About the asymptotic stability of positive semigroups

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Outline of the talk

1. Introduction
2. A version of Krein-Rutmann theorem
3. Existence of steady stade under subgeometric Lyapunov condition
4. Rate of convergence under Doeblin condition

Outline of the talk

Issues

Existence of steady state/invariance measure and its exponential or polynomial stability for a positive semigroup in a Banach lattice, say in $X = L^1$ or $X = M^1$.

• First eigenvalue problem under spectral gap condition *[Krein-Rutman Theorem]*

• Existence of steady state under subgeometric Lyapunov condition *[ergodic theorem of Birkhoff-Von Neuman]*

• Constructive rate of convergence under Doeblin condition.

▷ Natural PDE formulations / simple deterministic proofs

▷ All these results use a splitting structure:

$$\mathcal{L} = \mathcal{A} + \mathcal{B}, \quad \mathcal{A} \prec \mathcal{B} \ \ldots$$

• Semigroup school: Engel, Nagel, Gearhart, Metz, Diekmann, Prüss, Arendt, Greiner, Blake, Webb, Mokhtar-Kharoubi, Yao, ...

• Probability school: Doeblin, Harris, Meyn, Tweedie, Down, Douc, Fort, Guillin, Hairer, Mattingly, ...

Vocabulary

- Positive semigroup $\approx$ weak maximum principle $\approx$ Kato's inequality

- steady state $=$ invariance measure

- spectral gap $=$ geometric Lyapunov condition
  no spectral gap $\approx$ subgeometric Lyapunov condition

- strong positivity $\approx$ strong maximum principle $\approx$ Doeblin condition

Examples of operators - I

1) - Linear Boltzmann, e.g. $k(v, v_*) = \sigma(v, v_*)\, M(v_*)$, $\sigma(v_*, v) = \sigma(v, v_*)$,

$$\mathcal{L} f = \underbrace{\int k(v, v_*)\, f(v_*)\, dv_*}_{=:\mathcal{A} f} \underbrace{- \int k(v_*, v)\, dv_*\, f(v)}_{=:\mathcal{B} f}$$

2) - Fokker-Planck, with $E(v) \approx v\, |v|^{\gamma - 1}$, $\gamma \geq 1$ or $\gamma \in (0, 1)$

$$\mathcal{L} = \underbrace{\Delta_v + \mathrm{div}_v(E(v)\, \cdot) - M\, \chi_R}_{=:\mathcal{B}} + \underbrace{M\, \chi_R}_{=:\mathcal{A}}$$

3) - Nonhomogeneous/kinetic Fokker-Planck

$$\mathcal{L} = \underbrace{\mathcal{T} + \mathcal{C} - M\, \chi_R}_{=:\mathcal{B}} + \underbrace{M\, \chi_R}_{=:\mathcal{A}}$$

with

$$\mathcal{T} := -v \cdot \nabla_x + F \cdot \nabla_x, \quad \mathcal{C} f := \Delta_v f + \mathrm{div}_v(E(v)\, f)$$

4) - relaxation

$$\mathcal{L} = \underbrace{\mathcal{T} + \rho_f M \chi_R^c - f}_{=:\mathcal{B}} + \underbrace{\rho_f M \chi_R}_{=:\mathcal{A}}$$

with

$$\mathcal{T} := -v \cdot \nabla_x + F \cdot \nabla_x, \quad \rho_f := \int f \, dv$$

5) - Growth fragmentation

$$\mathcal{L} = \mathcal{F}^+ - \mathcal{F}^- + \mathcal{D} = \underbrace{\mathcal{F}^+_\delta}_{=:\mathcal{A}} + \underbrace{\mathcal{F}^{+,c}_\delta - \mathcal{F}^- + \mathcal{D}}_{=:\mathcal{B}}$$

with

$$\mathcal{D}f = -\tau(x)\partial_x f - \nu f, \quad (\tau, \nu) = (1, 0) \text{ or } (x, 2)$$

$$\mathcal{F}^+(f) := \int_x^\infty k(y, x)\, f(y)\, dy, \quad \mathcal{F}^- f := K(x)\, f$$

6) - Age structured population

$$\mathcal{L} = \underbrace{-\partial_x - a}_{=:\mathcal{B}} + \underbrace{K\delta_{x=0}}_{=:\mathcal{A}}, \quad K[f] = \int_0^\infty k(y) f(y)\, dy$$

Outline of the talk

Thm 1. (M. & Scher) Consider a generator $\mathcal{L}$ on a "Banach lattice" X such that
(1) $\mathcal{L}$ has a nice splitting structure associated to an abscissa $a^* \in \mathbb{R}$;
(2) $\exists b > a^*$ and $\psi \in D(\mathcal{L}^*) \cap X'_+ \backslash \{0\}$ such that $\mathcal{L}^* \psi \geq b\, \psi$;
(3) $-\mathcal{L}$ satisfies a weak maximum principle;
(4) $-\mathcal{L}$ satisfies a strong maximum principle.

There exist $\lambda_1 \geq b$, $0 < \phi \in D(\mathcal{L}^*)$ and $0 < G \in D(\mathcal{L})$ such that

$$\mathcal{L}^* \phi = \lambda_1\, \phi, \quad \mathcal{L} G = \lambda_1\, G.$$

Moreover, $\lambda_1 = s(\mathcal{L}) = \sup\{\Re e \mu;\ \mu \in \Sigma(\mathcal{L})\}$ (spectral bound), $\Sigma(\mathcal{L}) \cap \bar{\Delta}_{\lambda_1} = \{\lambda_1\}$, $N(\mathcal{L} - \lambda_1) = \mathrm{vect}(G)$ and G is asymptotically exponentially stable: $\exists a < \lambda$ (non constructive)

$$S_{\mathcal{L}}(t) f_0 = e^{\lambda t} G \langle f_0, \phi \rangle + \mathcal{O}(e^{at}).$$

Here $\Delta_\alpha := \{z \in \mathbb{C};\ \Re e z > \alpha\}$.

$\triangleright$ Variant of KR version in Arendt, Grabosch, Greiner, Groh, Lotz, Moustakas, Nagel, Neubrander and Schlotterbeck Lecture Notes (1986)

Splitting structure $\Rightarrow$ Spectral mapping & Weyl's theorem

In a general Banach space X, we assume

$$\mathcal{L} = \mathcal{A} + \mathcal{B}, \quad \mathcal{A} \prec \mathcal{B},$$

where $\mathcal{A}$ and $\mathcal{B}$ satisfy for some $a^* \in \mathbb{R}$ and any $a > a^*$

(h0) $\mathcal{A}$ is $\mathcal{B}^{\zeta'}$-bounded with $0 \le \zeta' < 1$,

(h1) $\forall\, \ell \ge 0$, $\|S_\mathcal{B} * (\mathcal{A} S_\mathcal{B})^{(*\ell)}\|_{X \to X} e^{-at} \in L^\infty_t$

(h2) $\exists\, n \ge 1$, $\|(\mathcal{A} S_\mathcal{B})^{(*n)}\|_{X \to Y} \, e^{-at} \in L^1_t$, with $Y \subset\subset X$, $Y \subset D(\mathcal{L}^\zeta)$, $\zeta > \zeta'$.

It is equivalent to

(h3) there exist $\xi_1, ..., \xi_J \in \bar\Delta_a$, there exist $\Pi_1, ..., \Pi_J$ some finite rank projectors, there exists $T_j \in \mathcal{B}(R\Pi_j)$ such that $\mathcal{L}\Pi_j = \Pi_j \mathcal{L} = T_j \Pi_j$, $\Sigma(T_j) = \{\xi_j\}$ such that

$$\emptyset \neq \Sigma(\mathcal{L}) \cap \bar\Delta_a = \{\xi_1, ..., \xi_J\} \subset \Sigma_d(\mathcal{L})$$

(because of assumption **(2)**) and

$$\|S_\mathcal{L}(t) - \sum_{j=1}^J e^{tT_j} \Pi_j\|_{X \to X} \lesssim C_a \, e^{at}, \quad \forall\, a > a^*$$

Taking the Laplace transform of the Duhamel formula

$$S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{L}} * \mathcal{A} S_{\mathcal{B}}$$

we get the resolvent identity

$$R_{\mathcal{L}} = R_{\mathcal{B}} - R_{\mathcal{L}} \mathcal{A} R_{\mathcal{B}}.$$

By iteration

$$R_{\mathcal{L}}(1-\mathcal{V}) = \mathcal{U}$$

with

$$\mathcal{U} := \sum_{\ell=0}^{N-1} (-1)^{\ell} R_{\mathcal{B}} (\mathcal{A} R_{\mathcal{B}})^{\ell}, \quad \mathcal{V} := (-1)^N (\mathcal{A} R_{\mathcal{B}})^N$$

For $N = n$, $\mathcal{V} : \Delta_a \to \mathbf{B}(X)$ is holomorphic $\Rightarrow R_{\mathcal{L}}$ has discrete singular points (from Ribarič, Vidav, Voigt theory of degenerate-meromorphic functions)

For $N = n + 1$, $\|\mathcal{V}(z)\|$ small for $|z|$ large $\Rightarrow R_{\mathcal{L}}$ is holomorphic in $\Delta_a \backslash B(0, R)$.

We deduce

$$\Sigma(\mathcal{L}) \cap \Delta_a = \{\xi_1, ..., \xi_J\} \subset \Sigma_d(\mathcal{L}) \cap B(0, R).$$

We split the semigroup into invariant linear sub-manifolds (eigenspaces)

$$S_{\mathcal{L}} = \Pi\, S_{\mathcal{L}} + \Pi^{\perp}\, S_{\mathcal{L}}$$

and write the (iterated) Duhamel formula

$$S_{\mathcal{L}} = \sum_{\ell=0}^{N-1} S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)} + S_{\mathcal{L}} * (\mathcal{A}S_{\mathcal{B}})^{(*N)}$$

These two identities together

$$S_{\mathcal{L}} \quad = \quad \Pi\, S_{\mathcal{L}} + \Pi^{\perp}\, \{ \sum_{\ell=0}^{N-1} S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)} \} + \Pi^{\perp}\, S_{\mathcal{L}} \, * (\mathcal{A}S_{\mathcal{B}})^{(*N)}$$

For the last term, we use the inverse Laplace transform formula

$$\Pi^{\perp} S_{\mathcal{L}} \, * (\mathcal{A}S_{\mathcal{B}})^{(*N)}(t) = \lim_{M\to\infty} \frac{i}{2\pi} \int_{a-iM}^{a+iM} e^{zt}\, \Pi^{\perp} R_{\mathcal{L}}(z) (\mathcal{A}R_{\mathcal{B}}(z))^{N}\, dz$$

and we conclude by showing

$$\|R_{\mathcal{L}}(z)(\mathcal{A}R_{\mathcal{B}}(z))^{N}\| \le C/|y|^2, \quad \forall\, z = a + iy,\, |y| \ge M,\, a > a_*$$

• Weak maximum principle for $-\mathcal{L}$ means that for some (and thus any) $a > \omega(\mathcal{L})$ and $g \in X_+$, there holds

$$f \in D(\mathcal{L}) \text{ and } (a - \mathcal{L})f = g \quad \text{imply} \quad f \geq 0,$$

or equivalently
(a) $S_\mathcal{L}$ is positive, that is $S_\mathcal{L}(t) \geq 0$ for any $t \geq 0$.
(b) $\mathcal{L}$ satisfies the (real) Kato's inequality

$$(\mathrm{sign}\, f)\mathcal{L}f \leq \mathcal{L}|f|, \quad \forall\, f \in X,$$

and there exists a strictly positive subeigenvector for the dual problem:

$$\exists\, \varphi \in X', \ \varphi > 0, \ \exists\, K \in \mathbb{R}, \quad \mathcal{L}^*\varphi \leq K\, \varphi.$$

• We assume a strong version of the strong maximum principle for $-\mathcal{L}$:

$$f \in X \backslash \{0\} \text{ and } (a - \mathcal{L})|f| \geq 0 \quad \text{imply} \quad |f| > 0 \text{ and } \exists u \in \mathbb{C}, \ f = u|f|.$$

We introduce the Jordan basis $\{g_{1,1}, ..., g_{J,L_J}\}$ of the eigenspace associated to $\{\xi_1, ..., \xi_J\}$ with $\Re e \xi_j = s(\mathcal{L}) =: \lambda_1$ so that

$$e^{\mathcal{L}t}\, g_{j,\ell} = e^{\xi_j t} g_{j,\ell} + ... + t^{\ell-1}\, e^{\xi_j t} g_{j,1}.$$

We take (for instance) $h := g_{1,1} = h^1 - h^2 + ih^3 - ih^4$ with $h^\beta \ge 0$. We may fix h^α such that $\Pi h^\alpha \neq 0$ and then we denote $k^* := \max\{k;\ \Pi_k h^\alpha \neq 0\}$, where

$$\Pi_{j,\ell} := \text{projection on}\, g_{j,\ell}, \quad \Pi_k := \text{projection on}\,\text{Vect}(g_{j,k},\, 1 \le j \le J\}.$$

We split the semigroup as

$$e^{\mathcal{L}t} h^\alpha \;\;=\;\; \sum_j \sum_\ell e^{\mathcal{L}t}\, \Pi_{j,\ell} h^\alpha + e^{\mathcal{L}t}\, (I - \Pi) h^\alpha.$$

Using $\Pi_{j,\ell} h^\alpha = (\pi_{j,\ell} h^\alpha)\, g_{j,\ell}$, $\pi_{j,\ell} h^\alpha \in \mathbb{C}$, we get

$$0 \le \frac{1}{t^{k^*-1}}\, e^{(\mathcal{L}-\lambda_1)t} h^\alpha = \sum_{j=1}^{J} (\pi_{j,k^*} h^\alpha) e^{(\xi_j - \lambda_1)t} g_{j,1} + o(1).$$

- Passing to the limit for a subsequence $t_k \to \infty$, we get

$$0 \le \sum_{j=1}^{J} (\pi_{j,k^*} h^\alpha)\, z_j\, g_{j,1} =: G,$$

for some $z_j \in \mathbb{C}$, $|z_j| = 1$.

- Because the vectors $g_{j,1}$, $1 \le j \le J$, are independent, $G \neq 0$.
- Applying the semigroup again, we get

$$0 \le e^{\mathcal{L}t} G = \sum_{j=1}^{J} e^{\xi_j t} [(\pi_{j,k^*} h^\alpha)\, z_j\, g_{j,1}] \quad \forall\, t \ge 0,$$

which in particular implies $\pi_{j,k^*} h^\alpha = 0$ if $\Im m\, \xi_j \neq 0$.

- We have proved

$$G \in X_+ \backslash \{0\}, \quad \mathcal{L} G = \lambda_1 G$$

and in a similar way

$$\exists \phi \in X'_+ \backslash \{0\}, \quad \mathcal{L}^* \phi = \lambda_1 \phi.$$

Outline of the talk

Subgeometric Foster-Lyapunov condition. There are two weight functions $m_0, m_1 : E = \mathbb{R}^d \to [1, \infty)$, $m_1 \geq m_0$, $m_0(x) \to \infty$ as $x \to \infty$, and two real constants $b, R > 0$ such that

$$\mathcal{L}^* m_1 \leq -m_0 + b\,\mathbf{1}_{B_R}.$$

Thm 2 Douc, Fort, Guillin ?

Any Feller-Markov semigroup (P_t) which fulfills the above Foster-Lyapunov condition has at least one invariant borelian measure $G \in M^1(m_0)$.

Remark.

- $m_0 = m_1$: geometric Foster-Lyapunov condition = spectral gap (the result is true, the proof is simpler)
- Feller-Markov semigroup acts on $C_0(E)$ and $P_t := S_{\mathcal{L}^*}(t)$.

We introduce the splitting

$$A := b\mathbf{1}_{B_R}, \quad B := \mathcal{L} - A.$$

We observe that

$$0 \le S_{\mathcal{B}} \in L^\infty_t(\mathcal{B}(M^1(m_i))); \quad \int_0^\infty \|S_{\mathcal{B}}(t) f_0\|_{M^1(m_0)}\, dt \le \|f_0\|_{M^1(m_1)}.$$

We write the Duhamel formula

$$S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{B}} * \mathcal{A} S_{\mathcal{L}},$$

and we consider the associated Cezaro means

$$U_T := \frac{1}{T}\int_0^T S_{\mathcal{L}}\, dt, \quad V_T := \frac{1}{T}\int_0^T S_{\mathcal{B}}\, dt, \quad W_T := \frac{1}{T}\int_0^T S_{\mathcal{B}} * \mathcal{A} S_{\mathcal{L}}\, dt.$$

We define $X := M^1(m_0)$ and we observe that

$$\|V_T\|_{X\to X} := \frac{1}{T}\Big\|\int_0^T S_{\mathcal{B}}\,dt\Big\|_{X\to X} \le 1$$

On the other hand, by Fubini and positivity

$$\begin{aligned}\|W_T f_0\|_{M^1(m_0)} &= \Big\|\frac{1}{T}\int_0^T S_{\mathcal{B}}(\tau)\int_0^{T-\tau}\mathcal{A}\,S_{\mathcal{L}}(s)\,f_0\,d\tau ds\Big\|_{M^1(m_0)} \\ &\le \frac{1}{T}\int_0^\infty\Big\|S_{\mathcal{B}}(\tau)\,\int_0^T\mathcal{A}\,S_{\mathcal{L}}(s)\,dsf_0\Big\|_{M^1(m_0)}d\tau \\ &\le \frac{1}{T}\Big\|\int_0^T\mathcal{A}\,S_{\mathcal{L}}(s)\,dsf_0\Big\|_{M^1(m_1)} \le C_{\mathcal{A}}\|f_0\|_{M^1(m_0)},\end{aligned}$$

We deduce $U_{T_k} f_0 \to G$ satisfies $\mathcal{L}G = G$ by writting

$$S_{\mathcal{L}}(s)G - G = \lim_{k\to\infty}\Big\{\frac{1}{T_k}\int_0^{T_k} S_{\mathcal{L}}(s)S_{\mathcal{L}}(t)f_0 - \frac{1}{T_k}\int_0^{T_k} S_{\mathcal{L}}(t)f_0\,dt\Big\} = 0,$$

Outline of the talk

We consider a positive semigroup $S_t = S_{\mathcal{L}}(t)$ defined in $X := L^1(E)$ and we assume

(H1) Subgeometric Foster-Lyapunov condition. There are two weight functions $m_0, m_1 : E \to [1,\infty)$, $m_1 \ge m_0$, $m_0(x) \to \infty$ as $x \to \infty$, and two real constants $b, R > 0$ such that

$$\mathcal{L}^* m_1 \le -m_0 + b\,\mathbf{1}_{B_R}.$$

(H2) Doeblin condition. $\exists\, T > 0\ \forall\, R > 0\ \exists\, \nu\ (= \nu_{T,R})$ such that

$$S_T f \ge \nu \int_{B_R} f, \quad \forall\, f \in X_+.$$

(H3) There are two other weight functions $m_2, m_3 : E \to [1,\infty)$,

$m_3 \ge m_2 \ge m_1$ such that

$$\mathcal{L}^* m_i \le -m_0 + b\,\mathbf{1}_{B_R}$$

and $m_2 \le m_0^\theta m_3^{1-\theta}$ with $\theta \in (1/2, 1]$.

Thm 3 Douc, Fort, Guillin, Hairer

Consider a Markov semigroup S on $X := L^1(m_2)$ which satisfies (H1), (H2). There holds

$$\|S_t f\|_{L^1} \le \Theta(t) \|f\|_{L^1(m_2)}, \quad \forall\, t \ge 0,\ \forall\, f \in X,\ \langle f \rangle = 0,$$

for the function Θ given by

$$\Theta(t) := \inf_{\lambda > \lambda_0} \left\{ e^{-\kappa t/\lambda} + \xi_\lambda \right\},$$

where $\xi_\lambda > 0$ is the smallest positive number associated to $\lambda > 0$ such that

$$m_1 \le \lambda m_0 + \xi_\lambda m_2.$$

S_T is bounded in $L^1(m_i)$, $i = 0, 2$

We fix $f_0 \in L^1(m_3)$, we denote $f_{\mathcal{B}t} := S_{\mathcal{B}}(t) f_0$.
From (H1) and (H3), we have

$$\frac{d}{dt} \|f_{\mathcal{B}t}\|_{m_3} \le -\|f_{\mathcal{B}t}\|_{m_0} \le 0$$

$$\frac{d}{dt} \|f_{\mathcal{B}t}\|_{m_2} \le -\|f_{\mathcal{B}t}\|_{m_0} \le -\|f_{\mathcal{B}t}\|_{m_2}^{1/\theta} \|f_0\|_{m_3}^{1-1/\theta} \le 0$$

so that $t \mapsto \|f_{\mathcal{B}t}\|_{m_2} \lesssim \langle t \rangle^{-\frac{\theta}{1-\theta}} \|f_0\|_{m_3} \in L^1(\mathbb{R}_+)$.

Using the splitting

$$S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{B}} * \mathcal{A} S_{\mathcal{L}}$$

we deduce

$$\|S_{\mathcal{L}}(t) f_0\|_{m_2} \le M_2 \, \|f_0\|_{m_2}.$$

Recalling that

$$\|S_{\mathcal{L}}(t) f_0\|_{L^1} \le \|f_0\|_{L^1},$$

we deduce by interpolation

$$\|S_{\mathcal{L}}(t) f_0\|_{m_0} \le M_0 \, \|f_0\|_{m_0}.$$

We fix $f_0 \in L^1(m_2)$, $\langle f_0 \rangle = 0$ and we denote $f_t := S_t f_0$. From (H1), we have

$$\frac{d}{dt}\|f_t\|_{m_1} \le -\|f_t\|_{m_0} + b\|f_t\|_{L^1},$$

from what we deduce

$$\|f_T\|_{m_1} + \int_0^T \|f_s\|_{m_0} \le \|f_0\|_{m_1} + bT\|f_0\|_{L^1}.$$

and then

$$\|S_T f_0\|_{m_1} + \frac{T}{M_0}\|S_T f_0\|_{m_0} \le \|f_0\|_{m_1} + bT\|f_0\|_{L^1}.$$

We fix $R > 0$ large enough such that $A := m(R)/4 \ge 3bM_0$ and we observe that the following altenative holds

$$\|f_0\|_{m_0} \le A\|f_0\|_{L^1} \quad \text{or} \quad \|f_0\|_{m_0} > A\|f_0\|_{L^1}.$$

We define

$$\|f\|_\beta := \|f\|_{L^1} + \beta\|f\|_{m_1}, \quad \beta > 0.$$

Step estimate and iteration

In both case and for $\beta > 0$ small enough, we have

$$\|S_T f_0\|_\beta + 3\alpha \|S_T f_0\|_{m_0} \le \|f_0\|_\beta + \alpha \|f_0\|_{m_0}, \quad \alpha > 0.$$

Using

$$\frac{1}{\lambda}\frac{1}{1+\beta}\|f\|_\beta \le \frac{1}{\lambda}\|f\|_{m_1} \le \|f\|_{m_0} + \frac{\xi}{\lambda}\|f\|_{m_2},$$

we deduce

$$\|S_T f_0\|_\beta + \frac{\delta}{\lambda}\|S_T f_0\|_\beta + 2\alpha \|S_T f_0\|_{m_0} \le \|f_0\|_\beta + \frac{\xi}{\lambda}\alpha \|S_T f_0\|_{m_2} + \alpha \|f_0\|_{m_0},$$

with $\delta := \alpha/(1+\beta)$.

We rewrite

$$Z\big(u_1 + \alpha v_1\big) \le (u_0 + \alpha v_0) + Z\frac{\xi}{\lambda}\alpha M_2 \|f_0\|_{m_2}$$

with

$$Z := 1 + \delta/\lambda \le 2.$$

By an induction argument, we deduce

$$Z^n(u_n + \alpha v_n) \le (u_0 + \alpha v_0) + \sum_{i=1}^{n} Z^i \, \frac{\xi}{\lambda}\alpha M_2 \|f_0\|_{m_2}$$