

Landau equation for Coulomb potentials near Maxwellians and related problems

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Results are picked up from

- Carrapatoso, M. *Landau equation for very soft and Coulomb potentials near Maxwellians*, submitted
- Kavian, M., *The Fokker-Planck equation with subcritical confinement force*, submitted
- M., *Semigroups in Banach spaces - factorization approach for spectral analysis and asymptotic estimates*, in progress

Generalize to a **weak dissipativity** framework some related previous works available in a **dissipativity** framework, in particular:

- Gualdani, M., Mouhot, *Factorization for non-symmetric operators and exponential H-Theorem*, arXiv 2010
- M., Scher, *Spectral analysis of semigroups and growth-fragmentation eqs*, Annales IHP 2016
- Carrapatoso, Tristani, Wu, *Cauchy problem and exponential stability for the inhomogeneous Landau equation*, ARMA 2016

which in turn formalize several reminiscent ideas from Bobylev 1975, Voigt 1980, Arkeryd 1988, Gallay-Wayne 2002, Mouhot 2006,

Outline of the talk

1 Introduction and main result

- Hypodissipativity vs weak hypodissipativity
- The Fokker-Planck equation with weak confinement
- Landau equation with Coulomb potential near Maxwellians

2 Weak hypodissipativity in an abstract setting

- From weak dissipativity to decay estimate
- From decay estimate to weak dissipativity
- Functional space extension (enlargement and shrinkage)
- Spectral mapping theorem
- Krein-Rutman theorem

3 About the proof for the Fokker-Planck equation

- $F = \nabla V$
- general forces

4 About the proof for the Landau equation

- Estimates on the nonlinear problem and natural large space
- Splitting trick, dissipativity and decay estimates on the linear operators

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Hypodissipative framework

With Mouhot, Gualdani and Scher, we have recently revisited the spectral theory of operators and semigroups in an hypodissipative and abstract general Banach framework, providing a set of results including:

- *Spectral mapping Theorem*
- *Weyl's Theorem* about distribution of eigenvalues under compact perturbation
- *Stability of the spectrum Theorem* under small perturbation
- *Krein-Rutman Theorem*
- *Functional space extension (enlargement and shrinkage) Theorem*

These results were motivated by linear and nonlinear evolution PDEs to which they have been applied

- Asymptotic behavior of linear PDEs in large space (Growth-Fragmentation, Kinetic Fokker-Planck, Run-and-Tumble in $W^{r,p}(m)$, $-1 \leq r \leq 1 \leq p \leq \infty$)
- Optimal (= linearized) exponential decay estimates for nonlinear PDE (homogeneous (inelastic) Boltzmann, Parabolic-elliptic Keller-Segel)
- Existence, uniqueness and stability results in perturbative regime (inhomogeneous (inelastic) Boltzmann, Parabolic-parabolic Keller-Segel, kinetic FitzHugh-Nagumo and others neuronal networks)

Main feature and next step 1

- Our theory is suitable for semigroups S_Λ which split as

$$\exists a \in \mathbb{R}, \quad S_\Lambda(t) = S_1(t) + S_2(t) = \sum_{\text{finite}} S_{\Lambda_j}(t) \Pi_j + \mathcal{O}(e^{at})$$

with S_1 has asymptotically dominante finite dimensional range and S_2 is asymptotically negligible

with $\Lambda_j = \Lambda|_{\Pi_j}$, $\Sigma(\Lambda_j) = \{\lambda_j\}$, $\Re \lambda_j > a$, Π_j commutes with Λ and $\dim \Pi_j < \infty$

We establish a suitable **characterisation** of such semigroups **with spectral gap**
 \simeq quantified principal spectral mapping theorem

- Next step 1 (open problems): take into account PDEs with boundary conditions such as the transport equation

$$\partial_t f + \partial_x f + a(x)f = 0, \quad f(t, 0) = \int_0^\infty a(y) f(t, y) dy, \quad a = \text{step function}$$

with applications to neurons network models as in Pakdaman-Perthame-Salort works, or kinetic PDEs

$$\partial_t f + v \cdot \nabla_x f = \mathcal{C}(f) \quad + \quad \text{Maxwell boundary condition}$$

as in Guo and Briant-Guo works.

Reminder about dissipativity and hypodissipativity

For a semigroup $S_{\mathcal{B}}$ with generator \mathcal{B} the following properties are “equivalent” :

(1) \mathcal{B} is **dissipative**:

$$\langle f^*, \mathcal{B}f \rangle_X \leq a \|f\|_X^2, \quad \text{for any } f \in X_1^{\mathcal{B}} \text{ and } f^* \in X' \text{ dual element};$$

(2) $S_{\mathcal{B}}$ satisfies the growth estimate

$$\|S_{\mathcal{B}}(t)\|_{X \rightarrow X} \leq e^{at};$$

(3) \mathcal{B} is **hypodissipative**:

$$\langle f^*, \mathcal{B}f \rangle_X \leq a \|f\|_X^2, \quad \text{for an equivalent norm } \|\cdot\|_X \text{ on } X;$$

(4) $S_{\mathcal{B}}$ satisfies the growth estimate

$$\|S_{\mathcal{B}}(t)\|_{X \rightarrow X} \leq C e^{at}, \quad C \geq 1.$$

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$$\|S_{\mathcal{B}}(t)\|_{X \rightarrow X} \leq C e^{at}, \quad C \geq 1.$$

(1) \Rightarrow (2): consequence of Gronwall lemma and the closed differential inequality

$$\frac{1}{2} \frac{d}{dt} \|f_t\|_X^2 = \langle f_t^*, \mathcal{B}f_t \rangle \leq a \|f_t\|_X^2, \quad f_t := S_{\mathcal{B}}(t)f_0.$$

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$$\|S_{\mathcal{B}}(t)\|_{X \rightarrow X} \leq e^{at};$$

(3) \mathcal{B} is **hypodissipative**:

$$\langle f^*, \mathcal{B}f \rangle_X \leq a \| \| f \| \|_X^2, \quad \text{for an equivalent norm } \| \| \cdot \| \|_X \text{ on } X;$$

(4) $S_{\mathcal{B}}$ satisfies the growth estimate

$$\|S_{\mathcal{B}}(t)\|_{X \rightarrow X} \leq C e^{at}, \quad C \geq 1.$$

(4) \Rightarrow (3): one may choose the equivalent handy norm defined by

$$\| \| f \| \|_X^2 := \eta \|f\|_X^2 + \int_0^\infty \|S_{\mathcal{B}}(\tau)f\|_X^2 e^{-b\tau} d\tau, \quad \eta > 0, b > a.$$

Next step 2: weakly hypodissipative framework

Possible extension to a **weakly dissipative** framework ?

We do not assume the dissipativity inequality (1) but the weaker inequality

$$\langle f^*, \mathcal{B}f \rangle_Y \leq a \|f\|_Z^2, \quad Y \subset Z, \quad a < 0.$$

▷ We cannot close a differential inequality with this only information.

However, assuming the additional (dissipativity) inequality

$$\langle f^*, \mathcal{B}f \rangle_X \leq 0, \quad X \subset Y,$$

we may exploit these two inequalities together with an interpolation argument in order to get some rate of decay to 0 (as for the Allen-Cahn equation)

That corresponds to the (no spectral gap) situation:

$$\Sigma_P(\mathcal{B}) \cap \bar{\Delta}_0 = \emptyset, \quad \Sigma(\mathcal{B}) \cap \bar{\Delta}_0 \neq \emptyset.$$

In this **weakly dissipative** framework, we will present:

- some (not all) abstract spectral analysis results
- some application to the Fokker-Planck equation with weak confinement force
- some application to the Landau equation for Coulomb potential near Maxwellians in the torus

The Fokker-Planck equation with weak confinement

Consider the Fokker-Planck equation

$$\partial_t f = \Lambda f = \Delta_v f + \operatorname{div}_v(F f)$$

on $f = f(t, v) \in \mathbb{R}$, $t \geq 0$, $v \in \mathbb{R}^d$, with a weak confinement force field term F such that

$$F(v) \approx v \langle v \rangle^{\gamma-2}, \quad \gamma \in (0, 1) \quad (\text{say } =)$$

and an initial datum

$$f(0) = f_0 \in W^{r,p}(m) \quad (\text{means } m f_0 \in W^{r,p}).$$

Here $p \in [1, \infty]$, $r = 0$ and m is a polynomial weight

$$m = \langle v \rangle^k, \quad k > k^*(p, r, \gamma),$$

or a exponential weight

$$m = e^{\kappa \langle v \rangle^s}, \quad s \in (0, \gamma], \quad \kappa > 0.$$

Statement of the decay theorem

Theorem 1. (Kavian & M.)

There exists a unique “smooth”, positive and normalized steady state f_∞ .
For any $f_0 \in L^p(m)$

$$\|f(t) - \langle f_0 \rangle f_\infty\|_{L^p} \leq \Theta(t) \|f_0 - \langle f_0 \rangle f_\infty\|_{L^p(m)},$$

with

$$\begin{aligned}\Theta(t) &= \frac{C}{\langle t \rangle^K}, \quad K \sim \frac{k - k^*(p)}{2 - \gamma} \quad \text{if } m = \langle x \rangle^k \\ &= Ce^{-\lambda t^\sigma}, \quad \sigma \sim \frac{s}{2 - \gamma} \quad \text{if } m = e^{\kappa \langle x \rangle^s}.\end{aligned}$$

▷ Improves (better rate and/or larger class of initial data) earlier results by Toscani, Villani, 2000 (based on log-Sobolev inequality) & Röckner, Wang, 2001 (based on weak Poincaré inequality).

Both works deal with a force field $F = \nabla V$ what is not necessary here.

Landau equation for Coulomb potentials

Consider the Landau equation

$$\begin{aligned}\partial_t f + v \cdot \nabla_x f &= Q(f, f) \\ f(0, \cdot) &= f_0\end{aligned}$$

on density of the plasma $f = f(t, x, v) \geq 0$, time $t \geq 0$, position $x \in \mathbb{T}^3$ (torus), velocity $v \in \mathbb{R}^3$

Q = the Landau (binary) collisions operator

$$Q(g, f) = \partial_j \int_{\mathbb{R}^3} a_{ij}(v - v_*) (g_* \partial_j f - f \partial_j g_*) dv_*$$

for the Coulomb potential cross section

$$a_{ij}(z) = |z|^{\gamma+2} \left(\delta_{ij} - \frac{z_i z_j}{|z|^2} \right), \quad \gamma = -3.$$

around the H-theorem

We recall that $\varphi = 1, v, |v|^2$ are collision invariants, meaning

$$\int_{\mathbb{R}^3} Q(f, f) \varphi \, dv = 0, \quad \forall f.$$

\Rightarrow laws of conservation

$$\int_{\mathbb{R}^6} f \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} = \int_{\mathbb{R}^6} f_0 \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

We also have the H-theorem, namely

$$\int_{\mathbb{R}^3} Q(f, f) \log f \begin{cases} \leq 0 \\ = 0 \end{cases} \Rightarrow f = \text{Maxwellian}$$

From both pieces of information, we expect

$$f(t, x, v) \xrightarrow{t \rightarrow \infty} \mu(v) := \frac{1}{(2\pi)^{3/2}} e^{-|v|^2/2}.$$

Theorem 2. (Carrapatoso, M.)

Take an “admissible” weight function m such that

$$\langle v \rangle^{2+3/2} \prec m \prec e^{|v|^2}.$$

There exists $\varepsilon_0 > 0$ such that if

$$\|f_0 - \mu\|_{H_x^2 L_v^2(m)} < \varepsilon_0,$$

there exists a unique global solution f to the Landau Coulomb equation and

$$\|f(t) - \mu\|_{H_x^2 L_v^2} \leq \Theta_m(t),$$

with

$$\Theta_m(t) \simeq \begin{cases} t^{-(k-2-3/2)/|\gamma|} & \text{if } m = \langle v \rangle^k \\ e^{-\lambda t^{s/|\gamma|}} & \text{if } m = e^{\kappa|v|^s} \end{cases}$$

Comments on the main Theorem 2

- Improves (larger space) Guo and Strain's results (CMP 2002, CPDE 2006, ARMA 2008) who proved a similar theorem in the higher order and strongly confinement Sobolev space $H_{x,v}^8(\mu^{-\theta})$, $\theta > 1/2$. Based on high order nonlinear (hypercoercivity) energy estimates.
- A corollary improves (faster rate) Desvillettes and Villani's result (Invent. Math 2005) who proved polynomial rate of convergence for a priori suitably bounded solutions in a space inhomogeneous setting. Based on entropy and hypocoercivity methods.
- A corollary improves (faster rate) Carrapatoso, Desvillettes and He result (arXiv 2015) who proved polynomial and exponential rate for weak solutions in a space homogeneous setting. Based on an entropy method.
- Our proof mixes
 - Simple nonlinear estimates and trap argument in large space (no self-adjointness)
 - Decay and dissipativity estimates in appropriate norms for the linearized equation

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For a given Banach space X , we want to develop a spectral analysis theory for operators Λ enjoying the splitting structure

$$\Lambda = \mathcal{A} + \mathcal{B}, \quad \mathcal{A} \prec \mathcal{B}, \quad \mathcal{B} \text{ weakly hypodissipative.}$$

We will

- clarify the links between dissipativity and decay;
 - present an extension of the decay estimate result;
 - present a possible version of spectral mapping theorem;
 - present a possible version of Krein-Rutman theorem.
-
- We do not present any version of Weyl's theorem or perturbation theorem.
 - Very few papers related to that topics. We may mention: Caflisch (CMP 1980), Toscani-Villani (JSP 2000), Röckner-Wang (JFA 2001), Lebeau & co-authors (1993 & after), Burq (Acta Math 1998), Batty-Duyckaerts (JEE 2008).

Prop 1.

Consider three “regular” Banach spaces $X \subset Y \subset Z$ and a generator Λ . Assume

$$\forall f \in Y_1^\Lambda, \quad \langle f_Y^*, \Lambda f \rangle_Y \lesssim -\|f\|_Z^2$$

$$\forall f \in X_1^\Lambda, \quad \langle f_X^*, \Lambda f \rangle_X \leq 0 \quad (\text{or } S_\Lambda \text{ is bounded } X)$$

$$\forall R > 0, \quad \varepsilon_R \|f\|_Y^2 \leq \|f\|_Z^2 + \theta_R \|f\|_X^2, \quad \varepsilon_R, \frac{\theta_R}{\varepsilon_R} \rightarrow 0.$$

There exists a decay function Θ such that

$$\|S_\Lambda(t)\|_{X \rightarrow Y} \leq \Theta(t) \rightarrow 0.$$

- We say that a Banach space E is regular if $\varphi : E \rightarrow \mathbb{R}, f \mapsto \|f\|_E^2/2$ is G-differentiable and

$$\{f^* \in E', \langle f^*, f \rangle_E = \|f\|_E^2 = \|f^*\|_{E'}^2\} = \{f_E^*\}, \quad f_E^* := D\varphi(f).$$

Hilbert spaces and L^p spaces, $1 < p < \infty$, are regular spaces.

- We denote $E_s^\Lambda := \{f \in E, \Lambda^s f \in E\}$ the abstract Sobolev spaces

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There exists a decay function Θ such that

$$\|S_\Lambda(t)\|_{X \rightarrow Y} \leq \Theta(t) \rightarrow 0.$$

- We say that m is an admissible if $m = \langle v \rangle^k$ or $m = e^{\kappa \langle v \rangle^s}$. We then write $m_0 \prec m_1$ or $m_1 \succ m_0$ or if $m_0/m_1 \rightarrow \infty$.
- For $X = L^p(m_1)$, $Y = L^p(m_0)$, $Z = L^p(m_0 \langle v \rangle^{\alpha/p})$, with $\alpha < 0$ and $m_1 \succ m_0$, we get

$$\Theta(t) \simeq \begin{cases} t^{-(k_1 - k_0)/|\alpha|} & \text{if } m_i = \langle v \rangle^{k_i} \\ e^{-\lambda t^s/|\alpha|} & \text{if } m_1 = e^{\kappa |v|^s} \end{cases}$$

Proof of Proposition 1

We define $f_t := S_\Lambda(t)f_0$, $f_0 \in X$, and we compute

$$\frac{d}{dt} \|f_t\|_X^2 \leq 0 \quad \Rightarrow \quad \|f_t\|_X \leq C \|f_0\|_X, \quad C \geq 1,$$

$$\begin{aligned} \frac{d}{dt} \|f_t\|_Y^2 &\lesssim -\|f_t\|_Z^2 \\ &\lesssim -\varepsilon_R \|f_t\|_Y^2 + \theta_R \|f_0\|_X^2, \end{aligned}$$

and from Gronwall lemma

$$\begin{aligned} \|f_t\|_Y^2 &\lesssim e^{-\varepsilon_R t} \|f_0\|_Y^2 + \frac{\theta_R}{\varepsilon_R} \|f_0\|_X^2 \\ &\lesssim \Theta(t)^2 \|f_0\|_X^2, \end{aligned}$$

with

$$\Theta(t)^2 := \inf_{R>0} \left(e^{-\varepsilon_R t} + \frac{\theta_R}{\varepsilon_R} \right).$$

Prop 2. Consider three “regular” Banach spaces $X \subset Y \subset Z$ and a generator \mathcal{L} . Assume

- $\|S_{\mathcal{L}}(t)\|_{X \rightarrow Z} \leq \Theta(t)$, with $\Theta \in L^2(\mathbb{R}_+)$ a decay function (i.e. which tends to 0)
- $\mathcal{L} = \mathcal{A} + \mathcal{B}$, $\mathcal{A} \prec \mathcal{B}$, with

$$\begin{aligned} \forall f \in X_1^{\mathcal{B}}, \quad \langle f^*, \mathcal{B}f \rangle_X &\lesssim -\|f\|_Y^2 \\ \forall f \in X_1^{\mathcal{A}}, \quad \langle f^*, \mathcal{A}f \rangle_X &\lesssim \|f\|_Z^2. \end{aligned}$$

Then, \mathcal{L} is weakly hypodissipative

$$\langle\langle f^*, \mathcal{L}f \rangle\rangle_X \lesssim -\|f\|_Y^2$$

for the duality product $\langle\langle \cdot, \cdot \rangle\rangle_X$ associated to the norm defined by

$$\|f\|^2 := \eta \|f\|_X^2 + \int_0^\infty \|S_{\mathcal{L}}(\tau)f\|_Z^2 d\tau,$$

for $\eta > 0$ small enough. That norm is equivalent to the initial norm in X .

We observe that $\|\cdot\| \sim \|\cdot\|_X$ because $\Theta \in L^2(\mathbb{R}_+)$.

We set $f_t := S_{\mathcal{L}}(t)f_0$ and we compute

$$\begin{aligned} \frac{d}{dt} \|\| f_t \|\|^2 &= 2\eta \langle f_t^*, \mathcal{L} f_t \rangle_X + \int_0^\infty \frac{d}{d\tau} \|S_{\mathcal{L}}(\tau + t)f_0\|_Z^2 d\tau \\ &= 2\eta \langle f_t^*, \mathcal{B} f_t \rangle_X + \eta \langle f_t^*, \mathcal{A} f_t \rangle_X - \|f_t\|_Z^2 \\ &\leq -2\eta C_1 \|f_t\|_Y^2 + (\eta C_2 - 1) \|f_t\|_Z^2 \\ &\lesssim -\|f_t\|_Y^2 \end{aligned}$$

as well as

$$\frac{d}{dt} \|\| f_t \|\|^2 \simeq \langle\langle f_t^*, \mathcal{L} f_t \rangle\rangle_X$$

Prop 3. Consider a decay function Θ such that

$$\Theta^{-1}(t) \lesssim \Theta^{-1}(t-s)\Theta^{-1}(s) \text{ for any } 0 < s < t.$$

We consider two sets of Banach spaces $X_1 \subset X_0$ and $Y_1 \subset Y_0$ and a generator Λ . We assume

- $\|S_\Lambda(t)\|_{X_1 \rightarrow X_0} \Theta^{-1} \in L^\infty$
- $\Lambda = \mathcal{A} + \mathcal{B}$, $\mathcal{A} \prec \mathcal{B}$, with

$$\forall \ell, \quad \|S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)}\|_{Y_1 \rightarrow Y_0} \Theta^{-1} \in L^\infty$$

$$\exists n, \quad \|(\mathcal{A}S_{\mathcal{B}})^{(*n)}\|_{Y_1 \rightarrow X_1} \Theta^{-1} \in L^1 \text{ if } X_0 \subset Y_0 \text{ (enlargement)}$$

$$\exists n, \quad \|(S_{\mathcal{B}}\mathcal{A})^{(*n)}\|_{X_0 \rightarrow Y_1} \Theta^{-1} \in L^1 \text{ if } Y_1 \subset X_1 \text{ (shrinkage)}$$

Then,

$$\|S_\Lambda(t)\|_{Y_1 \rightarrow Y_0} \Theta^{-1} \in L^\infty.$$

Enlargement result. We iterate the Duhamel formula

$$S_\Lambda = S_B + S_\Lambda * (\mathcal{A}S_B)$$

to get a “stopped Dyson-Phillips series” (the D-P series corresponds to $n = \infty$)

$$S_\Lambda = \sum_{\ell=0}^{n-1} S_B * (\mathcal{A}S_B)^{(*\ell)} + S_\Lambda * (\mathcal{A}S_B)^{(*n)} =: S_1 + S_2.$$

From the assumptions, we immediately have

$$\|S_\Lambda\|_{Y_1 \rightarrow Y_0} \Theta^{-1} \leq \|S_1\|_{Y_1 \rightarrow Y_0} \Theta^{-1} + \|S_\Lambda \Theta^{-1}\|_{X_1 \rightarrow X_0} * \|(\mathcal{A}S_B)^{(*n)} \Theta^{-1}\|_{Y_1 \rightarrow X_1} \in L^\infty$$

Shrinkage result. We argue similarly starting with the iterated the Duhamel formula / stopped Dyson-Phillips series

$$S_\Lambda = \sum_{\ell=0}^{n-1} S_B * (\mathcal{A}S_B)^{(*\ell)} + (S_B \mathcal{A})^{(*n)} * S_\Lambda.$$

Prop 4. (rough version) We consider two Banach spaces $X \subset Y$ and a generator Λ . We assume $X_\Lambda^1 \subset Y$ is compact and $\Theta(t) \approx e^{-\lambda t^{1/(1+j)}}$

- $\Sigma_P(\Lambda) \cap \bar{\Delta}_0 = \emptyset$, with $\Delta_0 := \{z \in \mathbb{C}; \Re z > 0\}$
- $\Lambda = \mathcal{A} + \mathcal{B}$, with $\mathcal{A} \in \mathbf{B}(Y, X)$, $\zeta \in (0, 1]$ and

$$(a1) \quad \forall \ell, \quad \|S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)}\|_{X \rightarrow Y} \Theta^{-1} \in L^\infty$$

$$(a2) \quad \forall \ell, \quad \sup_{z \in \bar{\Delta}_0} \|(R_{\mathcal{B}}(z))^\ell\|_{X \rightarrow Y} \leq C (\ell!)^j$$

$$(a3) \quad \forall \ell, \quad \sup_{z \in \bar{\Delta}_0} \|R_{\mathcal{B}}(z)\|_{Y \rightarrow X_\zeta^\Lambda} \leq C (\ell!)^j$$

Then,

$$\|S_\Lambda(t)\|_{X \rightarrow Y} \Theta^{-1} \in L^\infty.$$

Proof of Proposition 4

We start again with the stopped Dyson-Phillips series

$$S_\Lambda = \sum_{\ell=0}^{N-1} S_B * (\mathcal{A}S_B)^{(*\ell)} + S_\Lambda * (\mathcal{A}S_B)^{(*N)} = S_1 + S_2$$

The first $N - 1$ terms are fine. For the last one, we use the inverse Laplace formula

$$\begin{aligned} S_2(t)f &= \frac{i}{2\pi} \int_{\uparrow_0} e^{zt} R_\Lambda(z) (\mathcal{A}R_B(z))^N f \, dz \\ &\approx \frac{1}{t^k} \int_{\uparrow_0} e^{zt} \frac{d^k \Phi}{dz^k} \, dz \, f \\ &\lesssim \frac{C^k}{t^k} k! \int_{\uparrow_0} \sup_{|\alpha| \leq k} \underbrace{\|R_\Lambda^{1+\alpha_1}(z)\|_{X \rightarrow Y}}_{\in L^\infty(\uparrow_0)?} \underbrace{\|\mathcal{A}R_B^{1+\alpha_1} \dots \mathcal{A}R_B^{1+\alpha_N}(z)\|_{X \rightarrow X}}_{\in L^1(\uparrow_0)?} \, dz \|f\|_X, \end{aligned}$$

where $\uparrow_0 := \{z = 0 + iy, y \in \mathbb{R}\}$ and because

$$\frac{d^k \Phi}{dz^k} \approx \sum_{|\alpha| \leq k} \alpha! R_\Lambda^{1+\alpha_0} \mathcal{A}R_B^{1+\alpha_1} \dots \mathcal{A}R_B^{1+\alpha_N}$$

Key estimates

- Using (a2), (a3), the compact embedding $X_\Lambda^1 \subset Y$ and the fact that there is not punctual spectrum in $\bar{\Delta}_0$, we get

$$\sup_{z \in \bar{\Delta}_0} \|R_\Lambda(z)^\ell\|_{X \rightarrow Y} \leq C (\ell!)^j$$

- $\mathcal{A} \in \mathbf{B}(Y, X)$ and the resolvent identity

$$R_B(z) = \frac{1}{z} (R_B(z)\mathcal{B} - I) \in \mathbf{B}(X_1, X)$$

imply

$$\|\mathcal{A}R_B(z)\|_{X_1 \rightarrow X} \leq C/|z| \quad \forall z \in \bar{\Delta}_0.$$

Together with (a2) (where we assume that $\zeta = 1$ in order to make the proof simpler) we get

$$\|\mathcal{A}R_B(z)^{\ell_1} \mathcal{A}R_B(z)^{\ell_2}\|_{X \rightarrow X} \leq C (\ell_1!)^j (\ell_2!)^j \langle z \rangle^{-1}$$

- Choosing $N = 4$ and gathering the two estimates, we get

$$\left\| \frac{d^k \Phi}{dz^k}(z) \right\|_{X \rightarrow Y} \leq C^k (k!)^j \langle z \rangle^{-2} \in L^1(\uparrow_0).$$

Coming back to the term S_2 , we have

$$\begin{aligned} S_2(t) &\lesssim C^k k^{(1+j)k} t^{-k}. \\ &\lesssim e^{-\lambda t^{1/(1+j)}} = \Theta(t), \end{aligned}$$

by choosing appropriately $k = k(t)$

Prop 5.

- Consider a semigroup generator Λ on a Banach lattice X , and assume
- (1) Λ such as the spectral mapping Theorem holds (for $\|f\|_Y = \langle |f|, \phi \rangle$);
 - (2) $\phi \in D(\Lambda^*)$, $\phi \succ 0$ such that $\Lambda^* \phi = 0$;
 - (3) S_Λ is positive (and Λ satisfies Kato's inequalities);
 - (4) $-\Lambda$ satisfies a strong maximum principle.

There exists $0 < f_\infty \in D(\Lambda)$ such that

$$\Lambda f_\infty = 0, \quad \Sigma_P(\Lambda) \cap \bar{\Delta}_0 = \{0\}, \quad \Sigma_P(\Lambda_1) \cap \bar{\Delta}_0 = \emptyset$$

with $\Lambda_1 := \Lambda|_{X_1}$, $X_1 = R(I - \Pi_0) = (I - \Pi_0)X$,

$$\Pi_0 f = \langle f, \phi \rangle f_\infty \quad \forall f \in X.$$

Moreover the decay function Θ defined in the spectral mapping Theorem :

$$\|S_\Lambda(t)(I - \Pi_0)f_0\|_Y \lesssim \Theta(t) \|(I - \Pi_0)f_0\|_X \quad \forall t \geq 0, \forall f_0 \in X.$$

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- Hypodissipativity vs weak hypodissipativity
- The Fokker-Planck equation with weak confinement
- Landau equation with Coulomb potential near Maxwellians

2 Weak hypodissipativity in an abstract setting

- From weak dissipativity to decay estimate
- From decay estimate to weak dissipativity
- Functional space extension (enlargement and shrinkage)
- Spectral mapping theorem
- Krein-Rutman theorem

3 About the proof for the Fokker-Planck equation

- $F = \nabla V$
- general forces

4 About the proof for the Landau equation

- Estimates on the nonlinear problem and natural large space
- Splitting trick, dissipativity and decay estimates on the linear operators

- Weak Poincaré inequality

$$\langle \Lambda f, f \rangle_{E_0} \lesssim -\|f\|_{E_*}^2, \quad \forall f \in E_0, \quad \langle f \rangle = 0,$$

with $E_0 := L^2(f_\infty^{-1/2})$, $f_\infty := e^{-V}$, and $E_* := L^2(\langle v \rangle^{\gamma-1} f_\infty^{-1/2})$.

- By the generalized relative entropy inequality

$$\forall f, \forall p \geq 1, \quad \langle \Lambda f, (f/f_\infty)^{p-1} \rangle \leq 0,$$

and passing to the limit as $p \rightarrow \infty$, we deduce the semigroup (of contractions) estimate

$$\|f_t\|_{E_1} \leq \|f_0\|_{E_1}, \quad E_1 := L^\infty(f_\infty^{-1}).$$

- For any $f_0 \in E_1$, $\langle f_0 \rangle = 0$, both inequalities and an interpolation argument imply (as in Prop 1)

$$\|f_t\|_{E_0} \leq \Theta(t) \|f_0\|_{E_1}, \quad \Theta(t) \simeq e^{-t^{\frac{\gamma}{2-\gamma}}}.$$

We introduce the splitting $\Lambda = \mathcal{A} + \mathcal{B}$, with \mathcal{A} a multiplication operator

$$\mathcal{A}f = M\chi_R(v)f, \quad \chi_R(v) = \chi(v/R), \quad 0 \leq \chi \leq 1, \quad \chi \in \mathcal{D}(\mathbb{R}^d)$$

▷ $\mathcal{A} \in \mathbf{B}(X_0, X_1)$, $X_i = W^{r,p}(m_i)$, $m_1 \succeq m_0$

▷ \mathcal{B} is not a -dissipative in $X = W^{r,p}(m)$ with $a < 0$. However, it is weakly dissipative. For $p \in (1, \infty)$, and $M, R > 0$ large enough, we have

$$\langle f^*, \mathcal{B}f \rangle_{L^p} \lesssim -\|f\|_{L^p(m\langle v \rangle^{(\gamma-2+s)/p})}^2, \quad s := 0 \text{ for polynomial weight}$$

That is a consequence of the identity

$$\int (\Lambda f) f^{p-1} m^p = (1-p) \int |\nabla(fm)|^2 (fm)^{p-1} + \int (fm)^p \psi$$

$$\begin{aligned} \psi &= \left(\frac{2}{p} - 1\right) \frac{\Delta m}{m} + 2\left(1 - \frac{1}{p}\right) \frac{|\nabla m|^2}{m^2} + \left(1 - \frac{1}{p}\right) \operatorname{div} F - F \cdot \frac{\nabla m}{m} \\ &\sim -F \cdot \frac{\nabla m}{m} \sim -\langle v \rangle^{s+\gamma-2} \end{aligned}$$

- the estimate

$$(1) \quad \|S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)}\|_{X_1 \rightarrow X_0} \leq \Theta(t)$$

follows from Proposition 1.

- the estimate

$$(2) \quad \|(\mathcal{A}S_{\mathcal{B}})^{(*n)}\|_{\mathbf{B}(L^1(m_1), H^1(m_2))} \leq \Theta(t)$$

follows from (1) and the use a “Nash + regularity” trick for small time. More precisely, introducing

$$\mathcal{F}(t, h) := \|h\|_{L^1(m)}^2 + t^\bullet \|h\|_{L^2(m)}^2 + t^\bullet \|\nabla_\nu h\|_{L^2(m)}^2$$

we are able to prove (for convenient exponents $\bullet > 1$)

$$\frac{d}{dt} \mathcal{F}(t, S_{\mathcal{B}}(t)h) \leq 0 \quad \text{and then} \quad \|S_{\mathcal{B}}(t)h\|_{H^1(m)}^2 \leq \frac{1}{t^\bullet} \|h\|_{L^1(m)}^2$$

- In the case $F = \nabla V$, we conclude thanks to Prop 3 (enlargement argument)
- For the general case, we use the Krein-Rutman theory. The Fokker-Planck semigroup is obviously mass conservative and positive and the Fokker-Planck operator satisfies the strong maximum principle. The last point in order to apply Proposition 5 is to verify that assumption (a2) in Proposition 4 is satisfied.

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Strategy of proof of Theorem 2

The method consists in introducing the variation function $g = f - \mu$ and the corresponding Landau equation

$$\begin{aligned}\partial_t g &= \bar{\mathcal{L}}g + Q(g, g), \\ \bar{\mathcal{L}} &= -v \cdot \nabla_x + \mathcal{L}, \quad \mathcal{L} = Q(\cdot, \mu) + Q(\mu, \cdot)\end{aligned}$$

- As a starting point, we use the known weak dissipativity estimate

$$(\mathcal{L}g, g)_{L^2(\mu^{-1/2})} \lesssim -\|\Pi g\|_{H_*^1}^2 \simeq H^1(\mu^{1/2} \langle v \rangle^{(\gamma+2)/2}),$$

$\Pi := I - \Pi_0$, $\Pi_0 :=$ projector on $N(\mathcal{L})$,

- in order to prove the weak hypodissipativity estimate

$$(\bar{\mathcal{L}}g, g)_{\mathcal{H}_{x,v}^1(\mu^{-1/2})} \lesssim -\|\bar{\Pi}g\|_{H_x^1 H_{v*}^1}^2 \simeq H_{x,v}^1(\mu^{1/2} \langle v \rangle^{(\gamma+2)/2}),$$

$\bar{\Pi} := I - \bar{\Pi}_0$, $\bar{\Pi}_0 :=$ projector on $N(\bar{\mathcal{L}})$,

- and next factorization and semigroup tricks in order to get similar information in the space $\mathcal{X} := H_x^2 L^2(m)$.

A classical result asserts that for any weight functions m , $m_1 \succ \langle v \rangle^{2+3/2}$ and $m_0 \succ \langle v \rangle^2$

$$\langle Q(f, g), h \rangle_{L^2(m)} \lesssim \left(\|f\|_{L^2(m)} \|g\|_{H^1_*(m_1)} + \|f\|_{H^1(m_0)} \|g\|_{L^2(m)} \right) \|h\|_{H^1_*(m)}$$

with

$$\|f\|_{H^1_*(m)}^2 := \|f\|_{L^2(m\langle v \rangle^{(\gamma+\sigma)/2})}^2 + \|\tilde{\nabla} f\|_{L^2(m\langle v \rangle^{\gamma/2})}^2,$$

and

$$\tilde{\nabla}_v f = P_v \nabla_v f + \langle v \rangle (I - P_v) \nabla_v f, \quad P_v \xi = \left(\xi \cdot \frac{v}{|v|} \right) \frac{v}{|v|}.$$

As a consequence, we have

Prop 6.

for $m \succ \langle v \rangle^{2+3/2}$, defining $\mathcal{X} := H_x^2 L_v^2(m)$, $\mathcal{Y} := H_x^2 H_{v,*}^1(m)$, $\mathcal{Z} := H_x^2 H_{v,*}^{-1}(m)$, we have

$$\langle Q(f, g), h \rangle_{\mathcal{X}} \lesssim \left(\|f\|_{\mathcal{X}} \|g\|_{\mathcal{Y}} + \|f\|_{\mathcal{Y}} \|g\|_{\mathcal{X}} \right) \|h\|_{\mathcal{Y}}$$

$$\|Q(f, g)\|_{\mathcal{Z}} \lesssim \left(\|f\|_{\mathcal{X}} \|g\|_{\mathcal{Y}} + \|f\|_{\mathcal{Y}} \|g\|_{\mathcal{X}} \right).$$

A introduce the **equivalent norm** on $R\bar{\Pi}$

$$\|g\|_{\mathcal{X}}^2 := \eta \|g\|_{\mathcal{Y}}^2 + \int_0^\infty \|S_{\bar{\mathcal{L}}}(\tau)g\|_{\mathcal{X}_0}^2 d\tau,$$

with $\mathcal{X}_0 := H_x^2 L_v^2$, $\mathcal{Y}_0 := H_x^2 H_{v,*}^1$, $\mathcal{Z}_0 := H_x^2 H_{v,*}^{-1}$ (without weight!)

We consider a solution g to the Landau equation

$$\frac{d}{dt}g = \bar{\mathcal{L}}g + Q(g, g)$$

and we compute

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|g\|_{\mathcal{X}}^2 &= \langle \bar{\mathcal{L}}g, g \rangle_{\mathcal{X}} + \eta \langle Q(g, g), g \rangle_{\mathcal{X}} \\ &\quad + \int_0^\infty \langle S_{\bar{\mathcal{L}}}(\tau)Q(g, g), S_{\mathcal{L}}(\tau)g \rangle_{\mathcal{X}_0} d\tau =: T_1 + T_2 + T_3. \end{aligned}$$

From **Propositions 1, 2, 3**, we expect to have

$$T_1 \lesssim -\|g\|_{\mathcal{Y}}^2.$$

Thanks to the **choice of the norm and Proposition 6**, we have

$$T_2 \leq C \|g\|_{\mathcal{X}} \|g\|_{\mathcal{Y}}^2.$$

Nonlinear a priori estimate (continuation)

For the last term, thanks to Proposition 6, we have

$$\begin{aligned} T_3 &= \int_0^\infty \langle S_{\mathcal{L}}(\tau) Q(g, g), S_{\mathcal{L}}(\tau) g \rangle_{\mathcal{X}_0} d\tau \\ &\lesssim \int_0^\infty \|S_{\mathcal{L}}(\tau) Q(g, g)\|_{\mathcal{Z}_0} \|S_{\mathcal{L}}(\tau) g\|_{\mathcal{Y}_0} d\tau \\ &\lesssim \|Q(g, g)\|_{\mathcal{Z}} \|g\|_{\mathcal{Y}} \int_0^\infty \Theta(\tau)^2 d\tau \lesssim \|g\|_{\mathcal{X}} \|g\|_{\mathcal{Y}}^2, \end{aligned}$$

under the condition that

$$t \mapsto \|S_{\Lambda}(t)\|_{\mathcal{Y} \rightarrow \mathcal{Y}_0}, \|S_{\Lambda}(t)\|_{\mathcal{Z} \rightarrow \mathcal{Z}_0} \in L^2(\mathbb{R}_+).$$

We conclude with

$$\frac{d}{dt} \|g\|_{\mathcal{X}}^2 \lesssim \|g\|_{\mathcal{Y}}^2 (1 - C \|g\|_{\mathcal{X}})$$

We deduce

- ▷ a priori uniform estimate for $\|g_0\|_{\mathcal{X}}^2$ small, and then classically existence and uniqueness
- ▷ considering two weight functions $m \succ \tilde{m}$, the above a priori estimate implies

$$\frac{d}{dt} \|g\|_{\tilde{\mathcal{X}}}^2 \lesssim -\|g\|_{\tilde{\mathcal{Y}}}^2, \quad \frac{d}{dt} \|g\|_{\mathcal{X}}^2 \lesssim 0,$$

and we get decay estimate just repeating the proof of Proposition 1.

Splitting of the operator

We introduce the splitting $\tilde{\mathcal{L}} = \mathcal{A} + \mathcal{B}$

$$\mathcal{A}g := Q(g, \mu) + M\chi_R g = (a_{ij} * g)\partial_{ij}\mu - (c * g)\mu + M\chi_R g,$$

$$\mathcal{B}g := Q(\mu, g) - M\chi_R g - v \cdot \nabla_x g = (a_{ij} * \mu)\partial_{ij}g - (c * \mu)g - M\chi_R g - v \cdot \nabla_x g,$$

with

$$b_i(z) = \partial_j a_{ij}(z) = -2|z|^\gamma z_i, \quad c(z) = \partial_{ij} a_{ij}(z) = -8\pi\delta_0$$

We show

- Weak dissipativity of \mathcal{B} in many spaces (twisting trick, duality trick)

$$(\mathcal{B}f, f)_{H_x^2 L^2(m)} \lesssim -\|f\|_{H_x^2 H_{*,v}^1(m)}^2$$

$$(\mathcal{B}f, f)_{H_x^2 H_v^1(m)} \lesssim -\|f\|_{H_x^2 H^1(m \langle v \rangle^{(\gamma+2)/2})}^2$$

$$(\mathcal{B}^* f, f)_{H_x^2 H^1(m)} \lesssim -\dots$$

- Decay estimate of $S_{\mathcal{B}}$ in many spaces by Proposition 1.
- Regularization property of $S_{\mathcal{B}}$ in many spaces by using “Hormander-Hérau-Villani” hypoelliptic trick. More precisely, introducing

$$\mathcal{F}(t, h) := \|h\|_{L^2(m)}^2 + t^\bullet \|\nabla_v h\|_{L^2(m)}^2 + t^\bullet (\nabla_v h, \nabla_x h)_{L^2(m)} + t^\bullet \|\nabla_x h\|_{L^2(m)}^2$$

we get (for convenient exponents $\bullet \geq 1$)

$$\frac{d}{dt} \mathcal{F}(t, S_{\mathcal{B}}(t)h) \leq 0, \quad \forall t \in [0, 1].$$

- $\mathcal{A} \in \mathbf{B}(H_x^\alpha H_v^\beta(m_0), H_x^\alpha H_v^\beta(m_1))$ for any weight functions $m_1 \succeq m_0$.
- In the space of self-adjointness $L^2(\mu^{-1/2})$ we have the nice dissipativity estimate

$$\langle \mathcal{L}g, g \rangle_{L^2(\mu^{-1/2})} \lesssim - \|\Pi g\|_{H_*^1(\mu^{-1/2})}^2$$

from which we deduce thanks to the twisting hypocoercivity Nier-Hérau-Villini trick

$$\langle \bar{\mathcal{L}}g, g \rangle_{H_{x,v}^1(\mu^{-1/2})} \lesssim - \|\bar{\Pi}g\|_{H_x^1 H_{v*}^1(\mu^{-1/2})}^2$$

We deduce

- $S_{\bar{\mathcal{L}}}$ is bounded in many spaces because $S_{\bar{\mathcal{L}}}$ is bounded in one space and $\bar{\mathcal{L}}$ splits in a suitable way (Proposition 3 of extension).
- $S_{\bar{\mathcal{L}}}$ is fast decaying in one space $\mathbf{B}(H_{x,v}^1(\mu^{-3/2}), H_{x,v}^1(\mu^{-3/2}))$ because it is bounded in $H_{x,v}^1(\mu^{-3/2})$ and weakly dissipative in $H_{x,v}^1(\mu^{-1/2})$ (Proposition 1).
- $S_{\bar{\mathcal{L}}}$ is decaying in many space because $S_{\bar{\mathcal{L}}}$ is decaying in one space and $\bar{\mathcal{L}}$ splits in a suitable way (Proposition 3 of extension).

As a conclusion, we are able to prove

- On the one hand,

$$\|S_{\mathcal{L}}\|_{\mathcal{X} \rightarrow \mathcal{X}_0} \leq \Theta(t),$$

and $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with

$$\langle f, \mathcal{B}f \rangle_{\mathcal{X}} \lesssim -\|f\|_{\mathcal{Y}}^2, \quad \langle f, \mathcal{A}f \rangle_{\mathcal{X}} \lesssim -\|f\|_{\mathcal{X}_0}^2$$

in order to use Proposition 2 and define the weak dissipative norm $\|\cdot\|_{\mathcal{X}}$ for $\tilde{\mathcal{L}}$

- On the other hand,

$$t \mapsto \|S_{\mathcal{L}}\|_{\mathcal{Y} \rightarrow \mathcal{Y}_0}, \|S_{\mathcal{L}}\|_{\mathcal{Z} \rightarrow \mathcal{Z}_0} \in L^2(\mathbb{R}_+)$$

▷ That are the needed properties in order to get the a priori nonlinear estimate !

- Suitable spectral analysis theory in an abstract setting and a weakly dissipative framework ?
- What about the Boltzmann equation without Grad's cut-off (\sim fractional diffusion in the velocity variable)?
 - ▷ Work in progress by Hérau, Tonon, Tristani, ...
- What about the grazing collisions limit (from Boltzmann to Landau)?