Spectral analysis of semigroups in Banach spaces and Fokker-Planck equations

S. Mischler

(Paris-Dauphine)

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Results are picked up from

- Gualdani, M., Mouhot, *Factorization for non-symmetric operators and exponential H-Theorem*, arXiv 2010
- M., Mouhot, *Exponential stability of slowing decaying solutions to the kinetic Fokker-Planck equation*, arXiv 2014
- M., Scher, *Spectral analysis of semigroups and growth-fragmentation eqs*, to appear in Annales IHP
- Tristani, *Boltzmann equation for granular media with thermal force in a weakly inhomogeneous setting*, arXiv 2013
- Ndao, *Convergence to equilibrium for the Fokker-Planck equation with a general force field*, in progress
- Kavian, M., *The Fokker-Planck equation with subcritical confinement force*, in progress
- M., *Semigroups in Banach spaces - factorization approach for spectral analysis and asymptotic estimates*, in progress

Outline of the talk

1. Introduction
2. Spectral theory in an abstract setting
   - Spectral mapping theorem
   - Weyl's theorems and extension theorems
   - small perturbation theorem
   - Krein-Rutman theorem
3. The Fokker-Planck equations
   - Fokker-Planck equation with strong confinement
   - kinetic Fokker-Planck equation
   - Fokker-Planck equation with weak confinement
   - Discrete Fokker-Planck equation

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Revisit the spectral theory in an abstract setting

Spectral theory for general operator Λ and its semigroup $S_\Lambda(t) = e^{\Lambda t}$ in general (large) Banach space X which then only fulfills a growth estimate

$$\|S_\Lambda(t)\|_{\mathbf{B}(X)} \le C\, e^{bt}, \quad C \ge 1,\ b \in \mathbb{R}.$$

- *Spectral map Theorem* $\hookrightarrow$ $\Sigma(e^{t\Lambda}) \simeq e^{t\Sigma(\Lambda)}$ and $\omega(\Lambda) = s(\Lambda)$
- *Weyl's Theorems* $\hookrightarrow$ compact perturbation $\Sigma_{ess}(\mathcal{A} + \mathcal{B}) \simeq \Sigma_{ess}(\mathcal{B})$
  $\hookrightarrow$ distribution of eigenvalues $\sharp(\Sigma(\Lambda) \cap \Delta_a) \le N(a)$
- *Small perturbation* $\hookrightarrow$ $\Sigma(\Lambda_\varepsilon) \simeq \Sigma(\Lambda)$ if $\Lambda_\varepsilon \to \Lambda$
- *Krein-Rutman Theorem* $\hookrightarrow$ $s(\Lambda) = \sup \Re e \Sigma(\Lambda) \in \Sigma_d(\Lambda)$ when $S_\Lambda \ge 0$
- *functional space extension (enlargement and shrinkage)*
  $\hookrightarrow$ $\Sigma(L) \simeq \Sigma(\mathcal{L})$ when $L = \mathcal{L}_{|E}$
  $\hookrightarrow$ tide of (essential?) spectrum phenomenon

Structure: operator which splits as

$$\Lambda = \mathcal{A} + \mathcal{B}, \quad \mathcal{A} \prec \mathcal{B}, \quad \mathcal{B}\,\text{dissipative}$$

Examples: Boltzmann, Fokker-Planck, Growth-Fragmentation operators and $W^{\sigma,p}(m)$ weighted Sobolev spaces

• (1) Convergence rate in large Banach space for linear dissipative and hypodissipative PDEs (ex: Fokker-Planck, growth-fragmentation)

• (2) Long time asymptotic for nonlinear PDEs via the spectral analysis of linearized PDEs (ex: Boltzmann, Landau, Keller-Segel) in natural φ space

• (3) Existence, uniqueness and stability of equilibrium in "small perturbation regime" in large space (ex: inelastic Boltzmann, Wigner-Fokker-Planck, parabolic-parabolic Keller-Segel, neural network)

Is it new?

• Reminiscent ideas (e.g. Voigt 1980 on "power compactness", Bobylev 1975, Arkeryd 1988, Gallay-Wayne 2002 on the "enlargement" issue).

• first enlargement result in an abstract framework by Mouhot (CMP 2006)

• Unusual (and more quantitative) splitting

$$\Lambda = \underbrace{\mathcal{A}_0}_{compact} + \underbrace{\mathcal{B}_0}_{dissipative} = \underbrace{\mathcal{A}_\varepsilon}_{smooth} + \underbrace{\mathcal{A}^c_\varepsilon + \mathcal{B}_0}_{dissipative}$$

• Our set of results is the first systematic and general (semigroup and space) works on the "principal" part of the spectrum and the semigroup

• Fredholm, Hilbert, Weyl, Stone (Functional Analysis & semigroup Hilbert framework) $\leq$ 1932

• Hyle, Yosida, Phillips, Lumer, Dyson, Dunford, Schwartz, ...
(semigroup Banach framework & dissipative operator) 1940-1960

• Kato, Pazy, Voigt (analytic operator, positive operator) 1960-1975

• Engel, Nagel, Gearhart, Metz, Diekmann, Prüss, Arendt, Greiner, Blake, Mokhtar-Kharoubi, Yao, ... 1975-

Still active research field

• Semigroup school (≥ 0, bio): Arendt, Blake, Diekmann, Engel, Gearhart, Greiner, Metz, Mokhtar-Kharoubi, Nagel, Prüss, Webb, Yao, ...

• Schrodinger school / hypocoercivity and fluid mechanic: Batty, Burq, Duyckaerts, Gallay, Helffer, Hérau, Lebeau, Nier, Sjöstrand, Wayne, ...

• Probability school (diffusion equation): Bakry, Barthe, Bobkov, Cattiaux, Douc, Gozlan, Guillin, Fort, Ledoux, Roberto, Röckner, Wang, ...

• Kinetic school ($\sim$ Boltzmann):

▷ Guo, Strain, ..., in the spirit of Hilbert, Carleman, Grad, Ukai works (Spectral analysis of the linearized (in)homogeneous Boltzmann equation, existence and convergence to the equilibrium in "small spaces")

▷ Carlen, Carvalho, Toscani, Otto, Villani, ... (log-Sobolev inequality)

▷ Desvillettes, Villani, Mouhot, Baranger, Neuman, Strain, Dolbeault, Schmeiser, ... (Poincaré inequality & hypocoercivity)

▷ Arkeryd, Esposito, Pulvirenti, Wennberg, Mouhot, ... (Spectral analysis of the linearized (in)homogeneous Boltzmann equation, existence and convergence to the equilibrium in "large spaces")

A list of related papers

- Mouhot, *Rate of convergence to equilibrium for the spatially homogeneous Boltzmann equation with hard potentials*, CMP 2006
- M., Mouhot, *Stability, convergence to self-similarity and elastic limit for the Boltzmann equation for inelastic hard spheres*, CMP 2009
- Arnold, Gamba, Gualdani, M., Mouhot, Sparber, *The Wigner-Fokker-Planck equation: Stationary states and large time behavior*, M3AS 2012
- Cañizo, Caceres, M., *Rate of convergence to the remarkable state for fragmentation and growth-fragmentation equations*, JMPA 2011 & CAIM 2011
- Egaña, M., *Uniqueness and long time asymptotic for the Keller-Segel equation - the parabolic-elliptic case*, arXiv 2013
- Carrapatoso, *Exponential convergence ... homogeneous Landau equation*, arXiv 2013
- Tristani, *Boltzmann equation for granular media with thermal force in a weakly inhomogeneous setting*, arXiv 2013
- Carrapatoso, M., *Uniqueness and long time asymptotic for the parabolic-parabolic Keller-Segel equation*, arXiv 2014
- Briant, Merino-Aceituno, Mouhot, *From Boltzmann to incompressible Navier-Stockes in Sobolev spaces with polynomial weight*, arXiv 2014
- M., Quiñinao, Touboul, *On a kinetic FitzHugh-Nagumo model of neuronal network*, arXiv 2015
- Carrapatoso, Tristani, Wu, *On the Cauchy problem ... non homogeneous Landau equation*, arXiv 2015

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For a given operator Λ in a Banach space X, we want to prove

$$(1) \quad \Sigma(\Lambda) \cap \Delta_a = \{\xi_1\} \text{ (or } = \emptyset), \quad \xi_1 = 0$$

with $\Sigma(\Lambda)$ = spectrum, $\Delta_\alpha := \{z \in \mathbb{C},\ \Re e\, z > \alpha\}$

$$(2) \quad \Pi_{\Lambda,\xi_1} = \text{finite rank projection}, \quad \text{i.e. } \xi_1 \in \Sigma_d(\Lambda)$$

$$(3) \quad \|S_\Lambda(I - \Pi_{\Lambda,\xi_1})\|_{X \to X} \le C_a\, e^{at}, \quad a < \Re e \xi_1$$

Definition:
We say that Λ is a-hypodissipative iff $\|e^{t\Lambda}\|_{X \to X} \le C\, e^{at}$, $C \ge 1$
spectral bound $= s(\Lambda) := \sup \Re e \Sigma(\Lambda)$
growth bound $= \omega(\Lambda) := \inf\{a \in \mathbb{R}, \text{ s.t. } L - a \text{ is hypodissipative }\}$

Consider the generator Λ of a semigroup in several Banach spaces denoted by E, $\mathcal{E}$, X, $\mathcal{X}$, Y, $\mathcal{Y}$
We assume that Λ has the following splitting structure

$$\Lambda = \mathcal{A} + \mathcal{B},$$

and we make the following boundedness hypothesizes for a given $a \in \mathbb{R}$:
• $\mathcal{A}$ is $\mathcal{B}^{\zeta'}$-bounded with $0 \le \zeta' \le 1$.
• $\mathcal{B}$ is $\mathcal{A}$-power dissipative in $\mathcal{X}$

$$\forall\, \ell, \quad S_\mathcal{B} * (\mathcal{A} S_\mathcal{B})^{(*\ell)}(t)\, e^{-at} \in L^\infty(\mathbb{R}_+; \mathbf{B}(\mathcal{X})).$$

• $\mathcal{A}$ is right $S_\mathcal{B}$-power regular in $(\mathcal{X}, \mathcal{Y})$, $\mathcal{Y} \subset \mathcal{X}$

$$\exists\, n \ge 1, \qquad (\mathcal{A} S_\mathcal{B})^{(*n)}(t)\, e^{-at} \in L^1(\mathbb{R}_+; \mathbf{B}(\mathcal{X}, \mathcal{Y})).$$

or • $\mathcal{A}$ is left $S_\mathcal{B}$-power regular in $(\mathcal{X}, \mathcal{Y})$, $\mathcal{Y} \subset \mathcal{X}$

$$\exists\, n \ge 1, \qquad (S_\mathcal{B} \mathcal{A})^{(*n)}(t)\, e^{-at} \ \in L^1(\mathbb{R}_+; \mathbf{B}(\mathcal{X}, \mathcal{Y})).$$

Theorem 1. (Gearhart, Prüss, Gualdani, M., Mouhot, Scher)
Let $\Lambda \in \mathbf{G}(X)$ and $a^* \in \mathbb{R}$. The following equivalence holds:

(1) The operator Λ is a-hypodissipative in X $\forall\, a > a^*$;

(2) $L := \Lambda_{|Y}$ is a-hypodissipative in $Y \subset X$ $\forall\, a > a^*$, $\Lambda = \mathcal{A} + \mathcal{B}$, $\mathcal{B}$ is $\mathcal{A}$-power dissipative in X, $\mathcal{A}$ is right $S_{\mathcal{B}}$-power regular in (X, Y).

(2′) $\mathcal{L}_{|X} = \Lambda$ for some operator $\mathcal{L}$ which is a-hypodissipative in $Y \supset X$ for any $a > a^*$, $\Lambda = A + B$, B is A-power dissipative in X, A is left S_B-power regular in (Y, X).

(3) $\Sigma(\Lambda) \cap \Delta_{a^*} = \emptyset$ and Λ splits as $\Lambda = \mathcal{A} + \mathcal{B}$, $\mathcal{A} \preceq \mathcal{B}^{\zeta'}$ for some $0 \le \zeta' < \zeta \le 1$, $\mathcal{B}$ is $\mathcal{A}$-power dissipative in X, $\mathcal{A}$ is left $S_{\mathcal{B}}$-power regular in $(X, D(\mathcal{B}^\zeta))$.

(3′) $\Sigma(\Lambda) \cap \Delta_{a^*} = \emptyset$ and Λ splits as $\Lambda = \mathcal{A} + \mathcal{B}$, $\mathcal{A} \in \mathbf{B}(X, D(\mathcal{B}^{-\zeta'}))$ for some $0 \le \zeta' < \zeta \le 1$, $\mathcal{B}$ is $\mathcal{A}$-power dissipative in X, $\mathcal{A}$ is right $S_{\mathcal{B}}$-power regular in $(D(\mathcal{B}^{-\zeta}, X)$.

(4) if X is a Hilbert space, the resolvent R_Λ is uniformly bounded on Δ_a, $\forall\, a > a^*$.

We iterate the Duhamel formula

$$S_\Lambda = S_\mathcal{B} \quad + \quad S_\Lambda * (\mathcal{A} S_\mathcal{B})$$
$$\text{or} \quad + \quad (S_\mathcal{B} \mathcal{A}) * S_\Lambda$$

but stop the Dyson-Phillips series (the Dyson-Phillips series corresponds to the choice $n = \infty$)

$$S_\Lambda = \sum_{\ell=0}^{n-1} S_\mathcal{B} * (\mathcal{A} S_\mathcal{B})^{(*\ell)} \quad + \quad S_\Lambda * (\mathcal{A} S_\mathcal{B})^{(*n)}$$
$$\text{or} \quad + \quad (S_\mathcal{B} \mathcal{A})^{(*n)} * S_\Lambda.$$

We observe that the n terms are $\mathcal{O}(e^{at})$.

For $f \in D(\Lambda)$, we use the inverse Laplace formula

$$S_\Lambda(t)f \quad = \quad \frac{i}{2\pi} \int_{a-i\infty}^{a+i\infty} e^{zt} \, R_\Lambda(z) f \, dz$$

where R_Λ stands for the resolvent operator defined by

$$R_\Lambda(z) = (\Lambda - z)^{-1}.$$

and the resolvent identity

$$R_\Lambda(a + is) = (1 + (a - b) \, R_\Lambda(a + is)) \, R_\Lambda(b + is).$$

Using the Cauchy-Schwartz inequality and Plancherel's identity, we get

$$\begin{aligned} \|S_\Lambda(t) f\|_X &\lesssim e^{at} \left(\int_{-\infty}^{\infty} \|R_\Lambda(a + is) f\|_X^2 \, ds \right)^{1/2} \\ &\lesssim e^{at} (1 + (b - a) \|R_\Lambda\|_{L^\infty(\Delta_a)}) \left(\int_{-\infty}^{\infty} \|R_\Lambda(b + is) f\|_X^2 \, ds \right)^{1/2} \\ &\lesssim e^{at} (1 + (b - a) \|R_\Lambda\|_{L^\infty(\Delta_a)}) \left(\int_{-\infty}^{\infty} \|e^{(\Lambda - b)t}\|_{\mathbf{B}(X)}^2 \, ds \right)^{1/2} \|f\|_X \end{aligned}$$

Proof of the spectral mapping theorem (2) ⇒ (1)

We start again with the stopped Dyson-Phillips series

$$S_\Lambda = \sum_{\ell=0}^{N-1} S_\mathcal{B} * (\mathcal{A}S_\mathcal{B})^{(*\ell)} + S_\Lambda * (\mathcal{A}S_\mathcal{B})^{(*N)} = \mathcal{T}_1 + \mathcal{T}_2.$$

The first $N-1$ terms are fine. For the last one, we use the inverse Laplace formula

$$\begin{aligned} \mathcal{T}(t)f &= \frac{i}{2\pi} \int_{a-i\infty}^{a+i\infty} e^{zt}\, R_\Lambda(z)(\mathcal{A}R_\mathcal{B}(z))^N f\, dz \\ &\lesssim e^{at} \int_{a-i\infty}^{a+i\infty} \underbrace{\|R_\Lambda(z)\|}_{\in L^\infty(\uparrow_a)\,?}\ \underbrace{\|(\mathcal{A}R_\mathcal{B}(z))^N\|}_{\in L^1(\uparrow_a)\,?}\, dz\, \|f\|, \end{aligned}$$

where $\uparrow_a := \{z = a + iy,\ y \in \mathbb{R}\}$.

We assume (in order to make the proof simpler) that $\zeta = 1$, namely

$$\|(\mathcal{A}S_{\mathcal{B}})^{(*n)}\|_{X\to X_1} = \mathcal{O}(e^{at}) \quad \forall\, t \geq 0,$$

with $X_1 := D(\Lambda) = D(\mathcal{B})$, which implies

$$\|(\mathcal{A}R_{\mathcal{B}}(z))^n\|_{X\to X_1} \leq C_a \quad \forall\, z = a + iy,\ a > a_*.$$

We also assume (for the same reason) that $\zeta' = 0$, so that

$$\mathcal{A} \in \mathcal{L}(X) \ \text{ and } \ R_{\mathcal{B}}(z) = \frac{1}{z}(R_{\mathcal{B}}(z)\mathcal{B} - I) \in \mathcal{L}(X_1, X)$$

imply

$$\|\mathcal{A}R_{\mathcal{B}}(z)\|_{X_1\to X} \leq C_a/|z| \quad \forall\, z = a + iy,\ a > a_*.$$

The two estimates together imply

$$(*) \qquad \|(\mathcal{A}R_{\mathcal{B}}(z))^{n+1}\|_{X\to X} \leq C_a/\langle z\rangle \quad \forall\, z = a + iy,\ a > a_*.$$

- In order to deal with the general case $0 \leq \zeta' < \zeta \leq 1$ one has to use some additional interpolation arguments

We write

$$R_\Lambda(1-\mathcal{V}) = \mathcal{U}$$

with

$$\mathcal{U} := \sum_{\ell=0}^{n} (-1)^\ell \, R_\mathcal{B}(\mathcal{A}R_\mathcal{B})^\ell, \quad \mathcal{V} := (-1)^{n+1} \, (\mathcal{A}R_\mathcal{B})^{n+1}$$

For M large enough

$$(**) \qquad \|\mathcal{V}(z)\| \le 1/2 \quad \forall \, z = a + iy, \; |y| \ge M,$$

and we may write the Neuman series

$$R_\Lambda(z) = \underbrace{\mathcal{U}(z)}_{\text{bounded}} \underbrace{\sum_{j=0}^{\infty} \mathcal{V}(z)^j}_{\text{bounded}}$$

For $N = 2(n+1)$, we finally get from $(*)$ and $(**)$

$$\|R_\Lambda(z)(\mathcal{A}R_\mathcal{B}(z))^N\| \le C/\langle y\rangle^2, \quad \forall \, z = a + iy, \, |y| \ge M$$

Variant 1 of Theorem 1. (M., Scher)
(0) $\Lambda = \mathcal{A} + \mathcal{B}$, where $\mathcal{A}$ is $\mathcal{B}^{\zeta'}$-bounded with $0 \le \zeta' < 1$,
(1) $\|S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)}\|_{X\to X} e^{-at} \in L^\infty(\mathbb{R}_+)$, $\forall\, a > a^*$, $\forall\, \ell \ge 0$,
(2) $\exists\, n$ $\|S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*n)}\|_{X\to D(\Lambda^\zeta)} e^{-at} \in L^1(\mathbb{R}_+)$, $\forall\, a > a^*$, with $\zeta > \zeta'$,
(3) $\Sigma(\Lambda) \cap (\Delta_{a^{**}} \backslash \Delta_{a^*}) = \emptyset$, $a^* < a^{**}$,

is equivalent to

(4) there exists a projector Π which commutes with Λ such that
$\Lambda_1 := \Lambda_{|X_1} \in \mathcal{B}(X_1)$, $X_1 := R\Pi$, $\Sigma(\Lambda_1) \subset \bar{\Delta}_{a^{**}}$

$$\|S_\Lambda(t)\,(I - \Pi)\|_{X\to X} \le C_a\, e^{at}, \quad \forall\, a > a^*$$

In particular

$$\Sigma(e^{t\Lambda}) \cap \Delta_{e^{at}} = e^{t\Sigma(\Lambda)\cap\Delta_a} \quad \forall\, t \ge 0,\; a > a^*$$

and

$$\max(s(\Lambda), a^*) = \max(\omega(\Lambda), a^*)$$

Variant 2 of Theorem 1. (Gualdani, M. & Mouhot)
Assume for some $a \in \mathbb{R}$

$$\mathcal{L} = \mathcal{A} + \mathcal{B}, \ \ L = A + B, \ \ L = \mathcal{L}_{|E}, \ \ A = \mathcal{A}_{|E}, \ \ B = \mathcal{B}_{|E}, \ \ E \subset \mathcal{E}$$

(i) $\mathcal{B}$ is $\mathcal{A}$-power dissipative in $\mathcal{E}$, B is A-power dissipative in E,

(ii) $\mathcal{A}$ is right $S_{\mathcal{B}}$-power regular in $(\mathcal{E}, E)$, A is left S_B-power regular in $(\mathcal{E}, E)$.

Then the following for $(X, \Lambda) = (E, L)$, $(\mathcal{E}, \mathcal{L})$ are equivalent:
$\exists K_\Lambda \subset \Delta_a$ compact and a projector $\Pi_\Lambda \in \mathcal{B}(X)$ which commute with Λ and satisfy $\Sigma(\Lambda_{|\Pi_\Lambda}) = K_\Lambda$, so that

$$\forall\, t \ge 0, \quad \Big\| S_\Lambda(t) - S_\Lambda(t)\, \Pi_\Lambda \Big\|_{X \to X} \le C_{\Lambda,a}\, e^{a\,t}$$

In particular $K_L = \Sigma(L) \cap \Delta_a = \Sigma(\mathcal{L}) \cap \Delta_a = K_{\mathcal{L}}$ and $\Pi_L = \Pi_{\mathcal{L}}|_E$

Theorem 2. (Ribarič, Vidav, Voigt, M., Scher)
Assume
(0) $\Lambda = \mathcal{A} + \mathcal{B}$, where $\mathcal{A}$ is $\mathcal{B}^{\zeta'}$-bounded with $0 \le \zeta' < 1$,
(1) $\|S_\mathcal{B} * (\mathcal{A} S_\mathcal{B})^{(*\ell)}\|_{X \to X} e^{-a^* t} \in L^\infty(\mathbb{R}_+)$, $\forall\, \ell \ge 0$,
in particular $\Sigma(\mathcal{B}) \cap \Delta_{a^*} = \emptyset$,
(2) $\exists\, n$ $\|(\mathcal{A} S_\mathcal{B})^{(*n)}\|_{X \to X_\zeta}\, e^{-a^* t} \in L^1(\mathbb{R}_+)$, with $\zeta > \zeta'$,
(3′) $\exists\, m$ $\|(\mathcal{A} S_\mathcal{B})^{(*m)}\|_{X \to Y} e^{-a^* t} \in L^1(\mathbb{R}_+)$, with $Y \subset X$ compact.

Then, for any $a > a^*$ there exists a finite number of eigenvalues $\xi_1, \ldots, \xi_J$ with finite algebraic multiplicity such that

$$\Sigma(\Lambda) \cap \bar{\Delta}_a = \{\xi_1, ..., \xi_J\} \subset \Sigma_d(\Sigma).$$

In particular, we deduce a "principal" perturbation Weyl's theorem:

$$\Sigma_{ess}(\Lambda) \cap \Delta_{a^*} = \Sigma_{ess}(\mathcal{B}) \cap \Delta_{a^*} = \emptyset.$$

Corollary 2. (M., Scher)
(0) $\Lambda = \mathcal{A} + \mathcal{B}$, where $\mathcal{A}$ is $\mathcal{B}^{\zeta'}$-bounded with $0 \le \zeta' < 1$,
(1) $\|S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{B}})^{(*\ell)}\|_{X \to X} \le C_\ell \, e^{at}$, $\forall\, a > a^*$, $\forall\, \ell \ge 0$,
(2) $\exists\, n$ $\|(\mathcal{A} S_{\mathcal{B}})^{(*n)}\|_{X \to X_\zeta} \, e^{-at} \in L^1(\mathbb{R}_+)$, $\forall\, a > a^*$, with $\zeta > \zeta'$,
(3′) $\exists\, m$ $\|(\mathcal{A} S_{\mathcal{B}})^{(*m)}\|_{X \to Y} \in L^1(\mathbb{R}_+)$, $\forall\, a > a^*$, with $Y \subset X$ compact,

is equivalent to

(4′) there exist $\xi_1, \dots, \xi_J \in \bar{\Delta}_a$, there exist $\Pi_1, \dots, \Pi_J$ some finite rank projectors, there exists $T_j \in \mathcal{B}(R\Pi_j)$ such that $\Lambda \Pi_j = \Pi_j \Lambda = T_j \Pi_j$, $\Sigma(T_j) = \{\xi_j\}$, in particular

$$\Sigma(\Lambda) \cap \bar{\Delta}_a = \{\xi_1, \dots, \xi_J\} \subset \Sigma_d(\Sigma)$$

and there exists a constant C_a such that

$$\|S_\Lambda(t) - \sum_{j=1}^{J} e^{t T_j} \Pi_j\|_{X \to X} \le C_a \, e^{at}, \quad \forall\, a > a^*$$

Theorem 3. (M., Scher)

Assume

(0) $\Lambda = \mathcal{A} + \mathcal{B}$, where $\mathcal{A}$ is $\mathcal{B}^{\zeta'}$-bounded with $0 \le \zeta' < 1$,

(1) $\|S_\mathcal{B} * (\mathcal{A} S_\mathcal{B})^{(*\ell)}\|_{X \to X} \le C_\ell \, e^{at}$, $\forall \, \ell \ge 0$,

(2) $\|(\mathcal{A} S_\mathcal{B})^{(*n)}\|_{X \to X_\zeta} \, e^{-at} \in L^1(\mathbb{R}_+)$, with $\zeta > \zeta'$,

(3′) $\exists m \; \|(\mathcal{A} S_\mathcal{B})^{(*m)}\|_{X \to Y} \, e^{-at} \in L^1(\mathbb{R}_+)$, with $Y \subset X$ compact,

(3″) $\|(S_\mathcal{B} \mathcal{A})^{(*m)}\|_{X \to Y} \, e^{-at} \in L^1(\mathbb{R}_+)$, for the same m and Y,

(4) $\exists$ projectors (π_N) on X with rank N, $\exists$ positive real numbers (ε_N) with $\varepsilon_N \to 0$ and $\exists C$ such that

$$\forall \, f \in Y, \;\; \|\pi_N^\perp f\|_X \le \varepsilon_N \|f\|_Y.$$

Then, there exists a (constructive) constant N^* such that

$$\sharp(\Sigma(\Lambda) \cap \Delta_a) = \sharp(\Sigma_d(\Lambda) \cap \Delta_a) \le N^*$$

and the algebraic multiplicity of any eigenvalue is less than N^*.

Theorem 4. (M. & Mouhot; Tristani)
Consider a family (Λ_ε) of generators, $\varepsilon \geq 0$. Assume
(0) $\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ in X_i, $X_{-1} \subset\subset X_0 = X \subset\subset X_1$, $\mathcal{A}_\varepsilon \prec \mathcal{B}_\varepsilon$
(1) $\|S_{\mathcal{B}_\varepsilon} * (\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{(*\ell)}\|_{X_i \to X_i} e^{-at}$ bdd L^∞_t, $\forall\, a > a^*$, $\forall\, \ell \geq 0$, $i = 0, \pm 1$
(2) $\|(\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{(*n)}\|_{X_i \to X_{i+1}} e^{-at}$ bounded $L^1(\mathbb{R}_+)$, $\forall\, a > a^*$, $i = 0, -1$
(3) $X_{i+1} \subset D(\mathcal{B}_{\varepsilon|X_i}), D(\mathcal{A}_{\varepsilon|X_i})$ for $i = -1, 0$ and

$$\|\mathcal{A}_\varepsilon - \mathcal{A}_0\|_{X_i \to X_{i-1}} + \|\mathcal{B}_\varepsilon - \mathcal{B}_0\|_{X_i \to X_{i-1}} \leq \eta_1(\varepsilon) \to 0, \quad i = 0, 1,$$

(4) the limit operator satisfies (in both spaces X_0 and X_1)

$$\Sigma(\Lambda_0) \cap \Delta_a = \{\xi_1, ..., \xi_k\} \subset \Sigma_d(\Lambda_0).$$

Then

$$\Sigma(\Lambda_\varepsilon) \cap \Delta_a = \{\xi^\varepsilon_{1,1}, ..., \xi^\varepsilon_{1,d^\varepsilon_1}, ..., \xi^\varepsilon_{k,1}, ..., \xi^\varepsilon_{k,d^\varepsilon_k}\} \subset \Sigma_d(\Lambda_\varepsilon),$$
$$|\xi_j - \xi^\varepsilon_{j,j'}| \leq \eta(\varepsilon) \to 0 \quad \forall\, 1 \leq j \leq k,\ \forall\, 1 \leq j' \leq d_j;$$
$$\dim R(\Pi_{\Lambda_\varepsilon, \xi^\varepsilon_{j,1}} + ... + \Pi_{\Lambda_\varepsilon, \xi^\varepsilon_{j,d_j}}) = \dim R(\Pi_{\Lambda_0, \xi_j});$$

Theorem 5. (M. & Scher) Consider a semigroup generator Λ on a "nice" Banach lattice X, and assume

(1) Λ such as the semigroup Weyl's Theorem for some $a^* \in \mathbb{R}$;

(2) $\exists b > a^*$ and $\psi \in D(\Lambda^*) \cap X'_+ \backslash \{0\}$ such that $\Lambda^* \psi \ge b\, \psi$;

(3) S_Λ is positive (and Λ satisfies Kato's inequalities);

(4) $-\Lambda$ satisfies a strong maximum principle.

Defining $\lambda := s(\Lambda)$, there holds

$$a^* < \lambda = \omega(\Lambda), \qquad \lambda \text{ is simple},$$

and there exists $0 < f_\infty \in D(\Lambda)$ and $0 < \phi \in D(\Lambda^*)$ such that

$$\Lambda f_\infty = \lambda\, f_\infty, \quad \Lambda^* \phi = \lambda\, \phi, \quad R\Pi_{\Lambda,\lambda} = \text{Vect}(f_\infty),$$

and then

$$\Pi_{\Lambda,\lambda} f = \langle f, \phi \rangle\, f_\infty \quad \forall\, f \in X.$$

Moreover, there exist $\alpha \in (a^*, \lambda)$ and $C > 0$ such that for any $f_0 \in X$

$$\|S_\Lambda(t) f_0 - e^{\lambda t}\, \Pi_{\Lambda,\lambda} f_0\|_X \le C\, e^{\alpha t}\, \|f_0 - \Pi_{\Lambda,\lambda} f_0\|_X \qquad \forall\, t \ge 0.$$

• In the application of these Theorems one can take $n = 1$ in the simplest situations (most of space homogeneous equations in dimension $d \leq 3$), but one need to take $n \geq 2$ for the space inhomogeneous Boltzmann equation

• Open problem: (1) Beyond the "dissipative case"?

▷ example of the Fokker-Planck equation for "soft confinement potential" and relation with "weak Poincaré inequality" by Röckner-Wang

▷ Links with semi-uniform stability by Lebeau & co-authors, Burq, Liu-R, Bátkal-E-P-S, Batty-D, ...

▷ applications to the Boltzmann and Landau equations associated with "soft potential"

▷ Abstract theory in the "weak dissipative case"

(2) Spectral analysis for singular perturbation problems

Outline of the talk

1. Introduction
2. Spectral theory in an abstract setting
   - Spectral mapping theorem
   - Weyl's theorems and extension theorems
   - small perturbation theorem
   - Krein-Rutman theorem
3. The Fokker-Planck equations
   - Fokker-Planck equation with strong confinement
   - kinetic Fokker-Planck equation
   - Fokker-Planck equation with weak confinement
   - Discrete Fokker-Planck equation

Consider the Fokker-Planck equation

$$\partial_t f = \Lambda f = \Delta_v f + \mathrm{div}_v(F\, f)$$

with a force field term F such that

$$F(v) \approx v \langle v \rangle^{\gamma-2} \quad \gamma \geq 1$$

and an initial datum

$$f(0) = f_0 \in W^{\sigma,p}(m) \quad (\text{means } m f_0 \in W^{\sigma,p}).$$

Here $p \in [1, \infty]$, $\sigma \in \{-1, 0, 1\}$ and m is a polynomial weight

$$m = \langle v \rangle^k, \quad k > k^*(p, \sigma, \gamma), \quad \text{if } \gamma \geq 2,$$

or stretch exponential weight

$$m = e^{\kappa \langle v \rangle^s}, \quad s \in [2-\gamma, \gamma], \ \gamma \geq 1,$$

Theorem 6. Gualdani, M., Mouhot, Ndao

There exists a unique "smooth", positive and normalized steady state f_∞

$$\Lambda f_\infty = \Delta_v f_\infty + \mathrm{div}_v(F\, f_\infty) = 0.$$

That one is given by $f_\infty = \exp(-\Phi)$ is $F = \nabla\Phi$.
There exist $a = a_\sigma(p, m) < 0$, $C \geq 1$, such that for any $f_0 \in W^{\sigma,p}(m)$

$$\|f(t) - \langle f_0\rangle\, f_\infty\|_{W^{\sigma,p}(m)} \leq C\, e^{at}\, \|f_0 - \langle f_0\rangle\, f_\infty\|_{W^{\sigma,p}(m)}.$$

If $\gamma \in [2, 2 + 1/(d-1)]$,

$$W_1(f(t), \langle f_0\rangle\, f_\infty) \leq C\, e^{at}\, W_1(f_0, \langle f_0\rangle\, f_\infty)$$

Proof: We introduce the splitting $\Lambda = \mathcal{A} + \mathcal{B}$, with $\mathcal{A}$ a multiplicator operator

$$\mathcal{A}f = M\chi_R(v)f, \quad \chi_R(v) = \chi(v/R), \quad 0 \leq \chi \leq 1,\ \chi \in \mathcal{D}(\mathbb{R}^d),$$

so that $\mathcal{A}$ is bounded operator and $\mathcal{B}$ is a elliptic operator.

- the estimate

$$(1) \quad \|S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{B}})^{(*k)}\|_{\mathbf{B}(X)} \le C_k \, e^{-t}$$

is a consequence of the fact that
▷ $\mathcal{A} \in \mathbf{B}(X)$, $X = W^{\sigma,p}(m)$
▷ $\mathcal{B}$ is a-dissipative in $X = W^{\sigma,p}(m)$. For $\sigma = 0$, $p \in [1,\infty)$ that is a consequence of the estimates

$$\int (\Lambda f) f^{p-1} m^p = (p-1) \int |\nabla (fm)|^2 (fm)^{p-1} + \int (fm)^p \psi$$

$$\psi = (\frac{2}{p} - 1)\frac{\Delta m}{m} + 2(1-\frac{1}{p})\frac{|\nabla m|^2}{m^2} + (1-\frac{1}{p})\mathrm{div} F - F \cdot \frac{\nabla m}{m} \; (< 0)$$

- the estimate

$$(2) \quad \|S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{B}})^{(*n)}\|_{\mathbf{B}(L^1(m),H^1(m))} \le C_n \, e^{at}$$

use a “Nash + regularity” trick. More precisely, introducing

$$\mathcal{F}(t,h) := \|h\|^2_{L^1(m)} + t^{\bullet} \|h\|^2_{L^2(m)} + t^{\bullet} \|\nabla_v h\|^2_{L^2(m)}$$

we are able to prove (for convenient exponents $\bullet > 1$)

$$\frac{d}{dt}\mathcal{F}(t, S_{\mathcal{B}}(t)h) \le 0 \quad \text{and then} \quad \|S_{\mathcal{B}}(t) h\|^2_{H^1(m)} \le \frac{1}{t^{-\bullet}} \|h\|^2_{L^1(m)}$$

The kinetic Fokker-Planck equation (with strong confinement)

Consider the Fokker-Planck equation

$$\partial_t f = \Lambda f = -v \cdot \nabla_x f + \nabla_x \Psi \cdot \nabla_v f + \Delta_v f + \mathrm{div}_v(v\, f) \quad \text{in } \mathbb{R}^d \times \mathbb{R}^d$$

with a confinement potential

$$\Psi(x) \approx \frac{1}{\beta}|x|^\beta \quad \beta \geq 1, \quad H := 1 + |v|^2 + \Psi(x)$$

and an initial datum

$$f(0) = f_0 \in W^{\sigma,p}(m), \quad m = H^k \text{ or } = e^{\kappa H^s}.$$

Theorem 7. M. & Mouhot

There exist $a = a_\sigma(p,m) < 0$, $C \geq 1$, such that for any $f_0 \in W^{\sigma,p}(m)$

$$\|f(t) - \langle f_0\rangle\, f_\infty\|_{W^{\sigma,p}(m)} \leq C\, e^{at}\, \|f_0 - \langle f_0\rangle\, f_\infty\|_{W^{\sigma,p}(m)}.$$

We introduce

$$\Lambda = \mathcal{A} + \mathcal{B}, \quad \mathcal{A}h := M\chi_R(x,v)\,h$$

so that $\mathcal{A}$ is a bounded operator.

- For exhibiting the dissipativity properties of $\mathcal{B}$, we introduce the weight multiplier:

$$M(x,v) := m\,w, \quad w := 1 + \frac{1}{2}\frac{x\cdot v}{H_\alpha}, \quad H_\alpha := 1 + \alpha\frac{\langle x\rangle^\beta}{\beta} + \frac{1}{\alpha}\frac{|v|^2}{2},$$

and we show for instance

$$\int (\mathcal{B}f) f^{p-1} M^p \le a \int f^p M^p, \quad a < 0.$$

- For the regularizing estimate

$$(2) \quad \|S_\mathcal{B} * (\mathcal{A}S_\mathcal{B})^{(*n)}\|_{\mathbf{B}(L^1(m),H^1(mm_0))} \le C_n\, e^{at},$$

we use a "Nash-Hormander-Hérau-Villani" hypoelliptic trick. More precisely, introducing

$$\mathcal{F}(t,h) := \|h\|^2_{L^1(m)} + t^\bullet\|h\|^2_{L^2(m)} + t^\bullet\|\nabla_v h\|^2_{L^2(m)} + t^\bullet(\nabla_v h, \nabla_x h)_{L^2(m)} + t^\bullet\|\nabla_x h\|^2_{L^2(m)}$$

we are able to prove (for convenient exponents $\bullet \ge 1$)

$$\frac{d}{dt}\mathcal{F}(t, S_\mathcal{B}(t)h) \le 0, \quad \forall\, t \in [0,T].$$

Consider the Fokker-Planck equation

$$\partial_t f = \Lambda f = \Delta_v f + \mathrm{div}_v(F\, f)$$

with a weak force field term F

$$F(v) \approx v\langle v\rangle^{\gamma-2} \quad \gamma \in (0,1).$$

Theorem 8. Kavian & M.

There exists a unique "smooth", positive and normalized steady state f_∞.
For any $f_0 \in L^p(m)$

$$\|f(t) - \langle f_0\rangle\, f_\infty\|_{L^p} \le \Theta(t)\,\|f_0 - \langle f_0\rangle\, f_\infty\|_{L^p(m)},$$

with

$$\begin{aligned} \Theta(t) &= \frac{C}{\langle t\rangle^K}, \quad K \sim \frac{k - k^*(p)}{2-\gamma} \quad \text{if} \quad m = \langle x\rangle^k \\ &= Ce^{-\lambda t^\sigma}, \quad \sigma \sim \frac{s}{2-\gamma} \quad \text{if} \quad m = m = e^{\kappa\,\langle x\rangle^s}. \end{aligned}$$

▷ Improve Toscani, Villani, 2000 (based on log-Sobolev inequality)
& Röckner, Wang, 2001 (based on weak Poincaré inequality)

- We make the same splitting $\Lambda = \mathcal{A} + \mathcal{B}$, $\mathcal{A}f = M\chi_R f$, but now $\mathcal{B}$ is not a-dissipative anymore with $a < 0$.
- However, for $p \in [1, \infty)$, that is a consequence of the estimates

$$\int (\Lambda f) f^{p-1} m^p = (p-1) \int |\nabla (fm)|^2 (fm)^{p-1} + \int (fm)^p \psi$$

$$\text{if} \quad m = \langle v \rangle^k \text{ then } \psi \sim -F \cdot \frac{\nabla m}{m} \sim -\langle v \rangle^{\gamma - 2} \text{ is not uniformly negative !}$$

- We choose $E_j = L^1(\langle v \rangle^{k_j})$ with $k_0 < k_1 < k_2$, and we can prove

$$\frac{d}{dt} \|f_{\mathcal{L}}\|_{E_1} \le -\lambda \, \|f_{\mathcal{L}}\|_{E_0}, \qquad \frac{d}{dt} \|f_{\mathcal{L}}\|_{E_2} \le 0,$$

for some constant $\lambda > 0$. Since for some $\alpha \in (1, \infty)$, $C_\alpha \in (1, \infty)$

$$\|f\|_{E_1} \le C_\alpha \, \|f\|_{E_0}^{1/\alpha} \, \|f\|_{E_2}^{1-1/\alpha}, \quad \forall f \in E_2.$$

We immediately deduce the (closed) differential inequality

$$\frac{d}{dt} \|f_{\mathcal{L}}\|_{E_1} \le -\lambda \, C_\alpha^{-\alpha} \, \|f_0\|_{E_2}^{1-\alpha} \|f_{\mathcal{L}}\|_{E_1}^{\alpha},$$

that we readily integrate, and we end with

$$\|f_{\mathcal{L}}(t)\|_{E_1} \le \frac{C_\alpha^{\frac{\alpha}{\alpha-1}}}{((\alpha-1)\lambda)^{\frac{1}{\alpha-1}}} \, \frac{\|f_0\|_{E_2}}{t^{\frac{1}{\alpha-1}}}, \quad \forall t > 0.$$

Consider the discrete FP equation (associated to a rescaled random walk)

$$\partial_t f = \Lambda_\varepsilon f = \frac{1}{\varepsilon^2}(k_\varepsilon * f - f) + \mathrm{div}_v(v\, f)$$

for any $\varepsilon > 0$ and a given kernel $k_\varepsilon(v) = \varepsilon^{-d} k(\varepsilon^{-1} v)$,

$$\kappa\, \mathbf{1}_{B(0,r)} \le k \in W^{2,1}(\mathbb{R}^d) \cap L^1_{2q+4} \quad \int_{\mathbb{R}^d} k(v) \begin{pmatrix} 1 \\ v \\ v \otimes v \end{pmatrix} dx = \begin{pmatrix} 1 \\ 0 \\ 2 I_d \end{pmatrix}.$$

with $\kappa, r > 0$, $q > d/2 + 4$.

Theorem 8. M. & Tristani

For any $\varepsilon \in (0, \varepsilon_0)$, $\varepsilon_0 > 0$, there exists a unique "smooth", positive and normalized steady state G_ε.
For any $f_0 \in L^1(m)$, $m := \langle v \rangle^q$,

$$\| f_\varepsilon(t) - \langle f_0 \rangle\, G_\varepsilon \|_{L^1(m)} \le C\, e^{at}\, \| f_0 - \langle f_0 \rangle\, f_\infty \|_{L^1(m)}, \quad \text{uniformly in } \varepsilon > 0.$$

We split Λ_ε as

$$\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon.$$

- A first possible (naive) choice is

$$\mathcal{A}_\varepsilon f := \frac{1}{\varepsilon^2} k_\varepsilon * f \quad \text{compact}$$

and then $\mathcal{B}_\varepsilon$ is ε^{-2}-dissipative. Applying the Krein-Rutman that gives the existence, uniqueness and (ε dependent) exponential stability of a steady state G_ε.

- A second possible choice is

$$\mathcal{A}_\varepsilon f := M \chi_R (k_\varepsilon * f).$$

One can show that $\mathcal{B}_\varepsilon$ is still a-dissipative with $a < 0$. That choice is compatible with the splitting of the limit Fokker-Planck operator

$$\Lambda f = \Delta_v f + \mathrm{div}_v(vf), \quad \mathcal{A} f = M \chi_R f$$

Uniform smoothing effect on the product $\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}$

- The following elementary estimate holds

$$\|k_\varepsilon *_x f\|^2_{\dot H^1} \le K\, I_\varepsilon(f),$$

with

$$I_\varepsilon(f) := \frac{1}{2\varepsilon^2} \int_{\mathbb{R}^d\times\mathbb{R}^d} (f(x) - f(y))^2\, k_\varepsilon(x-y)\, dx\, dy.$$

- The energy estimate for the evolution equation

$$\partial_t f = \mathcal{B} f$$

writes

$$\begin{aligned} \frac{d}{dt}\|f_t\|^2_{L^2(m)} &\lesssim -I_\varepsilon(f_t) - \|f_t\|^2_{L^2(m)} \\ &\le 2a\|k_\varepsilon * f_t\|^2_{\dot H^1} + 2a\|f_t\|^2_{L^2(m)} \end{aligned}$$

which implies

$$\int_0^\infty \|\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}(t) f\|^2_{H^1} e^{-2at}\, dt \approx \int_0^\infty \|k_\varepsilon * f_t\|^2_{\dot H^1} e^{-2at}\, dt \lesssim \|f_0\|^2_{L^2}$$