Landau equation for Coulomb potentials near Maxwellians

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International Conference on Kinetic Equations and Related Topics
Wuhan, September 13-19, 2015

Results are picked up from

- Carrapatoso, M. *Landau equation for very soft and Coulomb potentials near Maxwellian*, in progress
- Kavian, M., *The Fokker-Planck equation with subcritical confinement force*, in progress
- M., *Semigroups in Banach spaces - factorization approach for spectral analysis and asymptotic estimates*, in progress

Generalize to a weak dissipativity framework some related previous works available in a dissipativity framework, in particular:

- Gualdani, M., Mouhot, *Factorization for non-symmetric operators and exponential H-Theorem*, arXiv 2010
- M., Scher, *Spectral analysis of semigroups and growth-fragmentation eqs*, to appear in Annales IHP
- Carrapatoso, Tristani, Wu, *Cauchy problem and exponential stability for the inhomogeneous Landau equation*, arXiv 2015

which in turn formalize several reminiscent ideas from Bobylev 1975, Voigt 1980, Arkeryd 1988, Gallay-Wayne 2002, Mouhot 2006, ...

Outline of the talk

1. Introduction and main result
2. Weak hypodissipativity in an abstract setting
   - From weak dissipativity to decay estimate
   - From decay estimate to weak dissipativity
   - Functional space extension (enlargement and shrinkage)
   - Spectral mapping theorem
   - Krein-Rutman theorem
3. Fokker-Planck equation with weak confinement
   - Statement of the decay theorem
   - Proof of the decay theorem
4. Landau equation with Coulomb potential
   - Estimate on the nonlinear operator and natural large space
   - A priori global nonlinear estimate
   - Splitting trick, dissipativity and decay estimates on the linear operators

Outline of the talk

With Mouhot, Gualdani and Scher, we have recently revisited the spectral theory of operators and semigroups in an hypodissipative and abstract general Banach framework, providing a set of results including:

• *Spectral mapping Theorem*

• *Weyl's Theorem* about distribution of eigenvalues under compact perturbation

• Stability of the spectrum Theorem under small perturbation

• *Krein-Rutman Theorem*

• *Functional space extension (enlargement and shrinkage) Theorem*

These results were motivated by linear and nonlinear evolution PDE to which they have been applied

• Asymptotic behavior of linear PDE in large space (Growth-Fragmentation, Kinetic Fokker-Planck in $W^{\sigma,p}(m)$, $-1 \le \sigma \le 1 \le p \le \infty$)

• Optimal (= linearized) exponential decay estimates for nonlinear PDE (homogeneous (inelastic) Boltzmann, Parabolic-elliptic Keller-Segel)

• Existence, uniqueness and stability results in perturbative regime (nonhomogeneous (inelastic) Boltzmann, Parabolic-parabolic Keller-Segel, kinetic FitzHugh-Nagumo and others neuronal networks)

In the present talk, we consider some possible extension to a weak hypodissipative framework. Namely, when we do not have (or we do not exploit) any estimate

$$\langle f^*, \Lambda f\rangle_X \lesssim -\|f\|_X^2,$$

but we have (and exploit)

$$\langle f^*, \Lambda f\rangle_X \lesssim -\|f\|_Y^2, \qquad X \subset Y.$$

When the estimate is sharp, = the operator is not "more dissipative", that corresponds to the situation

$$\Sigma_P(\Lambda) \cap \bar{\Delta}_0 = \emptyset, \quad \Sigma(\Lambda) \cap \bar{\Delta}_0 \neq \emptyset.$$

More specifically, we present

• some (not all) abstract spectral analysis results

• some application to the Fokker-Planck equation with weak confinement force

• some application to the Landau equation for Coulomb potential near Maxwellian in the torus

Landau equation

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f &= Q(f, f) \\ f(0, .) &= f_0 \end{aligned}$$

on density of the plasma $f = f(t, x, v) \geq 0$, time $t \geq 0$, position $x \in \mathbb{T}^3$ (torus), velocity $v \in \mathbb{R}^3$

$Q =$ the Landau (binary) collisions operator

$$Q(g, f) = \partial_j \int_{\mathbb{R}^3} a_{ij}(v - v_*)(g_* \partial_j f - f \partial_j g_*) \, dv_*$$

for the Coulomb potential cross section

$$a_{ij}(z) = |z|^{\gamma+3} \big(\delta_{ij} - \frac{z_i z_j}{|z|^2}\big), \quad \gamma = -3.$$

We recall that $\varphi = 1, v, |v|^2$ are collision invariants, meaning

$$\int_{R^3} Q(f,f)\varphi \, dv = 0, \quad \forall f.$$

$\Rightarrow$ laws of conservation

$$\int f \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} = \int f_0 \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

We also have the H-theorem, namely

$$\int Q(f,f) \log f \begin{cases} \leq 0 \\ = 0 \ \Rightarrow \ f = \text{Maxwellian} \end{cases}$$

From both information, we expect

$$f(t,x,v) \underset{t\to\infty}{\longrightarrow} \mu(v) := \frac{1}{(2\pi)^{3/2}} \, e^{-|v|^2/2}.$$

Theorem 1. (Carrapatoso, M.)

Take an "admissible" weight function m such that

$$\langle v\rangle^{2+3/2} \prec m \prec e^{|v|^2}.$$

There exists $\varepsilon_0 > 0$ such that if

$$\|f_0 - \mu\|_{H^2_x L^2_v(m)} < \varepsilon_0,$$

there exists a unique global solution f to the Landau Coulomb equation and

$$\|f(t) - \mu\|_{H^2_x L^2_v} \leq \Theta_m(t),$$

with

$$\Theta_m(t) \simeq \begin{cases} t^{-(k-2-3/2)/|\gamma|} & \text{if } m = \langle v\rangle^k \\ e^{-\lambda t^{s/|\gamma|}} & \text{if } m = e^{\kappa |v|^s} \end{cases}$$

comments on the main Theorem 1

• Improves Guo and Strain's results (CMP 2002, CPDE 2006, ARMA 2008) who proved a similar theorem in the higher order and strongly confinement Sobolev space $H^8_{x,v}(\mu^{-\theta})$, $\theta > 1/2$.

• The proof does not use high order nonlinear energy estimates, but

- Simple nonlinear estimates and trap argument

- Decay and dissipativity estimates for the linearized equation in the corresponding space

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- Simple nonlinear estimates and trap argument

- Decay and dissipativity estimates for the linearized equation in the corresponding space

• The method consists in introducing the variation function $g = f - \mu$ and the corresponding Landau equation

$$\partial_t g = \bar{\mathcal{L}} g + Q(g,g),$$
$$\bar{\mathcal{L}} = -v \cdot \nabla_x + \mathcal{L}, \quad \mathcal{L} = Q(\cdot,\mu) + Q(\mu,\cdot)$$

As a starting point, we use the known weak dissipativity estimate

$$(\mathcal{L}g, g)_{L^2(\mu^{-1/2})} \lesssim -\|(I - \Pi_0)g\|^2_{H^1_*(\mu^{1/2}\langle v\rangle^{(\gamma+2)/2})},$$

$\Pi_0 :=$ projector on $N(\mathcal{L})$, in order to prove the weak hypodissipativity estimate

$$(\bar{\mathcal{L}}g, g)_{\mathcal{H}^1_{x,v}(\mu^{-1/2})} \lesssim -\|(I - \bar{\Pi}_0)g\|^2_{H^1_{x,v}(\mu^{1/2}\langle v\rangle^{(\gamma+2)/2})},$$

$\bar{\Pi}_0 :=$ projector on $N(\bar{\mathcal{L}})$, and next factorization and semigroup tricks in order to get similar information in the space $H^2_x L^2(m)$.

Outline of the talk

For a given Banach space X, we want to develop a spectral analysis theory for operators Λ enjoying the splitting structure

$$\Lambda = \mathcal{A} + \mathcal{B}, \quad \mathcal{A} \prec \mathcal{B}, \quad \mathcal{B} \text{ weakly hypodissipative.}$$

We will

- clarify the links between dissipativity and decay;
- present an extension of the decay estimate result;
- present a possible version of spectral mapping theorem;
- present a possible version of Krein-Rutman theorem.

- We do not present any version of Weyl's theorem or perturbation theorem.
- Very few papers related to that topics. We may mention: Caflisch (CMP 1980), Toscani-Villani (JSP 2000), Röckner-Wang (JFA 2001), Lebeau & co-authors (1993 & after), Burq (Acta Math 1998), Batty-Duyckaerts (JEE 2008). That last is one of the only reference in a abstract Banach (in a more restrictive framework than ours).

Prop 1.

Consider three "regular" Banach spaces $X \subset Y \subset Z$ and a generator Λ. Assume

$$\begin{aligned} \forall\, f \in Y^\Lambda_1, \qquad & \langle f^*_Y, \Lambda f\rangle_Y \quad \lesssim \quad -\|f\|^2_Z \\ \forall\, f \in X^\Lambda_1, \qquad & \langle f^*_X, \Lambda f\rangle_X \quad \lesssim \quad 0 \quad \text{(or } S_\Lambda \text{ is bounded } X\text{)} \\ \forall\, R > 0, \qquad & \varepsilon_R \|f\|^2_Y \quad \le \quad \varepsilon_R \|f\|^2_Z + \theta_R \|f\|^2_X, \quad \varepsilon_R, \frac{\theta_R}{\varepsilon_R} \to 0. \end{aligned}$$

There exists a decay function Θ such that

$$\|S_\Lambda(t)\|_{X \to Y} \le \Theta(t) \to 0.$$

• We say that a Banach space E is regular if $\varphi : E \to \mathbb{R}$, $f \mapsto \|f\|^2_E/2$ is G-differentiable and

$$\{f^* \in E',\ \langle f^*, f\rangle_E = \|f\|^2_E = \|f^*\|^2_{E'}\} = \{f^*_E\}, \quad f^*_E := D\varphi(f).$$

Hilbert spaces and L^p spaces, $1 < p < \infty$, are regular spaces.

• We denote $E^\Lambda_s := \{f \in E,\ \Lambda^s f \in E\}$ the abstract Sobolev spaces

Prop 1.

Consider three "regular" Banach spaces $X \subset Y \subset Z$ and a generator Λ. Assume

$$
\begin{aligned}
\forall f \in Y_1^\Lambda, \qquad & \langle f_Y^*, \Lambda f\rangle_Y \lesssim -\|f\|_Z^2 \\
\forall f \in X_1^\Lambda, \qquad & \langle f_X^*, \Lambda f\rangle_X \lesssim 0 \quad \text{(or } S_\Lambda \text{ is bounded } X) \\
\forall R > 0, \qquad & \varepsilon_R \|f\|_Y^2 \le \varepsilon_R \|f\|_Z^2 + \theta_R \|f\|_X^2, \quad \varepsilon_R, \frac{\theta_R}{\varepsilon_R} \to 0.
\end{aligned}
$$

There exists a decay function Θ such that

$$\|S_\Lambda(t)\|_{X\to Y} \le \Theta(t) \to 0.$$

- We say that m is an admissible if $m = \langle v\rangle^k$ or $m = e^{\kappa \langle v\rangle^s}$. We then write $m_0 \prec m_1$ or $m_1 \succ m_0$ or if $m_0/m_1 \to \infty$.
- For $X = L^p(m_1)$, $Y = L^p(m_0)$, $Z = L^p(m_0 \langle v\rangle^{\alpha/p})$, with $\alpha < 0$ and $m_1 \succ m_0$, we get

$$\Theta(t) \simeq \begin{cases} t^{-(k_1-k_0)/|\alpha|} & \text{if } m_i = \langle v\rangle^{k_i} \\ e^{-\lambda t^{s/|\alpha|}} & \text{if } m_1 = e^{\kappa |v|^s} \end{cases}$$

We define $f_t := S_\Lambda(t) f_0$, $f_0 \in X$, and we compute

$$\frac{d}{dt}\|f_t\|_X^2 \le 0, \quad \Rightarrow \quad \|f_t\|_X \le \|f_0\|_X$$

$$\begin{aligned}\frac{d}{dt}\|f_t\|_Y^2 &\lesssim -\|f_t\|_Z^2 \\ &\lesssim -\varepsilon_R \|f_t\|_Y^2 + \theta_R \|f_0\|_X^2,\end{aligned}$$

and from Gronwall lemma

$$\begin{aligned}\|f_t\|_Y^2 &\lesssim e^{-\varepsilon_R t}\|f_0\|_Y^2 + \frac{\theta_R}{\varepsilon_R}\,\|f_0\|_X^2 \\ &\lesssim \Theta(t)^2\,\|f_0\|_X^2,\end{aligned}$$

with

$$\Theta(t)^2 := \inf_{R>0}\Big(e^{-\varepsilon_R t} + \frac{\theta_R}{\varepsilon_R}\Big).$$

Prop 2. Consider three "regular" Banach spaces $X \subset Y \subset Z$ and a generator $\mathcal{L}$. Assume

- $\|S_{\mathcal{L}}(t)\|_{X \to Z} \le \Theta(t)$, with $\Theta \in L^2(\mathbb{R}_+)$ a decay function (i.e. which tends to 0)
- $\mathcal{L} = \mathcal{A} + \mathcal{B}$, $\mathcal{A} \prec \mathcal{B}$, with

$$\begin{array}{rcl} \forall\, f \in X_1^{\mathcal{B}}, \qquad \langle f^*, \mathcal{B} f\rangle_X & \lesssim & -\|f\|_Y^2 \\ \forall\, f \in X_1^{\mathcal{A}}, \qquad \langle f^*, \mathcal{A} f\rangle_X & \lesssim & \|f\|_Z^2. \end{array}$$

Then, $\mathcal{L}$ is weakly hypodissipative

$$\langle\!\langle f^*, \mathcal{L} f\rangle\!\rangle_X \lesssim -\|f\|_Y^2$$

for the duality product $\langle\!\langle , \rangle\!\rangle_X$ associated to the norm defined by

$$|\!|\!| f |\!|\!|^2 := \eta \|f\|_X^2 + \int_0^\infty \|S_{\mathcal{L}}(\tau) f\|_Z^2 \, d\tau,$$

for $\eta > 0$ small enough. That norm is equivalent to the initial norm in X.

We observe that $\lVert\!| \cdot |\!\rVert \sim \| \cdot \|_X$ because $\Theta \in L^2(\mathbb{R}_+)$.

We set $f_t := S_{\mathcal{L}}(t) f_0$ and we compute

$$\begin{aligned}
\frac{d}{dt} \lVert\!| f_t |\!\rVert^2 &= \eta \langle f_t^*, \mathcal{L} f_t \rangle_X + \int_0^\infty \frac{d}{d\tau} \| S_{\mathcal{L}}(\tau + t) f_0 \|_Z^2 \, d\tau \\
&= \eta \langle f_t^*, \mathcal{B} f_t \rangle_X + \eta \langle f_t^*, \mathcal{A} f_t \rangle_X - \| f_t \|_Z^2 \\
&\leq -\eta C_1 \| f_t \|_Y^2 + (\eta C_2 - 1) \| f_t \|_Z^2 \\
&\lesssim -\| f_t \|_Y^2
\end{aligned}$$

as well as

$$\frac{d}{dt} \lVert\!| f_t |\!\rVert^2 \simeq \langle\!\langle f_t^*, \mathcal{L} f_t \rangle\!\rangle_X$$

Prop 3. Consider a decay function Θ such that

$$\Theta^{-1}(t) \lesssim \Theta^{-1}(t-s)\Theta^{-1}(s) \text{ for any } 0 < s < t.$$

We consider two sets of Banach spaces $X_1 \subset X_0$ and $Y_1 \subset Y_0$ and a generator Λ. We assume

- $\|S_\Lambda(t)\|_{X_1 \to X_0} \Theta^{-1} \in L^\infty$
- $\Lambda = \mathcal{A} + \mathcal{B}$, $\mathcal{A} \prec \mathcal{B}$, with

$$\begin{aligned} &\forall\, \ell, \quad \|S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{B}})^{(*\ell)}\|_{Y_1 \to Y_0} \Theta^{-1} \in L^\infty \\ &\exists\, n, \quad \|(\mathcal{A} S_{\mathcal{B}})^{(*n)}\|_{Y_1 \to X_1} \Theta^{-1} \in L^1 \text{ if } X_0 \subset Y_0 \text{ (enlargement)} \\ &\exists\, n, \quad \|(S_{\mathcal{B}} \mathcal{A})^{(*n)}\|_{X_0 \to Y_1} \Theta^{-1} \in L^1 \text{ if } Y_1 \subset X_1 \text{ (shrinkage)} \end{aligned}$$

Then,

$$\|S_\Lambda(t)\|_{Y_1 \to Y_0} \Theta^{-1} \in L^\infty.$$

Enlargement result. We iterate the Duhamel formula

$$S_\Lambda = S_\mathcal{B} + S_\Lambda * (\mathcal{A} S_\mathcal{B})$$

to get a "stopped Dyson-Phillips series" (the D-P series corresponds to $n = \infty$)

$$S_\Lambda = \sum_{\ell=0}^{n-1} S_\mathcal{B} * (\mathcal{A} S_\mathcal{B})^{(*\ell)} + S_\Lambda * (\mathcal{A} S_\mathcal{B})^{(*n)} =: S_1 + S_2.$$

From the assumptions, we immediately have

$$\|S_\Lambda\|_{Y_1 \to Y_0} \Theta^{-1} \le \|S_1\|_{Y_1 \to Y_0} \Theta^{-1} + \|S_\Lambda \Theta^{-1}\|_{X_1 \to X_0} * \|(\mathcal{A} S_\mathcal{B})^{(*n)} \Theta^{-1}\|_{Y_1 \to X_1} \in L^\infty$$

Shrinkage result. We argue similarly staring with the iterated the Duhamel formula / stopped Dyson-Phillips series

$$S_\Lambda = \sum_{\ell=0}^{n-1} S_\mathcal{B} * (\mathcal{A} S_\mathcal{B})^{(*\ell)} + (S_\mathcal{B} \mathcal{A})^{(*n)} * S_\Lambda.$$

Prop 4. (incorrect statement?) We consider two Banach spaces $X \subset Y$ and a generator Λ. We assume $X^1_\Lambda \subset Y$ is compact and $\Theta(t) \approx e^{-\lambda t^{1/(1+j)}}$

- $\Sigma_P(\Lambda) \cap \bar\Delta_0 = \emptyset$, with $\Delta_0 := \{z \in \mathbb{C};\, \Re e z > 0\}$
- $\Lambda = \mathcal{A} + \mathcal{B}$, with $\mathcal{A} \in \mathbf{B}(Y, X)$, $\zeta \in (0,1]$ and

$$
\begin{array}{lll}
(a1) & \forall\, \ell, & \|S_\mathcal{B} * (\mathcal{A} S_\mathcal{B})^{(*\ell)}\|_{X \to Y} \Theta^{-1} \in L^\infty \\
(a2) & \forall\, \ell, & \sup_{z \in \bar\Delta_0} \|(R_\mathcal{B}(z))^\ell\|_{X \to Y} \le C\, (\ell!)^j \\
(a3) & \forall\, \ell, & \sup_{z \in \bar\Delta_0} \|R_\mathcal{B}(z)\|_{Y \to X^\Lambda_\zeta} \le C\, (\ell!)^j
\end{array}
$$

Then,

$$\|S_\Lambda(t)\|_{X \to Y} \Theta^{-1} \in L^\infty.$$

We start again with the stopped Dyson-Phillips series

$$S_\Lambda = \sum_{\ell=0}^{N-1} S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{B}})^{(*\ell)} + S_\Lambda * (\mathcal{A} S_{\mathcal{B}})^{(*N)} = S_1 + S_2$$

The first $N-1$ terms are fine. For the last one, we use the inverse Laplace formula

$$\begin{aligned} S_2(t) f &= \frac{i}{2\pi} \int_{\uparrow_0} e^{zt}\, R_\Lambda(z) (\mathcal{A} R_{\mathcal{B}}(z))^N f \, dz \\ &\approx \frac{1}{t^k} \int_{\uparrow_0} e^{zt}\, \frac{d^k \Phi}{dz^k}\, dz\, f \\ &\lesssim \frac{C^k}{t^k} k! \int_{\uparrow_0} \sup_{|\alpha| \le k} \underbrace{\| R_\Lambda^{1+\alpha_1}(z) \|_{X \to Y}}_{\in L^\infty(\uparrow_0)\,?} \underbrace{\| \mathcal{A} R_{\mathcal{B}}^{1+\alpha_1} \ldots \mathcal{A} R_{\mathcal{B}}^{1+\alpha_N}(z) \|_{X \to X}}_{\in L^1(\uparrow_0)\,?} dz\, \|f\|_X, \end{aligned}$$

where $\uparrow_0 := \{z = 0 + iy,\ y \in \mathbb{R}\}$ and because

$$\frac{d^k \Phi}{dz^k} \approx \sum_{|\alpha| \le k} \alpha!\, R_\Lambda^{1+\alpha_0} \mathcal{A} R_{\mathcal{B}}^{1+\alpha_1} \ldots \mathcal{A} R_{\mathcal{B}}^{1+\alpha_N}$$

Key estimates

• Using (a2), (a3), the compact embedding $X^1_\Lambda \subset Y$ and the fact that there is not punctual spectrum in $\bar\Delta_0$, we get

$$\sup_{z\in\bar\Delta_0} \|R_\Lambda(z)^\ell\|_{X\to Y} \le C\,(\ell!)^j$$

• $\mathcal{A} \in \mathbf{B}(Y,X)$ and the resolvent identity

$$R_\mathcal{B}(z) = \frac{1}{z}(R_\mathcal{B}(z)\mathcal{B} - I) \in \mathbf{B}(X_1,X)$$

imply

$$\|\mathcal{A}R_\mathcal{B}(z)\|_{X_1\to X} \le C/|z| \quad \forall\, z \in \bar\Delta_0.$$

Together with (a2) (where we assume that $\zeta = 1$ in order to make the proof simpler) we get

$$\|\mathcal{A}R_\mathcal{B}(z)^{\ell_1}\mathcal{A}R_\mathcal{B}(z)^{\ell_2}\|_{X\to X} \le C\,(\ell_1!)^j(\ell_2!)^j\,\langle z\rangle^{-1}$$

• Choosing $N = 4$ and gathering the two estimates, we get

$$\|\frac{d^k\Phi}{dz^k}(z)\|_{X\to Y} \le C^k\,(k!)^j\,\langle z\rangle^{-2} \in L^1(\uparrow_0).$$

End of the proof

Coming back to the term S_2, we have

$$
\begin{aligned}
S_2(t) &\lesssim C^k k^{(1+j)k} t^{-k}. \\
&\lesssim e^{-\lambda t^{1/(1+j)}} = \Theta(t),
\end{aligned}
$$

by choosing appropriately $k = k(t)$

Prop 5.

Consider a semigroup generator Λ on a Banach lattice X, and assume
(1) Λ such as the spectral mapping Theorem holds (for $\|f\|_Y = \langle |f|, \phi\rangle$);
(2) $\phi \in D(\Lambda^*)$, $\phi \succ 0$ such that $\Lambda^*\phi = 0$;
(3) S_Λ is positive (and Λ satisfies Kato's inequalities);
(4) $-\Lambda$ satisfies a strong maximum principle.

There exists $0 < f_\infty \in D(\Lambda)$ such that

$$\Lambda f_\infty = 0, \quad \Sigma_P(\Lambda) \cap \bar{\Delta}_0 = \{0\}, \quad \Sigma_P(\Lambda_1) \cap \bar{\Delta}_0 = \emptyset$$

with $\Lambda_1 := \Lambda_{|X_1}$, $X_1 = R(I - \Pi_0) = (I - \Pi_0)X$,

$$\Pi_0 f = \langle f, \phi\rangle \, f_\infty \quad \forall\, f \in X.$$

Moreover the decay function Θ defining in the spectral mapping Theorem :

$$\|S_\Lambda(t)(I - \Pi_0) f_0\|_Y \lesssim \Theta(t)\, \|(I - \Pi_0) f_0\|_X \qquad \forall\, t \geq 0, \ \forall\, f_0 \in X.$$

Outline of the talk

Consider the Fokker-Planck equation

$$\partial_t f = \Lambda f = \Delta_v f + \mathrm{div}_v(F\, f)$$

with a weak confinement force field term F such that

$$F(v) \approx v \langle v \rangle^{\gamma-2} \quad \gamma \in (0,1)$$

and an initial datum

$$f(0) = f_0 \in W^{\sigma,p}(m) \quad (\text{means } m f_0 \in W^{\sigma,p}).$$

Here $p \in [1,\infty]$, $\sigma = 0$ and m is a polynomial weight

$$m = \langle v \rangle^k, \quad k > k^*(p,\sigma,\gamma),$$

or a exponential weight

$$m = e^{\kappa \langle v \rangle^s}, \quad s \in (0,\gamma],\ \kappa > 0.$$

For latter reference, we define $\sigma = 0$ if m is a polynomial and $\sigma = s$ if m is a exponential.

Theorem 2. (Kavian & M.)

There exists a unique "smooth", positive and normalized steady state f_∞.
For any $f_0 \in L^p(m)$

$$\|f(t) - \langle f_0 \rangle f_\infty\|_{L^p} \le \Theta(t) \, \|f_0 - \langle f_0 \rangle f_\infty\|_{L^p(m)},$$

with

$$\begin{aligned} \Theta(t) &= \frac{C}{\langle t \rangle^K}, \quad K \sim \frac{k - k^*(p)}{2 - \gamma} \quad \text{if} \quad m = \langle x \rangle^k \\ &= Ce^{-\lambda t^\sigma}, \quad \sigma \sim \frac{s}{2-\gamma} \quad \text{if} \quad m = e^{\kappa \langle x \rangle^s}. \end{aligned}$$

$\triangleright$ Improve by providing a better rate and/or a larger class of initial data earlier results by Toscani, Villani, 2000 (based on log-Sobolev inequality) & Röckner, Wang, 2001 (based on weak Poincaré inequality)

We introduce the splitting $\Lambda = \mathcal{A} + \mathcal{B}$, with $\mathcal{A}$ a multiplicator operator

$$\mathcal{A}f = M\chi_R(v)f, \quad \chi_R(v) = \chi(v/R), \quad 0 \le \chi \le 1, \ \chi \in \mathcal{D}(\mathbb{R}^d)$$

$\triangleright$ $\mathcal{A} \in \mathbf{B}(X_0, X_1)$, $X_i = W^{\sigma,p}(m_i)$, $m_1 \succeq m_0$
$\triangleright$ $\mathcal{B}$ is not a-dissipative in $X = W^{\sigma,p}(m)$ with $a < 0$. However, it is weakly dissipative. For $p \in (1,\infty)$, and $M, R > 0$ large enough, we have

$$\langle f^*, \mathcal{B}f\rangle_{L^p} \lesssim -\|f\|^2_{L^p(m\langle v\rangle^{(\gamma-2+\sigma)/p})}$$

That is a consequence of the identity

$$\int (\Lambda f) f^{p-1} m^p = (p-1)\int |\nabla(fm)|^2 (fm)^{p-1} + \int (fm)^p \psi$$

$$\begin{aligned} \psi &= \left(\frac{2}{p}-1\right)\frac{\Delta m}{m} + 2\left(1-\frac{1}{p}\right)\frac{|\nabla m|^2}{m^2} + \left(1-\frac{1}{p}\right)\mathrm{div}F - F\cdot\frac{\nabla m}{m} \\ &\sim -F\cdot\frac{\nabla m}{m} \sim -\langle v\rangle^{\sigma+\gamma-2} \end{aligned}$$

• the estimate

$$(1) \quad \|S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)}\|_{X_1 \to X_0} \le \Theta(t)$$

follows from Proposition 1.

• the estimate

$$(2) \quad \|(\mathcal{A}S_{\mathcal{B}})^{(*n)}\|_{\mathbf{B}(L^1(m_1),H^1(m_2))} \le \Theta(t)$$

follows from (1) and the use a "Nash + regularity" trick for small time. More precisely, introducing

$$\mathcal{F}(t,h) := \|h\|^2_{L^1(m)} + t^{\bullet}\|h\|^2_{L^2(m)} + t^{\bullet}\|\nabla_v h\|^2_{L^2(m)}$$

we are able to prove (for convenient exponents $\bullet > 1$)

$$\frac{d}{dt}\mathcal{F}(t, S_{\mathcal{B}}(t)h) \le 0 \quad \text{and then} \quad \|S_{\mathcal{B}}(t)h\|^2_{H^1(m)} \le \frac{1}{t^{\bullet}}\|h\|^2_{L^1(m)}$$

• The Fokker-Planck semigroup is obviously mass conservative and positive and the Fokker-Planck operator satisfies the strong maximum principle. The last point in order to apply Proposition 5 is to verify that assumption (a2) in Proposition 4 is satisfied.

Outline of the talk

A classical result ($\sim$ Guo?) sates that for any weight functions m, $m_1 \succ \langle v \rangle^{2+3/2}$ and $m_0 \succ \langle v \rangle^2$

$$\langle Q(f,g), h\rangle_{L^2(m)} \lesssim \left(\|f\|_{L^2(m)} \|g\|_{H^1_*(m_1)} + \|f\|_{H^1(m_0)} \|g\|_{L^2(m)} \right) \|h\|_{H^1_*(m)}$$

with

$$\|f\|^2_{H^1_*(m)} := \|f\|^2_{L^2(m\langle v\rangle^{(\gamma+\sigma)/2})} + \|\widetilde{\nabla} f\|^2_{L^2(m\langle v\rangle^{\gamma/2})},$$

and

$$\widetilde{\nabla}_v f = P_v \nabla_v f + \langle v\rangle (I - P_v) \nabla_v f, \quad P_v \xi = \left(\xi \cdot \frac{v}{|v|} \right) \frac{v}{|v|}.$$

As a consequence, we have

Prop 6.

for $m \succ \langle v\rangle^{2+3/2}$, defining $\mathcal{X} := H^2_x L^2_v(m)$, $\mathcal{Y} := H^2_x H^1_{v,*}(m)$, $\mathcal{Z} := H^2_x H^{-1}_{v,*}(m)$, we have

$$\langle Q(f,g), h\rangle_{\mathcal{X}} \lesssim \left(\|f\|_{\mathcal{X}} \|g\|_{\mathcal{Y}} + \|f\|_{\mathcal{Y}} \|g\|_{\mathcal{X}} \right) \|h\|_{\mathcal{Y}}$$

$$\|Q(f,g)\|_{\mathcal{Z}} \lesssim \left(\|f\|_{\mathcal{X}} \|g\|_{\mathcal{Y}} + \|f\|_{\mathcal{Y}} \|g\|_{\mathcal{X}} \right).$$

A introduce the equivalent norm

$$|||f|||^2_{\mathcal{X}} := \eta\|f\|^2_{\mathcal{X}} + \int_0^\infty \|S_{\mathcal{L}}(\tau)f\|^2_{\mathcal{X}_0}\,d\tau,$$

with $\mathcal{X}_0 := H^2_x L^2_v$, $\mathcal{Y}_0 := H^2_x H^1_{v,*}$, $\mathcal{Z}_0 := H^2_x H^{-1}_{v,*}$.
We consider a solution g to the Landau equation

$$\frac{d}{dt} g = \bar{\mathcal{L}} g + Q(g,g)$$

and we compute

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|||g|||^2_{X} &= \langle\langle \mathcal{L}g, g\rangle\rangle_{\mathcal{X}} + \eta\langle Q(g,g), g\rangle_{\mathcal{X}} \\ &\quad + \int_0^\infty \langle S_{\mathcal{L}}(\tau)Q(g,g), S_{\mathcal{L}}(\tau)g\rangle_{X_0}\,d\tau =: T_1 + T_2 + T_3. \end{aligned}$$

From Proposition 1, we expect to have

$$T_1 \lesssim -\|g\|^2_{\mathcal{Y}}.$$

Thanks to the choice of the norm and Proposition 6, we have

$$T_2 \le C\|g\|_{\mathcal{X}}\,\|g\|^2_{\mathcal{Y}}.$$

For the last term, thanks to Proposition 6, we have

$$\begin{aligned} T_3 &= \int_0^\infty \langle S_{\mathcal{L}}(\tau) Q(g,g), S_{\mathcal{L}}(\tau) g\rangle_{\mathcal{X}_0}\, d\tau \\ &\lesssim \int_0^\infty \|S_{\mathcal{L}}(\tau) Q(g,g)\|_{\mathcal{Z}_0} \|S_{\mathcal{L}}(\tau) g\|_{\mathcal{Y}_0}\, d\tau \\ &\lesssim \|Q(g,g)\|_{\mathcal{Z}} \|g\|_{\mathcal{Y}} \int_0^\infty \Theta(\tau)^2\, d\tau \lesssim \|g\|_{\mathcal{X}} \|g\|_{\mathcal{Y}}^2, \end{aligned}$$

under the condition that

$$t \mapsto \|S_\Lambda(t)\|_{\mathcal{Y}\to\mathcal{Y}_0}, \|S_\Lambda(t)\|_{\mathcal{Z}\to\mathcal{Z}_0} \in L^2(\mathbb{R}_+).$$

We conclude with

$$\frac{d}{dt} |||g|||_{\mathcal{X}}^2 \lesssim \|g\|_{\mathcal{Y}}^2 (1 - C|||g|||_{\mathcal{X}})$$

We deduce

▷ a priori uniform estimate for $|||g_0|||_{\mathcal{X}}^2$ small, and then classically existence and uniqueness

▷ considering two weight functions $m \succ \tilde{m}$, the above a priori estimate implies

$$\frac{d}{dt} |||g|||_{\tilde{\mathcal{X}}}^2 \lesssim -\|g\|_{\tilde{\mathcal{Y}}}^2, \quad \frac{d}{dt} |||g|||_{\mathcal{X}}^2 \lesssim 0,$$

and we get decay estimate just repeating the proof of Proposition 1.

We introduce the splitting $\bar{\mathcal{L}} = \mathcal{A} + \mathcal{B}$

$$\mathcal{A}g := Q(g,\mu) + M\chi_R g = (a_{ij} * g)\partial_{ij}\mu - (c * g)\mu + M\chi_R g,$$
$$\mathcal{B}g := Q(\mu,g) - M\chi_R g - v\cdot\nabla_x g = (a_{ij} * \mu)\partial_{ij} g - (c * \mu) g - M\chi_R g - v\cdot\nabla_x g,$$

with

$$b_i(z) = \partial_j a_{ij}(z) = -2\,|z|^\gamma\, z_i, \quad c(z) = \partial_{ij} a_{ij}(z) = -8\pi\delta_0$$

We show

• Weak dissipativity of $\mathcal{B}$ in many spaces (twisting trick, duality trick)

$$(\mathcal{B}f,f)_{H^2_x L^2(m)} \lesssim -\|f\|^2_{H^2_x H^1_{*,v}(m)}$$
$$(\mathcal{B}f,f)_{H^2_x H^1_v(m)} \lesssim -\|f\|^2_{H^2_x H^1(m\langle v\rangle^{(\gamma+2)/2})}$$
$$(\mathcal{B}^* f,f)_{H^2_x H^1(m)} \lesssim -...$$

• Decay estimate of $S_\mathcal{B}$ in many spaces by Proposition 1.

• Regularization property of $S_\mathcal{B}$ in many spaces by using "Hormander-Hérau-Villani" hypoelliptic trick. More precisely, introducing

$$\mathcal{F}(t,h) := \|h\|^2_{L^2(m)} + t^\bullet\|\nabla_v h\|^2_{L^2(m)} + t^\bullet(\nabla_v h, \nabla_x h)_{L^2(m)} + t^\bullet\|\nabla_x h\|^2_{L^2(m)}$$

we get (for convenient exponents $\bullet \geq 1$)

$$\frac{d}{dt}\mathcal{F}(t, S_\mathcal{B}(t)h) \leq 0, \quad \forall\, t \in [0,1].$$

• $\mathcal{A} \in \mathbf{B}(H^\alpha_x H^\beta_v(m_0), H^\alpha_x H^\beta_v(m_1))$ for any weight functions $m_1 \succeq m_0$.

• In the space of self-adjointness $L^2(\mu^{-1/2})$ we have the nice dissipativity estimate

$$\langle \mathcal{L} g, g \rangle_{L^2(\mu^{-1/2})} \lesssim - \|\Pi g\|^2_{H^1_*(\mu^{-1/2})}$$

from which we deduce thanks to the twisting hypocoercivity Nier-Hérau-Villini trick

$$\langle\!\langle \bar{\mathcal{L}} g, g \rangle\!\rangle_{H^1_{x,v}(\mu^{-1/2})} \lesssim - \|\bar{\Pi} g\|^2_{H^1_x H^1_{v*}(\mu^{-1/2})}$$

We deduce

• $S_{\bar{\mathcal{L}}}$ is bounded in many spaces because $S_{\bar{\mathcal{L}}}$ is bounded in one space and $\bar{\mathcal{L}}$ splits in a suitable way (Proposition 3 of extension).

• $S_{\bar{\mathcal{L}}}$ is fast decaying in one space $\mathbf{B}(H^1_{x,v}(\mu^{-3/2}, H^1_{x,v}(\mu^{-3/2})$ because it is bounded in $H^1_{x,v}(\mu^{-3/2})$ and weakly dissipative in $H^1_{x,v}(\mu^{-1/2})$ (Proposition 1).

• $S_{\bar{\mathcal{L}}}$ is decaying in many space because $S_{\bar{\mathcal{L}}}$ is decaying in one space and $\bar{\mathcal{L}}$ splits in a suitable way (Proposition 3 of extension).

As a conclusion, we are able to prove

• On the one hand,

$$\|S_{\mathcal{L}}\|_{\mathcal{X}\to\mathcal{X}_0} \le \Theta(t),$$

and $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with

$$\langle f, \mathcal{B} f\rangle_{\mathcal{X}} \lesssim -\|f\|^2_{\mathcal{Y}}, \quad \langle f, \mathcal{A} f\rangle_{\mathcal{X}} \lesssim -\|f\|^2_{\mathcal{X}_0}$$

in order to use Proposition 2 and define the weak dissipative norm for $\mathcal{L}$

• On the one hand,

$$t \mapsto \|S_{\mathcal{L}}\|_{\mathcal{Y}\to\mathcal{Y}_0}, \|S_{\mathcal{L}}\|_{\mathcal{Z}\to\mathcal{Z}_0} \in L^2(\mathbb{R}_+)$$

$\triangleright$ That are the need properties in order to get the a priori nonlinear estimate !