L^2-Hypocoercivity estimates (and variants of Korn's inequality)

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Outline of the talk

1. Introduction
   - Villani's program
   - Second step: quantitative hypocoercivity estimates
2. Relaxation equation (by Hérau, Dolbeault-Mouhot-Schmeiser)
   - The relaxation operator in the torus
   - The relaxation operator with confinement force
3. The linearized B/L operator with confinement (with Carrapatoso)

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Here is the program (Villani's Notes on 2001 IHP course, Section 8. Toward exponential convergence)

1. Find a constructive method for bounding below the spectral gap in $L^2(M^{-1})$, the space of self-adjointness, say for the Boltzmann operator with hard spheres.

3. Find a constructive argument to overcome the degeneracy in the space variable, to get an exponential decay for the linear semigroup associated with the linearized spatially <u>inhomogeneous</u> Boltzmann equation; something similar to hypo-ellipticity techniques.

2. Find a constructive argument to go from a spectral gap in $L^2(M^{-1})$ to a spectral gap in L^1, with all the subtleties associated with spectral theory of non-self-adjoint operators in infinite dimension ...

4. Combine the whole things with a perturbative and linearization analysis to get the exponential decay for the nonlinear equation close to equilibrium.

Consider the evolution equation

$$\partial_t F + v \cdot \nabla_x F + \cdots = Q(F)$$
$$F(0, .) = F_0$$

on the density of the particle $F = F(t, x, v) \geq 0$, time $t \geq 0$, velocity $v \in \mathbb{R}^3$, position $x \in \Omega$

$\Omega = \mathbb{T}^3$ (torus);

$\Omega \subset \mathbb{R}^3$ + boundary reflection conditions;

$\Omega = \mathbb{R}^3$ + confinement force field.

$Q =$ nonlinear (quadratic) Boltzmann or Landau collisions operator
: conservation of mass, momentum and energy

$Q =$ linear relaxation or Fokker-Planck collisions operator
: conservation of mass

First step in Villani's program: quantitative coercivity estimates

We define the linearized Boltzmann / Landau operator in the space homogeneous framework

$$\mathcal{S}f := Q(f, M) + Q(M, f)$$

and the orthogonal projection π in $L^2(M^{-1})$ on the eigenspace

$$\mathrm{Span}\{(1, v, |v|^2)M\}.$$

Theorem (..., Guo, Mouhot, Strain, Gressman, AMUXY)

There exist two Hilbert spaces $\mathfrak{h} = L^2(M^{-1})$ and $\mathfrak{h}_*$ and constructive constants $\lambda, K > 0$ such that

$$(-\mathcal{S}h, g)_{\mathfrak{h}} = (-\mathcal{S}g, h)_{\mathfrak{h}} \le K \|g\|_{\mathfrak{h}_*} \|h\|_{\mathfrak{h}_*}$$

and

$$(-\mathcal{S}h, h)_{\mathfrak{h}} \ge \lambda \, \|\pi^\perp h\|^2_{\mathfrak{h}_*}, \quad \pi^\perp = I - \pi$$

The space $\mathfrak{h}_*$ depends on the operator (linearized Boltzmann or Landau) and the interaction parameter $\gamma \in [-3, 1]$, $\gamma = 1$ corresponds to (Boltzmann) hard spheres interactions and $\gamma = -3$ corresponds to (Landau) Coulomb interactions.

In a Hilbert space $\mathcal{H}$, we consider now an operator

$$\mathcal{L} = \mathcal{S} + \mathcal{T}$$

with

$$\mathcal{S}^* = \mathcal{S} \le 0, \quad \mathcal{T}^* = -\mathcal{T}.$$

More precisely, $\mathcal{H} \supset \mathcal{H}_x \otimes \mathcal{H}_v$, $\mathcal{S}$ acts on the v variable space $\mathcal{H}_v$ with null space $N(\mathcal{S})$ of finite dimension, we denote π the projection on $N(\mathcal{S})$.
As a consequence, in the two variables space $\mathcal{H}$ the operator $\mathcal{S}$ is degenerately / partially coercive

$$(-\mathcal{S}f, f) \gtrsim \|f^\perp\|^2_{\mathcal{H}_*}, \quad f^\perp = f - \pi f$$

For the initial Hilbert norm, we get the same degenerate / partial positivity of the Dirichlet form

$$D[f] := (-\mathcal{L}, f) \gtrsim \|f^\perp\|^2_{\mathcal{H}_*}, \quad \forall f.$$

That information is not strong enough in order to control the longtime behavior of the dynamic of the associated semigroup !! We need to control $\pi f \in \mathcal{H}_x$!

What is hypocoercivity about - the twisted norm approach

▷ Find a new Hilbert norm by twisting

$$|||f|||^2 := \|f\|^2 + 2(Af, Bf)$$

such that the new Dirichlet form is coercive:

$$\begin{aligned} D[f] &:= ((-\mathcal{L}f, f)) \\ &= (-\mathcal{L}f, f) + (-A\mathcal{L}f, Bf) + (Af, -B\mathcal{L}f) \\ &\gtrsim \|f^\perp\|^2 + \|\pi f\|^2. \end{aligned}$$

▷ We destroy the nice symmetric / skew symmetric structure and we have also to be very careful with the "remainder terms".
▷ That functional inequality approach is equivalent (and more precise if constructive) to the other more dynamical approach (called "Lyapunov" or "energy" approach).

Theorem. (for strong coercive operators in both variables, in particular $\mathfrak{h}_* \subset \mathfrak{h}$)

There exist some new but equivalent Hilbert norm $|||\cdot|||$ and a (constructive) constant $\lambda > 0$ such that the associated Dirichlet form satisfies

$$D[f] \ge \lambda\, |||f|||^2, \quad \forall\, f,\ \langle \pi f \rangle = 0.$$

▷ It implies $|||e^{\mathcal{L}t} f||| \le e^{-\lambda t} |||f|||$ and then $\|e^{\mathcal{L}t} f\| \le C e^{-\lambda t} \|f\|$, $\forall\, f,\ \langle \pi f \rangle = 0$.

- Fourier approach and hypocoercivity : Kawashima
- Non constructive spectral analysis approach : Ukai (1974), Arkeryd, Esposito, Pulvirenti (1987), Wennberg (1995)
- Non constructive estimate and hypoellipticity : Eckmann, Pillet, Rey-Bellet (1999)
- Constructive entropy approach: Desvillettes-Villani (2001-2005)
- Energy (in high order Sobolev space) approach : Guo and Guo' school [2002-..]
- Micro-Macro approach : Shizuta, Kawashima (1984), Liu, Yu (2004), Yang, Guo, Duan, ...
- Constructive estimate and hypoelliptity : Hérau, Nier, Helffer, Eckmann, Hairer (2003-2005), Villani (2009)
- Constructive hypocoercivity estimates without hypoelliptity (2006–2015): Villani, Mouhot, Neumann, Dolbeault, Schmeiser, . . . , Briant, Merino-Aceituno, . . .
- Carrapatoso, M., Landau equation in the torus, 2017

- linearized B/L operators with boundary reflection condition uniformly in the Knudsen number (with I. Tristani)

  $\triangleright$ generalizes results by Guo, Briant-Guo and Briant-Merino-Aceituno-Mouhot

- "linearized" B/L operator with confinement (with K. Carrapatoso)

  $\triangleright$ improves results by Duhan, Duhan-Li

- starting with the case of the relaxation equation (Hérau, Dolbeault-Mouhot-Schmeiser)

We do not consider here :

- The case $\mathfrak{h}_* \not\subset \mathfrak{h}$
- The whole space with weak confinement
- The whole space without any confinement
- uniform estimate in the grazing collisions limit

Outline of the talk

We consider the "standard" relaxation kinetic operator

$$\mathcal{L} := \mathcal{S} + \mathcal{T}$$

where $\mathcal{S}$ is the "standard" relaxation operator

$$\mathcal{S} f := \rho_f M - f =: f^\perp, \quad \rho_f := \langle f \rangle.$$

and $\mathcal{T}$ is the transport operator

$$\mathcal{T} f := -v \cdot \nabla_x f + \dots.$$

We may assume

$$\begin{aligned}
&\text{(case 1)} \qquad \dots = 0, \quad \Omega := \mathbb{T}^d, \quad V := 0;\\
&\text{(case 2)} \qquad \dots = \nabla_x V \cdot \nabla_v f, \quad \Omega := \mathbb{R}^d, \quad V \sim |x|^\gamma, \ \gamma \ge 1;\\
&\text{(case 3)} \qquad \dots = 0, \quad \Omega \subset \mathbb{R}^d \ + \ \text{reflection}, \quad V := 0.
\end{aligned}$$

We introduce the probability measure

$$G := e^{-V} M(v), \quad M(v) := (2\pi)^{-d/2} e^{-|v|^2/2}.$$

We work in the Hilbert spaces $\mathfrak{h} := L^2_v(M^{-1})$ and $\mathcal{H} := L^2_{xv}(G^{-1})$. We observe that $f = G$ is the unique normalized positive steady state and the associated projector is

$$\pi f := (f, G)_{\mathfrak{h}} M = \langle f \rangle M \in L^2_x(e^V).$$

We introduce the Hilbert norm

$$|||f|||^2 := \|f\|_{\mathcal{H}}^2 - 2\eta(\nabla_x \Delta^{-1} \rho, j)$$

with $1 >> \eta > 0$ and then the Dirichlet form

$$\begin{aligned} D(f) &= ((-\mathcal{L}f, f)) \\ &= (-\mathcal{L}f, f) + \eta(\nabla_x \Delta^{-1} \rho_f, j[\mathcal{L}f]) + \eta(\nabla_x \Delta^{-1} \rho[\mathcal{L}f], j_f). \end{aligned}$$

Here

$$\begin{aligned} \rho &:= \rho_f = \rho[f] = \langle f \rangle, \\ j &:= j_f = j[f] = \langle f\, v \rangle. \end{aligned}$$

Theorem

For a convenient choice of $1 >> \eta > 0$ there holds (with explicit constant)

$$D(f) \gtrsim |||f|||^2 \simeq \|f\|_{\mathcal{H}}^2, \quad \forall\, f,\ \langle \pi f \rangle = 0.$$

$\Delta^{-1} :=$ solution to the Poisson equation with periodic condition.
We split $D = D_0 + D_1 + D_2$.

- We have

$$D_0 := (-\mathcal{L}f, f)_{L^2(M^{-1})} = \|f^\perp\|^2_{\mathcal{H}}$$

- We compute

$$\begin{aligned} j[\mathcal{L}f] &= \langle v\mathcal{T}\pi f\rangle + \langle v\mathcal{L}f^\perp\rangle \\ &= -\nabla_x\rho + \nabla_x\langle v\otimes v\, f^\perp\rangle + \langle vf^\perp\rangle, \end{aligned}$$

so that

$$\begin{aligned} D_1 &:= \eta(\nabla_x\Delta^{-1}\rho_f, j[\mathcal{L}f]) \\ &:= \eta\big(\nabla_x\Delta^{-1}\rho_f, -\nabla_x\rho + \nabla_x\langle v\otimes v\, f^\perp\rangle + \langle vf^\perp\rangle\big) \\ &\gtrsim \eta\|\rho_f\|^2_{L^2} - \eta\|\rho_f\|_{L^2}\,\|f^\perp\|_{\mathcal{H}}. \end{aligned}$$

- Similarly

$$\rho[\mathcal{L}f] = \langle\mathcal{T}\pi f\rangle + \langle\mathcal{L}f^\perp\rangle = -\nabla_x\langle vf^\perp\rangle,$$

so that

$$D_2 := \eta(\nabla_x\Delta^{-1}\rho[\mathcal{L}f], j_f) = -\eta(\nabla_x\Delta^{-1}\nabla_x\langle vf^\perp\rangle, j_f) \gtrsim -\eta\|f^\perp\|^2_{\mathcal{H}}$$

We rather define $\Delta^{-1} := \Delta_V^{*-1}$, Δ_V^* stands for the modified Laplacian operator

$$\Delta_V^* u \quad := \quad \Delta u - \nabla V \cdot \nabla u = e^V \nabla (e^{-V} \nabla u),$$

and the twisted L^2 scalar product

$$((f,g)) \quad = \quad (f,g)_{\mathcal{H}} - \eta(\nabla \Delta_V^{*-1}(\rho_f e^V), j_g)_{L^2} - \eta(j_f, \nabla \Delta_V^{*-1}(\rho_g e^V))_{L^2}.$$

We compute

$$\begin{aligned} j[-\mathcal{L}f] \quad &= \quad j[-\mathcal{T}\pi f] + \dots \\ &= \quad j[v \cdot \nabla_x \rho_f M - \nabla_x V \cdot \nabla_v \rho_f M] + \dots \\ &= \quad j[Mv \cdot (\nabla_x \rho_f + \nabla_x V \rho_f)] + \dots \\ &= \quad \nabla_x \rho_f + \nabla_x V \rho_f = e^{-V} \nabla (\rho_f e^V) + \dots \end{aligned}$$

We deduce that the leader term in D_1 is

$$\begin{aligned} D_{1,1} \quad &:= \quad -\eta(\nabla \Delta_V^{*-1}(\rho_f e^V), j[-\mathcal{T}\pi f])_{L^2} \\ &= \quad -\eta(\nabla \Delta_V^{*-1}(\rho_f e^V), e^{-V} \nabla (\rho_f e^V))_{L^2} \\ &= \quad \eta(e^V \nabla (e^{-V} \nabla \Delta_V^{*-1}(\rho_f e^V)), \rho_f)_{L^2} \\ &= \quad \eta \, \|\rho_f\|^2_{L^2(e^V)}. \end{aligned}$$

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We consider the same linearized kinetic Boltzmann/Landau operator

$$\mathcal{L} := \mathcal{S} + \mathcal{T}$$

where $\mathcal{S}$ is the linearized homogeneous Boltzmann/Landau operator defined through

$$\mathcal{S} f := Q(f, M) + Q(M, f)$$

and $\mathcal{T}$ is the transport operator

$$\mathcal{T} f := -v \cdot \nabla_x f + \nabla_x V \cdot \nabla_v f, \quad \Omega := \mathbb{R}^d, \; d := 3,$$

associated to a potential $V = \langle x \rangle^\gamma$, $\gamma \ge 1$.

The difficulty comes from the dimension $(= 5)$ of the null space $N(\mathcal{S})$. We define

$$\begin{aligned} a &:= a_f = a[f] = \langle f \rangle, \\ b &:= b_f = b[f] = \langle f v \rangle, \\ c &:= c_f = c[f] = \langle f (|v|^2 - 3)/6 \rangle, \end{aligned}$$

and the orthogonal projection operator on $N(\mathcal{S})$ by

$$\pi f := a M + b \cdot v M + c \, (|v|^2 - 3) M.$$

The macroscopic conservations are

$$\int f\,(1, v, x \wedge v, |v|^2)\, dvdx = 0,$$

and in particular

$$\int a\,dx = \int b\,dx = \int c\,dx = \int b \wedge x\,dx = 0.$$

We compute

$$\begin{cases} a[\mathcal{L}f] = -\nabla_x \cdot b \\ b[\mathcal{L}f] = -(\nabla_x a + \nabla_x V a) - \frac{1}{3}\nabla_x c - \nabla_x \cdot \Gamma[f^\perp] \\ c[\mathcal{L}f] = -2(\nabla_x \cdot b + \nabla_x V \cdot b) - \nabla_x E[f^\perp] \\ \Gamma[(\mathcal{L}f)^\perp] + \frac{1}{3} I_d c[\mathcal{L}f] = -\{\nabla_x^s b + (\nabla_x V \otimes b)^s\} + \Gamma[\mathcal{L}f^\perp] \\ E[(\mathcal{L}f)^\perp] = -\frac{1}{3}(\nabla_x c + \nabla_x V c) + E[\mathcal{L}f^\perp], \end{cases}$$

where we define

$$\Gamma_{ij}[f] := \langle (v_i v_j - 1) f \rangle, \quad E_i[f] := \langle v_i(|v|^2 - 5) f \rangle, \quad A^s := \frac{1}{2}(A + A^T),$$

We define the twisted L^2 using macroscopic correction norm

$$|||f|||^2 := \|f\|^2_{L^2(G^{-1})} + \eta_a(\nabla\Delta_V^{*-1}[ae^V], b)_{L^2_x}$$
$$+\eta_b(\nabla^s \widetilde{be^V}, \Gamma^\perp + \tfrac{c}{3} I_d)_{L^2_x} + \eta_c(\nabla\Delta_V^{*-1}[ce^V], E^\perp)_{L^2_x}$$

where for a given $u \in (L^2(e^{V/2}))^3$ we define $\tilde{u} \in \tilde{H}^1$ as the solution to the elliptic problem

$$(\nabla^s \tilde{u}, \nabla^s w)_{L^2_x(e^{-V})} = (u, w)_{L^2_x(e^{-V})}, \quad \forall w \in \tilde{H}^1,$$

with

$$\tilde{H}^1 := \{w \in (L^2(e^{-V}))^3,\ \nabla^s w \in (L^2(e^{-V}))^{3\times 3},\ \langle w_i e^{-V}\rangle = \langle (x_i w_j - x_j w_i) e^{-V}\rangle = 0,\ \forall i,j\}.$$

Theorem. ([Carrapatoso, M.])

For a convenient choice of (η_i) the associated Dirichlet form satisfies

$$D(f) \gtrsim |||f|||^2,$$

for any f satisfying the macroscopic conservations (with explicit constants).

Lemma. ([Carrapatoso, M.])

For any $u \in (L^2(e^V))^3$ such that $\langle u_i e^{-V} \rangle = \langle (x_i u_j - x_j u_i) e^{-V} \rangle = 0$ there exists a unique $\tilde{u} \in \tilde{H}^1$ solution to the elliptic problem

$$(\nabla^s \tilde{u}, \nabla^s w)_{L^2_x(e^{-V})} = (u, w)_{L^2_x(e^{-V})}, \quad \forall w \in \tilde{H}^1.$$

This one satisfies in particular

$$(\nabla^s \tilde{u}, \nabla^s u)_{L^2_x(e^{-V})} = \|u\|^2_{L^2_x(e^{-V})}.$$

Recalling that

$$(\Gamma^\perp + \tfrac{c}{3}\mathrm{Id})(\mathcal{L} f) \quad = \quad -\nabla^s (b e^V) e^{-V} + \mathcal{O}(\|f^\perp\|),$$

we have

$$\begin{aligned} D(f) \quad &= \quad \dots + \eta_b (\nabla^s \widetilde{be^V}, (\Gamma^\perp + \tfrac{c}{3}\mathrm{Id})(-\mathcal{L} f))_{L^2_x} + \dots \\ &= \quad \dots + \eta_b (\nabla^s \widetilde{be^V}, \nabla^s (b\, e^V))_{L^2_x(e^{-V})} + \dots \\ &\gtrsim \quad \dots + \eta_b \|b\|^2_{L^2_x(e^V)} + \dots \end{aligned}$$

The auxiliary result is based on a quantified Korn's inequality.

Lemma. ([Carrapatoso, M.])

There holds

$$\int |\nabla u|^2 e^{-V} \lesssim \int |\nabla^s u|^2 e^{-V},$$

for any u such that $\langle u_i e^{-V} \rangle = \langle (x_i u_j - x_j u_i) e^{-V} \rangle = 0$
or with explicit constants when u is such that $\langle u_i e^{-V} \rangle = \langle (u \otimes \nabla V)^a e^{-V} \rangle = 0$.

Remark. Let us recall

$$|\nabla^s u|^2 - |\nabla^a u|^2 = (\nabla \cdot u)^2 + \nabla[(u \cdot \nabla) u - u(\nabla \cdot u)], \quad |\nabla u|^2 = |\nabla^s u|^2 + |\nabla^a u|^2,$$

from what foolow

$$\int |\nabla u|^2 \leq 2 \int |\nabla^s u|^2$$