Quantitative, qualitative and uniform in time propagation of chaos

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A survey on recent papers by the "Kinetic French school":
Carrapatoso, Fournier, Guillin, Hauray, M., Mouhot and co-authors

Stochastic Limit Analysis for
Reacting Particle System workshop
Berlin, December 16-18, 2015

Aim of the talk

- General discussion about propagation of chaos
  - introduction to mean field limit / propagation of chaos
  - short discussion about chaos
  - quantitative/qualitative and uniform in time chaos
  - four different methods
- 2D viscous Vortex model and the nonlinear Martingale method
  - An example of “singular” McKean-Vlasov model
  - sketch of the proof
  - nonlinear Martingale method
  - Estimates thanks to Fisher information
  - Qualitative (entropic) chaos

Outlines of the talk

1. Introduction
2. Short discussion about chaos
3. Short discussion about methods
4. Nonlinear Martingale method and the vortex model
5. sketch of the proof - a priori estimates
6. sketch of the proof - probability argument
7. Sketch of the proof - functional analysis argument
8. Sketch of the proof - PDE/SDE argument
9. Sketch of the proof - entropy argument

Plan

Micro to macro

• How to go **rigorously** from a microscopic description to a statistical description?
How to derive (justify) the equation at the mesoscopic/macroscopic level ?
How to get something (simpler) from a microscopic description with a huge number of particles ?

• (Kac's) mean field limit ($\neq$ Boltzmann-Grad limit) in the sense that each particle interacts with all the other particles with an intensity of order $\mathcal{O}(1/N)$
$\Rightarrow$ statistical description = *law of large numbers limit* of a N-particle system

• at the formal level the identification of the limit is quite clear when one assumes the molecular chaos for the limit model

• main difficulty : propagation of chaos
 ▷ chaos for ∞ particles = Boltzmann's molecular chaos (stochastic independence)
 ▷ chaos for $N \to \infty$ particles = Kac's chaos (asymptotic stochastic independence)
 ▷ propagation of chaos: holds at time $t = 0$ implies holds for any $t > 0$
 ▷ propagation of chaos is necessary in order to identify the limit as $N \to \infty$

Consider a system of N indistinguishable (exchangeable) particles, each particle being described by its state (position, velocity) $\mathcal{Z}_1^N, ..., \mathcal{Z}_N^N \in E$, $E = \mathbb{R}^d$, which evolves according to

$$d\mathcal{Z}_i = \frac{1}{N} \sum_{j=1}^{N} K(\mathcal{Z}_i - \mathcal{Z}_j)\, dt + \sqrt{2\nu} d\mathcal{B}_i \qquad \text{(ODE or Brownian SDE)}$$

$$d\mathcal{Z} = \frac{1}{N} \sum_{i,j=1}^{N} \int_{S^{d-1}} (\mathcal{Z}'_{ij} - \mathcal{Z})\, B\, d\mathcal{N}_{i,j}(d\sigma) \qquad \text{(Boltzmann-Kac)}$$

$$d\mathcal{Z}_i = \frac{1}{N} \sum_{j=1}^{N} b(\mathcal{Z}_i - \mathcal{Z}_j)\, dt + \frac{1}{\sqrt{N}} \sum_{j=1}^{N} a^{1/2}(\mathcal{Z}_i - \mathcal{Z}_j)\, d\mathcal{B}_{i,j} \qquad \text{(Landau-Kac)}$$

where K is a pairwise interaction force field, $\mathcal{B}_i$ independent Brownian motions, $\nu \geq 0$, $\mathcal{N}$ Poisson measure, $\mathcal{Z}'_{ij} = (\mathcal{Z}_1, ..., \mathcal{Z}'_i, ..., \mathcal{Z}'_j, ..., \mathcal{Z}_N)$ represents the system after collision of the pair $(\mathcal{Z}_i, \mathcal{Z}_j)$, B cross-section, a Landau matrix kernel , $b = \mathrm{div} a$, $\mathcal{B}_{i,j}$ Brownian motions such that $\mathcal{B}_{j,i} = -\mathcal{B}_{i,j}$ and independent if $i < j$.

The law $G^N(t) := \mathcal{L}(\mathcal{Z}_t^N)$ satisfies the Master (Liouville or backward Kolmogorov) equation

$$\partial_t \langle G^N, \varphi\rangle = \langle G^N, \Lambda^N \varphi\rangle \qquad \forall\, \varphi \in C_b(E^N)$$

where the generator Λ^N writes

$$(\Lambda^N \varphi)(Z) := \frac{1}{N} \sum_{i,j=1}^{N} K(z_i - z_j) \cdot \nabla_i \varphi + \nu \sum_{i=1}^{N} \Delta_i \varphi \qquad \text{(ODE/SDE)}$$

$$(\Lambda^N \varphi)(Z) = \frac{1}{N} \sum_{1 \le i < j \le N}^{N} \int_{\mathbb{S}^{d-1}} \left[\varphi(Z'_{ij}) - \varphi(Z)\right] \, B(z_i - z_j, \sigma)\, \mathrm{d}\sigma \qquad \text{(Boltzmann-Kac)}$$

$$\begin{aligned}(\Lambda^N \varphi)(Z) \;&=\; \frac{1}{N} \sum_{1 \le i,j \le N}^{N} b(z_i - z_j) \cdot (\nabla_i \varphi - \nabla_j \varphi) \qquad \text{(Landau-Kac)} \\ &+\; \frac{1}{2N} \sum_{1 \le i,j \le N}^{N} a(z_i - z_j) : (\nabla^2_{ii} \varphi + \nabla^2_{jj} \varphi - \nabla^2_{ij} \varphi - \nabla^2_{ji} \varphi)\end{aligned}$$

Is it possible to identify the limit of the law $\mathcal{L}(\mathcal{Z}_1^N)$ of one typical particle?

More precisely, we want to show that $\mathcal{L}(\mathcal{Z}_1^N) \to f = f(t, dz)$ and that $f \in C([0,\infty); P(E))$ is a solution to

$$\partial_t f = \mathrm{div}_z[(K * f) f] + \nu \Delta f \qquad (McKean - Vlasov)$$

$$\partial_t f = \int_{\mathbb{R}^d \times S^{d-1}} [f(z')f(v') - f(z)f(v)]\, B\, dz d\sigma \qquad \text{(Boltzmann)}$$

$$\partial_t f = \nabla_z \int_{\mathbb{R}^d} a(z - z_*)[f(z_*)\nabla f(z) - f(z)\nabla f(z_*)]\, dz_* \qquad \text{(Landau)},$$

depending of the N-particle dynamics

Assuming that

$$\mathcal{L}(\mathcal{Z}_1^N) \to f = f(t, dz), \quad \mathcal{L}(\mathcal{Z}_1^N, \mathcal{Z}_2^N) \to g = g(t, dz, dv),$$

we easily (formally) show by taking $\varphi(Z) = \varphi(z_1)$ in the Master equation

$$\partial_t f = \mathrm{div}_z \Big[\int a(z - v) g(dz, dv) \Big] + \nu \Delta f$$

$$\partial_t f = \int_{\mathbb{R}^d \times S^{d-1}} [g(z', v') - g(z, v)] \, B \, dv d\sigma$$

$$\partial_t f = \nabla_z \int_{\mathbb{R}^d} a(z - v) [\nabla_z g(z, v) - \nabla_v g(z, v)] \, dv.$$

We obtain the McKean-Vlasov equation, the Boltzmann equation and the Landau equation if we make the additional

independence / molecular chaos assumption $g(v, z) = f(v) \, f(z)$.

• for a infinite system of indistinguishable particles: Boltzmann's (molecular) chaos means

$$\mathcal{L}(\mathcal{Z}_i, \mathcal{Z}_j) = f \otimes f$$

That is the stochastic independence (for a sequence of exchangeable random variables)

• for a system of N indistinguishable particles with $N \to \infty$: Kac's chaos means

$$\mathcal{L}(\mathcal{Z}_i^N, \mathcal{Z}_j^N) \to f \otimes f \quad \text{as} \quad N \to \infty$$

That is an asymptotically stochastic independence (of the coordinates of a sequence of random vectors with exchangeable coordinates)

Difficulty

• The above picture is not that easy because for N fixed particles the states $\mathcal{Z}_1(t)$, ..., $\mathcal{Z}_N(t)$ are **never independent** for positive time $t > 0$ even if the initial states $\mathcal{Z}_1(0), ..., \mathcal{Z}_N(0)$ are assumed to be independent : that is an inherent consequence of the fact that **particles do interact!**

• Equations are written in spaces with increasing dimension $N \to \infty$.
To overcome that difficulty we will work in **fixed spaces** using:
empirical probability measure

$$X \in E^N \mapsto \mu_X^N := \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i} \in \mathbb{P}(E)$$

and/or marginal densities

$$F^N \in \mathbb{P}_{sym}(E^N) \;\mapsto\; F_j^N := \int_{E^{N-j}} F^N dz_{j+1}...dz_N \in \mathbb{P}_{sym}(E^j)$$

• The nonlinear PDE can be obtained as a *"law of large numbers"* for a **not independent array of exchangeable random variables** in the mean-field limit.

• That is more demanding that the usual LLN. We need to **propagate** some asymptotic independence = Kac's stochastic chaos.

- We need at least
  - ▷ a priori estimates on the N-particle system
  - ▷ uniqueness for the limit nonlinear PDE

- Most of the works has been done in a **probability measures framework**.

In order that everything make sense, it is then needed that coefficients are not singular (they must be smooth enough, say C^0).

▷ propagation of chaos is even more difficult for singular models

- For numerical simulation purpose, one can introduce Nanbu-like stochastic dynamic

$$\begin{aligned} d\mathcal{Z}_i &= \frac{1}{N}\sum_{j=1}^{N}\int_{S^{d-1}} \tilde{B}(\mathcal{Z}_i, \mathcal{Z}_j, \sigma)\, d\mathcal{N}_i(d\sigma) \\ &= \frac{1}{N}\sum_{j=1}^{N} b(\mathcal{Z}_i - \mathcal{Z}_j)\, dt + \frac{1}{N}\sum_{j=1}^{N} a^{1/2}(\mathcal{Z}_i - \mathcal{Z}_j)\, d\mathcal{B}_i \end{aligned}$$

They are simpler to analyze but they are not energy conservative

• Kac (1956) defined the notion of chaos for sequences of random vectors. He proved the propagation of chaos for the *"Kac's caricature"* of Boltzmann model. He showed that the stochastic dynamic leaves invariant the Kac's sphere

$$\mathcal{KS}^N := \{V \in \mathbb{R}^N;\ |v_1|^2 + ... + |v_N|^2 = N\},$$

and, for any fixed $N \geq 2$, convergence to the equilibrium (stationary measure)

$$G_t^N = \mathcal{L}(\mathcal{V}_{1t}^N, ..., \mathcal{V}_{Nt}^N) \underset{t \to \infty}{\longrightarrow} \gamma^N = \text{ uniform measure on } \mathcal{KS}^N.$$

Kac's Program:

(Pb1) Establish propagation of chaos for more realistic (singular) models

(Pb2) Establish the convergence to the equilibrium as $t \to \infty$ with a uniform chaos rate with respect to the number N of particles

(Pb2′) Establish quantitative chaos estimate (rate) for Kac's chaos

(Pb3) Establish Boltzmann's H-theorem from a microscopic description (seems to be Kac's motivation)

Plan

Chaos is the asymptotic independence as $N \to \infty$ for a sequence $(\mathcal{Z}^N)$ of exchangeable random variables with values in E^N

$$\begin{array}{ccc} \mathcal{Z}^N = (\mathcal{Z}_1^N, ..., \mathcal{Z}_N^N) \in E^N & \to & F^N := \mathcal{L}(\mathcal{Z}^N) \in \mathbb{P}_{sym}(E^N) \\ \updownarrow & & \updownarrow \\ \mu^N_{\mathcal{Z}^N} := \dfrac{1}{N}\displaystyle\sum_{i=1}^N \delta_{\mathcal{Z}_i^N} \in \mathbb{P}(E) & \to & \hat{F}^N := \mathcal{L}(\mu^N_{\mathcal{Z}^N}) \in \mathbb{P}(\mathbb{P}(E)) \end{array}$$

For a random variable $\mathcal{Y}$ taking values in E with law $\mathcal{L}(\mathcal{Y}) = f \in \mathbb{P}(E)$ we say that $(\mathcal{Z}^N)$ is $\mathcal{Y}$-Kac's chaotic if

- $\mathcal{L}(\mathcal{Z}_1^N, ..., \mathcal{Z}_j^N) \rightharpoonup f^{\otimes j}$ weakly in $\mathbb{P}(E^j)$ as $N \to \infty$;
- $\mu^N_{\mathcal{Z}^N} \Rightarrow f$ in law as $N \to \infty$,
  meaning $\mathcal{L}(\mu^N_{\mathcal{Z}^N}) \to \delta_f$ in $\mathbb{P}(\mathbb{P}(E))$ as $N \to \infty$;
- $\mathbb{E}(|\mathcal{X}^N - \mathcal{Y}^N|) \to 0$ as $N \to \infty$ for a sequence $\mathcal{Y}^N$ of i.i.d.r.v with law f

Exchangeable means: $\mathcal{L}(\mathcal{Z}^N_{\sigma(1)}, ..., \mathcal{Z}^N_{\sigma(N)}) = \mathcal{L}(\mathcal{Z}_1^N, ..., \mathcal{Z}_N^N)$ for any permutation σ of the coordinates

For a given sequence (F^N) in $\mathbb{P}_{sym}(E^N)$ we define

• the marginals $F^N_j \in \mathbb{P}_{sym}(E^j)$ by

$$F^N_j = \int_{E^{N-j}} F^N dz_{j+1} ... dz_N$$

• the projection $\hat{F}^N \in \mathbb{P}(\mathbb{P}(E))$ by

$$\langle \hat{F}^N, \Phi \rangle = \int_{E^N} \Phi(\mu^N_X)\, F^N(dX) \quad \forall\, \Phi \in C_b(\mathbb{P}(E))$$

• the normalized MKW distance on $\mathbb{P}(E^j)$ by

$$W_1(F,G) := \inf_{\pi \in \Pi(F,G)} \int_{E^j \times E^j} \Big(\frac{1}{j} \sum_{i=1}^{j} |x_i - x_j| \wedge 1 \Big)\, \pi(dX, dY).$$

• the MKW distance $\mathcal{W}_1$ on $\mathbb{P}(\mathbb{P}(E))$ by

$$\mathcal{W}_1(\alpha, \beta) := \inf_{\pi \in \Pi(\alpha,\beta)} \int_{\mathbb{P}(E)\times\mathbb{P}(E)} W_1(\rho,\eta)\, \pi(d\rho, d\eta).$$

For a given sequence (F^N) in $\mathbb{P}_{sym}(E^N)$ we define

$-$ the marginals $F^N_j \in \mathbb{P}_{sym}(E^j)$,

$-$ the projection $\hat{F}^N \in \mathbb{P}(\mathbb{P}(E))$,

$-$ the normalized MKW distance W_1 on $\mathbb{P}(E^j)$,

$-$ the MKW distance $\mathcal{W}_1$ on $\mathbb{P}(\mathbb{P}(E))$,

and for $f \in \mathbb{P}(E)$ we say that (F^N) is f-Kac's chaotic if (equivalently)

- $\mathcal{D}_j(F^N; f) := W_1(F^N_j, f^{\otimes j}) = \mathbb{E}(|(\mathcal{X}^N_1, ..., \mathcal{X}^N_j) - (\mathcal{X}^N_1, ..., \mathcal{X}^N_j)|) \to 0$
- $\mathcal{D}_\infty(F^N; f) := \mathcal{W}_1(\hat{F}^N, \delta_f) = \mathbb{E}(W_1(\mu^N_{\mathcal{Z}^N}, f) \to 0$

More precisely, for $E = \mathbb{R}^d$

Lemma (Hauray, M.)

For given M and $k > 1$ there exist some constants $\alpha_i, C > 0$ such that $\forall\, f \in \mathbb{P}(E)$, $\forall\, F^N \in \mathbb{P}_{sym}(E^N)$ with $M_k(F^N_1), M_k(f) \le M$

$$\forall\, j, \ell \in \{1, ..., N, \infty\},\ \ell \neq 1 \quad \mathcal{D}_j(F^N; f) \le C\,\big(\mathcal{D}_\ell(F^N; f)^{\alpha_1} + \frac{1}{N^{\alpha_2}}\big).$$

Stronger chaos: entropy and Fisher's chaos

For $F^N \in \mathbb{P}_{sym}(E^N)$, $E = \mathbb{R}^d$, we define the normalized functionals

$$H(F^N) := \frac{1}{N} \int_{E^N} F^N \log F^N, \quad I(F^N) := \frac{1}{N} \int_{E^N} \frac{|\nabla F^N|^2}{F^N}.$$

Definition

Consider a sequence $F^N \in \mathbb{P}_{sym}(E^N)$ and $f \in \mathbb{P}(E)$

(F^N) is f-entropy chaotic if $F_1^N \rightharpoonup f$ weakly in $\mathbb{P}(E)$ and $H(F^N) \to H(f)$

(F^N) is f-Fisher's chaotic if $F_1^N \rightharpoonup f$ weakly in $\mathbb{P}(E)$ and $I(F^N) \to I(f)$

Theorem (Hauray, M.)

In the list below, each assertion implies the one which follows

(i) (F^N) *is Fisher's chaotic;*

(ii) (F^N) *is Kac's chaotic and* $I(F^N)$ *is bounded;*

(iii) (F^N) *is entropy chaotic;*

(iv) (F_j^N) *converges in* L^1 *for any* $j \geq 1$*;*

(v) (F^N) *is Kac's chaotic.*

Comments

Extensions by Carrapatoso, Fournier, Guillin, Hauray, M.

• Kac's chaos, entropic chaos and Fisher's chaos on Kac's spheres and on Boltzmann's spheres

• For a mixture of probability measures = without chaos hypothesis

• Optimal rate of convergence of $\mathcal{D}_\infty(f^{\otimes N}, f) \sim N^{1/d}$ for $f \in \mathbb{P}_q(\mathbb{R}^d)$, $d \geq 2$

Based on many previous works from Funct Analysis, Proba, Stat, Geo, ...

• Mixture: de Finetti (1937), Hewitt-Savage (1955), Robinson-Ruelle (1967)

• Functional and quantified LLN (Glivenko-Cantelli ... Rachev-Rüschendorf ... Barthe-Bordenave)

• local central limit theorem of Berry-Esseen

• HWI inequality of Otto and Villani

• Entropy inequalities: Carlen-Lieb-Loss (2004), Arstein-Ball-Barthe-Naor (2004)

• previous comparison, quantitative and qualitative results on chaos
Kac: Foundations of kinetic theory. (1956)
Sznitman: Topics in propagation of chaos. Saint-Flour -1989 (1991)
Carlen, Carvalho, Le Roux, Loss, Villani: Entropy and chaos ... (2010)

Plan

At least four strategies

- BBGKY method (introduced by BBGKY!)

$\oplus$ Quite easy to handle with and robust

$\ominus$ not quantitative, nor qualitative (?), two body interaction only

- semigroups method (introduced by Kac, McKean, Grünbaum)

$\oplus$ quantitative and uniform in time

$\ominus$ rates are not sharp (at all!), maybe not well adapted to singular model

- coupling method (introduced by McKean, popularized by Sznitman)

$\oplus$ best (and sharp) rate, uniform in time for some ("bounded") model

$\ominus$ non uniform in time for most models, not well adapted to singular model

- nonlinear Martingale method (introduced by Sznitman)

$\oplus$ propagation of chaos results for singular models

$\ominus$ no rate of convergence

For B bounded the Boltzmann-Kac operator is bounded and the semigroup writes

$$F^N(t) = e^{t\Lambda_N} F^N(0) = \sum_{k=1}^{\infty} \frac{t^k}{k!} \Lambda_N^k F^N(0).$$

Because Λ_N is self-adjoint, for any $\varphi \in C_b(E^j)$, $j \le N$, we have

$$\langle F^N(t), \varphi \rangle = \sum_{k=1}^{\infty} \frac{t^k}{k!} \langle F^N(0), \Lambda_N^k \varphi \rangle.$$

We first consider $\varphi \in C_b(E)$, so that

$$\langle F^N(0), \Lambda_N^k \varphi \rangle \to \langle f_0^{\otimes k+1}, \varphi_{k+1} \rangle, \quad \varphi_{k+1} = Z_k \varphi \in C(E^{k+1}),$$

and, then assuming $F^N \to \pi$ as $N \to \infty$, we get

$$\langle \pi_1(t), \varphi \rangle = \sum_{k=1}^{\infty} \frac{t^k}{k!} \langle f_0^{\otimes k+1}, \varphi_{k+1} \rangle.$$

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We first consider $\varphi \in C_b(E)$, so that

$$\langle F^N(0), \Lambda_N^k \varphi \rangle \to \langle f_0^{\otimes k+1}, \varphi_{k+1} \rangle, \quad \varphi_{k+1} = Z_k \varphi \in C(E^{k+1}),$$

and, then assuming $F^N \to \pi$ as $N \to \infty$, we get

$$\langle \pi_1(t), \varphi \rangle = \sum_{k=1}^{\infty} \frac{t^k}{k!} \langle f_0^{\otimes k+1}, \varphi_{k+1} \rangle.$$

We next consider $\gamma := \varphi \otimes \psi \in C_b(E^2)$, and in the infinite particles limit, we get

$$\langle \pi_2(t), \gamma \rangle = \sum_{k=1}^{\infty} \frac{t^k}{k!} \langle f_0^{\otimes k+1}, \gamma_{k+1} \rangle,$$

with

$$\gamma_{k+1} = \sum_{i=1}^{k} \varphi_i \, \psi_{k+1-i} \, \frac{k!}{i!(k+1-i)!}.$$

We recognize

$$\begin{aligned} \langle \pi_2(t), \gamma \rangle &= \sum_{k,i}^{\infty} \frac{t^{k-i}\, t^i}{i!(k+1-i)!} \langle f_0^{\otimes i}, \varphi_i \rangle \langle f_0^{\otimes k+1-i}, \psi_{k+1-i} \rangle, \\ &= \langle \pi_1(t) \otimes \pi_1(t), \varphi \otimes \psi \rangle \end{aligned}$$

We conclude that $\pi_2 = \pi_1 \otimes \pi_1$ and

$$\pi_1(t) = \sum_{k=1}^{\infty} \frac{t^k}{k!} Z_k f_0^{\otimes k+1} \quad (\text{Wild Sum } = f(t)).$$

$\triangleright$ In general we cannot write such an explicit formula and we have to write the all family of equations (for a two body problem)

$$\partial_t F_j^N = (\Lambda_N F^N)_j = \Lambda_{N,j+1} F_{j+1}^N \quad \underset{N \to \infty}{\longrightarrow} \quad \partial_t \pi_j = \bar{\Lambda}_{j+1} \pi_{j+1} \quad \forall\, j \geq 1.$$

$\triangleright$ biblio: Bogolioubov (?), Born & Green (1946), Kirkwood (1935), Yvon (1935)
Lanford: Time evolution of large classical systems. (1974)
Spohn: On the Vlasov hierarchy (1981)
Arkeryd-Caprino-Ianiro: The homogeneous Boltzmann hierarchy ... (1991)
Gallagher-Saint-Raymond-Texier (2013), Bodineau-Gallagher-Saint-Raymond (2015)

We split

$$
\begin{aligned}
\langle F^N_{kt} - f_t^{\otimes k}, \varphi \rangle &= \langle F^N_t - f_t^{\otimes N}, \varphi \otimes 1^{\otimes N-k} \rangle = \\
&= \langle F^N_t, \varphi \otimes 1^{\otimes N-k} - R_\varphi(\mu^N_V) \rangle \qquad (= T_1) \\
&\quad + \langle F^N_t, R_\varphi(\mu^N_V) \rangle - \langle F^N_0, R_\varphi(S^{NL}_t \mu^N_V) \rangle \qquad (= T_2) \\
&\quad + \langle F^N_0, R_\varphi(S^{NL}_t \mu^N_V) \rangle - \langle f_t^{\otimes k}, \varphi \rangle \qquad (= T_3)
\end{aligned}
$$

where R_φ is the "polynomial function" on $\mathbb{P}(\mathbb{R}^3)$ defined by

$$
R_\varphi(\rho) = \int_{E^k} \varphi \, \rho(dv_1) \dots \rho(dv_k)
$$

and S^{NL}_t is the nonlinear semigroup associated to the nonlinear mean-field limit equation by $g_0 \mapsto S^{NL}_t g_0 := g_t$.

$$
\begin{aligned}
|T_1| &= \left| \left\langle F_t^N, \varphi \otimes 1^{\otimes(N-k)}(V) - R_\varphi(\mu_V^N) \right\rangle \right| \\
&= \left| \left\langle F_t^N, \varphi \otimes \widetilde{1^{\otimes(N-k)}}(V) - R_\varphi(\mu_V^N) \right\rangle \right| \\
&\leq \left\langle F_t^N, \frac{2\,k^2}{N} \, \|\varphi\|_{L^\infty(E^k)} \right\rangle = \mathcal{O}\Big(\frac{1}{N}\Big)
\end{aligned}
$$

where we use that F^N is symmetric and a probability and we introduce the symmetrization function associated to $\varphi \otimes 1^{\otimes(N-k)}$ by

$$
\varphi \otimes \widetilde{1^{\otimes(N-k)}}(V) = \frac{1}{\sharp \mathfrak{S}_N} \sum_{\sigma \in \mathfrak{S}_N} \varphi \otimes 1^{\otimes(N-k)}(V_\sigma).
$$

$$\begin{aligned}
|T_3| &= \left|\left\langle F_0^N, R_\varphi(S_t^{NL}\mu_V^N) - R_\varphi(S_t^{NL} f_0)\right\rangle\right| \\
&\le [R_\varphi]_{C^{0,1}} \left\langle F_0^N, W_1(S_t^{NL}\mu_V^N, S_t^{NL} f_0)\right\rangle \\
&\le k\,\|\nabla\varphi\|_{L^\infty(E^k)}\, C_T \left\langle F_0^N, W_1(\mu_V^N, f_0)\right\rangle \\
&\le k\,\|\nabla\varphi\|_{L^\infty(E)}\, C_T\, \mathcal{W}_{W_1}(\hat F_0^N, \delta_{f_0}) \\
&= \mathcal{O}\Big(\mathcal{D}_\infty(F_0^N, f_0)\Big) \quad \text{but in fact} \quad \stackrel{!}{=} \mathcal{O}\Big(\frac{1}{\log N}\Big)
\end{aligned}$$

where

$$[R_\varphi]_{C^{0,1}} := \sup_{W_1(\rho,\eta)\le 1} |R_\varphi(\eta) - R_\varphi(\rho)| = k\,\|\nabla\varphi\|_{L^\infty}$$

and we have to prove that the nonlinear flow satisfies

$$(A5) \qquad W_1(f_t, g_t) \le C_T\, W_1(f_0, g_0) \quad \forall f_0, g_0 \in \mathbb{P}(E)$$

T_2 : We write

$$T_2 \quad = \quad \left\langle F_t^N, R_\varphi(\mu_V^N) \right\rangle - \left\langle F_0^N, R_\varphi(S_t^{NL} \mu_V^N) \right\rangle$$

T_2 : We write

$$\begin{aligned} T_2 &= \left\langle F_t^N, R_\varphi(\mu_V^N)\right\rangle - \left\langle F_0^N, R_\varphi(S_t^{NL}\mu_V^N)\right\rangle \\ &= \left\langle F_0^N, T_t^N(R_\varphi \circ \mu_V^N) - (T_t^\infty R_\varphi)(\mu_V^N)\right\rangle \end{aligned}$$

with

- T_t^N = dual semigroup (acting on $C_b(E^N)$) of the N-particle flow $F_0^N \mapsto F_t^N$;
- T_t^∞ = pushforward semigroup (acting on $C_b(\mathbb{P}(E))$) of the nonlinear semigroup S_t^{NL} defined by $(T^\infty \Phi)(\rho) := \Phi(S_t^{NL}\rho)$;

T_2 : We write

$$\begin{aligned} T_2 &= \left\langle F_t^N, R_\varphi(\mu_V^N) \right\rangle - \left\langle F_0^N, R_\varphi(S_t^{NL}\mu_V^N) \right\rangle \\ &= \left\langle F_0^N, T_t^N(R_\varphi \circ \mu_V^N) - (T_t^\infty R_\varphi)(\mu_V^N) \right\rangle \\ &= \left\langle F_0^N, (T_t^N \pi_N - \pi_N T_t^\infty)\, R_\varphi \right\rangle \end{aligned}$$

with

- T_t^N = dual semigroup (acting on $C_b(E^N)$) of the N-particle flow $F_0^N \mapsto F_t^N$;
- T_t^∞ = pushforward semigroup (acting on $C_b(\mathbb{P}(E))$) of the nonlinear semigroup S_t^{NL} defined by $(T^\infty \Phi)(\rho) := \Phi(S_t^{NL}\rho)$;
- π_N = projection $C(\mathbb{P}(E)) \to C(E^N)$ defined by $(\pi_N \Phi)(V) = \Phi(\mu_V^N)$.

$$
\begin{aligned}
T_2 &= \left\langle F_0^N, (T_t^N \pi_N - \pi_N T_t^\infty)\, R_\varphi \right\rangle \\
&= \left\langle F_0^N, \int_0^T T_{t-s}^N \,(\Lambda^N \pi_N - \pi_N \Lambda^\infty)\, T_s^\infty \, ds \, R_\varphi \right\rangle \\
&= \int_0^T \left\langle F_{t-s}^N, (\Lambda^N \pi_N - \pi_N \Lambda^\infty)\,(T_s^\infty \, R_\varphi) \right\rangle ds = \mathcal{O}\left(\frac{1}{N^\bullet}\right)
\end{aligned}
$$

where

- Λ^N is the generator associated to T_t^N and Λ^∞ is the generator associated to T_t^∞.

Now we have to make some assumptions

- (A1) F_t^N has enough bounded moments;
- (A2) $\Lambda^\infty \Phi(\rho) = \langle Q(\rho), D\Phi(\rho) \rangle$;
- (A3) $(\Lambda^N \pi^N \Phi)(V) = \langle Q(\mu_V^N), D\Phi(\mu_V^N) \rangle + \mathcal{O}([\Phi]_{C^{1,a}}/N)$
- (A4) $S_t^{NL} \in C^{1,a}(\mathbb{P}(E); \mathbb{P}(E))$ "uniformly" in time $t \in [0, T]$

Example of result : Uniform in time propagation of chaos for the hard spheres Boltzmann-Kac model and time relaxation to the equilibrium uniformly in the number of particles

Theorem (M., Mouhot, 2013, a possible answer to Kac's problems)

For any $f_0 \in \mathbb{P}(E)$ + conditions, there exists a sequence $\mathcal{V}^N(0)$ of initial conditions for the Boltzmann-Kac process for hard spheres such that

$$\sup_{t\geq 0} \mathbb{E}(W_1(\mu^N_{\mathcal{V}^N(t)}, f(t)) \leq \frac{C}{\log N}$$

$$H(\mathcal{V}^N(t)|\gamma^N) \to H(f(t)|\gamma)$$

$$\sup_{N\geq 1} W_1(F^N(t), \gamma^N) \leq \frac{C}{\log t}.$$

▷ biblio: Kac: Foundations of kinetic theory (1956)
McKean: An exponential formula for Boltzmann eq. for a Maxwellian gas (1967)
Grünbaum: Propagation of chaos for the Boltzmann equation (1971)
Kolokoltsov: Nonlinear Markov Processes and Kinetic Equations (book, 2010)
M., Mouhot: Kac's program in kinetic theory (2013)
M., Mouhot, Wennberg: A new approach to quantitative chaos propagation ... (2015)
Carrapatoso: Propagation of chaos for .. Landau eq. for Maxwellian molecules (2016)

Consider two solutions to a smooth coefficients nonlinear Brownian SDE

$$\begin{aligned} d\mathcal{Z} &= (K * f)(\mathcal{Z})\, dt + d\mathcal{B}, \quad f = \mathcal{L}(Z) \\ d\bar{\mathcal{Z}} &= (K * \bar{f})(\bar{\mathcal{Z}})\, dt + d\mathcal{B}, \quad \bar{f} = \mathcal{L}(\bar{Z}), \end{aligned}$$

with same (synchronous) Brownian motion. We have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\mathcal{Z} - \bar{\mathcal{Z}}|^2 &= [(K * f)(\mathcal{Z}) - (K * f)(\bar{\mathcal{Z}}) + K * (f - \bar{f})(\bar{\mathcal{Z}})](\mathcal{Z} - \bar{\mathcal{Z}}) \\ &\lesssim |\mathcal{Z} - \bar{\mathcal{Z}}|^2 + \mathbb{E}(|\mathcal{Z} - \bar{\mathcal{Z}}|^2) \end{aligned}$$

because $\|K * (f - \bar{f})\|_\infty \le \mathbb{E}(|\mathcal{Z} - \bar{\mathcal{Z}}|^2)$. Taking the expectation, we deduce from the Gronwall lemma

$$\mathbb{E}(|\mathcal{Z}_t - \bar{\mathcal{Z}}_t|^2) \le e^{Lt}\, \mathbb{E}(|\mathcal{Z}_0 - \bar{\mathcal{Z}}_0|^2).$$

Consider a solution to a N-particle system of Brownian SDE

$$\begin{aligned} d\mathcal{Z}_i &= \frac{1}{N}\sum_{j=1}^N K(\mathcal{Z}_i - \mathcal{Z}_j)\, dt + d\mathcal{B}_i \\ &= (K * \mu^N_{\mathcal{Z}})(\mathcal{Z}_i) + d\mathcal{B}_i. \end{aligned}$$

Consider a solutions to the associated the nonlinear Brownian SDE

$$\begin{aligned} d\bar{\mathcal{Z}}_i &= (K * \bar{f})(\bar{\mathcal{Z}}_i)\, dt + d\mathcal{B}_i \\ &= (K * \mu^N_{\bar{\mathcal{Z}}})(\bar{\mathcal{Z}}_i)\, dt + d\mathcal{B}_i + \{(K * \bar{f})(\bar{\mathcal{Z}}_i) - (K * \mu^N_{\bar{\mathcal{Z}}})(\bar{\mathcal{Z}}_i)\} \end{aligned}$$

with same (synchronous and independent) Brownian motions.
Using a functional LLN estimate, we similarly get

$$\frac{d}{dt}\frac{1}{N}\sum_{i=1}^N \mathbb{E}(|\mathcal{Z}_i - \bar{\mathcal{Z}}_i|^2) \;\lesssim\; \frac{1}{N}\sum_{i=1}^N \mathbb{E}(|\mathcal{Z}_i - \bar{\mathcal{Z}}_i|^2) + \mathcal{O}\Big(\frac{1}{N}\Big).$$

Coupling method - other ideas : distances, truncation, not synchronous coupling

• Write a differential inequality on an appropriate distance and use Gronwall lemma

$$\mathbb{E}(|\mathcal{Z}^N - \bar{\mathcal{Z}}^N|^q) \quad \text{or} \quad \mathcal{D}_\infty(\mu^N_{\mathcal{Z}^N}, \bar{f}) \sim \mathcal{D}_N(\mathcal{Z}^N, \bar{f}) \sim \mathbb{E}(W_2^2(\mu^N_{\mathcal{Z}}, \bar{f}))^{1/2}$$

• Uniqueness for Boltzmann and Landau equation

$$\frac{d}{dt} W_2^2(f_t, g_t) \le 0 \quad \text{or even} \qquad \le -W_2^2(f_t, g_t)^{1+\bullet}$$

for Maxwellian molecules ($\gamma = 0$) and then

$$\frac{d}{dt} W_q(f_t, g_t) \lesssim R^\gamma W_q(f_t, g_t) + e^{-R^2}$$

when f_t has exponential moment bounds (but not g_t) for hard potentials ($\gamma > 0$).

• Use a more convenient coupling that the same synchronous coupling or even more than one coupling ...

Theorem (Fournier, Guillin, 2015)

For any $f_0 \in \mathbb{P}(\mathbb{R}^3)$ + conditions, there exists a sequence $(\mathcal{V}_0^N)$ of initial conditions for the Landau-Kac process for hard potential such that

$$\sup_{[0,T]} \mathbb{E}[W_2^2(\mu^N_{\mathcal{V}^N(t)}, f(t))] \le C_T \left(\mathbb{E}[W_2^2(\mu^N_{\mathcal{V}_0^N}, f_0)] + \frac{1}{N^{1/3}}\right)^{1-\bullet}$$

▷ biblio: McKean: Propagation of chaos for a class of non-linear parabolic eq. (1967)
Dobrushin: Vlasov equations (1979)
Tanaka: Probabilistic treatment of Boltzmann eq. for Maxwellian molecules (1978/79)
Sznitman: Topics in propagation of chaos. Saint-Flour -1989 (1991)
Graham-Méléard: Convergence rate for approximations to the Boltzmann eq. (1996)
Malrieu; Convergence to equilibrium for ... and their Euler schemes. (2003)
Fontbona-Guérin-Méléard: Convergence rate for Landau particle systems (2009)
Fournier: Particle approximation of some Landau equations (2009)
Fournier-M.: Rate of convergence for Nanbu particle system (arX 2013)
Cortez-Fontbona: Quantitative propagation of chaos for Kac particle systems (arX 2014)

Plan

Consider a system of N indistinguishable (exchangeable) particles, each particle being described by its position $\mathcal{X}_1^N, ..., \mathcal{X}_N^N \in \mathbb{R}^2$, which evolves according to

$$d\mathcal{X}_i = \frac{1}{N} \sum_{j=1}^{N} K(\mathcal{X}_i - \mathcal{X}_j)\, dt + \sqrt{2\nu} d\mathcal{B}_i \qquad \text{(Brownian SDE)}$$

where $\nu > 0$ is the viscosity and $K : \mathbb{R}^2 \to \mathbb{R}^2$ is the Biot-Savart kernel defined by

$$\forall\, x = (x_1, x_2) \in \mathbb{R}^2, \quad K(x) = \frac{x^\perp}{|x|^2} = \Big(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2} \Big) = \nabla^\perp \log |x|,$$

The associated mean field limit is the $2D$ Navier-Stokes equation written in vorticity formulation

$$\partial_t w_t(x) = (K \star w_t)(x) \cdot \nabla_x w_t(x) + \nu \Delta_x w_t(x), \tag{1}$$

where $w : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}_+$ is the vorticity function

All that can be done for vortices which turn in both (trigonometric and reverse) senses and thus $w : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$

Theorem (Fournier, Hauray, M., 2014, first version)

(1) If $\mathcal{X}_0^N$ is w_0-Kac's chaotic and "appropriately bounded" then $\mathcal{X}_t^N$ is w_t-Kac's chaotic for any time $t > 0$.

(2) If $\mathcal{X}_0^N$ is w_0-entropy chaotic and has bounded moment of order $k \in (0, 1]$ then $\mathcal{X}_t^N$ is w_t–entropy chaotic for any time $t > 0$.

▷ biblio: Sznitman: Equations de type de Boltzmann, spatialement homogenes. (1984)
Osada: Propagation of chaos for the 2D Navier-Stokes equation (1985)–(1987)
Fournier-Hauray: Chaos propag for Landau eq with moderate soft potentials (arX 2015)
Fournier-Jourdain. Stochastic particle approximation of Keller-Segel equation (arX 2015)

Nonlinear SDE for viscous vortex or stochastic NS vortex equation

We say that $\mathcal{X} = (\mathcal{X}_t)_{t>0}$ a continuous stochastic process with values in $\mathbb{R}^2$ is a solution to the stochastic NS vortex equation if it satisfies the nonlinear Brownian SDE

$$d\mathcal{X}_t = (K * w_t)(\mathcal{X}_t) + \sqrt{2\nu}\, d\mathcal{B}_t$$

for some given brownian motion $\mathcal{B}$ and where $w_t = \mathcal{L}(\mathcal{X}_t)$ is the law of $\mathcal{X}_t$.

It is important to point out that (thanks to Ito formula) the law w_t of $\mathcal{X}_t$ then satisfies the NS vortex equation

$$\partial_t w_t = (K * w_t) \cdot \nabla_x w_t + \nu \Delta_x w_t.$$

Theorem (Fournier, Hauray, M., 2014, second version)

Consider $w_0 \geq 0$ a function such that

$$\int_{\mathbb{R}^2} w_0 \, (1 + |x|^k + |\log w_0|) \, dx < \infty, \quad k \in (0,1],$$

the vortices trajectories $\mathcal{X}^N = (\mathcal{X}^N_t)_{t \geq 0}$ associated to an i.c. $\mathcal{X}^N_0 \sim w_0^{\otimes N}$ and $\mathcal{X}$ the solution to the stochastic NS vortex equation associated to an i.c. $\mathcal{X}_0 \sim w_0$. There holds

$$\mu^N_{\mathcal{X}^N} \;\Rightarrow\; \mathcal{X} \quad \text{in law in } \mathbb{P}(C([0,\infty);\mathbb{R}^2)) \text{ as } N \to \infty$$

$$\mathcal{L}(\mathcal{X}^N_1(t)) \to w_t = \mathcal{L}(\mathcal{X}_t) \quad \text{strongly in } L^1(\mathbb{R}^2) \text{ as } N \to \infty$$

The first convergence means

$$\mathcal{L}(\mu^N_{\mathcal{X}^N}) \;\rightharpoonup\; \delta_{\mathcal{L}(\mathcal{X})} \quad \text{weakly in } \mathbb{P}(\mathbb{P}(C([0,\infty);\mathbb{R}^2)) \text{ as } N \to \infty$$

and the second can be improved into

$$\mathcal{L}(\mathcal{X}^N_1(t), ..., \mathcal{X}^N_j(t)) \to w_t^{\otimes j} \quad \text{strongly in } L^1(\mathbb{R}^2)^j \text{ as } N \to \infty$$

The proof follow the by-now well-known "weak stability on nonlinear martingales" approach, which goes back to Sznitman 1984.
Everything is standard except the fact that we use the Fisher information bound in each step.

• A priori estimates (on entropy, moment and Fisher information)

• tightness of the law Q^N of the empirical process $\mu^N_{\mathcal{X}^N}$ in $\mathbb{P}(\mathbb{P}(E))$

• pass to the limit and identify the set of limit points $\mathcal{S}$ as the probability measures $q \in \mathbb{P}(E)$ associated to a process $\mathcal{X}$ which solves the (Martingale problem associated to the) stochastic NS vortex equation and has finite Fisher information.

• if $q \in \mathcal{S}$ and $q = \mathcal{L}(\mathcal{X})$ then $w_t := \mathcal{L}(\mathcal{X}_t)$ is the unique solution to the NS vortex PDE

• the Martingale problem has a unique solution $\bar{\mathcal{X}}$ and then $\mathcal{S} = \{\bar{q}\}$ where $\mathcal{L}(\bar{q}) = \bar{\mathcal{X}}$.

In conclusion, $Q^N \rightharpoonup \delta_{\bar{q}}$ in $\mathbb{P}(\mathbb{P}(E))$. (that Kac's chaos)

Plan

Using $\mathrm{div} K = 0$, we get the entropy identity

$$H(F_t^N) + \nu \int_0^t I(F_s^N)\, ds = H(F_0^N).$$

As usually we need a control of a moment of F_t^N in order to take advantage of the entropy bound (we need a lower bound on H).

We define the moment M_k of order $k \in (0,1]$ by

$$M_k(F^N) = \int_{\mathbb{R}^{2N}} F^N \frac{1}{N} \sum_{j=1} \langle x_j \rangle^k = \int_{\mathbb{R}^2} F_1^N \langle x \rangle^k dx$$

We then compute

$$\begin{aligned} \frac{d}{dt} M_k(F_t^N) &= \nu \int_{\mathbb{R}^2} F_{1t}^N \, \Delta \langle x \rangle^k + \int_{\mathbb{R}^4} F_{2t}^N \, K(x_1 - x_2) \cdot \nabla_1 \langle x_1 \rangle^k \\ &\leq C_1 \int_{\mathbb{R}^2} F_{1t}^N + C_2 \int_{\mathbb{R}^4} F_{2t}^N \frac{1}{|x_1 - x_2|} \end{aligned}$$

Define $g^N := \mathcal{L}(X_2 - X_1)$ and use (Carlen 1991) Fisher information inequalities

$$\frac{1}{2} I_1(g^N) \le I_2(F_2^N) \le I_N(F^N)$$

as well as Gagliardo-Niremberg type inequalities in 2D

$$\forall\, g \in \mathbb{P}(\mathbb{R}^2),\ \forall\, p \in [1,\infty) \quad \|g\|_{L^p} \le C_p\, I(g)^{1-1/p}.$$

Coming back to the singular term in the moment equation, we compute

$$\begin{aligned}
\int_{\mathbb{R}^4} \frac{F_{2t}^N}{|x_1 - x_2|}\, dx_1 dx_2 &= \sqrt{2}\int_{B_1} \frac{g_t^N(x)}{|x|}\, dx + \sqrt{2}\int_{B_1^c} \frac{g_t^N(x)}{|x|}\, dx \\
&\le \sqrt{2}\, \||\cdot|^{-1}\|_{L^{3/2}(B_1)} \|g_t^N\|_{L^3(B_1)} + \sqrt{2}\, \|g_t^N\|_{L^1(B_1^c)} \\
&\le C_3\, I(g_t^N)^{2/3} + C_4 \\
&\le \frac{\nu}{4 C_2}\, I(g_t^N) + C_5 \\
&\le \frac{\nu}{2 C_2}\, I(F_t^N) + C_5
\end{aligned}$$

Summing up the two equations on the entropy and on the moment of order k, we find

Lemma (a priori estimates)

Uniformly in N

$$H(F_t^N) + M_k(F_t^N) + \frac{\nu}{2}\int_0^t I(F_s^N)\,ds$$
$$\le H(F_0^N) + M_k(F_0^N) + (C_1 + C_2)t$$

Plan

We denote

$$\mathcal{X}^N := (\mathcal{X}_1^N, ..., \mathcal{X}_N^N) \text{ the exchangeable r.v. with value in } E^N$$

where $\mathcal{X}_i^N = (\mathcal{X}_i^N(t))_{t \ge 0} \in E := C([0,\infty); \mathbb{R}^2)$ solution to the SDE

$$\mathcal{X}_i(t) = \mathcal{X}_i(0) + \int_0^t (K * \mu^N_{\mathcal{X}(s)})(\mathcal{X}_i(s))\, ds + \sqrt{2\nu}\, \mathcal{B}_i(t)$$

and we want to show that each particle behaves asymptotically like N independent copies of the same process $\mathcal{X} = (\mathcal{X}(t))_{t \ge 0}$ defined as the solution to the nonlinear SDE

$$\mathcal{X}(t) = \mathcal{X}(0) + \int_0^t (K * w_s)(\mathcal{X}(s))\, ds + \sqrt{2\nu}\, \mathcal{B}(t),$$

where $w_s := \mathcal{L}(X(s))$ and then is a solution (Ito formula) to the NS vortex equation

$$\partial_t w = (K \star w) \cdot \nabla_x w + \nu \Delta_x w.$$

Lemma

the family of laws $\mathcal{L}(\mu^N_{\mathcal{X}^N})_{N\geq 1}$ *is tight in* $\mathbb{P}(\mathbb{P}(E))$

From classical compactness criterium (Sznitman 1984) it is enough to prove that the family of laws $\mathcal{L}(\mathcal{X}^N_1)_{N\geq 1}$ is tight in $\mathbb{P}(E)$. That is a consequence of

Lemma

For all $T > 0$, $\theta \in (0, 1/2)$

$$\mathbb{E}\Big[\sup_{0<s<t<T} \frac{|\mathcal{X}^N_1(t) - \mathcal{X}^N_1(s)|}{(t-s)^\theta}\Big] \leq C\Big(1 + \int_0^T I(G^N_u)\,du\Big)$$

By Prokhorov, we get

Lemma

There exists $Q \in \mathbb{P}(\mathbb{P}(E))$ *such that*

$$Q^N \rightharpoonup Q \quad in \quad \mathbb{P}(\mathbb{P}(E)).$$

Identification of the the limit thanks to "Sznitman" argument.

Lemma

Assume that $Q \in \mathbb{P}(\mathbb{P}(C([0,+\infty),\mathbb{R}^2)) = \mathbb{P}(\mathbb{P}(E))$ *is a mixture measure obtained as a limit point of some subsequence of* Q^N*. Then* $\operatorname{supp} Q \subset \mathcal{S}$

$$\mathcal{S} := \left\{ \begin{array}{l} q \text{ is the law of some } \mathcal{X} \text{ solution to stoch. NS vortex eq.} \\ \forall T > 0, \quad \int_0^T I(\mathcal{L}(X_s))\, ds < +\infty \end{array} \right\} = \mathcal{S}_0 \cap \mathcal{S}_1$$

- $q \approx \mathcal{X}$ solves the stoch. NS vortex eq. iif for all times s, t, ψ, φ...

$$\mathcal{F}(q) := \iint_{E^2} q(dx)q(dy)\psi(x(s \le t))\Big[\varphi(x(t)) - \varphi(x(s)) - \int_s^t K(x(u) - y(u)) \cdot \nabla\varphi(x(u))du - \nu \int_s^t \Delta\varphi(x(u))du\Big] = 0$$

- Q concentrated on $\mathcal{S}_0 \iff \mathbb{E}_Q[|\mathcal{F}(\cdot)|^2] = 0$ for all s, t, ψ, φ.
- $\mathbb{E}_{Q^N}[|\mathcal{F}(\cdot)|^2] \to 0$ as $N \to +\infty$.
- Continuity $\mathbb{P}(\mathbb{P}(E)) \ni R \mapsto \mathbb{E}_R[|\mathcal{F}(\cdot)|^2]$ under the condition $\mathbb{E}_R\big[\int_0^t I(\cdot_s)\, ds\big] < +\infty$.
- $\mathbb{E}_P\big[\int_0^t I(\cdot_s)\, ds\big] < +\infty$, which is equivalently $P \in \mathcal{S}_1$.

Plan

Consider $\pi \in \mathbb{P}(\mathbb{P}(E))$, $E = \mathbb{R}^2$, and define

$$\mathcal{I}(\pi) := \int_{\mathbb{P}(E)} I(\rho)\,\pi(d\rho), \quad \mathcal{I}'(\pi) := \sup_{j \ge 1} I(\pi_j) = \lim_{j \to \infty} I(\pi_j)$$

where π_j is given by (the easy part of) Hewitt and Savage theorem

$$\pi_j := \int_{\mathbb{P}(E)} \rho^{\otimes j}\,\pi(d\rho) \in \mathbb{P}_{sym}(E^j).$$

From $I(f^{\otimes j}) = I(f)$ (good normalization), I is lsc, convex, proper and ≥ 0 on $\mathbb{P}_{sym}(E^j)$, $\forall\, j \ge 1$, and $\mathcal{I}'$ is linear on disjoint convex combination, we deduce

Theorem (representation formula, Hauray-M.)

$$\forall\, \pi \in \mathbb{P}(\mathbb{P}(E)) \quad \mathcal{I}(\pi) = \mathcal{I}'(\pi).$$

A similar formula is known for the entropy (Robinson-Ruelle, 1967)

Application: the Fisher information is Γ-lsc in the sense

$$\mathbb{P}_{sym}(E^N) \ni F^N \rightharpoonup \pi \in \mathbb{P}(\mathbb{P}(E)) \text{ implies } \mathcal{I}(\pi) \le \liminf I(F^N).$$

One line proof: for any $j \ge 1$ by lsc of I_j: $I_j(\pi_j) \le \liminf I_j(F^N_j) \le \liminf I_N(F^N)$.

We know (from tightness) that

$$\mathcal{L}(\mu^N_{\mathcal{X}}) \rightharpoonup Q \quad \text{weakly in } \mathbb{P}(\mathbb{P}(E))$$

with here $E := C([0,\infty);\mathbb{R}^2)$. We define $Q_t :=$ projection on the section $\mathbb{P}(\mathbb{P}(\{t\} \times \mathbb{R}^2))$ so that

$$G^N_t = \mathcal{L}(\mathcal{X}^N_t), \mathcal{L}(\mu^N_{\mathcal{X}^N_t}) \rightharpoonup Q_t \quad \text{weakly in } \mathbb{P}(\mathbb{P}(\mathbb{R}^2))$$

As a consequence, by Fubini, Γ-lsc property of the Fisher information and Fatou

$$\begin{aligned}
\int_{\mathbb{P}(E)} \int_0^T I(q_t) dt\, Q(dq) &= \int_0^T \int_{\mathbb{P}(E))} I(q_t)\, Q(dq)\, dt \\
&= \int_0^T \mathcal{I}(Q_t)\, dt \\
&\le \int_0^T \liminf_N I(G^N_t) dt \le \liminf_N \int_0^T I(G^N_t) dt.
\end{aligned}$$

This last quantity is finite, so that $\int_0^T I(q_t) dt < \infty$ Q-a.s. and $\operatorname{supp} Q \subset \mathcal{S}_1$.

Plan

We claim that

$$\forall\, q \in \mathcal{S},\ q = \mathcal{L}(\mathcal{X}), \quad w_t := \mathcal{L}(\mathcal{X}_t) = \bar{w}_t := \text{ unique solution of NS vortex.}$$

• First, for $q \in \mathcal{S}$, it is clear that $w_t := \mathcal{L}(\mathcal{X}_t)$ satisfies

$$w \in C([0,T);\mathbb{P}(R^2)), \quad I(w) \in L^1(0,T)$$

and w is a weak solution to (take $\nu = 1$)

$$\partial_t w = \Delta w + (K * w) \cdot \nabla w.$$

• Second, the uniqueness is known (Ben-Artzi 1994, Brézis 1994, improved by Gallagher-Gallay 2005) in the class of function

$$t^{1/4}\, \|w(t,.)\|_{L^{4/3}} \to 0 \text{ as } t \to 0.$$

• We have to prove by a "regularity argument" (through a renormalization trick) that w satisfies the Ben-Artzi & Brézis criterium

If $q \in \mathcal{S}$ we consider the associated linear SDE

$$\mathcal{X}_t = \mathcal{X}_0 + \int_0^t u_s(\mathcal{X}_s)\, ds + \nu B_t, \qquad u_s = K * \bar{w}_s,$$

Lemma

Strong uniqueness for the previous linear SDE holds (and thus weak uniqueness by Yamada-Watanabe theorem). In other words, $\mathcal{S} = \{\bar{q}\}$.

Sketch of the proof

- Use argument used by Crippa-De Lellis for uniqueness in ODE with low regularity.
- Two solutions $\mathcal{X}$ and $\mathcal{Y}$ satisfies

$$\forall \delta > 0, \ \ \mathbb{E}\left[\ln\Big(1 + \frac{1}{\delta} \sup_{s \le t} |\mathcal{X}_s - \mathcal{Y}_s|\Big)\right] \le \mathbb{E}\left[\int_0^t [M\nabla u_s(\mathcal{X}_s) + M\nabla u_s(\mathcal{Y}_s)]\, ds\right]$$

  where M stands for maximal function.
- Standard estimates + bounds on Fischer information helps to bound the r.h.s.
- A variant of Chebichev ineq. allows to conclude.

Plan

From

$$H(F_t^N) + \int_0^t I(F_s^N)\, ds = H(F_0^N)$$

and

$$H(w_t) + \int_0^t I(w_s)\, ds = H(w_0),$$

as well as the Γ-lsc of H and I we get if

$$H(F_0^N) \to H(w_0),$$

the conclusion

$$\begin{aligned} H(w_t) + \int_0^t I(w_s)\, ds &\leq \liminf_{N\to\infty} \left\{ H(F_t^N) + \int_0^t I(F_s^N)\, ds \right\} \\ &\leq \limsup_{N\to\infty} \left\{ H(F_t^N) + \int_0^t I(F_s^N)\, ds \right\} \\ &= \limsup_{N\to\infty} H(F_0^N) = H(w_0) \end{aligned}$$

and then

$$H(F_t^N) \to H(w_t) \quad \forall\, t > 0$$