On a linear runs and tumbles equation

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Plan

The Runs and Tumbles equation (RTeq) writes

We consider the Runs and Tumbles linear equation

$$\partial_t f = \mathcal{L} f = -v \cdot \nabla_x f + \int_{\mathcal{V}} \{K' f' - K f\}\, dv',$$

on the *density of cells* $f = f(t, x, v) \geq 0$ which at time $t \geq 0$, position $x \in \mathbb{R}^d$ move with velocity $v \in \mathcal{V} := B(0, V_0)$, $|\mathcal{V}| = 1$. Shorthands $f' = f(t, x, v')$, ...
The turning kernel is assumed to be

$$K = K(x, v) := 1 + \chi\zeta = 1 \pm \chi, \quad \chi \in (0, 1), \quad \zeta = \mathrm{sign}(x \cdot v).$$

The evolution equation is complemented with an initial condition

$$f(0, .) = f_0 \quad \text{in} \quad \mathbb{R}^d \times \mathcal{V}.$$

▷ We aim to provide a complete description of the long time behavior of solutions: they are attracted by a unique associated steady state.

▷ Our result generalizes to any dimension $d \geq 1$ a similar result due to Calvez, Raoul and Schmeiser (KRM 2015) for $d = 1$.

▷ Biologically motivated by a nonlinear chemotaxis model.

▷ Mathematically motivated by the unusual confinement mechanism (in the $x \in \mathbb{R}^d$ variable) due to the specific form of K.

For any test function φ, we have

$$(\mathcal{L}^*\varphi)(x,v) := v \cdot \nabla_x \varphi + K \int_{\mathcal{V}} \{\varphi' - \varphi\}\, dv'.$$

With the choice $\varphi = 1$, we deduce the mass conservation

$$\langle\langle f(t,.)\rangle\rangle = \langle\langle f_0\rangle\rangle, \quad \forall\, t \geq 0,$$

where the mass is defined through

$$\langle\langle g\rangle\rangle := \int_{\mathbb{R}^d \times \mathcal{V}} g\, dvdx.$$

For the functions $\beta(s) = s_\pm$ and $\beta(s) = |s|$, we have Kato's inequalities

$$\mathcal{L}\beta(f) \geq \Re e \beta'(f)(\mathcal{L} f).$$

Using $\beta(s) = s_-$, we deduce that the equation preserves positivity

$$f(t,.) \geq 0 \quad \forall\, t \geq 0 \quad \text{if} \quad f_0 \geq 0.$$

And there is nothing simple to say more !!

In a chemical agent environment S, the turning kernel is given by

$$K[S](v) := 1 - \chi\, \mathrm{sign}(v \cdot \nabla_x S(x)), \quad \chi \in (0,1).$$

When the chemical agent is produced by the microorganisms themselves, it is given by the damped Poisson equation

$$-\Delta S + S = \varrho := \int_{\mathcal{V}} f \, dv.$$

When ϱ is radially symmetric and decreasing then S satisfies the same properties (thanks to the maximum principle) and

$$K[S] = 1 - \chi \mathrm{sign}(v \cdot (-x)) = 1 + \chi \mathrm{sign}(x \cdot v) = K.$$

Functional framework

For a weight function

$$m = m(x) = e^{\gamma \langle x \rangle}, \ \gamma > 0, \ \langle x \rangle^2 = 1 + |x|^2,$$

we define the weighted Lebesgue space $L^p(m)$, $1 \le p \le \infty$, through its norm

$$\|f\|_{L^p(m)} := \|f \, m\|_{L^p}.$$

Theorem 1.

$\exists\, \gamma^* > 0$, $\exists!\; G > 0$ radially symmetric and normalized stationary state

$$0 < G \in L^1(m_0) \cap L^\infty, \quad \langle\!\langle G \rangle\!\rangle = 1,$$
$$-v \cdot \nabla_x G + \int_{\mathcal{V}} \big\{ K' G' - K G \big\}\, dv' = 0,$$

where $m_0(x) := \exp(\gamma_* \langle x \rangle)$.

For any (exponential) weight function m and $0 \le f_0 \in L^1(m)$, there exists a unique solution $f \in C([0,\infty); L^1(m))$ to the Runs and Tumbles equation associated to the initial datum f_0 and

$$\| f(t) - \langle\!\langle f_0 \rangle\!\rangle G \|_{L^1(m)} \le \Theta_m(t)\, \| f_0 \|_{L^1(m)}, \quad \forall\, t \ge 0,$$

with $\Theta_m(t) = e^{at}$, $a < 0$.

Plan

Moment estimate

Lemma 1. (Moment estimate)

There exists $\gamma^* > 0$ and for any $\gamma \in (0, \gamma^*)$

$$\exists \tilde m, \quad \tilde m \sim m := e^{\gamma \langle x \rangle}, \quad \exists C > 0,$$

such that

$$\sup_{t \ge 0} \|f(t)\|_{L^1(\tilde m)} \le \max\Big(C\, \|f_0\|_{L^1}, \|f_0\|_{L^1(\tilde m)} \Big).$$

Proof: We prove that there exists $C, \tilde\beta > 0$ such that

$$\mathcal{L}^*(\tilde m) \le \tilde\beta C - \tilde\beta \tilde m,$$

so that for any solution

$$\frac{d}{dt} \int f(t)\, \tilde m = \int f(t) \mathcal{L}^* \tilde m \le \tilde\beta C \int f_0 - \tilde\beta \int f(t)\, \tilde m.$$

We conclude by integrating that differential inequality.

We successively compute

$$\mathcal{L}^* \, e^{\gamma\langle x\rangle} = v\cdot\nabla_x\Big[...\Big] = \gamma(v\cdot\frac{x}{\langle x\rangle})e^{\gamma\langle x\rangle},$$

$$\begin{aligned}\mathcal{L}^*\Big[(v\cdot\frac{x}{\langle x\rangle})e^{\gamma\langle x\rangle}\Big] &= v\cdot\nabla_x\Big[...\Big] - K\Big[...\Big]\\ &= \Big(\mathcal{O}\Big(\frac{1}{\langle x\rangle}+\gamma\Big) - v\cdot\frac{x}{\langle x\rangle} - \chi\,\frac{|v\cdot x|}{\langle x\rangle}\Big)e^{\gamma\langle x\rangle},\end{aligned}$$

$$\begin{aligned}\mathcal{L}^*\Big[-\frac{|v\cdot x|}{\langle x\rangle}\,e^{\gamma\langle x\rangle}\Big] &= v\cdot\nabla_x\Big[...\Big] - K(V_1\frac{|x|}{\langle x\rangle} - \frac{|v\cdot x|}{\langle x\rangle})e^{\gamma\langle x\rangle}\\ &\le \Big(\mathcal{O}\Big(\frac{1}{\langle x\rangle}+\gamma\Big) - (1-\chi)V_1\frac{|x|}{\langle x\rangle} + (1+\chi)\frac{|v\cdot x|}{\langle x\rangle}\Big)\,e^{\gamma\langle x\rangle}.\end{aligned}$$

Proof of the moment estimate

We successively compute

$$\mathcal{L}^* e^{\gamma\langle x\rangle} = \gamma(v \cdot \frac{x}{\langle x\rangle})\, e^{\gamma\langle x\rangle},$$

$$\mathcal{L}^* \Big[\gamma(v \cdot \frac{x}{\langle x\rangle}) e^{\gamma\langle x\rangle}\Big] \quad = \quad \Big(\mathcal{O}\Big(\frac{\gamma}{\langle x\rangle} + \gamma^2\Big) - \gamma v \cdot \frac{x}{\langle x\rangle} - \gamma\chi\,\frac{|v\cdot x|}{\langle x\rangle}\Big) e^{\gamma\langle x\rangle}.$$

$$\mathcal{L}^* \Big[-\beta\frac{|v\cdot x|}{\langle x\rangle}\, e^{\gamma\langle x\rangle}\Big] \quad \le \quad \Big(\mathcal{O}\Big(\frac{\gamma}{\langle x\rangle} + \gamma^2\Big) - \beta(1-\chi)V_1\frac{|x|}{\langle x\rangle} + \beta(1+\chi)\frac{|v\cdot x|}{\langle x\rangle}\Big)\, e^{\gamma\langle x\rangle}.$$

Introducing then the weight function

$$\widetilde{m} := \Big(1 + \gamma(v\cdot\frac{x}{\langle x\rangle}) - \beta\,\frac{|v\cdot x|}{\langle x\rangle}\Big) e^{\gamma\langle x\rangle} \sim m, \quad \beta(1+\chi) = \gamma\chi, \quad \gamma > 0 \text{ small},$$

we conclude with

$$\mathcal{L}^*(\tilde{m}) \le \Big(\mathcal{O}\Big(\frac{\beta}{\langle x\rangle} + \beta^2\Big) - 2\tilde{\beta}\Big)\, e^{\gamma\langle x\rangle} \le \tilde{\beta}C - \tilde{\beta}\tilde{m}, \quad 2\tilde{\beta} := \beta(1-\chi)V_1.$$

A remark to conclude (without rate and slightly abusing ...)

▷ That is the cornerstone estimate of the proof !!

▷ From that weighted L^1 bound one may deduce the existence of a probability measure which is a steady state thanks to a (Brouwer) fixed point theorem

▷ Because the "PDE is linear and positive", we may use the generalized entropy method and conclude to the uniqueness of the steady state for any given mass and the convergence (without rate) of any solution to the associated steady state (with same mass).

▷ In order to prove our result, we may apply a Krein-Rutman theorem established in a paper by M. & Scher (Annals IHP 2016). For pedagogical reason, we rather split the proof is several steps.

Plan

We introduce the generator $\mathcal{B}_0$ defined by

$$\mathcal{B}_0 f := -v \cdot \nabla_x f - Kf$$

and the averaging operator $A = A_\varphi : L^q_x L^1_v \to L^q_x$ defined by

$$A_\varphi f(x) := \int_{\mathcal{V}} \varphi(x, v')\, f(x, v')\, dv', \quad \varphi \in L^\infty_{xv}.$$

Lemma 2.

$\exists\, a^* < 0$, $\exists\, n \geq 1$, such that for any $m = e^{\gamma \langle x \rangle}$, $\gamma \in [0, \gamma^*)$, $\forall\, a > a^*$,

$$\begin{array}{lll} (1) & S_{\mathcal{B}_0}(t) : L^p(m) \to L^p(m) \quad \text{bdd of order} & \mathcal{O}(e^{at}), \\ (2) & A_\varphi S_{\mathcal{B}_0}(t) : L^1_x L^\infty_v(m) \to L^\infty_{xv} \quad \text{bdd of order} & \mathcal{O}(t^{-d}\, e^{at}), \\ (3) & (A_\varphi S_{\mathcal{B}_0})^{(*n)}(t) : L^1_{xv}(m) \to L^\infty_{xv} \quad \text{bdd of order} & \mathcal{O}(e^{at}). \end{array}$$

Proof: For (2), we use Bardos & Degond's dispersion lemma.

For (3), we observe that

$$\left.\begin{array}{l} u : L^1_x L^\infty_v(m) \to L^1_x L^\infty_v(m) \text{ as } \mathcal{O}(e^{at}) \\ u : L^\infty \to L^\infty \text{ as } \mathcal{O}(e^{at}) \\ u : L^1_x L^\infty_v(m) \to L^\infty \text{ as } \mathcal{O}(t^{-d} e^{at}) \end{array}\right\} \quad \text{imply} \quad u^{(*d)} : L^1_x L^\infty_v(m) \to L^\infty \text{ as } \mathcal{O}(e^{at}),$$

and we take $n = d + 1$.

L^p estimate for $S_{\mathcal{B}_1}$

We introduce the generator $\mathcal{B}_1$ defined by

$$\mathcal{B}_1 f := -v \cdot \nabla_x f - Kf + (1-\phi_R) \int_{\mathcal{V}} K' f' \, dv',$$

with $\phi_R(x) := \phi(x/R)$ for a given radially symmetric function $\phi \in \mathcal{D}(\mathbb{R}^d)$ such that $\mathbf{1}_{B(0,1)} \le \phi \le \mathbf{1}_{B(0,2)}$.

Lemma 3.

$\exists R > 0$, $\exists n \ge 1$ such that for any $m = e^{\gamma \langle x \rangle}$, $\gamma \in [0, \gamma^*)$, $\forall\, a > a^*$,

$$\begin{aligned}
&(1) \quad S_{\mathcal{B}_1}(t) : L^1(m) \to L^1(m) \quad \text{bdd of order } \mathcal{O}(e^{at}),\\
&(2) \quad S_{\mathcal{B}_1}(t) : L^1(m) \cap L^p(m^{1/p}) \to L^1(m) \cap L^p(m^{1/p}) \quad \text{bdd of order } \mathcal{O}(e^{at}),\\
&(3) \quad A_\varphi S_{\mathcal{B}_1}(t) : L^1_x L^\infty_v(m) \to L^\infty_{xv} \quad \text{bdd of order } \mathcal{O}(t^{-d} e^{at}),\\
&(4) \quad (A_\varphi S_{\mathcal{B}_1})^{(*n)}(t) : L^1_{xv}(m) \to L^\infty_{xv} \quad \text{bdd of order } \mathcal{O}(e^{at}).
\end{aligned}$$

Proof: For (1), we use $\mathcal{B}_1^* = \mathcal{L}^* - \phi_R\, K$ and Lemma 1.

For (2), (3) and (4), we write $\mathcal{B}_1 = \mathcal{B}_0 + \mathcal{A}_0^c$ and we use the iterated Duhamel formula

$$S_{\mathcal{B}_1} = \left\{ S_{\mathcal{B}_0} + \ldots + S_{\mathcal{B}_0} * (\mathcal{A}_0^c S_{\mathcal{B}_0})^{(*n)} \right\} + S_{\mathcal{B}_0} * (\mathcal{A}_0^c S_{\mathcal{B}_0})^{(*n)} * \mathcal{A}_0^c S_{\mathcal{B}_1}.$$

L^∞ estimate for $S_\mathcal{L}$ and existence of a steady state

Lemma 3.

$S_\mathcal{L}$ is a bounded semigroup in $X = L^1(m) \cap L^\infty$.

Proof: We observe that

$$\mathcal{L} = \mathcal{A}_1 + \mathcal{B}_1, \quad \mathcal{A}_1 := A_\psi, \quad \psi := \phi_R K(x,v),$$

and we write

$$S_\mathcal{L} = \left\{ S_{\mathcal{B}_1} + ... + S_{\mathcal{B}_1} * (\mathcal{A}_1 S_{\mathcal{B}_1})^{(n*)} \right\} + S_{\mathcal{B}_1} * (\mathcal{A}_1 S_{\mathcal{B}_1})^{(n*)} * \mathcal{A}_1 S_\mathcal{L}.$$

Corollary.

There exists at least one nonnegative with unit mass steady state $G \in X$.

Proof: We define the equivalent norm

$$\forall f \in X, \quad |||f||| := \sup_{t \geq 0} \|S_\mathcal{L}(t) f\|_X.$$

For $C \geq 1$ large enough, the set

$$\mathcal{C} := \Big\{ f \in L^1(m) \cap L^\infty;\ f \geq 0,\ \langle\langle f \rangle\rangle = 1,\ f_R = f, \forall R \in SO(d),\ |||f||| \leq C \Big\}$$

is convex, weakly $*$ compact and left invariant by the action of $S_\mathcal{L}$. Because $S_\mathcal{L}$ is weakly $*$ continuous, we may use a Brouwer type fixed point theorem to conclude.

Plan

Lemma 3 (strong MP).

For any solution to $\mathcal{L}f = 0$, $f \geq 0$, there holds $f \equiv 0$ or $f > 0$.

Proof: When $f \not\equiv 0$, we have

$$v \cdot \nabla_x f + (1+\chi) f \geq (1-\chi)\varrho, \quad \varrho := \int_{\mathcal{V}} f \, dv,$$

and we spread out positivity.

Corollary (uniqueness).

The null space $N(\mathcal{L})$ is $\mathbb{R}G$.

Proof: Consider f another steady state. We may reduce to the case when f is nonnegative and has unit mass. Then $g := f - G$ satisfies $\mathcal{L}g = 0$. In particular,

$$\mathcal{L}g_+ \geq (\mathrm{sign}_+ g)\mathcal{L}g = 0 \quad \text{and} \quad \int \mathcal{L}g_+ = \int g_+ \mathcal{L}^* 1 = 0,$$

so that $\mathcal{L}g_+ = 0$. From the strong MP, we deduce $g_+ = 0$ or $g_+ > 0$. In the second case, we get $g > 0$ and then

$$1 = \langle |f| \rangle \geq \langle f \rangle > \langle G \rangle = 1 \quad \text{absurd!}$$

In a similar way, we have $g_- = 0$ and we conclude with $f - G = g = 0$.

Lemma (Strong Kato's inequality).

The case of saturation in Kato's inegality

$$\mathcal{L}|f| = \Re e(\mathrm{sign} f)\, \mathcal{L} f$$

implies $\exists u \in \mathbb{C}$ such that $f = u|f|$.

Corollary (about the spectrum on the imaginary axis).

There is no other eignevalue on $i\mathbb{R}$: $\Sigma(\mathcal{L}) \cap i\mathbb{R} = \{0\}$ and 0 is (algebraically) simple.

Proof: If $\mathcal{L} f = \mu f$ with $\Re e \mu = 0$, we write

$$0 = (\Re e \mu)|f| = \Re e(\mathrm{sign} f)\, \mathcal{L} f \leq \mathcal{L}|f|$$

and then $\mathcal{L}|f| = 0$ by integration. We may applies the strong Kato's inequality to get $f = u|f|$ and then $\mathcal{L} f = 0$. That implies $\mu = 0$.

Plan

We introduce

$$\mathcal{L} = \mathcal{A} + \mathcal{B}, \quad \mathcal{A}f := \int_{\mathcal{V}} K'_{R,\delta_i} f' \, dv',$$

where

$$K_{R,\delta_i} = \phi_{\delta_2,R}(x)\, \psi_{\delta_1}(v)\, K_{\delta_3}(x,v), \quad K_{\delta_3}(x,v) = 1 + \chi\zeta_{\delta_3}(x\cdot v),$$

for some real numbers $R > 1$, $\delta_1, \delta_2, \delta_3 \in (0,1)$ to be fixed, and where $\phi_{\delta_2,R}$, ψ_{δ_1}, ζ_{δ_3} are smooth versions of

$$\mathbf{1}_{\delta_2<|x|<R}, \quad \mathbf{1}_{\delta_1<|v|<V_0-\delta_1}, \quad \mathrm{sign}(x\cdot v).$$

As a consequence

$$\mathcal{A} : W^{1,p}(m) \to W^{1,p}(\nu)!$$

We shall perform a spectral analysis of the generator $\mathcal{L}$ (Weyl's theorem) and of the associated semigroup $S_{\mathcal{L}}$ (spectral mapping theorem) in the Banach space

$$X := L^1(m) \cap L^2(m^{1/2}), \quad m = e^{\gamma\langle x\rangle}, \quad \gamma \in (0,\gamma^*).$$

Lemma (Dissipativity of $\mathcal{B}$).

There exist some $R, \delta_1, \delta_2, \delta_3 > 0$ such that

$$\|S_{\mathcal{B}}(t)\|_{X\to X} \lesssim e^{at}, \quad \forall\, t \geq 0, \quad \forall\, a > a^*.$$

Proof: We introduce the new equivalent norm

$$\begin{aligned} \|f\|_X^2 &:= \|f\|^2_{L^1(m)} + \|f\|^2_{L^2(m^{1/2})}, \\ |||f|||^2 &:= \eta_1\, \|f\|^2_{L^2(\widetilde{m}_0^{1/2})} + \eta_2 \|f\|_X^2 + \int_0^\infty \|S_{\mathcal{B}_1}(\tau) f\|_X^2\, d\tau, \end{aligned}$$

for some constants $\eta_1, \eta_2 \in (0,1)$ to be fixed, for the weight function

$$\widetilde{m}_0 := \left(1 - \frac{x}{|x|^{1/2}} \cdot \frac{v}{|v|}\right) \phi_{1/2}(x), \quad \phi_{1/2} \sim \mathbf{1}_{|x|\leq 1/2}.$$

We define $f_{\mathcal{B}}(t) := S_{\mathcal{B}}(t) f_0$, we compute

$$\frac{1}{2}\frac{d}{dt} |||f_{\mathcal{B}}(t)|||^2 = \eta_1 T_1 + \eta_2 T_2 + T_3$$

and we estimates each term T_i.

• Using Lions-Perthame's multiplicator trick which gives a nice version of "Perthame's third moment lemma", we get

$$T_1 \lesssim -\|f_{\mathcal{B}}\|^2_{L^2(\widetilde{m}_1^{1/2})} + \|f_{\mathcal{B}}\|^2_X, \quad \widetilde{m}_1(x,v) := \frac{|v|}{|x|^{1/2}} \mathbf{1}_{|x|\le 1}.$$

• $T_2 \lesssim \|f_{\mathcal{B}}\|^2_X$.

• We introduce $\mathcal{B} = \mathcal{B}_1 + \mathcal{A}_1^c + \mathcal{A}_2^c + \mathcal{A}_3^c$, with

$$\mathcal{A}_2^c f = \phi_{\delta_2}(x) \int_{\mathcal{V}} K' f' \psi_{\delta_1}(v')\, dv', \quad \mathcal{A}_3^c f = \phi_{\delta_2,R}(x) \int_{\mathcal{V}} K^c_{\delta_3}(x\cdot v') f' \psi_{\delta_1}(v')\, dv'.$$

We observe that for $p \in \{1,2\}$

$$\|\mathcal{A}_2^c f\|^2_{L^p(m^{1/p})} \lesssim \frac{\delta_2^{1/2}}{\delta_1} \|f_{\mathcal{B}}\|^2_{L^2(\widetilde{m}_1^{1/2})} \quad \|\mathcal{A}_3^c f\|^2_{L^p(m^{1/p})} \lesssim \frac{\delta_3}{\delta_2} \|f\|^2_{L^2},$$

and we deduce

$$T_3 \lesssim \left(\delta_1 + \frac{\delta_3}{\delta_2} - 1\right) \|f_{\mathcal{B}}\|^2_X + \frac{\delta_2^{1/2}}{\delta_1} \|f_{\mathcal{B}}\|^2_{L^2(\widetilde{m}_1^{1/2})}.$$

Lemma (Regularity estimate).

$$\int_0^\infty \|\mathcal{A}S_\mathcal{B}(t)\,f\|_Y^2\,e^{-2at}\,dt \le C_a\,\|f\|_X^2, \quad \forall\, f \in X,$$

with

$$Y := \{f \in L^2(\mathbb{R}^d \times \mathcal{V});\ \mathrm{supp}\, f \subset B(0,R) \times \mathcal{V},\ \ f \in H^{1/2}\}.$$

Proof: Step 1. We introduce the damped free transport equation and its associated semigroup

$$\partial_t f = \mathcal{T} f := -v \cdot \nabla_x f - f, \quad f_{|t=0} = f_0, \quad [S_\mathcal{T}(t) f_0](x,v) := f_0(x - vt, v)\,e^{-t}$$

Following Bouchut-Desvillettes' version of the "averaging moment lemma", we define the Fourier transform on the x variable $\hat f = \hat f(t,\xi,v)$ and starting from

$$\partial_t \hat f + iv\cdot\xi \hat f - \hat f = 0, \quad \hat f_{|t=0} = \hat f_0,$$

we deduce that for any $\varphi \in L^2(\mathcal{V})$, there holds

$$\int_0^\infty \|A_\varphi S_\mathcal{T}(t) f_0\|^2_{H^{1/2}_x}\,e^{2t}\,dt \lesssim \|\varphi\|^2_{L^2(\mathcal{V})}\|f_0\|^2_{L^2}, \quad \forall\, f_0 \in L^2.$$

Step 2. Expanding the smooth kernel in Fourier series

$$K_{R,\delta_i}(x,v) = \sum_{k,\ell \in \mathbb{Z}^d} a_{k,\ell}\, e^{i\, x\cdot k}\, e^{i\, v\cdot \ell}$$

and using Step 1, we deduce

$$\int_0^\infty \|\mathcal{A}S_{\mathcal{T}}(t) f_0\|^2_{X^{1/2}_{\mathcal{B}}}\, e^{2t}\, dt \lesssim \|f_0\|^2_{L^2}.$$

Step 3. We split $\mathcal{B} = \mathcal{T} + \mathcal{C}$ (with $\mathcal{C}$ of order 0) and we write the Duhamel formula as

$$S_{\mathcal{B}} = S_{\mathcal{T}} + S_{\mathcal{T}} * \mathcal{C} S_{\mathcal{B}},$$

from which we deduce

$$\mathcal{A}S_{\mathcal{B}} = \mathcal{A}S_{\mathcal{T}} + \mathcal{A}S_{\mathcal{T}} * \mathcal{C} S_{\mathcal{B}}.$$

Lemma (spectral gap).

There is $a^* < 0$ such that $\Sigma(\mathcal{L}) \cap \Delta_{a^*} = \{0\}$.

Proof. For an generator L we define the resolvent operator

$$R_L(z) = (L - z)^{-1} = -\int_0^\infty S_L(t)\, e^{-zt}\, dt.$$

From $\mathcal{L} = \mathcal{A} + \mathcal{B}$, we get

$$R_{\mathcal{L}} = R_{\mathcal{B}} - R_{\mathcal{L}} \mathcal{A} R_{\mathcal{B}} = R_{\mathcal{B}} - R_{\mathcal{B}} \mathcal{A} R_{\mathcal{B}} + R_{\mathcal{L}} (\mathcal{A} R_{\mathcal{B}})^2$$

from what we deduce

$$R_{\mathcal{L}}(z)(1 - (\mathcal{A} R_{\mathcal{B}}(z))^2) = R_{\mathcal{B}}(z) - R_{\mathcal{B}}(z) \mathcal{A} R_{\mathcal{B}}(z).$$

- From

$$\|\mathcal{A} R_{\mathcal{B}}(z) f_0\|_Y^2 \le \int_0^\infty \|\mathcal{A} S_{\mathcal{B}}(t)\, f_0\|_Y^2\, e^{-2at}\, dt \le C_a\, \|f_0\|_X^2, \quad \forall\, f_0 \in X,\ z \in \Delta_a,$$

we get the estimate

$$\mathcal{A} R_{\mathcal{B}}(z) : X \to Y \text{ as } \mathcal{O}(1), \quad \forall\, z \in \Delta_a, \quad a < 0.$$

End of the proof of the spectral gap

• On the one hand, together with the interpolation estimate

$$\left.\begin{array}{l} R_{\mathcal{B}}(z) : X_1 \to X \text{ as } \mathcal{O}(\langle z\rangle^{-1}) \\ R_{\mathcal{B}}(z) : X \to X \text{ as } \mathcal{O}(1) \end{array}\right\} \quad \text{imply} \quad R_{\mathcal{B}}(z) : X_{1/2} \to X \text{ as } \mathcal{O}(\langle z\rangle^{-1/2}),$$

and observing that $Y \subset X_{1/2}$, we deduce

$$(\mathcal{A}R_{\mathcal{B}}(z))^2 = \mathcal{A}\, R_{\mathcal{B}}(z)(\mathcal{A}R_{\mathcal{B}}(z)) : X \to X \text{ as } \mathcal{O}(\langle z\rangle^{-1/2}).$$

In particular, $I - (\mathcal{A}R_{\mathcal{B}}(z))^2$ is invertible in $\Delta_a \cap B(0, M)^c$ for $M > 1$ large.

• On the other hand, because $Y \subset X$ with compact embedding, the operator $I - (\mathcal{A}R_{\mathcal{B}}(z))^2$ is an analytic and compact perturbation of the identity, and the Ribarič-Vidav-Voigt's version of Weyl's theorem implies that

$$\Sigma(\mathcal{L}) \cap \Delta_a = \Sigma_d(\mathcal{L}) \cap \Delta_a = \text{discrete set}.$$

• Both information together, we have

$$\Sigma(\mathcal{L}) \cap \Delta_a = \Sigma_d(\mathcal{L}) \cap \Delta_a = \text{finite set}.$$

We conclude by using that $\Sigma(\mathcal{L}) \cap \bar{\Delta}_0 = \{0\}$.

Lemma (semigroup decay in X).

Defining $\Pi g := G\langle g\rangle$, there holds

$$\|S_{\mathcal{L}}(t)(I-\Pi)\|_{X\to X} \lesssim e^{at}, \quad \forall\, t\ge 0,\ \forall\, a > a^*.$$

Proof. We set $\Pi^\perp = I - \Pi$ and we write

$$\begin{aligned} S_{\mathcal{L}}(t)\Pi^\perp &= \Pi^\perp\{S_{\mathcal{B}} + ... + S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*5)} + S_{\mathcal{L}} * (\mathcal{A}S_{\mathcal{B}})^{(*6)})\} \\ &\simeq \Pi^\perp\{S_{\mathcal{B}} + ... + S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*5)}\} + \int_{\uparrow_a} \Pi^\perp R_{\mathcal{L}}(z)(\mathcal{A}R_{\mathcal{B}})^6\, e^{zt}\, dz. \end{aligned}$$

Because $\|\Pi^\perp R_{\mathcal{L}}(z)\|$ is uniformly bounded on $\bar\Delta_a$, and $\|(\mathcal{A}R_{\mathcal{B}})^6(z)\| \lesssim \langle z\rangle^{-3/2}$, we obtain that each term is of order $\mathcal{O}(e^{at})$

Lemma (semigroup decay in $L^1(m)$).

For any $m = e^{\gamma\langle x\rangle}$, $\gamma \in (0, \gamma^*)$, there holds

$$\|S_{\mathcal{L}}(t)(I - \Pi)\|_{L^1(m)\to L^1(m)} \lesssim e^{at}, \quad \forall\, t \geq 0,\ \forall\, a > a^*.$$

Proof. For n large enough, we have

$$S_{\mathcal{L}}(t)\Pi^\perp \quad = \quad \Pi^\perp\{S_{\mathcal{B}_1} + ... + S_{\mathcal{B}_1} * (\mathcal{A}_1 S_{\mathcal{B}_1})^{(*n-1)}\} + (\Pi^\perp S_{\mathcal{L}}) * (\mathcal{A}_1 S_{\mathcal{B}_1})^{(*n)})\},$$

where each term is of order $\mathcal{O}(e^{at})$. Indeed, for the last term, we have $(\mathcal{A}_1 S_{\mathcal{B}_1})^{(*n)} : L^1(m) \to X$ with rate $\mathcal{O}(e^{at})$ and we may use the previous estimate.