Neuronal Network: overarching framework and examples

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Results are picked up from

- M., Quiñinao, Touboul, *On a kinetic FitzHugh-Nagumo model of neuronal network*, arXiv 2015, to appear in Comm. Math. Phys.
- M., Weng, *Relaxation in time elapsed neuron network models in the weak connectivity regime*, arXiv 2015
- M., Quiñinao, Touboul, *A survey on kinetic models and methods for neuronal networks*, in progress

Outline of the talk

1. Overarching framework for Neuronal Network
   - Models
   - Qualitative analyze and weak connectivity regime
2. Relaxation in time elapsed neuron network models
   - Model
   - Spectral analysis
3. On a FitzHugh-Nagumo statistical model for neural networks
   - Well-posedness and existence of steady states
   - Spectral analysis for vanishing connectivity
   - Spectral analysis for small connectivity

Outline of the talk

$z \simeq$ state variable of a neuron (membrane potential, elapsed time since last discharge, ...), $\mathbf{Z}$ the set of state variables.

The state $Z_t \in \mathbf{Z}$ of one neuron is a time dependent random variable and evolves accordingly to the SDE

$$dZ_t = F(Z_t, M_t, d\mathcal{L}_t), \tag{1}$$

$M_t :=$ given neuron network activity,
$d\mathcal{L}_t :=$ noise process (Brownian, Poissonian)

We are only saying that the neuronal network environment is here known and then one neuron evolves according to a general Markov (no autonomous) process.

Problem 1. Mathematical analyse of equation (1). That is a job for probabilists. We will come back on that issue later.

Consider an finite assembly $(Z^1, ..., Z^N)$ of neurons in interaction.
The evolution equation for each neuron Z^i_t is exactly the same

$$(2a) \qquad dZ^i_t = F(Z^i_t, M_t, d\mathcal{L}^i_t),$$

excepted that the neuronal network activity M_t is determined by the electric activity of every neurons:

$$(2b) \qquad M_t = \mathcal{M}\Big[\frac{1}{N}\sum_{i=1}^N \delta_{Z^i_{[0,t]}}\Big]$$

and $\mathcal{L}^i_t$ are independent stochastic noise processes.

$\Rightarrow$ Neurons are indistinguishable. Simple and quite weak interaction (possibly with delay) between neurons through a same quantity $M(t)$

Problem 2. Mathematical analyse of equation (2) is really a job for probabilists. We will not consider that issue here.

When N becomes very large, in the mean field limit, we expect

$$\mathcal{L}(Z_s^i) \underset{N\to\infty}{\longrightarrow} f_s, \quad \text{same limit,}$$

$$\mathcal{L}(Z_s^i, Z_s^j) \underset{N\to\infty}{\longrightarrow} f_s \otimes f_s, \quad \text{asymptotic independence (chaos),}$$

and

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{Z_s^i} \underset{N\to\infty}{\longrightarrow} f_s, \quad \text{functional law of large numbers.}$$

As a consequence,

$$f_t = \mathcal{L}(Z_t) = \text{law of a typical neuron}$$

and Z_t evolves according to the same (but now nonlinear) SDE

In the mean field limit

$$\frac{1}{N}\sum_{i=1}^{N}\delta_{Z^i_{|[0,t]}} \to f_{|[0,t]} := \mathcal{L}(Z_{|[0,t]}) = \text{law of a typical neuron}$$

where Z_t evolves according to the mean field SDE

$$dZ_t = F(Z_t, M_t, d\mathcal{L}_t), \quad M_t = \mathcal{M}\Big[f_{|[0,t]}\Big]. \tag{3}$$

Problem 3. Establish the mean field limit $N \to \infty$. That is a large number law + the proof of asymptotic independence between pairs of neurons (using a propagation of chaos argument). That can be done using several strategies

- BBGKY method (BBGKY, ...)
- Semigroup method (Kac, McKean, Grünbaum, ...)
- Coupling method (Tanaka, Sznitman, ...)
- Nonlinear Martingale method (Sznitman, ...)

▷ For the first (elapsed time) model: see the recent papers by Fournier, Löcherbach, Quiñinao, Robert, Touboul.

For any test function $\varphi : \mathbf{Z} \to \mathbb{R}$ and from Itô formula, one deduces

$$\mathbb{E}[\varphi(Z_t)] - \mathbb{E}[\varphi(Z_0)] = \int_0^t \mathbb{E}[(\mathcal{L}^*_{M_s}\varphi)(Z_s)]\,ds,$$

for a suitable integro-differential linear operator $\mathcal{L}^*_m$.

As a consequence, the law $f := \mathcal{L}(Z)$ is a solution to the evolution PDE

$$(4) \qquad \partial_t f = \mathcal{L}_{M(t)} f, \quad M(t) = \mathcal{M}(f_{|[0,t]}), \quad f(0,\cdot) = f_0.$$

Other possible definition/equation on the network activity are

$$M(t,x) = \mathcal{M}(f_{|[0,t]}, x), \quad M(t) = \mathcal{M}(f_{|[0,t]}, M(t)).$$

Problem 4. Well-posedness of equation (4) and perform a qualitative analyze of the solutions.

From the fact that Z_t is a stochastic process, we find

$$\langle f_t \rangle := \int_{\mathbf{Z}} f_t = \mathbb{E}[1] \equiv 1, \quad \forall\, t \geq 0.$$

Number of neurons is conserved (that is good!) and it is the only general available qualitative information on the solutions.

Under general and mild assumptions on the operators F and $\mathcal{M}$

Theorem 1. For any $0 \leq f_0 \in L^1$, there exists (at least) one global solution $f \in C([0,\infty); L^1)$ to the PDE (4).

$\triangleright$ Be careful with Noisy Leaky Integrate and Fire model for which blow up can occur

Theorem 2. There exists (at least) one stationary solution $0 \leq G \in L^1$ to the evolution PDE (4), that is

$$0 = \mathcal{L}_M G, \quad M = \mathcal{M}(G, M). \tag{5}$$

$\triangleright$ proof: intermediate value theorem or Brouwer fixed theorem

We introduce a small parameter $\varepsilon > 0$ corresponding to the strength of the connectivity of neurons with each other, and thus to the nonlinearity of the model:

$$(4_\varepsilon) \qquad \partial_t f = \mathcal{L}_{\varepsilon M(t)} f, \quad M(t) = \mathcal{M}_\varepsilon(f_{|[0,t]}, M(t)), \quad f(0,\cdot) = f_0.$$

In the not connected regime $\varepsilon = 0$, the equation is linear

$$(4_0) \qquad \partial_t f = \mathcal{L}_0 f, \quad f(0,\cdot) = f_0.$$

Theorem 3 (Krein-Rutman).

- There exists a unique normalized and positive stationary state G_0 to the evolution PDE (4_0), that is $\mathcal{L}_0 G_0 = 0$.
- G_0 is stable for the associated semigroup: $\exists\, a < 0$, $C \geq 1$,

$$\|S_{\mathcal{L}_0}(t)\, f_0\| \leq C\, e^{at}\, \|f_0\|, \quad \forall\, t \geq 0,\ \forall\, f_0,\ \langle f_0 \rangle = 0.$$

$\triangleright$ proof: KR $\Rightarrow \exists! G_0 \geq 0$, $\langle G_0 \rangle = 1$, $\mathcal{L}_0 G_0 = \lambda G_0$, but $\lambda = 0$ because $\mathcal{L}_0^* 1 = 0$ (mass conservation) or because of Theorem 2.

Theorem 4.1. There exists $\varepsilon_0 > 0$ such that the normalized and positive stationary state G_ε is unique for any $\varepsilon \in (0, \varepsilon_0)$.

$\triangleright$ $\mathcal{L}_0^{-1}$ exists and use (half of the) implicit function theorem

Theorem 4.2. There exists $\varepsilon_1 > 0$ such that G_ε is exponentially linearly stable for the associated semigroup: $\exists\, a < 0,\ C \ge 1,$

$$\|S_{\mathcal{L}_\varepsilon}(t)\, f_0\| \le C\, e^{at}\, \|f_0\|, \quad \forall\, t \ge 0,\ \forall\, f_0,\ \langle f_0 \rangle = 0,\ \forall\, \varepsilon \in (0, \varepsilon_1).$$

$\triangleright$ $\Sigma(\mathcal{L}_\varepsilon) \cap \Delta_a = \{0\}$ for $\varepsilon > 0$ small by a perturbation trick and then use the spectral mapping theorem.

Theorem 4.3. There exists $\varepsilon_2 > 0$ such that G_ε is exponentially nolinearly stable : $\exists\, a < 0,\ C \ge 1,$

$$\|f(t) - G_\varepsilon\| \le C_{f_0}\, e^{at}, \quad \forall\, t \ge 0,\ \forall\, f_0,\ \langle f_0 \rangle = 1,\ \forall\, \varepsilon \in (0, \varepsilon_2).$$

• What ever is the complexity of the model: asynchronous spiking holds in the small connectivity regime. Synchronization comes from nonlinearity?

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• State of a neuron: local time (or internal clock) $x \geq 0$ corresponding to the elapsed time since the last discharge;

• Dynamic of the neuron network: (age structured) evolution equation

$$\partial_t f = -\partial_x f - a(x, \varepsilon\, m(t)) f =: \mathcal{L}_{\varepsilon m(t)} f, \quad f(t,0) = p(t)$$

on the density number of neurons $f = f(t,x) \geq 0$.

• $a(x, \varepsilon\, \mu) \geq 0$: firing rate of a neuron in the state x for a network activity $\mu \geq 0$ and a network connectivity parameter $\varepsilon \geq 0$.

• $p(t)$: total density of neurons undergoing a discharge at time t given by

$$p(t) := \mathcal{P}_\varepsilon[f(t); m(t)], \quad \mathcal{P}_\varepsilon[g, \mu] := \int_0^\infty a(x, \varepsilon\mu) g(x) \mathrm{d}x.$$

• $m(t)$: network activity at time $t \geq 0$ resulting from earlier discharges given by

$$m(t) := \int_0^\infty p(t-y) b(\mathrm{d}y),$$

b delay distribution taking into account the persistence of electric activity

- *Case without delay*, when $b = \delta_0$ and then $m(t) = p(t)$.
- *Case with delay*, when b is a smooth function.

Monotony and smoothness assumptions

$$\partial_x a \geq 0, \quad a' = \partial_\mu a \geq 0,$$

$$0 < a_0 := \lim_{x \to \infty} a(x,0) \leq \lim_{x,\mu \to \infty} a(x,\mu) =: a_1 < \infty,$$

$$a \in W^{2,\infty}(\mathbb{R}^2_+).$$

$$b = \delta_0 \quad \text{or} \quad \exists \delta > 0, \quad \int_0^\infty e^{\delta y} \left(b(y) + |b'(y)| \right) \mathrm{d}y < \infty.$$

Theorems 1,2. For any $0 \leq f_0 \in L^1$, there exists (at least) one global solution $f \in C([0,\infty); L^1)$. There exists (at least) one normalized and positive stationary solution G_ε:

$$\mathcal{L}_{\varepsilon M_\varepsilon} G_\varepsilon = -\partial_x G_\varepsilon - a(x, \varepsilon\, M_\varepsilon) G_\varepsilon = 0, \quad G_\varepsilon(0) = M_\varepsilon,$$

$$M_\varepsilon = \mathcal{P}_\varepsilon[G_\varepsilon; M_\varepsilon] = \int_0^\infty a(x, \varepsilon M_\varepsilon)\, G_\varepsilon(x) \mathrm{d}x.$$

See also

Theorems 3, 4a, 4b. $\exists\,\varepsilon_1 > 0$ such that for any $\varepsilon \in (0, \varepsilon_1)$ the stationary solution G_ε is unique and exponentially stable for the associated linear semigroup.

$\triangleright$ About the proof. The linearized equation for the variation

$$(g, n, q) = (f, m, p) - (G_\varepsilon, M_\varepsilon, M_\varepsilon)$$

around a stationary state $(G_\varepsilon, M_\varepsilon, M_\varepsilon)$, which writes

$$\partial_t g = -\partial_x g - a(x, \varepsilon\, M_\varepsilon) g - n(t)\, \varepsilon (\partial_\mu a)(x, \varepsilon M_\varepsilon) F_\varepsilon, \quad g(t, 0) = q(t),$$

with

$$q(t) = \int_0^\infty a(x, \varepsilon\, M_\varepsilon) g \,\mathrm{d}x + n(t)\, \varepsilon \int_0^\infty (\partial_\mu a)(x, \varepsilon M_\varepsilon) F_\varepsilon \,\mathrm{d}x$$

$$n(t) := \int_0^\infty q(t - y) b(\mathrm{d}y).$$

We introduce an intermediate evolution equation

$$\partial_t v + \partial_y v = 0, \quad v(t,0) = q(t), \quad v(0,y) = 0,$$

where $y \geq 0$ represent the local time for the network activity.
That last equation can be solved with the characteristics method

$$v(t,y) = q(t-y)\mathbf{1}_{0 \leq y \leq t}.$$

The equation on the variation $n(t)$ of network activity writes

$$n(t) = \mathcal{D}[v(t)], \quad \mathcal{D}[v] := \int_0^\infty v(y) b(\mathrm{d}y),$$

and the equation on the variation $q(t)$ of discharging neurons writes

$$q(t) = \mathcal{O}_\varepsilon[g(t), v(t)],$$

with

$$\mathcal{O}_\varepsilon[g,v] := \mathcal{N}_\varepsilon[g] + \kappa_\varepsilon \, \mathcal{D}[v],$$
$$\mathcal{N}_\varepsilon[g] := \int_0^\infty a_\varepsilon(M_\varepsilon) g \, \mathrm{d}x, \quad \kappa_\varepsilon := \int_0^\infty a'_\varepsilon(M_\varepsilon) F_\varepsilon \, \mathrm{d}x.$$

For the new unknown (g, v) the equation writes

$$\partial_t(g, v) = \Lambda_\varepsilon(g, v) = \mathcal{A}_\varepsilon(g, v) + \mathcal{B}_\varepsilon(g, v),$$

where the operator $\Lambda_\varepsilon = (\Lambda^1_\varepsilon, \Lambda^2_\varepsilon)$ is defined by

$$\begin{aligned} \Lambda^1_\varepsilon(g, v) &:= -\partial_x g - a_\varepsilon g - a'_\varepsilon F_\varepsilon \mathcal{D}[v] + \delta_{x=0}\mathcal{O}_\varepsilon[g, v], \\ \mathcal{L}^2_\varepsilon(g, v) &:= -\partial_y v + \delta_{y=0}\mathcal{O}_\varepsilon[g, v], \end{aligned}$$

▷ $\mathcal{B}_\varepsilon$ is dissipative;

▷ $S_{\mathcal{B}_\varepsilon} * \mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon}$ has a smoothing effect

▷ We may apply the spectral theory in general Banach space (Weyl's Theorem, spectral mapping Theorem, Krein-Rutman Theorem, perturbation Theorem) developped by M., Scher, Tristani.

Outline of the talk

$$\partial_t f = \mathcal{Q}_\varepsilon(\mathcal{J}_f, f) = \partial_x(Af) + \partial_v(Bf) + \partial^2_{vv} f \quad \text{on } (0,\infty)\times\mathbb{R}^2$$

complemented withy an initial condition

$$f(0,.) = f_0 \ge 0 \quad \text{in } \mathbb{R}^2.$$

where

$$\begin{cases} A = A(x,v) = ax - bv, \quad B = B_\varepsilon[\mathcal{J}_f] = B(x,v;\mathcal{J}_f) \\ B(x,v;\mu) = v^3 - v + x + \varepsilon\,(v-\mu), \quad \mathcal{J}_f := \int_{\mathbb{R}^2} v\, f(x,v)\, dvdx \end{cases}$$

- $t \ge 0$ is the time variable, $v \in \mathbb{R}$ is the membrane potential of one neuron, $x \in \mathbb{R}$ is an auxiliary variable
- $f = f(t,x,v) \ge 0$ is the time-dependent density of neurons in state $(x,v) \in \mathbb{R}^2$
- a, b, ε are positive parameters and ε is the connectivity of the network

The equation being in divergence form the number of neurons is a constant along time (that's better!):

$$\int_{\mathbb{R}^2} f(t,x,v) dxdv = \int_{\mathbb{R}^2} f_0 dxdv \equiv 1.$$

• As a simplification of the Hodgin-Huxley 4d ODE, FitzHugh-Nagumo 2d ODE describes the electric activity of one neuron and writes

$$\dot{v} = v - v^3 - x + I_{ext} = -B_0 + I_{ext}$$
$$\dot{x} = bv - ax = -A,$$

with $I_{ext} = i(t) + \sigma \dot{W}$ exterior input split as a deterministic part + a stochastic noise. We assume $i(t) \equiv 0$.

• For a network of N coupled neurons, the associated model writes for the state $\mathcal{Z}_t^i := (\mathcal{X}_t^i, \mathcal{V}_t^i)$ of the neuron labeled $i \in \{1, ..., N\}$

$$d\mathcal{V}^i = [-B_0(\mathcal{X}^i, \mathcal{V}^i) - \sum_{j=1}^{N} \varepsilon_{ij} (\mathcal{V}^i - \mathcal{V}^j)]dt + \sigma \, d\mathcal{W}^i$$
$$d\mathcal{X}^i = -A(\mathcal{X}^i, \mathcal{V}^i)dt$$

where $\varepsilon_{ij} > 0$ corresponds to the connectivity between the two neurons labeled i and j. The model takes into account an intrinsic deterministic dynamic + mean field interaction + stochastic noise.

Motivation: to a statistical description (mean field limit)

We assume $\varepsilon_{ij} := \varepsilon/N$, $(\mathcal{Z}_0^1 ..., \mathcal{Z}_0^N)$ are i.i.d. random variables with same law f_0 and we pass to the limit $N \to \infty$.
We get that $(\mathcal{Z}_t^1, ..., \mathcal{Z}_t^N)$ is chaotic which means that any two neurons $\mathcal{Z}_t^i$ and $\mathcal{Z}_t^j$ are asymptotically independent and $\mathcal{Z}_t^i \to \bar{\mathcal{Z}}_t = (\bar{\mathcal{X}}_t, \bar{\mathcal{Y}}_t)$ which is a solution to the nonlinear ODS

$$\bar{\mathcal{V}} = [-B_0(\bar{\mathcal{X}}, \bar{\mathcal{V}}) - \varepsilon\,(\bar{\mathcal{V}} - \mathbb{E}(\bar{\mathcal{V}}))]dt + \sigma\, d\mathcal{W}$$
$$\bar{\mathcal{X}} = -A(\bar{\mathcal{X}}, \bar{\mathcal{V}})dt.$$

From Ito calculus we immediately see that the law $f(t,.) := \mathcal{L}(\bar{\mathcal{X}}_t, \bar{\mathcal{V}}_t)$ satisfies the associated backward Kolmogrov equation which is nothing but the FHN nonlinear statistical equation (here and below we make the choice $\sigma := \sqrt{2}$ for the sake of simplification of notations).

We introduce the weight function $m_0 = m_0(x,v) := 1 + x^2/2 + v^2/2$ and the weighted Lebesgue spaces $L^p(m)$ associated to the norm

$$\|f\|_{L^p(m)} = \|fm\|_{L^p}, \quad \|f\|_{W^{1,p}(m)} = \|f\|_{L^p(m)} + \|\nabla f\|_{L^p(m)},$$

and the shorthand $L^p_k := L^p(m_0^{k/2})$.

Th 8. M., Quininao, Touboul

For any $f_0 \in \mathcal{E}_0 := L^1_2 \cap L^1 \log L^1 \cap \mathbb{P}(\mathbb{R}^2)$ there exists a unique global solution $f \in C([0,\infty); L^1 \cap \mathbb{P})$ to the FHN statistical equation. It also satisfies

$$\int f_t m \le \max(C_m, \int f_0 m), \quad \|f_t\|_{\mathcal{H}^1(m)} \le \max(C_2, \|f_0\|_{\mathcal{H}^1(m)}).$$

It depends continuously in the initial datum: $f_{n,t} \to f_t$ in L^1_2 for any time $t \ge 0$ if $f_{n,0} \to f_0$ in L^1_2 and $\|f_{n,0}\|_{L^1_4} + H(f_{n,0}) \le C$.
For any $\tau > 0$ there exists C_τ such that

$$\sup_{t \ge \tau} \|f_t\|_{H^1} \le C_\tau.$$

Th 9. M., Quininao, Touboul

There exists at least one stationary solution G to the FHN statistical equation:

$$\exists\, G \in H^1(m) \cap \mathbb{P}(\mathbb{R}^2), \quad 0 = \partial_x(AG) + \partial_v(B_\varepsilon[\mu_G]G) + \partial^2_{vv} G \quad \text{in } \mathbb{R}^2.$$

Th 10. M., Quininao, Touboul

There exists $\varepsilon^* > 0$ such that in the small connectivity regime $\varepsilon \in (0, \varepsilon^*)$ the stationary solution is unique and exponentially stable: there exist $\eta^*_\varepsilon > 0$, $a < 0$ such that $\eta^*_\varepsilon \to \infty$ when $\varepsilon \to 0$ and

$$\forall\, f_0 \in L^1_2 \cap \mathbb{P}, \quad \|f_0 - G\|_{L^1_2} \le \eta^*_\varepsilon \text{ there holds } \|f(t) - G\|_{L^1_2} \le C\, e^{at}\ \forall\, t \ge 0$$

- We follow a strategy introduced in M., Mouhot (CMP 2009) for the inelastic homogenous Boltzmann equation and improved in Tristani (arXiv 2013) in a weakly inhomogeneous setting.
- But we fundamentally use the fact that the limit equation (for $\varepsilon = 0$) is positive and it is then exponentially asymptotically stable thanks to the Krein-Rutmann theorem (Theorem 4)
- We also use some "hypocoercive" calculus tricks developed by Hérau and Villani for the kinetic Fokker-Planck equation

Proof - L^1_k estimate

The vector field (A, B) does not derive from a potential (even in the case $\varepsilon = 0$) but has the following "confinement property"

$$\begin{aligned} -x\,A - v\,B &= -ax^2 + bxv - v^4 - (1+\varepsilon)xv + \varepsilon\mu x \\ &\leq C(a,b,\varepsilon) - \frac{a}{2}x^2 - \frac{1}{2}v^4 + 2\frac{\varepsilon^2}{a}\mu^2. \end{aligned}$$

Also observe (Cauchy-Schwarz inequality)

$$\mathcal{J}_f^2 \leq \int f\, v^2 dxdv \quad \forall f \in \mathbb{P}(\mathbb{R}^2).$$

Lemma (uniform in time L^1_k estimate, $k \geq 2$)

For $m_0 := 1 + x^2/2 + v^2/2$ and any $f \in \mathbb{P}(\mathbb{R}^2)$, there holds for $C_i > 0$

$$\int \mathcal{Q}_\varepsilon[\mu, f] m_0 \leq C_1\,(1+\mu^2) - C_2 \int f\,(1 + x^2 + v^4).$$

As a consequence, for any $f \in \mathbb{P}(\mathbb{R}^2)$

$$\int \mathcal{Q}_\varepsilon[\mathcal{J}_f, f] m_0 \leq C_3 - C_2 \int f\, m_0,$$

and for any $f_0 \in \mathbb{P}(\mathbb{R}^2)$

$$\mathcal{J}^2_{f(t)} \leq \int f_t\, m_0 \leq \max\Big(\frac{C_3}{C_2}, \int f_0\, m_0\Big) \quad \forall t \geq 0, \quad m = m_0^{k/2}.$$

Proof - $\mathcal{H}^1$ estimate

In the same way for $m = e^{\kappa\, m_0}$

$$\frac{d}{dt}\int f^2 m^2 = 2\int \mathcal{Q}_\varepsilon[\mu,f]\, f\, m^2 \le C_1 \int f^2 - C_2 \int f^2\, m^2 m_0 - \int |\partial_v f|^2 m^2,$$

but we do not know how to conclude (in order to get uniform in times bound) !?

We introduce the (equivalent) twisted norm (reminiscent of hypocoercivity theory)

$$\|f\|^2_{\mathcal{H}^1(m)} := \|f\|^2_{L^2(m)} + \|\nabla_x f\|^2_{L^2(m)} + \alpha^{5/6}(\nabla_x f, \nabla_v f)_{L^2(m)} + \alpha \|\nabla_v f\|^2_{L^2(m)}$$

for $\alpha > 0$ small enough. For the associated scalar product $\langle \cdot, \cdot \rangle$

$$\begin{aligned} \langle Q_\varepsilon[\mu,f], f\rangle &\le C_1 \int f^2 - C_2 \int f^2\, m^2 m_0 - \alpha \int |\partial_v f|^2 m^2 - \alpha^{5/6} \int |\partial_x f|^2 m^2 \\ &\le K_1 \|f\|_{\mathcal{H}^1} - K_2 \|f\|^2_{\mathcal{H}^1} \end{aligned}$$

by using Nash inequality

$$\|f\|^2_{L^2} \le C \|f\|_{L^1} \|f\|_{H^1}.$$

Lemma uniform in times $\mathcal{H}^1$ estimate)

For any $f_0 \in \mathbb{P}(\mathbb{R}^2)$

$$\|f_t\|_{\mathcal{H}^1(m)} \le \max\Big(\frac{K_1}{K_2}, \|f_0\|_{\mathcal{H}^1(m)}\Big) \quad \forall\, t \ge 0.$$

We compute

$$\begin{aligned}\frac{d}{dt}\int f\log f &= \int(\partial_{vv}f)\log f + \int(\partial_x(Af)+\partial_v(Bf))\log f\\ &= -\int\frac{(\partial_v f)^2}{f} + \int(\partial_x A+\partial_v B)f\\ &\leq -\mathcal{I}_v(f)+\int m_0 f, \qquad \mathcal{I}_v(f):=\int\frac{(\partial_v f)^2}{f}.\end{aligned}$$

We conclude by standard (weak L^1 compacteness) argument to the existence of a solution $f \in C([0,\infty); L^1)$ such that

$$\sup_{[0,T]}\int f\,(m_0^2+\log f)+\int_0^T \mathcal{I}_v(f)\,dt \leq C_T \quad \forall\, T>0$$

for any $f_0 \in L^1_4 \cap \mathbb{P} \cap L^1\log L^1$.

More about the proof - uniqueness

For any two solutions f_1 et f_2 to the FHN equation

$$\partial_t f_i = \partial_x(A f_i) + \partial_v(\mathcal{B}_i f_i) + \partial^2_{vv} f_i$$

with $\mathcal{B}_i := B_0 + \varepsilon(v - \mathcal{J}_{f_i})$, the difference $f = f_2 - f_1$ satisfies

$$\partial_t f = \partial_x(A f) + \partial_v(\mathcal{B}_1 f) + \varepsilon\, \mathcal{J}_f \partial_v f_2 + \partial^2_{vv} f.$$

As a consequence, by Kato's inequality

$$\partial_t |f| \le \partial_x(A|f|) + \partial_v(\mathcal{B}_1 |f|) + \varepsilon\, |\partial_v f_2|\, |\mathcal{J}_f| + \partial^2_{vv}|f|.$$

Using the inequality

$$\int |\partial_v f_2| m_0^{1/2} \le \Big(\int f_2\, m_0\Big)^{1/2} \Big(\int \frac{|\partial_v f_2|^2}{f_2}\Big)^{1/2} \le C\, \mathcal{I}_v(f_2)^{1/2}$$

we get

$$\begin{aligned} \frac{d}{dt}\int |f| m_0^{1/2} \quad &\le \quad \int |f|(-A\partial_x m_0^{1/2} - B\partial_v m_0^{1/2}) + \varepsilon\, C\, \mathcal{I}_v(f_2)^{1/2} \int |f| m_0^{1/2} + \int |f| \\ &\le \quad (C + \varepsilon \mathcal{I}_v(f_2)) \int |f| m_0^{1/2}. \end{aligned}$$

We conclude to the uniqueness by Gronwall lemma.

Define

$$\mathcal{Z} := \{f \in \mathcal{H}^1(m) \cap \mathbb{P}; \quad \|f\|_{\mathcal{H}^1(m)} \leq K_1/K_2\}$$

and

$$S = (S_t) \text{ by } S_t f_0 := f_t \text{ solution of the FHN equation.}$$

- $\mathcal{Z}$ is a convex and strongly compact subset of the Banach space L^1_2;
- S leaves $\mathcal{Z}$ invariant and it is a L^1_2-continuous semigroup.

A direct application of the Schauder fixed point theorem implies

$$\exists\, G \in \mathcal{Z} \text{ such that } S_t G = G \quad \forall\, t \geq 0$$

or equivalently

$$G \text{ is a stationary solution to the FHN equation (for any given } a, b, \varepsilon > 0).$$

We may simplify that existence part by working in the space of symmetric solutions $\mathcal{S}$ (i.e. $f \in \mathcal{S}$ iff $f(-x,-v) = f(x,v)$) in which space the FHN equation is linear.

For any stationary state $G_\varepsilon \in \mathcal{Z}$, we define the linearized operator

$$\mathcal{L}_\varepsilon h := \partial_x(Ah) + \partial_v((B_0 + \varepsilon(v - \mu_{G_\varepsilon}))h) - \varepsilon\mu_h\partial_v G_\varepsilon + \partial^2_{vv} h$$

We write

$$\mathcal{L}_\varepsilon = \mathcal{A} + \mathcal{B}_\varepsilon, \quad \mathcal{A}h := M\chi_R(x,v)\, h$$

and we have

$$(1) \quad \|S_\mathcal{B} * (\mathcal{A}S_\mathcal{B})^{(*k)}\|_{\mathbf{B}(X)} \le C_k\, e^{-t}$$

and

$$(2) \quad \|S_\mathcal{B} * (\mathcal{A}S_\mathcal{B})^{(*n)}\|_{\mathbf{B}(L^1(m),H^1(mm_0))} \le C_n\, e^{-t}$$

As a consequence, the Weyl theorem (Theorem 2) implies

$$\Sigma(\mathcal{L}) \cap \Delta_{-1} = \text{finite} \subset \Sigma_d(\mathcal{L}).$$

Proof of estimates (1) and (2)

• the estimate

$$(1) \quad \|S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*k)}\|_{\mathbf{B}(X)} \le C_k\, e^{-t}$$

is a consequence of the fact that
$\triangleright$ $\mathcal{A} \in \mathbf{B}(X)$, $X = L^1(m)$, $L^2(m)$, $\mathcal{H}^1(m)$;
$\triangleright$ $\mathcal{B}$ is -1-dissipative in $X = L^1(m)$, $L^2(m)$, $\mathcal{H}^1(m)$ as a consequence of the already established estimates

$$\int \mathcal{Q}_\varepsilon[\mu, f] f^{p-1} m^p \le C_1 \int f^p - C_2 \int f^p m^p m_0$$

and the similar estimate in $\mathcal{H}^1(m)$.

• the estimate

$$(2) \quad \|S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*n)}\|_{\mathbf{B}(L^1(m), H^1(mm_0))} \le C_n\, e^{-t}$$

is similar to the Nash argument in the proof of the stability of $\mathcal{Z}$. More precisely, introducing

$$\mathcal{F}(t,h) := \|h\|^2_{L^1(m)} + t^\bullet \|h\|^2_{L^2(m)} + t^\bullet \|\nabla_v h\|^2_{L^2(m)} + t^\bullet (\nabla_v h, \nabla_x h)_{L^2(m)} + t^\bullet \|\nabla_x h\|^2_{L^2(m)}$$

we are able to prove (for convenient exponents $\bullet \ge 1$)

$$\frac{d}{dt} \mathcal{F}(t, S_{\mathcal{B}}(t) h) \le 0, \quad \forall\, t \in [0, T].$$

We observe that in $X = L^p(m)$

$$\mathcal{L}_0 h = \partial_x(Ah) + \partial_v((B_0 h) + \partial^2_{vv} h$$

is such that

(1) $\mathcal{L} = \mathcal{A} + \mathcal{B}_0$ as above with $a^* = -1$;

(2) $\exists G_0 \in \mathcal{Z}$, $\mathcal{L}_0 G_0 = 0$ and $\mathcal{L}_0^* 1 = 0$;

(3) $\mathcal{L}_0$ is strongly positive, in the sense that

$\triangleright$ $S_{\mathcal{L}_0}$ is a positive semigroup : $f_0 \geq 0$ implies $S_{\mathcal{L}_0}(t) f_0 \geq 0$;

$\triangleright$ $\mathcal{L}_0$ satisfies a weak maximum principle: $(\mathcal{L}_0 - a)f \leq 0$ and a large imply $f \geq 0$;

$\triangleright$ $\mathcal{L}_0$ satisfies Kato inequality : $\mathcal{L}_0 \theta(f) \geq \theta'(f) \mathcal{L}_0 f$, $\theta(s) = |s|, s_+$;

$\triangleright$ $\mathcal{L}_0$ satisfies a strong maximum principle: $(\mathcal{L}_0 - \mu) f \leq 0$ and $f \in X_+ \backslash \{0\}$ imply $f > 0$.

The Peron-Frobenius-Krein-Rutman theorem asserts

$$G_0 \in \mathbb{P} \quad \mathcal{L}_0 G_0 = 0, \quad G_0 \text{ is unique and stable.}$$

More precisely

(1) $\exists a < 0$ such that $\Sigma(\mathcal{L}_0) \cap \Delta_a = \{0\}$;

(2) 0 is simple and $\ker \mathcal{L}_0 = \mathrm{vect}\, G_0$;

(3) $\Pi_0 h = \langle h \rangle G_0$ and $\mathcal{L}_0$ is invertible from $R(I - \Pi_0)$ onto X.

From the the Krein-Rutman theorem, for any solution $\mathcal{L}_0 f = g \in L^2(m)$ with $\langle g \rangle = 0$

$$\|f\|_{L^2(m)} \le C\, \|g\|_{L^2(m)}.$$

Using the additional estimate

$$\forall f \quad \int (\mathcal{L}_0 f) f\, m_0 m^2 \le C_1 \int f^2 m^2 - \kappa_1 \int f^2 m_0^2 m^2 - \kappa_1 \int (\partial_v f)^2 m_0 m^2,$$

we deduce the stronger bound

$$\|f\|_{\mathcal{V}} := \|f\|_{L^2(mH)} + \|\nabla_v f\|_{L^2(mH^{1/2})} \le C\, \|g\|_{L^2(m)}.$$

For any two stationary solutions, we now write

$$\begin{aligned} G_\varepsilon - F_\varepsilon &= \mathcal{L}_0^{-1} \Big[\mathcal{L}_0\, G_\varepsilon - \mathcal{L}_0\, F_\varepsilon \Big] \\ &= \varepsilon\, \mathcal{L}_0^{-1} \Big[\partial_v \Big((v - \mathcal{J}(F_\varepsilon)) F_\varepsilon - (v - \mathcal{J}(G_\varepsilon)) G_\varepsilon \Big) \Big] \end{aligned}$$

and then

$$\begin{aligned} \|F_\varepsilon - G_\varepsilon\|_{\mathcal{V}} &\le \varepsilon\, C \Big\| \partial_v \Big((v - \mathcal{J}(F_\varepsilon))(F_\varepsilon - G_\varepsilon) + (\mathcal{J}(F_\varepsilon) - \mathcal{J}(G_\varepsilon)) G_\varepsilon \Big) \Big\|_{L^2(m)} \\ &\le \varepsilon\, C\, \|F_\varepsilon - G_\varepsilon\|_{\mathcal{V}}. \end{aligned}$$

which in turn implies that necessarily $\|F_\varepsilon - G_\varepsilon\|_{\mathcal{V}} = 0$ for $\varepsilon > 0$ small enough.

The above Krein-Rutman theorem on $\mathcal{L}_0$ and the following properties on $\mathcal{L}_\varepsilon$

$$\mathcal{L}_\varepsilon \to \mathcal{L}_0 \quad \text{and} \quad \mathcal{L}_\varepsilon^* 1 = 0$$

imply (thanks to Theorem 5)

$$\Sigma(\mathcal{L}_\varepsilon) \cap \Delta_a = \{0\}, \quad a < 0, \ \varepsilon \text{ small } > 0.$$

For any solution f the function $h := f - G_\varepsilon$ satisfies

$$\partial_t h = \mathcal{L}_\varepsilon h - \varepsilon \partial_v [\mu_h \, h].$$

From the spectral mapping theorem, we may compute (rigorously at the level of the Duhamel formulation)

$$\begin{aligned} \frac{d}{dt} \|h\|_{L^2}^2 &\leq 2a\|h\|_{L^2}^2 + 2a\|\partial_v h\|_{L^2}^2 + \varepsilon \, |\mu_h| \, \|h\|_{L^2} \, \|\partial_v h\|_{L^2} \\ &\leq 2a\|h\|_{L^2}^2 + C \, \|h\|_{L^2}^4. \end{aligned}$$

As a consequence, the set $\mathcal{C} := \{\|h\|_{L^2}^2 \leq |a|/C\}$ is stable. Then for any $h_0 \in \mathcal{C}$, we get

$$\|h(t)\|_{L^2} \leq C \, e^{at}.$$

• What about the "large" connectivity regime: ε is not small?

▷ unstability of "the" steady state?

▷ periodic solutions? local stability of one of them?

• What about a Hodgin-Huxley statistical model based on the Hodgin-Huxley 4d ODE sytem?