Régimes de connectivité faible et forte dans un modèle de réseau de neurones

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27ème Journée Interne du
Laboratoire Jacques-Louis Lions

References

- Voigt, *A perturbation theorem for the essential spectral radius of strongly continuous semigroups*, Monatsh. Math. (1980)
- Pakdaman, Perthame, Salort, *Dynamics of a structured neuron population*, Nonlinearity (2010)
- Pakdaman, Perthame, Salort, *Relaxation and self-sustained oscillations in the time elapsed neuron network model*, SIAM J. Appl. Math. (2013)
- M., Weng, *Relaxation in time elapsed neuron network models in the weak connectivity regime*, arXiv 2015
- M., Scher, *Spectral analysis of semigroups and growth-fragmentation equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire (2016)
- Weng, *General time elapsed neuron network model: well posedness and strong connectivity regime*, arXiv 2016
- M., Quiñinao, Weng, *Weak and strong connectivity regimes for a general time elapsed neuron network model*, Weng's phD thesis and in progress

Outline of the talk

1. Introduction
   - Model
   - Assumptions
   - Main results

2. Linear and nonlinear stability
   - An auxiliary linear equation (without delay)
   - Coming back to the nonlinear equation
   - The case with delay

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• State of a neuron: local time (or internal clock) $x \geq 0$ corresponding to the elapsed time since the last discharge;

• Dynamic of the neuron network: (age structured) evolution equation

$$\partial_t f = -\partial_x f - a(x, \lambda\, m(t)) f =: \mathcal{L}_{\lambda m(t)} f, \quad f(t,0) = p(t)$$

on the density number of neurons $f = f(t,x) \geq 0$.

• $a(x, \lambda\, \mu) \geq 0$: firing rate of a neuron in the state x for a network activity $\mu \geq 0$ and a network connectivity parameter $\lambda \geq 0$.

• $p(t)$: total density of neurons undergoing a discharge at time t given by

$$p(t) := \mathcal{P}_\lambda[f(t); m(t)], \quad \mathcal{P}_\lambda[g, \mu] := \int_0^\infty a(x, \lambda\mu) g(x) \mathrm{d}x.$$

• $m(t)$: network activity at time $t \geq 0$ resulting from earlier discharges given by

$$m(t) := \int_0^\infty p(t-y) b(\mathrm{d}y),$$

b delay distribution taking into account the persistence of electric activity

- *Case without delay*, when $b = \delta_0$ and then $m(t) = p(t)$.
- *Case with delay*, when b is a smooth function.

Hypothesis

- Monotony and smoothness of the firing rate

$$\partial_x a \geq 0, \quad a' = \partial_\mu a \geq 0,$$

$$0 < a_0 := \lim_{x \to \infty} a(x,0) \leq \lim_{x,\mu \to \infty} a(x,\mu) =: a_1 < \infty,$$

$$a \in Lip_\mu L^1_x :$$

$\forall\, \xi > 0$ small, $\forall\, \mu_0 > 0$, $\forall\, \mu_\infty > 0$, $\exists\, \lambda_0 > 0$, $\exists\, \lambda_\infty > 0$

$$\int_0^\infty |a(\lambda\mu_2) - a(\lambda\mu_1)|\, dx \leq \xi\, |\mu_2 - \mu_1|, \quad \forall \mu_1, \mu_2 \in (0, \mu_0),\ \forall\, \lambda \in (0, \lambda_0)$$

$$\int_0^\infty |a(\lambda\mu_2) - a(\lambda\mu_1)|\, dx \leq \xi\, |\mu_2 - \mu_1|, \quad \forall \mu_1, \mu_2 \in (\mu_\infty, \infty),\ \forall\, \lambda \in (\lambda_\infty, \infty)$$

- Without delay or smooth delay

$$b = \delta_0 \quad \text{or} \quad \exists \delta > 0, \quad \int_0^\infty e^{\delta y} \left(b(y) + |b'(y)|\right) \mathrm{d}y < \infty.$$

- Bounded and positive initial datum

$$f_0 \in L^1_q \cap L^\infty,\ q > 0, \quad \kappa_0 := \int_0^\infty a(x,0) f_0(x)\, dx > 0.$$

Example

$$a(x,\mu) = \mathbf{1}_{x>\sigma(\mu)}, \quad \sigma' \le 0,$$

with

$$\sigma_+ := \sigma(0), \quad \sigma_- := \sigma(\infty), \quad \sigma_- < \sigma_+ < 1,$$

and σ satisfies the regularity condition

$$\sigma, \sigma^{-1} \in W^{1,\infty}(\mathbb{R}_+)$$

together with

$$s\sigma'(s) \to 0 \quad \text{as} \quad s \to \infty.$$

Theorem 1. There exists a weak solution $f \in C([0,\infty); L^1) \cap L^\infty(0,\infty; L^\infty)$
- in the delay case;
- in the case without delay in the strong/weak connectivity regime.

The solution is unique in the strong/weak connectivity regime.

Theorem 2. There exists (at least) one normalized and positive stationary solution F_λ:

$$\mathcal{L}_{\lambda M_\lambda} F_\lambda = -\partial_x F_\lambda - a(x, \lambda\, M_\lambda) F_\lambda = 0, \quad F_\lambda(0) = M_\lambda,$$

$$M_\lambda = \mathcal{P}_\lambda[F_\lambda; M_\lambda] = \int_0^\infty a(x, \lambda M_\lambda)\, F_\lambda(x) \mathrm{d}x.$$

Theorem 3. The stationary solution F_λ is unique and exponentially stable in the in the strong/weak connectivity regime:

$$\exists\, a < 0, \qquad \|f(t) - F_\lambda\|_{L^1} \le C_{f_0}\, e^{at}, \quad \forall\, t \ge 0$$

when $\lambda \in (0, \lambda_0) \cup (\lambda_\infty, \infty)$.

Outline of the talk

We introduce an auxiliary linear evolution equation

$$\partial_t g + \partial_x g + a_\lambda g = 0, \quad g(t,0) = \mathcal{P}_\lambda[g(t,.)],$$

with

$$a_\lambda := a(x, \lambda M_\lambda), \quad \mathcal{P}_\lambda[g] := \int_0^\infty a_\lambda g \, dx.$$

The equation also writes

$$\frac{d}{dt} g = \mathcal{L} g := -\partial_x g - a_\lambda g + \delta_0 \mathcal{P}_\lambda[g].$$

Theorem 4 (Krein-Rutman). $\forall \lambda \geq 0$, $\exists \alpha < 0$, $\exists C \geq 1$ such that $\Sigma(\mathcal{L}) \cap \Delta_\alpha = \{0\}$ and

$$\|S_{\mathcal{L}}(t) g_0\|_X \leq C\, e^{\alpha t} \|g_0\|_X, \quad \forall t \geq 0,$$

for any $g_0 \in X := L^1_q$, $q > 0$.

Lemma 1 (positivity). $g_0 \geq 0 \Longrightarrow S_{\mathcal{L}}(t) g_0 \geq 0$.

Lemma 2 (strong macimum principle).

$$g \in D(\mathcal{L}),\ g \in X_+ \backslash \{0\},\ (\eta - \mathcal{L}) g \geq 0 \ \Longrightarrow \ g > 0.$$

Corollary $\Sigma(\mathcal{L}) \cap \bar{\Delta}_0 = \{0\}$ and $N(\mathcal{L}) = \mathrm{span}(F_\lambda)$.

We split $\mathcal{L}$ as

$$\mathcal{L} = \mathcal{A} + \mathcal{B}, \quad \mathcal{A} g := \delta_0 \mathcal{P}_\lambda[g], \quad \mathcal{B} g := -\partial_x g - a_\lambda g.$$

We have for $a < 0$:

(1) $(S_{\mathcal{B}} \mathcal{A})^{(*\ell)} * S_{\mathcal{B}} = \mathcal{O}(e^{at})$;

(2) $(R_{\mathcal{B}} \mathcal{A})^2 = \mathcal{O}(1/|z|)$, $R_{\mathcal{B}}(z) := (\mathcal{B} - z)^{-1} = (\widehat{S}_{\mathcal{B}})(z)$;

(3) $(R_{\mathcal{B}} \mathcal{A})^2 \in \mathcal{B}(X, Y)$, $Y := D(\mathcal{L}) \cap L^1_{q+1} \subset\subset X$.

Theorem 5. The conclusions of Weyl's theorem and the spectral mapping theorem apply to $\mathcal{L}$. Roughly $\Sigma(\mathcal{L}) \cap \Delta_a \subset \Sigma_d(\mathcal{L})$ the discrete spectrum and $\Sigma(S_{\mathcal{L}}(t)) = e^{t\Sigma(\mathcal{L})}$.

We compute

$$(S_{\mathcal{B}}\mathcal{A})^{(*2)}(t)g = \varphi_t \, \mathcal{P}_\lambda[g],$$

with

$$\varphi_t(x) = \psi(t-x)\exp(-A_\lambda(x)), \quad \psi(u) := \mathbf{1}_{u\geq 0}\, a_\lambda(u)\, e^{-A_\lambda(u)}, \quad A_\lambda' = a_\lambda.$$

Since $\psi \in BV$, we deduce

$$\|(R_{\mathcal{B}}\mathcal{A})^2(z)g\|_X = \|e^{-zx-A_\lambda(x)}\|_X \, |\hat{\psi}(z)| \, |\mathcal{P}_\lambda[g]| \lesssim \frac{1}{|z|} \, \|g\|_X,$$

which is nothing but (2).

For proving Weyl's theorem and the spectral mapping theorem, we use the itareted Duhamel formula and its resolvent contrepart

$$\mathcal{R}_{\mathcal{L}} = \mathcal{R}_{\mathcal{B}} - \cdots + (-1)^{n-1}(\mathcal{R}_{\mathcal{B}}\mathcal{A})^{n-1}\mathcal{R}_{\mathcal{B}} + (-1)^n(\mathcal{R}_{\mathcal{B}}\mathcal{A})^n\mathcal{R}_{\mathcal{L}}$$

$$S_{\mathcal{L}} = S_{\mathcal{B}} + \ldots + (S_{\mathcal{B}}\mathcal{A})^{(*(n-1))} S_{\mathcal{B}} + \frac{i(-1)^n}{2\pi} \int_{\uparrow_a} (\mathcal{R}_{\mathcal{B}}\mathcal{A})^n \mathcal{R}_{\mathcal{L}} e^{zt} \, dz$$

Coming back to the nonlinear equation

In the weak/strong connectivity regime, we may write

$$\begin{aligned}\partial_t f &= -\partial_x f - a(\lambda\varphi[f])f + \delta_0\varphi[f],\\ 0 &= -\partial_x F + a(\lambda\varphi[F])F + \delta_0\varphi[F],\end{aligned}$$

with $g \mapsto \varphi[g]$ is the unique solution to the constraint equation

$$m = \mathcal{P}_\lambda[g, m] = \int_0^\infty a_\lambda(x, \lambda m) g \, dx.$$

Setting $g := f - F$, we deduce

$$\partial_t g = \mathcal{L} g + Z[g],$$

with the nonlinear term

$$Z[g] := -Q[g] + \delta_0 \langle Q[g] \rangle, \quad Q[g] := (a_\lambda(\varphi[f]) - a_\lambda(\varphi(F)))f.$$

We conclude by obsering $\|Q[g]\|_X \le \varepsilon(\lambda)\|g\|_X$ with $\varepsilon(\lambda) \to 0$ as $\lambda \to 0$ or $\lambda \to \infty$.

We introduce an auxiliary unknown $u = u(t, y)$ and we rewrite the problem as an autonnomous system

$$\partial_t f = -\partial_x f - a_\lambda(\mathcal{D}[u]) f + \delta_0 p$$
$$\partial_t u = -\partial_y u + \delta_0 p,$$

with

$$p := \mathcal{P}(f, \mathcal{D}[u]), \quad \mathcal{D}[w] := \int_0^\infty w(y)\, b(\mathrm{d}y).$$

Similarly, we rewrite the stationary equation as

$$0 = -\partial_x F - k_\lambda(M) F + \delta_0 M$$
$$0 = -\partial_y U + \delta_0 M,$$

with $M = \mathcal{P}(F, M)$, $M = \mathcal{D}[U]$.

The variations $g := f - F$, $v := u - U$ satisfy

$$\begin{aligned} \partial_t g &= -\partial_x g - a_\lambda(M) g + \delta_0 \mathcal{P}_\lambda[g, M] + Z_1[g, v] \\ \partial_t v &= -\partial_x v + \delta_0 \mathcal{P}_\lambda[g, M] + Z_2[g, v], \end{aligned}$$

and we proceed similarly as in the case without delay.