On the long time asymptotic for the Growth-Fragmentation equation

a brief survey and a spectral analysis approach

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Outline of the talk

1. Introduction and brief survey
   - List of papers
   - Growth and fragmentation equation
   - Long time asymptotic
   - Main result: exponential rate of convergence
   - Several mathematical techniques
2. The spectral analysis approach
   - Spectral gap and semigroup decay
   - A Lyapunov condition
   - Some strong positivity conditions
   - the constant case
3. Open problems

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List of papers from the PDE community

- Escobedo, M., Rodriguez, *On self-similarity and stationary pb for frag. and coagulation models*, Annales IHP (2005)
- Michel, M., Perthame, *GRE inequality: an illustration on growth models*, JMPA (2005)
- Perthame, Ryzhik, *Exp. decay for the frag. or cell-division equation*, JDE (2005)
- Michel, *Existence of a solution to the cell division eigenproblem*, M3AS (2006)
- Laurençot, Perthame, *Exp. decay for the GF/cell-division equation*, CMS (2009)
- Doumic, Gabriel, *Eigenelements of a gal aggregation-fragmentation model*, M3AS (2010)
- Caceres, Cañizo, M., *Rate of convergence to an asymptotic profile for the self-similar fragmentation and GF equations*, JMPA (2011), CAIM (2011)
- Balagué, Cañizo, Gabriel, *Fine asymptotics of profiles and relaxation to equilibrium for growth-fragmentation equations with variable drift rates*, KRM (2013)
- M., Scher, *Spectral analysis of semigroups and GF equations*, Ann. IHP (2016)
- Doumic, Escobedo, *Time asymptotics for a critical case in fragmentation and GF equations*, KRM (2016)
- Bernard, Doumic, Gabriel, *Cyclic asymptotic behaviour of a population reproducing by fission into two equal parts*, arXiv 2016

And other papers by Banasiak, Bourgeron, Calvez, Doumic, Escobedo, Gabriel, Salvarani, ...

List of papers from the Probability community

- J. Bertoin, *Around self-similar asymptotic behavior of fragmentation processes*, PTRF (2001), Annales IHP (2002), JEMS (2003)
- B. Haas, *Around dust formation (or shattering) of fragmentation processes*, SPA (2003), Annales IHP (2004), C. Math. Sc. (2004), Ann. Appl. Probab (2010)
- J.-B. Bardet, A. Christen, A. Guillin, F. Malrieu, P.-A. Zitt, *Total variation estimates for the TCP process*, Electron. J. Probab. (2013)
- C. Goldschmidt, B. Haas, *Behavior near the extinction time in self-similar fragmentations I & II*, Annales IHP (2010), Ann. Probab. (2016)
- J. Bertoin, A.R. Watson, *Probabilistic aspects of critical growth-fragmentation equations*, Adv. Appl. Probab. (2016)
- J. Bertoin, A.R. Watson, *A probabilistic approach to spectral analysis of growth-fragmentation equations*, arXiv 2017

And other papers by Berestycki, Bertoin, Miermont, Stephenson, Watson, ...

We will consider

$$\partial_t f = \Lambda f = \mathcal{D} f + \mathcal{F} f$$

on $f = f(t, x) \geq 0$ the *number density of particles* (or *cells, polymers, organisms, individuals*),
$t \geq 0$ is the *time* variable,
$x \in (0, \infty)$ is the *size* (or *mass, age*)

We take into account a *fragmentation* mechanism through

$$(\mathcal{F} f)(x) := \int_x^\infty k(y, x) f(y) dy - K(x) f(x),$$

and possibly a *growth* mechanism by choosing $\mathcal{D} = 0$ or

$$(\mathcal{D} f)(x) := -\partial_x (\tau(x) f(x)) - \nu(x)\, f(x)$$

Fragmentation mechanism

The *fragmentation* operator writes

$$\mathcal{F} := \mathcal{F}^+ - \mathcal{F}^-, \quad (\mathcal{F}^+ f)(x) := \int_x^\infty k(y,x) f(y) dy, \quad (\mathcal{F}^- f)(x) := K(x) f(x),$$

with fragmentation kernel k and total rate of fragmentation K related by

$$K(x) = \int_0^x k(x,y) \, \frac{y}{x} \, dy.$$

Modeling the division (breakage) of a single *mother particle* of size $x > 0$ into two or more pieces (*daughter particles, offspring*) of size $x_i > 0$, conserving the mass

$$\{x\} \ \xrightarrow{k} \ \{x_1\} + ... + \{x_i\} + ... , \qquad x = \sum x_i.$$

We observe that

$$(\mathcal{F}^* \phi)(x) = \int_0^x k(x,y) \, [\phi(y) - \frac{y}{x} \, \phi(x)] \, dy.$$

As a consequence, for $\varphi_\alpha = x^\alpha$, there hold

$$\mathcal{F}^* \varphi_0 > 0, \quad \mathcal{F}^* \varphi_1 = 0, \quad \mathcal{F}^* \varphi_2 < 0.$$

$\Rightarrow$ Mass is conserved and particles are produced.

The *growth* operator models the growth (for particles and cells) or the aging (for individuals) and the death. It writes

$$(\mathcal{D}f)(x) := -\partial_x(\tau(x)f(x)) - \nu(x)\,f(x),$$

with drift speed (or *growth rate*) function $\tau : [0,\infty) \to \mathbb{R}$ and a damping rate $\nu : [0,\infty) \to [0,\infty)$. Schematically

$$\{x\} \stackrel{e^{-\nu(x)}}{\longrightarrow} \{x + \tau(x)\,dx\}.$$

Observing

$$(\mathcal{D}^*\phi)(x) = \tau(x)\partial_x\phi(x) - \nu(x)\,\phi(x),$$

the only invariant is

$$\phi(x) = \phi_0 e^{\int^x \nu(y)/\tau(y)dy} \neq \varphi_1$$

except when $\tau(x)/\nu(x) = x$. For instance: $(\mathcal{D}f)(x) := -\partial_x(xf(x)) - f(x)$

We will take $\tau(x) = 1$ (constant growth) or $\tau(x) = x$ (self-similar growth).

The growth-fragmentation operator generates a

positive and C_0-semigroup S_Λ in L^1.

From now, we exclude singular fragmentation kernels at the origine ($\Rightarrow$ shattering phenomenon = lost of mass) of the discussion

Questions:

- Is S^*_Λ Markov? = conservation law?
- $\exists$ of invariant measure for S^*_Λ? = $\exists$ of steady state for (GF) equation?

First answers:

- In general, no trivial conservation law (except φ_1 when $\mathcal{D} = 0$).
- In general, no explicit steady state.
- For the pure fragmentation equation ($\mathcal{D} = 0$) no steady state. But for a total rate $K(x) = x^\gamma$, $\gamma > 0$, we may change variables into "self-similar" variables (which adds a new growth operator $\mathcal{D}f := -\partial_x(xf) - f$) such that the new equation have a steady state (a self-similar profile for the initial equation). We denote this model as the (SSF) equation.

A complete answer thanks to Krein-Rutman (Perron-Frobenius) theory

Krein-Rutmann theory says

$$\exists\,(\lambda, G, \phi), \quad \lambda \in \mathbb{R},\ G > 0,\ \phi > 0,\ (\Lambda - \lambda)G = 0,\ (\Lambda^* - \lambda)\phi = 0.$$

- Finite dimensional approximation and compactness argument
  - Perthame, Ryzhik (2005), Michel (2006), Doumic, Gabriel (2010)
- Semigroup and compactness argument
  - Escobedo, M., Rodriguez (2005), M., Scher (2016)

Up to a change of unknown, we may then assume

$$\exists\,(G, \phi),\ G > 0,\ \phi > 0,\ \Lambda G = 0,\ \Lambda^*\phi = 0.$$

As a consequence, any solution f to the GF equation satisfies

$$\frac{d}{dt}\int f\phi = 0, \quad \frac{d}{dt}\int j(f/G)G\phi = -\mathcal{D}_j(f) \le 0$$

for any $j : \mathbb{R} \to \mathbb{R}$ convex function, with

$$\mathcal{D}_j(f) := \iint k_* G_* \phi (j(u) - j(u_*) - j'(u_*)(u - u_*))\, dx dx_*, \quad u := \frac{f}{G}.$$

Main result: exponential rate of convergence

Assume (for simplicity)

$$k(x,y) = K(x)\,\wp(y/x)/x, \quad \int_0^1 z\,\wp(dz) = 1;$$

$$K(x) \sim x^\gamma,\ \gamma \ge 0;$$

$$(GF) \qquad \tau(x) = 1 \quad \text{and} \quad \wp \text{ smooth and positive} \quad \text{or} \quad \wp = \delta_{1/2};$$

$$(SSF) \qquad \tau(x) = x \quad \text{and} \quad \wp \text{ smooth and positive.}$$

Theorem

There exist $a < 0$, $C \ge 1$ and a weight function $m : [0,\infty) \to [1,\infty)$ such that

$$\|S_\Lambda(t) f_0 - \Pi_{\Lambda,0} f_0\|_{L^1_m} \le C\, e^{at} \|f_0 - \Pi_{\Lambda,0} f_0\|_{L^1_m}$$

for any $f_0 \in L^1_m$, with $\Pi_{\Lambda,0} f := G\,\langle f, \phi\rangle$.

Furthermore, a, C are constructive when $K = cst$ as well as in the (SSF) case.

Several mathematical techniques for convergence and rate of convergence

- compactness argument + Lyapunov/dissipation of entropy
  - Escobedo, M., Rodriguez (2005); Michel, M., Perthame (2005), Bernard, Doumic, Gabriel (arXiv 2016)
- ad hoc W_1 distance when $K \sim cst^*$
  - Perthame, Ryzhik (2005), Laurençot, Perthame (2009)
- dissipation of entropy-entropy inequality*
  - Caceres, Cañizo, M. (2011), Balagué, Cañizo, Gabriel (2013)
- spectral analysis of semigroup*
  - M., Scher (2016)
- direct Laplace / Mellin analysis
  - Doumic, Escobedo (2016); Bertoin, Watson (arXiv 2017)

$*$ with constructive constants

Outline of the talk

In the spirit of Perron-Frobenius ($\sim$1907) and Krein-Rutman (1948) theory
$\simeq$ Positive semigroup theory by "german school" (Arendt, Engel, Grabosch, Greiner, Groh, Nagel, Voigt, ... $\sim$ 80's)
$\simeq$ The Harris-Meyn-Tweedie-Down theory about Markov chain and semigroup (1956–90's) (revisited by Hairer-Mattingly in 2011)

From positivity and conservation, we know that S_Λ is a SG of contractions

$$\Rightarrow \Sigma(\Lambda) \cap \Delta_0 = \emptyset, \quad \Delta_a := \{z \in \mathbb{C}, \ \Re e\, z > a\}.$$

We also know that $0 \in \Sigma(\Lambda)$ and $G \in E_0 :=$ eigenspace associated to the eigenvalue 0.

Spectral analysis issues:

▷ $E_0 = \mathrm{Vect}(G)$? $\Sigma(\Lambda) \cap \Delta_a = \{0\}$ for some $a < 0$?

Semigroup issue:

▷ deduce the corresponding semigroup decay ($\sim$ quantified spectral mapping theorem)?

Λ satisfies a "Lyapunov condition"

Proposition

There exist $m : [0,\infty) \to [1,\infty)$, $m(x) \to \infty$ as $x \to \infty$, and $a < 0$, $M \ge 0$, such that

$$\Lambda^* m \le a m + M$$

For the GF operator, assuming

$$K_0\, x^\gamma \le K(x) \le K_1\, x^\gamma, \quad \forall\, x \ge x_1,$$

and taking $\beta > \beta^*$ such that

$$\wp_{\beta^*} = K_0/K_1 \in (0,1], \quad \wp_\beta := \int_0^1 z^\beta\, \wp(dz),$$

we may choose $m(x) = e^{-\mathcal{K}(x)} \mathbf{1}_{x \le x_1} + x^\beta\, \mathbf{1}_{x \ge x_1}$.

For the SSF operator, me may choose $m(x) = x^\alpha \mathbf{1}_{x \le 1} + \eta x^\beta \mathbf{1}_{x \ge 1}$, whatever are $\alpha < 1 < \beta$ and $\eta > 0$ small enough.

Key point: for "large" particles fragmentation dominates growth while the inverse holds for "small" particles.

Lemma (strong MP).

For any solution to $\Lambda f = 0$, $f \geq 0$, there holds $f \equiv 0$ or $f > 0$.

Proof: When $f \not\equiv 0$, we have

$$\tau(x)\partial_x f + (K(x) + \mu) f \geq \mathcal{F}^+ f \geq 0, \ \not\equiv 0,$$

and we spread out positivity.

Corollary (uniqueness).

$E_0 = \mathrm{Vect}(G)$

Proof: Consider f another steady state. We may reduce to the case when f is nonnegative and has unit mass. Then $g := f - G$ satisfies $\Lambda g = 0$. In particular,

$$\Lambda g_+ \geq (\mathrm{sign}_+ g)\Lambda g = 0 \quad \text{and} \quad \int (\Lambda g_+)\phi = \int g_+ (\Lambda^* \phi) = 0,$$

so that $\Lambda g_+ = 0$. From the strong MP, we deduce $g_+ = 0$ or $g_+ > 0$. In the second case, we get $g > 0$ and then

$$1 = \langle |f| \rangle \geq \langle f \rangle > \langle G \rangle = 1 \quad \text{absurd!}$$

In a similar way, we have $g_- = 0$ and we conclude with $f - G = g = 0$.

Lemma (Strong Kato's inequality).

The case of saturation in Kato's inequality

$$\Lambda|f| = \Re e(\mathrm{sign} f)\,\Lambda f$$

implies $\exists u \in \mathbb{C}$ such that $f = u|f|$.

Fails in the case $\tau(x) = x$ and $\wp = \delta_{1/2}$!

Corollary (about the spectrum on the imaginary axis).

There is no other eigenvalue on $i\mathbb{R}$: $\Sigma(\Lambda) \cap i\mathbb{R} = \{0\}$ and 0 is (algebraically) simple.

Proof: If $\Lambda f = \mu f$ with $\Re e\mu = 0$, we write

$$0 = (\Re e\mu)|f| = \Re e(\mathrm{sign} f)\,\Lambda f \le \Lambda|f|$$

and then $\Lambda|f| = 0$ by integration. We may applies the strong Kato's inequality to get $f = u|f|$ and then $\Lambda f = 0$. That implies $\mu = 0$.

In $X := L^1_m$, we split

$$\Lambda = \mathcal{A} + \mathcal{B}, \quad \mathcal{A} := \mathcal{F}^+_R,$$

and we proved (using in particular Proposition 1) that for some $a^* < 0$

$$t \mapsto S_\mathcal{B} * (\mathcal{A} S_\mathcal{B})^{(*k)}(t)\, e^{-a^* t} \in L^\infty(\mathbb{R}_+; \mathcal{B}(X)), \quad \forall\, k \geq 0,$$

$$t \mapsto (\mathcal{A} S_\mathcal{B})^{(*2)}(t)\, e^{-a^* t} \in L^\infty(\mathbb{R}_+; \mathcal{B}(X, Y)), \quad Y \subset D(\Lambda^{1/2}),\ Y \subset\subset X.$$

From Voigt's power compact version of Weyl's theorem, we have

$$Y \subset\subset X \quad \text{implies} \quad \Sigma(\Lambda) \cap \Delta_{a^*} \text{ is discrete}.$$

Similarly, we have

$$Y \subset D(\Lambda^{1/2}) \quad \text{implies} \quad \Sigma(\Lambda) \cap \Delta_{a^*} \text{ is bounded, and thus finite!}$$

Conclusion: $\exists\, a < 0$ such that $\Sigma(\Lambda) \cap \Delta_a = \{0\}$ and $\Pi := \langle \cdot, \phi \rangle G$.

Lemma (spectral gap).

There is $a < 0$ such that $\Sigma(\Lambda) \cap \Delta_a = \{0\}$.

Proof. For an generator L we define the resolvent operator

$$R_L(z) = (L - z)^{-1} = -\int_0^\infty S_L(t)\, e^{-zt}\, dt.$$

From $\Lambda = \mathcal{A} + \mathcal{B}$, we get

$$R_\Lambda = R_\mathcal{B} - R_\Lambda \mathcal{A} R_\mathcal{B} = R_\mathcal{B} - R_\mathcal{B} \mathcal{A} R_\mathcal{B} + R_\Lambda (\mathcal{A} R_\mathcal{B})^2$$

from what we deduce

$$R_\Lambda(z)(1 - (\mathcal{A} R_\mathcal{B}(z))^2) = R_\mathcal{B}(z) - R_\mathcal{B}(z) \mathcal{A} R_\mathcal{B}(z).$$

- From

$$\|\mathcal{A} R_\mathcal{B}(z) f_0\|_Y^2 \le \int_0^\infty \|\mathcal{A} S_\mathcal{B}(t)\, f_0\|_Y^2\, e^{-2at}\, dt \le C_a\, \|f_0\|_X^2, \quad \forall\, f_0 \in X,\ z \in \Delta_a,$$

we get the estimate

$$\mathcal{A} R_\mathcal{B}(z) : X \to Y \text{ as } \mathcal{O}(1), \quad \forall\, z \in \Delta_a, \quad a < 0.$$

End of the proof of the spectral gap

• On the one hand, together with the interpolation estimate

$$\left.\begin{array}{l} R_{\mathcal{B}}(z) : X_1 \to X \text{ as } \mathcal{O}(\langle z\rangle^{-1}) \\ R_{\mathcal{B}}(z) : X \to X \text{ as } \mathcal{O}(1) \end{array}\right\} \quad \text{imply} \quad R_{\mathcal{B}}(z) : X_{1/2} \to X \text{ as } \mathcal{O}(\langle z\rangle^{-1/2}),$$

and observing that $Y \subset X_{1/2}$, we deduce

$$(\mathcal{A}R_{\mathcal{B}}(z))^2 = \mathcal{A}\, R_{\mathcal{B}}(z)(\mathcal{A}R_{\mathcal{B}}(z)) : X \to X \text{ as } \mathcal{O}(\langle z\rangle^{-1/2}).$$

In particular, $I - (\mathcal{A}R_{\mathcal{B}}(z))^2$ is invertible in $\Delta_a \cap B(0, M)^c$ for $M > 1$ large.

• On the other hand, because $Y \subset X$ with compact embedding, the operator $I - (\mathcal{A}R_{\mathcal{B}}(z))^2$ is an analytic and compact perturbation of the identity, and the Ribarič-Vidav-Voigt's version of Weyl's theorem implies that

$$\Sigma(\Lambda) \cap \Delta_a = \Sigma_d(\Lambda) \cap \Delta_a = \text{discrete set}.$$

• Both information together, we have

$$\Sigma(\Lambda) \cap \Delta_a = \Sigma_d(\Lambda) \cap \Delta_a = \text{finite set}.$$

We conclude by using that $\Sigma(\Lambda) \cap \bar{\Delta}_0 = \{0\}$.

Lemma (semigroup decay in L^1_m).

Defining $\Pi g := G\langle g, \phi\rangle$, there holds

$$\|S_\Lambda(t)(I-\Pi)\|_{X\to X} \lesssim e^{at}, \quad \forall\, t \geq 0,\ \forall\, a > a^*.$$

Proof. We set $\Pi^\perp = I - \Pi$ and we write

$$\begin{aligned} S_\Lambda(t)\Pi^\perp &= \Pi^\perp\{S_\mathcal{B} + ... + S_\mathcal{B} * (\mathcal{A}S_\mathcal{B})^{(*n-1)} + S_\Lambda * (\mathcal{A}S_\mathcal{B})^{(*n)}\} \\ &\simeq \Pi^\perp\{S_\mathcal{B} + ... + S_\mathcal{B} * (\mathcal{A}S_\mathcal{B})^{(*n-1)}\} + \int_{\uparrow_a} \Pi^\perp R_\Lambda(z)(\mathcal{A}R_\mathcal{B})^n\, e^{zt}\, dz. \end{aligned}$$

Because $\|\Pi^\perp R_\Lambda(z)\|$ is uniformly bounded on $\bar\Delta_a$, and $\|(\mathcal{A}R_\mathcal{B})^n(z)\| \lesssim \langle z\rangle^{-3/2}$, we obtain that each term is of order $\mathcal{O}(e^{at})$

2nd strong positivity condition and dissipation of entropy-entropy inequality

We may prove (with constructive constants in the (SSF) case)

$$A_1 e^{-\kappa_1 \Lambda(x)} \leq G(x) \leq A_0 e^{-\kappa_0 \Lambda(x)}, \quad \kappa_i, A_i > 0,$$

with $\Lambda(x) = x^\gamma$ in the (SSF) case and $\Lambda(x) = x^{\gamma+1}$ in the (GF) case. We also recall that $\phi(x) = x$ in the (SSF) case and in the (GF) case, we may prove

$$C_\alpha (1+x)^\alpha \leq \phi(x) \leq C\,(1+x), \quad \forall\, \alpha \in (0,1).$$

We recall

$$\begin{aligned} H_2(f|G) \quad &:= \quad \int (u-1)^2 G\phi = \|f - G\|^2_{L^2_\Omega}, \quad u := \frac{f}{G}, \\ -\mathcal{D}_2(f|G) \quad &:= \quad -\frac{d}{dt} H_2(f|G) = \iint k_* G_* \phi\,(u - u_*)^2\, dx dx_*. \end{aligned}$$

Proposition In both (GF) and (SSF) cases, when $\gamma \in (0,2)$ and $\wp$ is smooth

$$\exists\, a < 0, \quad D_2(f|G) \geq (-a) H_2(f|G).$$

Corollary In both (GF) and (SSF) cases, when $\gamma \in (0,2)$ and $\wp$ is smooth

$$\|S_\Lambda(t) f_0 - \Pi f_0\|^2_{L^2_\Omega} \leq e^{at} \|f_0 - \Pi f_0\|^2_{L^2_\Omega}, \quad \forall\, t \geq 0.$$

3rd strong positivity condition and constructive spectral gap

For a normalized eigenvalue-eigenfunction (ξ, f) with $\xi \in \Delta_a \cap \Sigma(\Lambda) \subset B(0, R)$, we have

$$\langle |f|, \phi \rangle = 1, \quad \|f\|_{L^1_m} \le C, \quad \|f\|_{W^{1,\infty}(\delta,\delta^{-1})} \le C.$$

When $\wp$ is smooth, we deduce

$$D_1(f|G) := \Re e \langle \Lambda |f| - (\Lambda f) \overline{\mathrm{sign}} f, \phi \rangle \ge \kappa := -a^{**} > 0,$$

with constructive constant when $\phi(x) = x$.
As a consequence:

$$\begin{aligned} \Re e \xi \, \langle |f|, \phi \rangle &= \Re e \langle \xi f \, \overline{\mathrm{sign}} f, \phi \rangle \\ &= \Re e \langle \Lambda f \, \overline{\mathrm{sign}} f, \phi \rangle \\ &\le \langle \Lambda |f|, \phi \rangle + a^{**} \end{aligned}$$

and then $\Re e \xi \le a^{**}$. As a consequence, $\Delta_{a^{**}} \cap \Sigma(\Lambda) = \{0\}$.

Corollary

Constructive rate of convergence for (SSF) case when $\gamma > 0$ and $\wp$ is smooth.

Outline of the talk

- Constructive exponential rate of convergence in the case $\gamma \geq 0$

$\triangleright$ Generalize M. & Scher's spectral analysis approach to the more general framework of Balagué, Cañizo, Doumic, Gabriel?

$\triangleright$ Make all the constants constructive in the upper and lower bound of G ?

$\triangleright$ Restriction on γ? Prove first $D(f|G) \geq \|f - G\|_*^2$ for a weaker norm?

$\triangleright$ Make all the constants constructive in the positivity and regularity estimates on an eigenvector f when the associated eigenvalue $\xi \in \Delta_a$?

- Meyn-Tweedie approach: Is is true

$$\forall\, C, R > 0,\ \exists\, T, \kappa > 0, \quad (S_\Lambda(T) f_0)(x) \geq \kappa,\ \forall\, x \in (0, R)$$

for any $f_0 \geq 0$, $\langle f_0, \phi \rangle = 1$, $\|f_0\|_{L^1_m} \leq C$?

- Beyond spectral gap

$\triangleright$ polynomial rate of convergence when $\gamma < 0$? (subgeometric framework)

$\triangleright$ rate of convergence for the ergodic behavior in the critical case (SSF) equation with $\wp = \delta_{1/2}$?