mathbb{R}^d) \cap \mathcal{S}(\mathbb{R}^d)$ solution to

$$(E) \qquad Q_e(G_e, G_e) + (1-e)\,\Delta G_e = 0, \quad \int G_e\, v\, dv = 0$$

- Q_e Boltzmann kernel associated to $e \in [0,1)$ inelastic coefficient
- elastic collision: $e = 1$
- ΔG_e diffuse forcing
- $G_e \approx e^{-|v|^{3/2}} \notin L^2(G^{-1})$!
- See also Gamba, Panferov, Villani & Bobylev, Gamba, Panferov (2004)

Step 1 : uniqueness of the steady state G_e ...

- $G_e \to G_1$ when $e \to 1$ with

$$G_1 \in P(\mathbb{R}^d) \cap \mathcal{S}(\mathbb{R}^d), \quad Q(G_1, G_1) = 0, \quad \int G_1 \, v \, dv = 0.$$

$$G_1(v) = (2\pi\theta)^{-d/2} \, e^{-\frac{|v|^2}{2\theta}} \text{ for some } \theta > 0.$$

- $(E) \times |v|^2$ implies

$$-(1 - e^2) \, D_{\mathcal{E}}(G_e) + (1 - e) \, 2d \int G_e \, dv = 0$$

and in the limit $e \to 1$:

$$D_{\mathcal{E}}(G_1) := \int \int |v - v_*|^3 \, G_1(v) \, G_1(v_*) \, dv dv_* = d \qquad \theta = \bar\theta.$$

- We prove more: $\exists! \bar G_1$ for "any" strong norm $\| \cdot \|$ $\exists \, C$

$$\forall \, G_e \text{ solution} \quad \|G_e - \bar G_1\| \le C \, \eta(1 - e) \to 0$$

Step 1 : ... by a "implicit function argument"

- $\Phi(e, G_e) = 0$ when we define

$$\Phi(e,g) := (D_{\mathcal{E}}(g) - \frac{2d}{1+e}, Q_e(g,g) + (1-e)\,\Delta g).$$

- We define $A : \mathcal{E} \to \mathbb{R} \times \mathcal{E}_0$ invertible, $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$, by

$$Ah := D_2\Phi(1, \bar{G}_1)\, h = [2\, D_{\mathcal{E}}(g, \bar{G}_1), \mathcal{L}\, h], \quad \mathcal{L} h := 2\, Q(\bar{G}_1, h).$$

- For two given solutions G_e and H_e of (E) :

$$\begin{aligned} G_e - H_e &= A^{-1}\,[A\, G_e - \Phi(e, G_e) + \Phi(e, H_e) - A\, H_e] \\ \Rightarrow \|G_e - H_e\| &\leq \|A^{-1}\|\, \eta(1-e)\, \|G_e - H_e\| \end{aligned}$$

$$\|G_e - \bar{G}_1\| = 0 \quad \text{if} \quad \|A^{-1}\|\, \eta(1-e) < 1 \ \text{ we note it}$$

Step 2 : linear and nonlinear stability of $\bar{G}_e$

- Define the inelastic linearized operator

$$\mathcal{L}_e h \;\; := \;\; 2\, Q_e(\bar{G}_e, h) + (1-e)\,\Delta h \approx 2\, Q_1(\bar{G}_1, h) = \mathcal{L}_1 h$$

- Introduce a decomposition $\mathcal{L} = \mathcal{A} + \mathcal{B}$, $\mathcal{B}(\xi) = \mathcal{B} - \xi$, $L_1(\xi) = L_1 - \xi$, and $\mathcal{U}(\xi) := \mathcal{B}(\xi)^{-1} - L_1(\xi)^{-1}\,\mathcal{A}\,\mathcal{B}(\xi)^{-1}$, we get

$$(\mathcal{L}_e - \xi)\,\mathcal{U}(\xi) \;\; = \;\; Id - (\mathcal{L}_e - \mathcal{L}_1)\, L_1(\xi)\, A\, \mathcal{B}(\xi) \approx Id$$

if $\mathcal{A}\, h := Q^{+,*}_{e,\delta}(\bar{G}_e, h)$, $\mathcal{B}\, h := r_{e,\delta}(h) - \nu(\bar{G}_e)\, h - (1-e)\,\Delta h$

- We conclude with
  - $\Sigma(\mathcal{L}_e) \cap \Delta_a = \{\lambda_{\mathcal{E}}(e), 0\}, \quad \lambda_{\mathcal{E}}(e) \approx -(1-e)\,\bar{\lambda}_{\mathcal{E}} < 0$
  - $e^{t\,\mathcal{L}_e}\,(Id - \Pi_{\mathcal{L}_e,a}) = \mathcal{O}(e^{a\,t})$