Introduction to spectral theory for weakly hypodissipative generators

S. Mischler

(Université Paris-Dauphine)

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Results are picked up from

- Carrapatoso, M. *Landau equation for very soft and Coulomb potentials near Maxwellians*, submitted
- Kavian, M., *The Fokker-Planck equation with subcritical confinement force*, submitted
- M., *Semigroups in Banach spaces - factorization approach for spectral analysis and asymptotic estimates*, in progress

Outline of the talk

1. Introduction and main result
   - Hypodissipativity vs weak hypodissipativity
   - The Fokker-Planck equation with weak confinement
2. Weak hypodissipativity in an abstract setting
   - From weak dissipativity to decay estimate
   - From decay estimate to weak dissipativity
   - Functional space extension (enlargement and shrinkage)
   - Spectral mapping theorem
   - Krein-Rutman theorem
3. About the proof for the Fokker-Planck equation
   - $F = \nabla V$
   - general forces

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Reminder about dissipativity and hypodissipativity

For a semigroup $S_{\mathcal{B}}$ with generator $\mathcal{B}$ the following properties are "equivalent" :

(1) $\mathcal{B}$ is dissipative:

$$\langle f^*, \mathcal{B} f\rangle_X \le a \|f\|_X^2, \quad \text{for any } f \in X_1^{\mathcal{B}} \text{ and } f^* \in X' \text{ dual element;}$$

(2) $S_{\mathcal{B}}$ satisfies the growth estimate

$$\|S_{\mathcal{B}}(t)\|_{X\to X} \le e^{at};$$

(3) $\mathcal{B}$ is hypodissipative:

$$\langle f^*, \mathcal{B} f\rangle_X \le a \, |||f|||_X^2, \quad \text{for an equivalent norm } |||\cdot|||_X \text{ on } X;$$

(4) $S_{\mathcal{B}}$ satisfies the growth estimate

$$\|S_{\mathcal{B}}(t)\|_{X\to X} \le C\, e^{at}, \quad C \ge 1.$$

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$$\|S_{\mathcal{B}}(t)\|_{X\to X} \le C\, e^{at}, \quad C \ge 1.$$

(1) $\Rightarrow$ (2): consequence of Gronwall lemma and the closed differential inequality

$$\frac{1}{2}\frac{d}{dt}\|f_t\|_X^2 = \langle f_t^*, \mathcal{B} f_t\rangle \le a\, \|f_t\|_X^2, \quad f_t := S_{\mathcal{B}}(t) f_0.$$

For a semigroup $S_\mathcal{B}$ with generator $\mathcal{B}$ the following properties are "equivalent" :

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(2) $S_\mathcal{B}$ satisfies the growth estimate

$$\|S_\mathcal{B}(t)\|_{X\to X} \le e^{at};$$

(3) $\mathcal{B}$ is hypodissipative:

$$\langle f^*, \mathcal{B} f\rangle_X \le a \, |\!|\!| f |\!|\!|_X^2, \quad \text{for an equivalent norm } |\!|\!| \cdot |\!|\!|_X \text{ on } X;$$

(4) $S_\mathcal{B}$ satisfies the growth estimate

$$\|S_\mathcal{B}(t)\|_{X\to X} \le C\, e^{at}, \quad C \ge 1.$$

(4) $\Rightarrow$ (3): one may choose the equivalent handy norm defined by

$$|\!|\!| f |\!|\!|_X^2 := \eta \|f\|_X^2 + \int_0^\infty \|S_\mathcal{B}(\tau) f\|_X^2 \, e^{-b\tau}\, d\tau, \quad \eta > 0,\ b > a.$$

Possible extension to a weakly dissipative framework ?

We do not assume the dissipativity inequality (1) but the weaker inequality

$$\langle f^*, \mathcal{B} f \rangle_Y \le a \|f\|_Z^2, \qquad Y \subset Z, \quad a < 0.$$

$\triangleright$ We cannot close a differential inequality with this only information.

However, assuming the additional (dissipativity) inequality

$$\langle f^*, \mathcal{B} f \rangle_X \le 0, \quad X \subset Y,$$

we may exploit these two inequalities together with an interpolation argument in order to get some rate of decay to 0 (as for the Allen-Cahn equation)

That corresponds to the (no spectral gap) situation:

$$\Sigma_P(\mathcal{B}) \cap \bar{\Delta}_0 = \emptyset, \quad \Sigma(\mathcal{B}) \cap \bar{\Delta}_0 \ne \emptyset.$$

In this weakly dissipative framework, we will present:

- some (not all) abstract spectral analysis results
- some application to the Fokker-Planck equation with weak confinement force
- some application to the Landau equation for Coulomb potential near Maxwellians in the torus

Consider the Fokker-Planck equation

$$\partial_t f = \Lambda f = \Delta_v f + \mathrm{div}_v(F\, f)$$

on $f = f(t,v) \in \mathbb{R}$, $t \geq 0$, $v \in \mathbb{R}^d$, with a weak confinement force field term F such that

$$F(v) \approx v\langle v\rangle^{\gamma-2}, \quad \gamma \in (0,1) \qquad (\text{say} =)$$

and an initial datum

$$f(0) = f_0 \in W^{r,p}(m) \quad (\text{means } m\, f_0 \in W^{r,p}).$$

Here $p \in [1,\infty]$, $r = 0$ and m is a polynomial weight

$$m = \langle v\rangle^k, \quad k > k^*(p,r,\gamma),$$

or a exponential weight

$$m = e^{\kappa\langle v\rangle^s}, \quad s \in (0,\gamma],\ \kappa > 0.$$

Theorem 1. (Kavian & M.)

There exists a unique "smooth", positive and normalized steady state f_∞.
For any $f_0 \in L^p(m)$

$$\|f(t) - \langle f_0 \rangle f_\infty\|_{L^p} \le \Theta(t) \, \|f_0 - \langle f_0 \rangle f_\infty\|_{L^p(m)},$$

with

$$\begin{aligned} \Theta(t) &= \frac{C}{\langle t \rangle^K}, \quad K \sim \frac{k - k^*(p)}{2-\gamma} \quad \text{if} \quad m = \langle x \rangle^k \\ &= C e^{-\lambda t^\sigma}, \quad \sigma \sim \frac{s}{2-\gamma} \quad \text{if} \quad m = e^{\kappa \langle x \rangle^s}. \end{aligned}$$

$\triangleright$ Improves (better rate and/or larger class of initial data) earlier results by Toscani, Villani, 2000 (based on log-Sobolev inequality)
& Röckner, Wang, 2001 (based on weak Poincaré inequality).

Both works deal with a force field $F = \nabla V$ what is not necessary here.
See however Bakry-Cattiaux-Guillin.

Outline of the talk

For a given Banach space X, we want to develop a spectral analysis theory for operators Λ enjoying the splitting structure

$$\Lambda = \mathcal{A} + \mathcal{B}, \quad \mathcal{A} \prec \mathcal{B}, \quad \mathcal{B} \text{ weakly hypodissipative.}$$

We will

- clarify the links between dissipativity and decay;
- present an extension of the decay estimate result;
- present a possible version of spectral mapping theorem;
- present a possible version of Krein-Rutman theorem.

• We do not present any version of Weyl's theorem or perturbation theorem.

• Very few papers related to that topics. We may mention: Caflisch (CMP 1980), Toscani-Villani (JSP 2000), Röckner-Wang (JFA 2001), Lebeau & co-authors (1993 & after), Burq (Acta Math 1998), Batty-Duyckaerts (JEE 2008).

Prop 1.

Consider three "regular" Banach spaces $X \subset Y \subset Z$ and a generator Λ. Assume

$$\begin{aligned} \forall\, f \in Y^\Lambda_1, \qquad \langle f^*_Y, \Lambda f \rangle_Y \quad &\lesssim \quad -\|f\|^2_Z \\ \forall\, f \in X^\Lambda_1, \qquad \langle f^*_X, \Lambda f \rangle_X \quad &\le \quad 0 \quad \text{(or } S_\Lambda \text{ is bounded } X) \\ \forall\, R > 0, \qquad \varepsilon_R \|f\|^2_Y \quad &\le \quad \|f\|^2_Z + \theta_R \|f\|^2_X, \quad \varepsilon_R, \frac{\theta_R}{\varepsilon_R} \to 0. \end{aligned}$$

There exists a decay function Θ such that

$$\|S_\Lambda(t)\|_{X \to Y} \le \Theta(t) \to 0.$$

• We say that a Banach space E is regular if $\varphi : E \to \mathbb{R}$, $f \mapsto \|f\|^2_E/2$ is G-differentiable and

$$\{f^* \in E',\ \langle f^*, f \rangle_E = \|f\|^2_E = \|f^*\|^2_{E'}\} = \{f^*_E\}, \quad f^*_E := D\varphi(f).$$

Hilbert spaces and L^p spaces, $1 < p < \infty$, are regular spaces.

• We denote $E^\Lambda_s := \{f \in E,\ \Lambda^s f \in E\}$ the abstract Sobolev spaces

Prop 1.

Consider three "regular" Banach spaces $X \subset Y \subset Z$ and a generator Λ. Assume

$$\begin{array}{lrcl} \forall\, f \in Y_1^\Lambda, & \langle f_Y^*, \Lambda f \rangle_Y & \lesssim & -\|f\|_Z^2 \\ \forall\, f \in X_1^\Lambda, & \langle f_X^*, \Lambda f \rangle_X & \leq & 0 \quad \text{(or } S_\Lambda \text{ is bounded } X\text{)} \\ \forall\, R > 0, & \varepsilon_R \|f\|_Y^2 & \leq & \|f\|_Z^2 + \theta_R \|f\|_X^2, \quad \varepsilon_R, \dfrac{\theta_R}{\varepsilon_R} \to 0. \end{array}$$

There exists a decay function Θ such that

$$\|S_\Lambda(t)\|_{X \to Y} \leq \Theta(t) \to 0.$$

- We say that m is an admissible if $m = \langle v \rangle^k$ or $m = e^{\kappa \langle v \rangle^s}$. We then write $m_0 \prec m_1$ or $m_1 \succ m_0$ or if $m_0/m_1 \to \infty$.
- For $X = L^p(m_1)$, $Y = L^p(m_0)$, $Z = L^p(m_0 \langle v \rangle^{\alpha/p})$, with $\alpha < 0$ and $m_1 \succ m_0$, we get

$$\Theta(t) \simeq \begin{cases} t^{-(k_1 - k_0)/|\alpha|} & \text{if } m_i = \langle v \rangle^{k_i} \\ e^{-\lambda t^{s/|\alpha|}} & \text{if } m_1 = e^{\kappa |v|^s} \end{cases}$$

We define $f_t := S_\Lambda(t) f_0$, $f_0 \in X$, and we compute

$$\frac{d}{dt}\|f_t\|_X^2 \le 0 \quad \Rightarrow \quad \|f_t\|_X \le C \|f_0\|_X, \; C \ge 1,$$

$$\begin{aligned} \frac{d}{dt}\|f_t\|_Y^2 &\lesssim -\|f_t\|_Z^2 \\ &\lesssim -\varepsilon_R \|f_t\|_Y^2 + \theta_R \|f_0\|_X^2, \end{aligned}$$

and from Gronwall lemma

$$\begin{aligned} \|f_t\|_Y^2 &\lesssim e^{-\varepsilon_R t} \|f_0\|_Y^2 + \frac{\theta_R}{\varepsilon_R} \|f_0\|_X^2 \\ &\lesssim \Theta(t)^2 \|f_0\|_X^2, \end{aligned}$$

with

$$\Theta(t)^2 := \inf_{R>0} \Big(e^{-\varepsilon_R t} + \frac{\theta_R}{\varepsilon_R} \Big).$$

Prop 2. Consider three "regular" Banach spaces $X \subset Y \subset Z$ and a generator $\mathcal{L}$. Assume

- $\|S_{\mathcal{L}}(t)\|_{X \to Z} \le \Theta(t)$, with $\Theta \in L^2(\mathbb{R}_+)$ a decay function (i.e. which tends to 0)
- $\mathcal{L} = \mathcal{A} + \mathcal{B}$, $\mathcal{A} \prec \mathcal{B}$, with

$$\begin{array}{lrcl} \forall\, f \in X_1^{\mathcal{B}}, & \langle f^*, \mathcal{B} f \rangle_X & \lesssim & -\|f\|_Y^2 \\ \forall\, f \in X_1^{\mathcal{A}}, & \langle f^*, \mathcal{A} f \rangle_X & \lesssim & \|f\|_Z^2. \end{array}$$

Then, $\mathcal{L}$ is weakly hypodissipative

$$\langle\!\langle f^*, \mathcal{L} f \rangle\!\rangle_X \lesssim -\|f\|_Y^2$$

for the duality product $\langle\!\langle , \rangle\!\rangle_X$ associated to the norm defined by

$$|\!|\!| f |\!|\!|^2 := \eta \|f\|_X^2 + \int_0^\infty \|S_{\mathcal{L}}(\tau) f\|_Z^2 \, d\tau,$$

for $\eta > 0$ small enough. That norm is equivalent to the initial norm in X.

We observe that $|\!|\!| \cdot |\!|\!| \sim \| \cdot \|_X$ because $\Theta \in L^2(\mathbb{R}_+)$.

We set $f_t := S_{\mathcal{L}}(t) f_0$ and we compute

$$
\begin{aligned}
\frac{d}{dt} |\!|\!| f_t |\!|\!|^2 &= 2\eta \langle f_t^*, \mathcal{L} f_t \rangle_X + \int_0^\infty \frac{d}{d\tau} \| S_{\mathcal{L}}(\tau + t) f_0 \|_Z^2 \, d\tau \\
&= 2\eta \langle f_t^*, \mathcal{B} f_t \rangle_X + \eta \langle f_t^*, \mathcal{A} f_t \rangle_X - \| f_t \|_Z^2 \\
&\le -2\eta C_1 \| f_t \|_Y^2 + (\eta C_2 - 1) \| f_t \|_Z^2 \\
&\lesssim -\| f_t \|_Y^2
\end{aligned}
$$

as well as

$$
\frac{d}{dt} |\!|\!| f_t |\!|\!|^2 \simeq \langle\!\langle f_t^*, \mathcal{L} f_t \rangle\!\rangle_X
$$

Prop 3. Consider a decay function Θ such that

$$\Theta^{-1}(t) \lesssim \Theta^{-1}(t-s)\Theta^{-1}(s) \text{ for any } 0 < s < t.$$

We consider two sets of Banach spaces $X_1 \subset X_0$ and $Y_1 \subset Y_0$ and a generator Λ. We assume

- $\|S_\Lambda(t)\|_{X_1 \to X_0} \Theta^{-1} \in L^\infty$
- $\Lambda = \mathcal{A} + \mathcal{B}$, $\mathcal{A} \prec \mathcal{B}$, with

$$\begin{aligned}
&\forall\, \ell, \quad \|S_\mathcal{B} * (\mathcal{A} S_\mathcal{B})^{(*\ell)}\|_{Y_1 \to Y_0} \Theta^{-1} \in L^\infty \\
&\exists\, n, \quad \|(\mathcal{A} S_\mathcal{B})^{(*n)}\|_{Y_1 \to X_1} \Theta^{-1} \in L^1 \text{ if } X_0 \subset Y_0 \text{ (enlargement)} \\
&\exists\, n, \quad \|(S_\mathcal{B} \mathcal{A})^{(*n)}\|_{X_0 \to Y_1} \Theta^{-1} \in L^1 \text{ if } Y_1 \subset X_1 \text{ (shrinkage)}
\end{aligned}$$

Then,

$$\|S_\Lambda(t)\|_{Y_1 \to Y_0} \Theta^{-1} \in L^\infty.$$

Enlargement result. We iterate the Duhamel formula

$$S_\Lambda = S_\mathcal{B} + S_\Lambda * (\mathcal{A} S_\mathcal{B})$$

to get a "stopped Dyson-Phillips series" (the D-P series corresponds to $n = \infty$)

$$S_\Lambda = \sum_{\ell=0}^{n-1} S_\mathcal{B} * (\mathcal{A} S_\mathcal{B})^{(*\ell)} + S_\Lambda * (\mathcal{A} S_\mathcal{B})^{(*n)} =: S_1 + S_2.$$

From the assumptions, we immediately have

$$\|S_\Lambda\|_{Y_1 \to Y_0} \Theta^{-1} \le \|S_1\|_{Y_1 \to Y_0} \Theta^{-1} + \|S_\Lambda \Theta^{-1}\|_{X_1 \to X_0} * \|(\mathcal{A} S_\mathcal{B})^{(*n)} \Theta^{-1}\|_{Y_1 \to X_1} \in L^\infty$$

Shrinkage result. We argue similarly staring with the iterated the Duhamel formula / stopped Dyson-Phillips series

$$S_\Lambda = \sum_{\ell=0}^{n-1} S_\mathcal{B} * (\mathcal{A} S_\mathcal{B})^{(*\ell)} + (S_\mathcal{B} \mathcal{A})^{(*n)} * S_\Lambda.$$

Prop 4. (rough version) We consider two Banach spaces $X \subset Y$ and a generator Λ. We assume $X^1_\Lambda \subset Y$ is compact and $\Theta(t) \approx e^{-\lambda t^{1/(1+j)}}$

- $\Sigma_P(\Lambda) \cap \bar\Delta_0 = \emptyset$, with $\Delta_0 := \{z \in \mathbb{C};\, \Re e z > 0\}$
- $\Lambda = \mathcal{A} + \mathcal{B}$, with $\mathcal{A} \in \mathbf{B}(Y,X)$, $\zeta \in (0,1]$ and

$$\begin{array}{lll} (a1) & \forall\, \ell, & \|S_\mathcal{B} * (\mathcal{A} S_\mathcal{B})^{(*\ell)}\|_{X\to Y}\Theta^{-1} \in L^\infty \\ (a2) & \forall\, \ell, & \sup_{z\in\bar\Delta_0} \|(R_\mathcal{B}(z))^\ell\|_{X\to Y} \le C\,(\ell!)^j \\ (a3) & \forall\, \ell, & \sup_{z\in\bar\Delta_0} \|R_\mathcal{B}(z)\|_{Y\to X^\Lambda_\zeta} \le C\,(\ell!)^j \end{array}$$

Then,

$$\|S_\Lambda(t)\|_{X\to Y}\Theta^{-1} \in L^\infty.$$

We start again with the stopped Dyson-Phillips series

$$S_\Lambda = \sum_{\ell=0}^{N-1} S_\mathcal{B} * (\mathcal{A} S_\mathcal{B})^{(*\ell)} + S_\Lambda * (\mathcal{A} S_\mathcal{B})^{(*N)} = S_1 + S_2$$

The first $N-1$ terms are fine. For the last one, we use the inverse Laplace formula

$$\begin{aligned} S_2(t)f &= \frac{i}{2\pi}\int_{\uparrow_0} e^{zt}\, R_\Lambda(z)(\mathcal{A}R_\mathcal{B}(z))^N f\, dz \\ &\approx \frac{1}{t^k}\int_{\uparrow_0} e^{zt}\,\frac{d^k\Phi}{dz^k}\, dz\, f \\ &\lesssim \frac{C^k}{t^k} k! \int_{\uparrow_0} \sup_{|\alpha|\le k} \underbrace{\|R_\Lambda^{1+\alpha_1}(z)\|_{X\to Y}}_{\in L^\infty(\uparrow_0)\,?} \underbrace{\|\mathcal{A}R_\mathcal{B}^{1+\alpha_1} ... \mathcal{A}R_\mathcal{B}^{1+\alpha_N}(z)\|_{X\to X}}_{\in L^1(\uparrow_0)\,?} dz\, \|f\|_X, \end{aligned}$$

where $\uparrow_0 := \{z = 0 + iy,\, y\in\mathbb{R}\}$ and because

$$\frac{d^k\Phi}{dz^k} \approx \sum_{|\alpha|\le k} \alpha!\, R_\Lambda^{1+\alpha_0}\mathcal{A}R_\mathcal{B}^{1+\alpha_1} ... \mathcal{A}R_\mathcal{B}^{1+\alpha_N}$$

• Using (a2), (a3), the compact embedding $X^1_\Lambda \subset Y$ and the fact that there is not punctual spectrum in $\bar\Delta_0$, we get

$$\sup_{z \in \bar\Delta_0} \|R_\Lambda(z)^\ell\|_{X \to Y} \le C\,(\ell!)^j$$

• $\mathcal{A} \in \mathbf{B}(Y, X)$ and the resolvent identity

$$R_\mathcal{B}(z) = \frac{1}{z}(R_\mathcal{B}(z)\mathcal{B} - I) \in \mathbf{B}(X_1, X)$$

imply

$$\|\mathcal{A}R_\mathcal{B}(z)\|_{X_1 \to X} \le C/|z| \quad \forall\, z \in \bar\Delta_0.$$

Together with (a2) (where we assume that $\zeta = 1$ in order to make the proof simpler) we get

$$\|\mathcal{A}R_\mathcal{B}(z)^{\ell_1}\mathcal{A}R_\mathcal{B}(z)^{\ell_2}\|_{X \to X} \le C\,(\ell_1!)^j(\ell_2!)^j\,\langle z\rangle^{-1}$$

• Choosing $N = 4$ and gathering the two estimates, we get

$$\|\frac{d^k\Phi}{dz^k}(z)\|_{X \to Y} \le C^k\,(k!)^j\,\langle z\rangle^{-2} \in L^1(\uparrow_0).$$

Coming back to the term S_2, we have

$$\begin{aligned} S_2(t) &\lesssim C^k k^{(1+j)k} t^{-k}. \\ &\lesssim e^{-\lambda t^{1/(1+j)}} = \Theta(t), \end{aligned}$$

by choosing appropriately $k = k(t)$

Prop 5.

Consider a semigroup generator Λ on a Banach lattice X, and assume
(1) Λ such as the spectral mapping Theorem holds (for $\|f\|_Y = \langle |f|, \phi \rangle$);
(2) $\phi \in D(\Lambda^*)$, $\phi \succ 0$ such that $\Lambda^* \phi = 0$;
(3) S_Λ is positive (and Λ satisfies Kato's inequalities);
(4) $-\Lambda$ satisfies a strong maximum principle.

There exists $0 < f_\infty \in D(\Lambda)$ such that

$$\Lambda f_\infty = 0, \quad \Sigma_P(\Lambda) \cap \bar{\Delta}_0 = \{0\}, \quad \Sigma_P(\Lambda_1) \cap \bar{\Delta}_0 = \emptyset$$

with $\Lambda_1 := \Lambda_{|X_1}$, $X_1 = R(I - \Pi_0) = (I - \Pi_0)X$,

$$\Pi_0 f = \langle f, \phi \rangle \, f_\infty \quad \forall\, f \in X.$$

Moreover the decay function Θ defined in the spectral mapping Theorem :

$$\|S_\Lambda(t)(I - \Pi_0) f_0\|_Y \lesssim \Theta(t) \, \|(I - \Pi_0) f_0\|_X \qquad \forall\, t \geq 0, \; \forall\, f_0 \in X.$$

Outline of the talk

• Weak Poincaré inequality

$$\langle \Lambda f, f \rangle_{E_0} \lesssim -\|f\|^2_{E_*}, \quad \forall\, f \in E_0,\ \langle f \rangle = 0,$$

with $E_0 := L^2(f_\infty^{-1/2})$, $f_\infty := e^{-V}$, and $E_* := L^2(\langle v \rangle^{\gamma-1} f_\infty^{-1/2})$.

• By the generalized relative entropy inequality

$$\forall\, f,\ \forall\, p \geq 1, \quad \langle \Lambda f, (f/f_\infty)^{p-1} \rangle \leq 0,$$

and passing to the limit as $p \to \infty$, we deduce the semigroup (of contractions) estimate

$$\|f_t\|_{E_1} \leq \|f_0\|_{E_1}, \quad E_1 := L^\infty(f_\infty^{-1}).$$

• For any $f_0 \in E_1$, $\langle f_0 \rangle = 0$, both inequalities and an interpolation argument imply (as in Prop 1)

$$\|f_t\|_{E_0} \leq \Theta(t) \|f_0\|_{E_1}, \quad \Theta(t) \simeq e^{-t^{\frac{\gamma}{2-\gamma}}}.$$

We introduce the splitting $\Lambda = \mathcal{A} + \mathcal{B}$, with $\mathcal{A}$ a multiplication operator

$$\mathcal{A}f = M\chi_R(v)f, \quad \chi_R(v) = \chi(v/R), \quad 0 \le \chi \le 1, \ \chi \in \mathcal{D}(\mathbb{R}^d)$$

$\triangleright$ $\mathcal{A} \in \mathbf{B}(X_0, X_1)$, $X_i = W^{r,p}(m_i)$, $m_1 \succeq m_0$
$\triangleright$ $\mathcal{B}$ is not a-dissipative in $X = W^{r,p}(m)$ with $a < 0$. However, it is weakly dissipative. For $p \in (1, \infty)$, and $M, R > 0$ large enough, we have

$$\langle f^*, \mathcal{B}f \rangle_{L^p} \lesssim -\|f\|^2_{L^p(m\langle v\rangle^{(\gamma-2+s)/p})}, \quad s := 0 \text{ for polynomial weight}$$

That is a consequence of the identity

$$\int (\Lambda f) f^{p-1} m^p = (1-p) \int |\nabla(fm)|^2 (fm)^{p-1} + \int (fm)^p \psi$$

$$\begin{aligned} \psi &= \left(\frac{2}{p} - 1\right)\frac{\Delta m}{m} + 2\left(1 - \frac{1}{p}\right)\frac{|\nabla m|^2}{m^2} + \left(1 - \frac{1}{p}\right)\mathrm{div} F - F \cdot \frac{\nabla m}{m} \\ &\sim -F \cdot \frac{\nabla m}{m} \sim -\langle v\rangle^{s+\gamma-2} \end{aligned}$$

• the estimate

$$(1) \quad \|S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)}\|_{X_1 \to X_0} \le \Theta(t)$$

follows from Proposition 1.

• the estimate

$$(2) \quad \|(\mathcal{A}S_{\mathcal{B}})^{(*n)}\|_{\mathbf{B}(L^1(m_1),H^1(m_2))} \le \Theta(t)$$

follows from (1) and the use a "Nash + regularity" trick for small time. More precisely, introducing

$$\mathcal{F}(t,h) := \|h\|^2_{L^1(m)} + t^\bullet \|h\|^2_{L^2(m)} + t^\bullet \|\nabla_v h\|^2_{L^2(m)}$$

we are able to prove (for convenient exponents $\bullet > 1$)

$$\frac{d}{dt}\mathcal{F}(t, S_{\mathcal{B}}(t)h) \le 0 \quad \text{and then} \quad \|S_{\mathcal{B}}(t)h\|^2_{H^1(m)} \le \frac{1}{t^\bullet}\|h\|^2_{L^1(m)}$$

• In the case $F = \nabla V$, we conclude thanks to Prop 3 (enlargement argument)

• For the general case, we use the Krein-Rutman theory. The Fokker-Planck semigroup is obviously mass conservative and positive and the Fokker-Planck operator satisfies the strong maximum principle. The last point in order to apply Proposition 5 is to verify that assumption (a2) in Proposition 4 is satisfied.

• Suitable spectral analysis theory in an abstract setting and a weakly dissipative framework ?

• What about the Boltzmann equation without Grad's cut-off ($\sim$ fractional diffusion in the velocity variable)?

$\triangleright$ Work in progress by Hérau, Tonon, Tristani, ...

• What about the grazing collisions limit (from Boltzmann to Landau)?