Villani's program on constructive rate of convergence to the equilibrium

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Outline of the talk

1. Introduction and main result
   - Boltzmann and Landau equation
   - Linearization and entropy approaches
   - Villani's program
2. First step: quantitative coercivity estimates
   - Boltzmann and Landau operators
   - A flavour of the proof : the Fokker-Planck equation
3. Second step: quantitative hypocoercivity estimates
   - H^1 hypocoercivity
   - L^2 hypocoercivity
4. Third step: Change of the functional space
5. Fourth step: the nonlinear problem

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Consider the Boltzmann/Landau equation

$$\partial_t F + v \cdot \nabla_x F = Q(F,F)$$
$$F(0,.) = F_0$$

on the density of the particle $F = F(t,x,v) \geq 0$, time $t \geq 0$, velocity $v \in \mathbb{R}^3$, position $x \in \Omega$

$\Omega = \mathbb{T}^3$ (torus);

$\Omega \subset \mathbb{R}^3$ + boundary conditions;

$\Omega = \mathbb{R}^3$ + force field confinement.

Q = nonlinear (quadratic) Boltzmann or Landau collisions operator
: conservation of mass, momentum and energy

Around the H-theorem

We recall that $\varphi = 1, v, |v|^2$ are collision invariants, meaning

$$\int_{\mathbb{R}^3} Q(F,F)\varphi\, dv = 0, \quad \forall F.$$

$\Rightarrow$ laws of conservation

$$\int_{\mathbb{T}^3\times\mathbb{R}^3} F \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} = \int_{\mathbb{T}^3\times\mathbb{R}^3} F_0 \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$$

We also have the H-theorem on the Boltzmann's entropy, namely

$$\int_{\mathbb{R}^3} Q(F,F)\log F \begin{cases} \leq 0 \\ = 0 \;\Rightarrow\; F = \text{Maxwellian}, \end{cases}$$

so that in particular the Boltzmann's entropy

$$\mathcal{H}(F) = \int_{\mathbb{T}^3\times\mathbb{R}^3} F \log F\, dvdx$$

is a Lyapunov functional for the Boltzmann equation.

From both pieces of information, we expect

$$F(t,x,v) \underset{t\to\infty}{\longrightarrow} M(v) := \frac{1}{(2\pi)^{3/2}}\, e^{-|v|^2/2}.$$

• space homogeneous Boltzmann equation by Hilbert, Weyl, Carleman, Grad (1912-58)
Spectral properties in $L^2(M^{-1})$ of the linear(ized) operator

$$\mathcal{S}f := Q(f, M) + Q(M, f).$$

It is self-adjoint and a compact perturbation of a dissipative operator. One gets non constructive spectral gap, exponential stability at the linear level and then exponential stability at the nonlinear level in a perturbative (close to the equilibrium) regime

At the linearized level the H-theorem writes

$$D(f) := \int_{\mathbb{R}^3} (-\mathcal{S}f) f \, M^{-1} \, dv \geq 0,$$

that we can prove directly from the expression of $\mathcal{S}$ or by passing to the limit in the H-Theorem

$$-\int_{\mathbb{R}^3} Q(M + \varepsilon f, M + \varepsilon f) \log(M + \varepsilon f) \, dv \geq 0$$

• space inhomogeneous Boltzmann equation by Ukai, Vidav, Arekeryd, Espositio, Pulvirenti (1974-87).
Spectral properties in $L^2(\mathbb{T}^3 \times \mathbb{R}^3, M^{-1} dvdx)$ of the linear(ized) operator

$$\mathcal{L}f := \underbrace{-v \cdot \nabla_x f}_{=:\mathcal{T}f} + \underbrace{Q(f,M) + Q(M,f)}_{=:\mathcal{S}f}$$

It is not self-adjoint anymore and but still one gets non constructive spectral gap, exponential stability at the linear level and then exponential stability at the nonlinear level in a perturbative (close to the equilibrium) regime.
The proof is not really clean!

• space homogeneous Boltzmann equation by Arkeryd and next sweeden, french and italian schools with new impulse given by Carlen-Carvalho and Toscani-Villani works (1970-2000')

For any initial datum $F_0 \geq 0$ with finite mass, energy and entropy there exists a unique solution $F(t)$ to the homogeneous Boltzmann equation and this one satisfies

$$\frac{d}{dt}\mathcal{H}(F(t)|M) = -\mathcal{D}(F(t)) \leq -\Theta(\mathcal{H}(F(t)|M)) \leq 0,$$

with $\mathcal{H}(F|M) := \mathcal{H}(F) - \mathcal{H}(M)$, which in turns implies

$$(1) \qquad \|F(t) - M\|_{L^1}^2 \lesssim \mathcal{H}(F(t)|M) \lesssim t^{-1/\varepsilon}, \quad \forall\, t \geq 0, \forall\, \varepsilon > 0,$$

with constructive constants.

▷ relies on convexity and log-Sobolev inequality trick

• space inhomogeneous Boltzmann equation by DiPerna-Lions, ..., Desvillettes-Villani (1989-2000')
For any initial datum $F_0 \geq 0$ with finite mass, energy and entropy there exists at least one renormalized solution $F(t)$ to the space inhomogeneous Boltzmann equation.
If furthermore this one satitisfies the (unverified) strong estimates

$$\sup_{t\geq 0} \left(\|F_t\|_{H^k} + \|F_t\|_{L^1(1+|v|^s)} \right) \leq C_{s,k} < \infty,$$

then (1) is true again with constructive constants.
$\triangleright$ relies on previous entropy-dissipation entropy inequality together with "nonlinear hypocoercivity" trick (in order to track the mixing effect in space variable).

• Issues:

$\triangleright$ (1) is really weaker from exponential convergence established by linearization;
$\triangleright$ even in the space homogeneous framework $L^2(M^{-1})$ is not an appropriate space for the well-posedness thoery in a far from equilibrium regime;
$\triangleright$ the $L^2(M^{-1})$ is not based on constructive estimates.

Here is Villani's program (Notes on 2001 IHP course, Section 8. Toward exponential convergence)

1. Find a constructive method for bounding below the spectral gap in $L^2(M^{-1})$, the space of self-adjointness, say for the Boltzmann operator with hard spheres.

3. Find a constructive argument to overcome the degeneracy in the space variable, to get an exponential decay for the linear semigroup associated with the linearized spatially inhomogeneous Boltzmann equation; something similar to hypo-ellipticity techniques.

2. Find a constructive argument to go from a spectral gap in $L^2(M^{-1})$ to a spectral gap in L^1, with all the subtleties associated with spectral theory of non-self-adjoint operators in infinite dimension ...

4. Combine the whole things with a perturbative and linearization analysis to get the exponential decay for the nonlinear equation close to equilibrium.

Other problems (not dealt here):

- The case $\mathfrak{h}_* \not\subset \mathfrak{h}$
- Many collision models (Boltzmann, Landau, ...) and geometry (torus, ...)
- The whole space with weak confinement
- The whole space without any confinement
- uniform estimate in the macroscopic limit
- uniform estimate in the grazing collisions limit

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We define the linearized Boltzmann / Landau operator in the space homogeneous framework

$$\mathcal{S}f := \frac{1}{2}\Big\{Q(f,M) + Q(M,f)\Big\}$$

and the orthogonal projection π in $L^2(M^{-1})$ on the eigenspace

$$\mathrm{Span}\{(1, v, |v|^2)M\}.$$

Theorem 1. (..., Guo, Mouhot, Strain)

There exist two Hilbert spaces $\mathfrak{h} = L^2(M^{-1})$ and $\mathfrak{h}_*$ and <u>constructive</u> <u>constants</u> $\lambda, K > 0$ such that

$$(-\mathcal{S}h, g)_{\mathfrak{h}} = (-\mathcal{S}g, h)_{\mathfrak{h}} \leq K \|g\|_{\mathfrak{h}_*} \|h\|_{\mathfrak{h}_*}$$

and

$$(-\mathcal{S}h, h)_{\mathfrak{h}} \geq \lambda \, \|\pi^\perp h\|^2_{\mathfrak{h}_*}, \quad \pi^\perp = I - \pi$$

The space $\mathfrak{h}_*$ depends on the operator (linearized Boltzmann or Landau) and the interaction parameter $\gamma \in [-3, 1]$, $\gamma = 1$ corresponds to (Boltzmann) hard spheres interactions and $\gamma = -3$ corresponds to (Landau) Coulomb interactions.

Consequence of the coercivity estimate

We define

$$f^\perp = \pi^\perp f = (I - \pi) f$$

and we compute

$$\partial_t f^\perp = \mathcal{S} f^\perp.$$

From the positivity of the Dirichlet form, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|f^\perp\|^2 &= (\mathcal{S} f^\perp, f^\perp) \\ &\le -\lambda\, \|f^\perp\|^2. \end{aligned}$$

From the Gronwall lemma, we deduce the exponential stability of the equilibrium

$$\|f(t) - \pi f_0\| \le e^{-\lambda t} \|f_0 - \pi f_0\|.$$

Comments on Theorem 1

• Takes roots in Hilbert, Weyl, Carleman and Grad (non constructive) spectral analysis for the linearized Boltzmann operator

• Degond-Lemou (non constructive) spectral analysis for the linearized Landau operator

• Constructive by Wang Chang et al & Bobylev for Boltzmann operator ($\gamma = 0$) through Hilbert basis decomposition

• Constructive by Desvillettes-Villani for Landau operator ($\gamma = 0$) through log-Sobolev inequality and linearization of the entropy-dissipation of entropy inequality.

• Proved by Mouhot and collaborators (Baranger, Strain) in any cases $\gamma \in [-3, 1]$

• Our aim is to present a new and comprehensive proof :

- Integration by part for Landau operator when $\gamma = 0$
- Integration along the Ornstein-Uhlenbeck flow when $\gamma \sim 0$ (a trick already used by Toscani & Villani in a nonlinear context)
- strictly positive (but not sharp) estimates
- sharp (but not strictly positive) estimates

Comments on Theorem 1 - Previous proof

• Linearized Boltzmann operator (first)

[1] Wang Chang et al 70, Bobylev 88, $\gamma = 0$, L^2 estimate (direct Fourier analysis).

[2] Baranger-Mouhot 05, $\gamma > 0$, L^2 estimate (from [1] - intermediate collisions).

[3] Mouhot 06, $\gamma \in (-3, 1]$, L^2_γ estimate (from [1] for $\gamma < 0$ and [2] for $\gamma > 0$).

• Linearized Landau operator (next)

[4] Desvillettes-Villani 01, $\gamma = 0$, $H^1_{*,0}$ estimate (directly by linearization of nonlinear log-Sobolev inequality).

[5] Baranger-Mouhot 05, $\gamma \geq 0$, L^2 estimate (from [2] - grazing collisions).

[6] Mouhot 06, $\gamma \in (-3, 1]$, H^1_γ estimate (from [4,5] for $\gamma < 0$ and [5] for $\gamma > 0$).

[7] Mouhot-Strain 07, $\gamma \in (-3, 1]$, $H^1_{\gamma,*}$ estimate (from [6]).

Comments on Theorem 1 - scheme of our proof

- Linearized Landau operator (first)

(1) $\gamma = 0$, identity

(2) $\gamma > 0$, from (1) and splitting argument

(3) $\gamma < 0$, from (1) and splitting argument

- Linearized Boltzmann operator (next)

(4) $\gamma \in [0, \gamma^*]$, $\gamma^* > 0$, from (3) associated to $\gamma - 2$ by integration along the flow of the Ornstein-Uhlenbeck semigroup

(5) $\gamma > \gamma^*$, from (4) and splitting argument

(6) $\gamma < 0$, from (4) and splitting argument

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(5) $\gamma > \gamma^*$, from (4) and splitting argument

(6) $\gamma < 0$, from (4) and splitting argument

For the Fokker-Planck equation

$$\partial_t f = \mathcal{S} f := \Delta f + \mathrm{div}(v f)$$

one can show:

- mass conservation;
- equilibrium are Maxwellian;
- the operator is self-adjoint;
- the Dirichlet form is positive or zero
  $\Leftrightarrow$ the $L^2(M^{-1})$ norm is a Lyapunov function;
- the Dirichlet form is ("strictly") positive
  $\Rightarrow$ the $L^2(M^{-1})$ norm is a strict Lyapunov function
  $\Leftrightarrow$ $\mathcal{S}$ has a spectral gap.

Outline of the talk

In a Hilbert space $\mathcal{H}$, we consider now an operator

$$\mathcal{L} = \mathcal{S} + \mathcal{T}$$

with

$$\mathcal{S}^* = \mathcal{S} \le 0, \quad \mathcal{T}^* = -\mathcal{T}.$$

More precisely, $\mathcal{H} \supset \mathcal{H}_x \otimes \mathcal{H}_v$, $\mathcal{S}$ acts on the v variable space $\mathcal{H}_v$ with null space $N(\mathcal{S})$ of finite dimension, we denote π the projection on $N(\mathcal{S})$.
As a consequence, in the two variables space $\mathcal{H}$ the operator $\mathcal{S}$ is degenerately / partially coercive

$$(-\mathcal{S}f, f) \gtrsim \|f^\perp\|_*^2, \quad f^\perp = f - \pi f$$

For the initial Hilbert norm, we get the same degenerate / partial positivity of the Dirichlet form

$$D[f] := (-\mathcal{L}, f) \gtrsim \|f^\perp\|_*^2, \quad \forall\, f.$$

That information is not strong enough in order to control the longtime behavior of the dynamic of the associated semigroup !!

▷ Find a new Hilbert norm by twisting

$$|||f|||^2 := \|f\|^2 + 2(Af, Bf)$$

such that the new Dirichlet form is coercive:

$$\begin{aligned} D[f] &:= ((-\mathcal{L}f, f)) \\ &= (-\mathcal{L}f, f) + (A\mathcal{L}f, Bf) + (Af, B\mathcal{L}f) \\ &\gtrsim \|f^\perp\|^2 + \|\pi f\|^2. \end{aligned}$$

▷ We destroy the nice symmetric / skew symmetric structure and we have also to be very careful with the "remainder terms".
▷ That functional inequality approach is equivalent (and more precise if constructive) to the other more dynamical approach (called "Lyapunov" or "energy" approach).

Theorem 2. (for strong coercive operators in both variables, in particular $\mathfrak{h}_* \subset \mathfrak{h}$)

There exist some new but equivalent Hilbert norm $|||\cdot|||$ and a (constructive) constant $\lambda > 0$ such that the associated Dirichlet form satisfies

$$D[f] \ge \lambda\, |||f|||^2, \quad \forall\, f,\ \langle \pi f\rangle = 0.$$

▷ It implies $|||e^{\mathcal{L}t} f||| \le e^{-\lambda t} |||f|||$ and then $\|e^{\mathcal{L}t} f\| \le C e^{-\lambda t}\|f\|$, $\forall\, f,\ \langle \pi f\rangle = 0$.

- Fourier approach and hypocoercivity : Kawashima
- Non constructive spectral analysis approach : Ukai (1974), Arkeryd, Esposito, Pulvirenti (1987), Wennberg (1995)
- Non constructive estimate and hypoellipticity : Eckmann, Pillet, Rey-Bellet (1999)
- Constructive entropy approach: Desvillettes-Villani (2001-2005)
- Energy (in high order Sobolev space) approach : Guo and Guo' school [2002-..]
- Micro-Macro approach : Shizuta, Kawashima (1984), Liu, Yu (2004), Yang, Guo, Duan, ...
- Constructive estimate and hypoellipticty : Hérau, Nier, Helffer, Eckmann, Hairer (2003-2005), Villani (2009)

  2006–2015: hypocoercivity estimates with Neumann, Dolbeault and Schmeiser
- Carrapatoso, M., Landau equation in the torus, 2017
- Carrapatoso, M., Landau equation in the whole space, work in progress

Several issues

Geometry of the domain:

- the torus
- the whole space with confinement force
- bounded domain

Collisions operator

- elliptic operator (Fokker-Planck operator)
- relaxation operator (no additional derivative)
- linearized Boltzmann/Landau : more than one invariant (velocity)

Steps

- H^1 estimate : torus and Fokker-Planck in the whole space
- macroscopic projection : domain and relaxation operator in the whole space
- H^1+ micro-macro decomposition : Boltzmann in the whole space
- micro-macro decomposition : Boltzmann in the whole space

We consider

$$\mathcal{L} := \mathcal{S} + \mathcal{T}$$

for "any" linear collision term $\mathcal{S}$ "of hard potential type" and

$$\mathcal{T} g := -v \cdot \nabla_x g, \quad \Omega := \mathbb{T}^d.$$

We work in the flat space L^2. We define the twisted H^1 norm

$$|||g|||^2 := \|g\|_{L^2}^2 + \eta_x \|\nabla_x g\|_{L^2}^2 + 2\eta(\nabla_v g, \nabla_x g)_{L^2} + \eta_v \|\nabla_v g\|_{L^2}^2,$$

by choosing $\eta^2 < \eta_x \eta_v$ and then the Dirichlet form

$$\begin{aligned} D(g) &= ((-\mathcal{L} g, g)) \\ &= (-\mathcal{L} g, g) - \eta_x (\nabla_x \mathcal{L} g, \nabla_x g) \\ &\quad -\eta(\nabla_v \mathcal{L} g, \nabla_x g) - \eta(\nabla_v g, \nabla_x \mathcal{L} g) - \eta_v (\nabla_v \mathcal{L} g, \nabla_v g). \end{aligned}$$

Theorem 2′. ([Villani 2009] after [Mouhot, Neuman 2006])

For convenient choices of $1 \ge \eta_x > \eta > \eta_v > 0$ there holds (with explicit constants)

$$D(g) \gtrsim \|g\|_{H^1_{xv}}^2 \gtrsim |||g|||^2, \quad \forall g, \ \langle \pi g \rangle = 0.$$

A possible choice is $\eta_x = 1$, $\eta = \varepsilon^2$, $\eta_v = \varepsilon^3$, $\varepsilon > 0$ small enough.

- The crucial information comes from the third term (in blue). More precisely, throwing away the contribution of the collision operator $\mathcal{S}$, we compute:

$$
\begin{aligned}
D_{3,1} &:= -\eta(\nabla_v \mathcal{T} g, \nabla_x g) - \eta(\nabla_v g, \nabla_x \mathcal{T} g) \\
&= -\eta(\nabla_v \mathcal{T} g, \nabla_x g) - \eta(\nabla_v g, \mathcal{T} \nabla_x g) \quad \text{because } [\mathcal{T}, \nabla_x] = 0 \\
&= \eta([\nabla_v, -\mathcal{T}] g, \nabla_x g) \\
&= \eta(\nabla_x g, \nabla_x g) \\
&= \eta \|\nabla_x g\|^2.
\end{aligned}
$$

- Another key remark is that for any g such that $\langle \pi g \rangle = 0$, we have

$$D_{3,1} = \|\nabla_x g\|^2 \gtrsim \|\nabla_x \pi g\|^2 \gtrsim \|\pi g\|^2,$$

where we have used the Poincaré(-Wirtinger) inequality in the torus in the last inequality. Together with the first term

$$D_1 = (-\mathcal{L} g, g) = (-\mathcal{S} g, g) \geq \|g^\perp\|_*^2 \geq \|g^\perp\|^2,$$

we get

$$D(g) \gtrsim \ldots + \|g^\perp\|^2 + \|\pi g\|^2 = \ldots + \|g\|_{L^2}^2.$$

We consider the linearized Boltzmann/Landau operator

$$\mathcal{L} := \mathcal{S} + \mathcal{T}$$

where

$$\mathcal{T} f := -v \cdot \nabla_x f, \quad x \in \Omega \subset \mathbb{R}^3 \text{ bounded},$$

with boundary condition
- diffusion reflection;
- specular reflection;
- Maxwell reflection (a mix of both).

For simplicity, we rather consider the case of the torus but the proof may be adapted to a pure diffusion or a Maxwell reflection (not clear for a pure specular reflection).

The difficulty comes from the dimension $(= 5)$ of the null space $N(\mathcal{S})$. We define

$$\begin{aligned} a &:= a_f = a[f] = \langle f \rangle =: \bar\pi_0 f = \bar\pi_0, \\ b &:= b_f = b[f] = \langle f v \rangle =: (\bar\pi_\beta f)_{1 \le \beta \le 3} = (\bar\pi_\beta)_{1 \le \beta \le 3}, \\ c &:= c_f = c[f] = \langle f (|v|^2 - 3)/6 \rangle =: \bar\pi_4 f = \bar\pi_4, \end{aligned}$$

and the orthogonal projection operator on $N(\mathcal{S})$ by

$$\pi f := aM + b \cdot vM + c\,(|v|^2 - 3) M = \sum_{\beta=0}^{4} \hat\varphi_\beta \bar\pi_\beta, \quad \hat\varphi_\beta = \varphi_\beta M.$$

We define the twisted L^2 norm

$$|||f|||^2 := \|f\|^2_{\mathcal{H}} + 2\eta(\bar\pi[f], \Delta^{-1}\nabla\widetilde\pi[f])_{L^2}$$

where the last term is a shorthand for

$$\sum_{\alpha,k} 2\eta_\alpha(\bar\pi_\alpha, \Delta^{-1}\partial_{x_k}\widetilde\pi_{\alpha k})$$

and the macroscopic quantities

$$\bar\pi_\alpha := \langle f\varphi_\alpha M\rangle, \qquad \widetilde\pi_{\alpha k} := \langle f\widetilde\varphi_{\alpha k}\rangle.$$

We define the Dirichlet form

$$D(f) \quad = \quad (-\mathcal{L}f, f) - \eta(\widetilde\pi[\mathcal{L}f], \nabla\Delta^{-1}\bar\pi[f]) - \eta(\widetilde\pi[f], \nabla\Delta^{-1}\bar\pi[\mathcal{L}f]).$$

Theorem 2″. (inspired from [Guo, Briant 2010, 2016])

For a convenient choice of $(\widetilde\varphi_{\alpha k})$ and (η_α) there holds (with explicit constants)

$$D(f) \gtrsim \|f\|^2_{\mathcal{H}} \gtrsim |||f|||^2, \quad \forall\, f,\ \langle\pi f\rangle = 0.$$

We consider the kinetic Fokker-Planck operator in the torus

$$\mathcal{L} := \mathcal{S} + \mathcal{T}$$

where $\mathcal{T}$ is the free transport operator

$$\mathcal{T} f := -v \cdot \nabla_x f, \quad \Omega := \mathbb{T}^d,$$

and $\mathcal{S}$ is the "standard" Fokker-Planck operator or relaxation operator

$$\mathcal{S} f := \Delta_v f - \mathrm{div}(v f) \quad \text{or} \quad \mathcal{S} f = \rho M - f.$$

We introduce the Hilbert norm

$$|||f|||^2 := \|f\|^2_{\mathcal{H}} + 2\eta(\rho, \nabla_x \Delta_x^{-1} j)_{L^2(e^{V/2})}$$

with $1 >> \eta > 0$ and then the Dirichlet form

$$\begin{aligned} D(f) &= ((-\mathcal{L} f, f)) \\ &= (-\mathcal{L} f, f) - \eta(\rho_f, \Delta_x^{-1} \nabla_x j[\mathcal{L} f]) - (\rho[\mathcal{L} f], \Delta_x^{-1} \nabla_x j_f). \end{aligned}$$

Here

$$\begin{aligned} \rho &:= \rho_f = \rho[f] = \langle f \rangle, \\ j &:= j_f = j[f] = \langle f\, v \rangle. \end{aligned}$$

We compute

$$\begin{aligned} j[\mathcal{L}f] &= \langle v\mathcal{T}\pi f\rangle + \langle v\mathcal{L}f^\perp\rangle \\ &= -\nabla\rho_f + \langle v\mathcal{L}f^\perp\rangle. \end{aligned}$$

Coming back to the Dirichlet term we get

$$\begin{aligned} D(f) &= (-\mathcal{L}f, f) - \eta(\rho_f, \Delta_x^{-1}\nabla_x j[\mathcal{L}f]) + \ \dots \\ &\geq \lambda\|f^\perp\|^2 - \eta(\rho_f, \Delta_x^{-1}\nabla_x(-\nabla\rho_f)) + \ \dots \\ &= \lambda\|f^\perp\|^2 + \eta\|\rho_f\|^2 + \ \dots \end{aligned}$$

and the other terms are $\mathcal{O}(\|\rho_f\|\|f^\perp\| + \|f^\perp\|^2)$.

Theorem 2‴. ([Dolbeault, Mouhot, Schmeiser 2015] after [Hérau 2006])

For a convenient choice of $1 >> \eta > 0$ there holds (with explicit constants)

$$D(f) \gtrsim \|f\|^2_{\mathcal{H}} \gtrsim |||f|||^2, \quad \forall f,\ \langle\pi f\rangle = 0.$$

Outline of the talk

Consider E Banach space, L the generator of a semigroup S_L

Spectral analysis of L
- localization of the spectrum $\Sigma(L)$
- eigenspace + eigenspace projector
- growth estimate on semigroup e^{tL}

$=$ L and e^{tL} have some spectral gap

$=$ L satisfies principal spectral mapping theorem

(spectral mapping theorem $\Sigma(e^{tL}) = e^{t\Sigma(L)}$ holds FP equation)

Main question:

prove the same "spectral properties" on a
larger Banach space $\mathcal{E} \supset E$ or smaller Banach space $\mathcal{E} \subset E$?

We split the operator as

$$\mathcal{L} = \mathcal{A} + \mathcal{B}, \quad \mathcal{A} \prec \mathcal{B}$$

We write the Duhamel formula

$$S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{L}} * \mathcal{A} S_{\mathcal{B}} \quad \text{or} \quad S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{B}} \mathcal{A} * S_{\mathcal{L}},$$

with the following definition of the convolution

$$U * V = (U * V)(t) := \int_0^t U(t-s) V(s)\, ds.$$

When $E \subset \mathcal{E}$ (extension), we iterate the first Duhamel formula to get

$$S_{\mathcal{L}} = S_{\mathcal{B}} + ... + S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{B}})^{(n-1)} + S_{\mathcal{L}} * (\mathcal{A} S_{\mathcal{B}})^{(*n)}$$

with

$$U^{(*1)} = U, \quad U^{(*\ell)} = U^{(*(\ell-1))} * U.$$

For some $a < 0$, we establish dissipation estimates

$$\forall\, \ell \geq 0, \quad \| S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{B}})^{(*(\ell-1))}(t) \|_{\mathcal{B}(\mathcal{E})} = \mathcal{O}(e^{at})$$

and a regularisation estimate

$$\exists\, n \geq 1, \quad \| (\mathcal{A} S_{\mathcal{B}})^{(*n)}(t) \|_{\mathcal{B}(\mathcal{E}, E)} = \mathcal{O}(e^{at}).$$

Denoting by Π_E the projection on the principale eigenspace associated to the eigenvalue 0 of L in $\mathcal{E}$, we prove that it extends to $\mathcal{E}$: there exists $\Pi : \mathcal{E} \to E$ such that

$$\Pi_E = \Pi_{|E}.$$

In the modified iterated Duhamel formula

$$\Pi^\perp S_{\mathcal{L}} = \Pi^\perp (S_{\mathcal{B}} + ... + S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{B}})^{(n-1)}) + \Pi_E S_L * (\mathcal{A} S_{\mathcal{B}})^{(*n)}$$

all the terms are

$$\mathcal{O}(\max(e^{at}, e^{-\lambda_L t}).$$

Outline of the talk

Existence, uniqueness and stability in small perturbation regime in large space and with constructive rate

Theorem 4. (Briant, Carrapatoso, Gualdani, Guo and Guo's school, M., Mouhot)

Take an "admissible" weight function m such that

$$\langle v \rangle^{2+3/2} \prec m \prec e^{|v|^2}.$$

There exist some Lebesgue or Sobolev space $\mathcal{E}$ associated with the weight m and some $\varepsilon_0 > 0$ such that if

$$\|F_0 - M\|_{\mathcal{E}(m)} < \varepsilon_0,$$

there exists a unique global solution F to the Boltzmann/Landau equation and

$$\|F(t) - M\|_{\mathcal{E}(\tilde{m})} \le \Theta_m(t),$$

with optimal rate

$$\Theta_m(t) \simeq e^{-\lambda t^\sigma} \text{ or } t^{-K}$$

with $\lambda > 0$, $\sigma \in (0,1]$, $K > 0$ depending on m and whether the interactions are "hard" or "soft".

Conditionally (up to time uniform strong estimate) exponential H-Theorem

Corollary. (Desvillettes-Villani + Gualdani-M.-Mouhot)

$\exists\, s_1, k_1$ s.t. for any $a > \lambda_2$ exists C_a

$$\forall\, t \geq 0 \qquad \int_{\Omega\times\mathbb{R}^3} F_t \log \frac{F_t}{M(v)}\, dvdx \leq C_a\, e^{\frac{a}{2}\, t},$$

with $\lambda_2 < 0$ (2^{nd} eigenvalue of the linearized Boltzmann eq. in $L^2(M^{-1})$).

Existence near the equilibrium and trend to the equilibrium (a general picture) :

- Ukai (1974), Arkeryd, Esposito, Pulvirenti (1987), Wennberg (1995): non-constructive method for HS Boltzmann equation in the torus
- Desvillettes, Villani (2001 & 2005) if-theorem by entropy method
- Villani, 2001 IHP lectures on "Entropy production and convergence to equilibrium" (2008)
- Guo and Guo' school (issues 1,2,3,4)

  2002–2008: high energy (still non-constructive) method for various models

  2010–...: Villani's program for various models and geometries
- Mouhot and collaborators (issues 1,2,3,4)

  2005–2007: coercivity estimates with Baranger and Strain

  2006–2015: hypocoercivity estimates with Neumann, Dolbeault and Schmeiser

  2006–2013: $L^p(m)$ estimates with Gualdani and M.
- Carrapatoso, M., Landau equation for Coulomb potentials, 2017

We write $F = M + f$ and we compute

$$\frac{1}{2}\frac{d}{dt}|||f|||^2_{\mathcal{X}} = \langle\langle \mathcal{L}f + Q(f,f), f^* \rangle\rangle_{\mathcal{X}},$$

for any nice norm $|||\cdot|||_{\mathcal{X}}$ and $f(0) = f_0$ such that $\Pi f_0 = 0$.

$\triangleright$ Q is quadratic and "unbounded" (lost of derivative or moment);
$\triangleright$ $\mathcal{L}$ is not disipative but it is hypodissipative.
We have

$$\langle\langle Q(f,f), f^* \rangle\rangle_{\mathcal{X}} \le \|Q(f,f)\|_{\mathcal{Z}} \|f\|_{\mathcal{Y}}, \quad \mathcal{Z} := \mathcal{Y}',$$
$$\|Q(f,g)\|_{\mathcal{Z}} \lesssim \|f\|_{\mathcal{X}} \|g\|_{\mathcal{Y}} + \|f\|_{\mathcal{Y}} \|g\|_{\mathcal{X}},$$

and

$$\langle\langle \mathcal{L}f, f^* \rangle\rangle_{\mathcal{X}} \lesssim -\|f\|^2_{\mathcal{Y}}$$

by changing the usual norm by an equivalent norm

$$|||f|||^2_{\mathcal{X}} = \|f\|^2_X + \int_0^\infty \|\Pi^\perp S_{\mathcal{L}}(\tau) f\|^2_X \, d\tau$$

All together, we have

$$\frac{d}{dt}|||f|||^2_{\mathcal{X}} \lesssim \|f\|^2_{\mathcal{Y}} (C|||f|||_{\mathcal{X}} - 1)$$

From

$$\frac{d}{dt}|||f|||_{\mathcal{X}}^2 \lesssim \|f\|_{\mathcal{Y}}^2 (C|||f|||_{\mathcal{X}} - 1),$$

we deduce

▷ the a priori uniform estimate $|||f|||_{\mathcal{X}} \le |||f_0|||_{\mathcal{X}}$ when $|||f_0|||_{\mathcal{X}}$ small, and then classically existence and uniqueness

▷ when $\mathcal{Y} \subset \mathcal{X}$ the equation simplifies

$$\frac{d}{dt}|||f|||_{\mathcal{X}}^2 \lesssim -\|f\|_{\mathcal{X}}^2,$$

and we obtain an exponential convergence to 0;

▷ considering two weight functions $m \succ \tilde{m}$, the above a priori estimate implies

$$\frac{d}{dt}|||f|||_{\tilde{\mathcal{X}}}^2 \lesssim -\|f\|_{\tilde{\mathcal{Y}}}^2, \quad \frac{d}{dt}|||f|||_{\mathcal{X}}^2 \lesssim 0,$$

and we get decay estimate (but not exponential).