Semigroups, large time behavior, hypodissipativity and weak dissipativity

S. Mischler

(Paris-Dauphine)

in collaboration with J. A. Cañizo, O. Kavian

International workshop on kinetic theory and related topics
Tsinghua Sanya International Mathematics Forum
June 25th-29th, 2018

We will mainly consider the longtime asymptotic of the solution $f = f(t,x)$, $t \geq 0$, $x \in \mathbb{R}^d$, to the Fokker-Planck equation

$$\partial_t f = \Delta f + \text{div}\,(\mathbf{E} f) =: \mathcal{L} f$$

for a weakly confinement vectors field

$$\mathbf{E} \simeq x|x|^{\gamma-2} \ = \nabla\Big(\frac{|x|^\gamma}{\gamma}\Big), \quad \gamma \in (0,1),$$

and an initial datum in a weighted Lebesgue space

$$f(0,.) = f_0 \in L^p_m \subset L^1.$$

The equation is mass conservative

$$\langle f(t,\cdot)\rangle = \langle f_0\rangle, \quad \langle g\rangle := \int_{\mathbb{R}^d} g\,dx$$

and it generates a semigroup $S_t = S_{\mathcal{L}}(t)$ which is positive

$$S_t f_0 = f(t,\cdot) \geq 0 \quad \text{if} \quad f_0 \geq 0.$$

Theorem 1

(1) $\exists!$ stationary state $G \geq 0$, $\langle G \rangle = 1$, $\mathcal{L} G = 0$. It is smooth and positive.

(2) For any $f_0 \in L^p_m$, $\langle f_0 \rangle = 0$, there holds,

$$\|f(t,\cdot)\|_{L^p} \leq \Theta(t) \|f_0\|_{L^p_m}, \quad \forall\, t \geq 0,$$

with

$$\Theta(t) \simeq t^{-\frac{k-k^*}{2-\gamma}}, \quad \text{if } m = \langle x \rangle^k,\ k = k^*(E,p) = \frac{d}{p'}$$

$$\Theta(t) \simeq e^{-\lambda t^{\frac{s}{2-\gamma}}}, \quad \text{if } m = e^{\kappa \langle x \rangle^s},\ s \in (0,\gamma],\ \kappa > 0.$$

(3) As a consequence, for any $f_0 \in L^p(m)$, there holds,

$$\|f(t,\cdot) - \langle f_0 \rangle G\|_{L^p} \leq \Theta(t) \|f_0 - \langle f_0 \rangle G\|_{L^p_m}, \quad \forall\, t \geq 0.$$

We use the notations $\langle x \rangle := (1 + |x|^2)^{1/2}$ and $\|f\|_{L^p_m} = \|fm\|_{L^p}$ for any weight function $m : \mathbb{R}^d \to [1,\infty)$

Outline of the talk

1. Introduction
2. Weak Poincaré inequality
3. Existence of steady stade under subgeometric Lyapunov condition
4. Rate of convergence under Doeblin-Harris condition
5. Weakly hypocoercivity equations

Outline of the talk

1. Introduction
2. Weak Poincaré inequality
3. Existence of steady stade under subgeometric Lyapunov condition
4. Rate of convergence under Doeblin-Harris condition
5. Weakly hypocoercivity equations

We want to understand the longtime asymptotic behavior

$$f(t) \quad \text{as} \quad t \to \infty$$

of the solution to an evolution equation

$$\partial_t f = \mathcal{L} f, \quad f(0) = f_0,$$

when $\mathcal{L}$ is a linear operator acting on a Banach space X.
We wish establish that the semigroup $S_{\mathcal{L}}$, defined by $S_{\mathcal{L}}(t) f_0 := f(t)$, splits as

$$S_{\mathcal{L}}(t) = S_0(t) + S_1(t), \quad S_1(t) \text{ ``simple''},\ S_0(t) = o(S_1(t)).$$

The simplest situation is $S_1(t) = P$ projection on $N(\mathcal{L})$ of finite dimension, and the issue is

$$\|S_{\mathcal{L}}(t) - P\| = \Theta(t) \to 0?\ \Theta?$$

For the Fokker-Planck equation, $Pf = \langle f \rangle G$, $\dim P = 1$.

Long history and still active domain of research

• Kinetic school: Hilbert, Weyl, Carleman, Grad, Vidav, Ukai, Arkeryd's school, french school, Guo's school, chinese school, ...

• Semigroup school: Phillips, Dyson, Krein-Rutman, Vidav, Voigt, Engel, Nagel, Gearhart, Metz, Diekmann, Prüss, Arendt, Greiner, Blake, Webb, Mokhtar-Kharoubi, Yao, Batty, ...

• Probability school - Markovian approach / coupling method: Doeblin, Harris, Meyn, Tweedie, Down, Douc, Fort, Guillin, Hairer, Mattingly, Eberle, ...

• Probability school - Functional inequalities: Toulouse school, Rockner, Wang, Wu, Guillin, Bolley, ...

• Spectral analysis approach: Gallay-Wayne, Nier, Helffer, Hérau, Lerner, Burq, Lebeau, ...

The classical framework

The classical equivalent notions are coercive (in Hilbert space) / dissipative (in Banach space) operators and semigroup of contractions:

- $\mathcal{L}$ is coercive if $(\mathcal{L}f, f)_H \le 0, \ \forall f$;
- $S_\mathcal{L}$ is a contraction if $\|S_\mathcal{L}(t)\|_{H \to H} \le 1$.

We are rather interested here by the two equivalent more accurate estimates

- $\mathcal{L}$ is coercive if $(\mathcal{L}f, f)_H \le a\|f\|_H^2, \ a < 0, \ \forall f \in N(\mathcal{L})^\perp$;
- $\|S_\mathcal{L}(t) - P\|_{H \to H} \le \Theta(t) = Ce^{at}, \ C = 1, \ a < 0.$

The classical proofs to get such estimates are
• $\mathcal{L}^* = \mathcal{L} \le 0$ & compactness argument $\Rightarrow \Sigma(\mathcal{L}) \subset \mathbb{R}$ and discrete;
• $S_\mathcal{L} > 0$ & compactness argument $\Rightarrow \Sigma(\mathcal{L}) = \{\lambda_1\} \cup \Sigma'$, $\sup \Re e \Sigma' < \lambda_1$;
• $\mathcal{L} = \mathcal{A} + \mathcal{B}$, $\mathcal{A}$ small and $\mathcal{B}$ known.
The three points give us the spectral description of $\mathcal{L}$. We get a growth description of $S_\mathcal{L}$ thanks to the spectral mapping theorem
• Alternatively, we may use Doeblin-Harris argument giving convergence under recurrence assumption.

These tools give satisfactory answer for the FP equation with Harmonic potential. More precisely in $X = H = L^2(G^{-1})$, $G := e^{-|x|^2/2}$, we get

$$\exists\, \lambda_1 \in \mathbb{R},\ S_1(t) = e^{\lambda_1 t}\ P,\ S_0(t) = \mathcal{O}(e^{at}),\ a < \lambda_1 = 0.$$

Around 2000's at least four new (or more insistently) problems arise:

(1) Explicit / constructive growth estimates ?

(2) How to deal with operators $\mathcal{L} = \mathcal{S} + \mathcal{T}$, $\mathcal{S}^* = \mathcal{S}$, $\mathcal{T}^* = -\mathcal{T}^*$?
$\to$ hypocoercivity

(3) How to deal with the case without spectral gap ? $\to$ weak dissipativity

(4) How to change the functional space in which the spectral analysis / growth estimate is obtained in order to fit with the nonlinear theory ?

Some comments

(1) Exclude compactness argument but rather use robust constructive functional inequalities or tractable dynamic (semigroup) arguments. Goes back to Bakry-Emery Γ_2 theory ?

(2) Hypocoercivity : change (by twisting) the norm in order that $\mathcal{L}$ is coercive/dissipative or equivalently accept (in the spectral gap case)

$$\Theta(t) = Ce^{at}, \quad C > 1.$$

New name (and new techniques) but quite old idea !

(3) Weak dissipativity : Use two (in fact at least three) norms and Θ does not decay exponentially fast. Motivated by
- Landau equation for Coulomb interaction (Guo-Strain, Carrapatoso-M., ...)
- Damped wave equation (Lebeau, Burq, Lerner, Léautaud, Anantharaman, ...)
- Free transport equation with Maxwellian reflexion (Aoki-Golse, ...)

(4) Explicit (basis decomposition) for Boltzmann (Bobylev) and harmonic FP (Gallay-Wayne). Abstract version (Mouhot, Gualdani-M.-Mouhot) based on a splitting $\mathcal{L} = \mathcal{A} + \mathcal{B}$, the (iterated) Duhamel formula

$$S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{L}} * (\mathcal{A}S_{\mathcal{B}}) = S_{\mathcal{B}} + ... + S_{\mathcal{L}} * (\mathcal{A}S_{\mathcal{B}})^{(*n)},$$

providing that $(\mathcal{A}S_{\mathcal{B}})^{(*n)}$ has some smoothing property.

• Constructive rate of convergence through weak Poincaré inequality (L^2 approach)

• Existence of steady state under subgeometric Lyapunov condition *[ergodic theorem of Birkhoff-Von Neuman]*

• Constructive rate of convergence under Doeblin-Harris condition (L^1 approach)

• Perspective: weakly hypodissipativity equations

▷ Natural PDE formulations / simple deterministic proofs

▷ All these results use a splitting structure:

$$\mathcal{L} = \mathcal{A} + \mathcal{B}, \quad \mathcal{A} \prec \mathcal{B}, \quad \mathcal{B} \text{ weakly dissipative},$$

and in particular, the subgeometric Foster-Lyapunov condition

$$\mathcal{L}^* w \le -\xi + b\mathbf{1}_{\text{ball}}, \quad \xi << w$$

(geometric Lyapunov condition corresponds to $\xi \sim w$)

Vocabulary / notations

• Positive semigroup $\approx$ weak maximum principle $\approx$ Kato's inequality

• steady state = invariance measure

• spectral gap = geometric Lyapunov condition
no spectral gap $\approx$ subgeometric Lyapunov condition

• strong positivity $\approx$ strong maximum principle $\approx$ Doeblin-Harris recurrent condition

• A possible definition of weakly coercivity is

$$(\mathcal{L}f, f)_H \le a\|f\|^2_{\mathcal{H}}, \quad a < 0, \quad \mathcal{H} \not\subset H,$$

but I do not know any kind of equivalent characterization in terms of semigroup decay.

• We define the convolution

$$(U * V)(t) = \int_0^t U(t-s)V(s)ds.$$

Outline of the talk

Theorem 1 is true.
(Toscani-Villani 00, Rochner-Wang 01, Bakry-Cattiaux-Guillin 08, Kavian-M.)

The proof is based on 4 ideas.

Idea 1. We can prove the estimate for one value of $p \in [1, \infty]$ and m. Here $p = 2$ and $m = G^{-1-\bullet}$. In the next part, we will choose $p = 1$.

Idea 2. Subgeometric Lyapunov condition. When $p = 1$, it is nothing but

$$\mathcal{L}^* m \le -\nu |x|^{s+\gamma-2} \, m + b \, \mathbf{1}_{B_R},$$

with $m = \langle x \rangle^k$ ($s = 0$) and $m = \exp(\kappa \langle x \rangle^s)$. Here $b, R, \nu > 0$ are constants.

Idea 3. Dissipation by local Poincaré inequality. In the next part, dissipation is given by the Doeblin-Harris recurrente condition.

Idea 4. A system of differential inequalities + interpolation (in contrast with the only one differential inequality in the spectral gap case).

We assume $E = \nabla V$, $G = e^{-V}$, $V = |x|^\gamma/\gamma$. We fix $f \in L^2(G^{-1})$, $\langle f \rangle = 0$.

$$\begin{aligned} \int (\mathcal{L} f)\, f\, G^{-1} &= -\int |\nabla (f/G)|^2 G \\ &= -\int |\nabla (f/G^{1/2})|^2 + \int f^2 G^{-1} \psi \quad \text{(Idea 2)} \end{aligned}$$

with $\psi \lesssim -|\nabla V|^2 + \mathbf{1}_{B_R}$. Be careful with $|\nabla V|^2 \sim |x|^{2(\gamma-1)} \to 0$ as $x \to \infty$.

Both together, with $h = f/G$, we get

$$\int h^2 |\nabla V|^2 G \lesssim \int |\nabla h|^2 G + \int_{B_R} h^2 G$$

We use Poincaré-Wirtinger inequality (Idea 3) in order to bound the red color term

$$\begin{aligned} \int_{B_R} h^2 G &\lesssim \int_{B_R} |\nabla h|^2 G + \Big(\int_{B_R} hG \Big)^2 \\ &= \int_{B_R} |\nabla h|^2 G + \Big(\int_{B_R^c} hG \Big)^2 \\ &\lesssim \int_{B_R} |\nabla h|^2 G + \int_{B_R^c} h^2 |\nabla V|^2 G \int_{B_R^c} |\nabla V|^{-2} G \quad \to 0 \text{ as } R \to \infty. \end{aligned}$$

All together and for R large enough, we get the weak Poincaré inequality

$$\int h^2 |\nabla V|^2 G \lesssim \int |\nabla h|^2 G$$

or equivalently

$$\int f^2 |\nabla V|^2 G^{-1} \lesssim (\mathcal{L} f, f)_{L^2(G^{-1})}$$

The consequence on the solution to the FP equation is the differential inequality

$$\frac{d}{dt} \int f^2 G^{-1} \ \lesssim - \int f^2 |\nabla V|^2 G^{-1}$$

• When $\gamma \geq 1$, then $|\nabla V|^2 \gtrsim 1$, and we may close the equation on the above quantity (denoted by u), namely

$$\frac{d}{dt} u \leq au, \ a < 0, \quad \Rightarrow \quad u(t) \leq e^{at} u_0.$$

• When $\gamma \in (0,1)$ we need another information

We may prove the additional bound

$$(A) \qquad \int (f_t/G)^p G \le \int (f_0/G)^p G, \quad \forall\, p \in [1,\infty], \text{ take } p > 2;$$

as well as

$$(B) \qquad \int f_t^2 m^2 \le C \int f_0^2 m^2, \quad \forall\, f_0 \in L^2_m.$$

As a consequence, we have

$$\begin{cases} u_1' \lesssim -u_0, \qquad u_2 \lesssim u_2(0) \\ (C)\ u_1 \lesssim u_0^{\frac{\alpha}{1+\alpha}} u_2^{\frac{1}{1+\alpha}} \text{ or } (D)\ u_1 \lesssim \varepsilon_R^{-1} u_0 + \eta_R u_2, \end{cases}$$

with $\alpha > 0$, $\varepsilon_R, \eta_R \to 0$ as $R \to \infty$.

• In case (C), we then have

$$u_1' \lesssim -u_1^{1+1/\alpha} u_2(0)^{-1/\alpha} \quad \Rightarrow \quad u_1 \lesssim \frac{u_2(0)}{t^\alpha}.$$

• In case (D), we then have

$$u_1' \lesssim -\varepsilon_R u_1 + \varepsilon_R \eta_R u_2(0) \quad \Rightarrow \quad u_1 \lesssim \Theta(t) u_2(0), \quad \Theta(t) := \inf_R \{\, e^{-\varepsilon_R t} + \eta_R \,\}.$$

- We get (A) by writing the FP equation in gradient flow form

$$\partial_t f = \mathrm{div}(G\nabla(f/G)),$$

from what we have

$$\frac{1}{p}\frac{d}{dt}\int (f/G)^p G = -\int G\nabla(f/G)^{p-1} \cdot \nabla(f/G) \le 0$$

- The proof of (B) is more tricky. It is similar to the Step 4 (4th idea) about the change of functional space.

- To prove (D), we write

$$\int f^2 G^{-1} \le R^{2(1-\gamma)} \int_{B_R} f^2 G^{-1} |\nabla V|^2 + \|f/G\|^2_{L^\infty} \int_{B_R^c} G|\nabla V|^2.$$

- From a rough version of (D) we deduce (C) by optimising over $R \in (0,\infty)$.

We assume (in particular, that we have yet established point (1) in Theorem 1)

$$x \cdot \mathbf{E} \sim |x|^\gamma \quad \text{and} \quad \exists! \, G \text{ stationary state, } G \sim e^{-|x|^\gamma}.$$

We observe that for any weight function* $W : \mathbb{R}^d \to [1, \infty]$ we have

$$D[f] := \int (-Lh) h \, WG = \int |\nabla h|^2 GW - \frac{1}{2} \int h^2 \, (\mathcal{L}^* W) \, G.$$

For the choice $W := w + \lambda^*$ with w a Lyapunov function associated to $\mathcal{L}$ in the sense that

$$\mathcal{L}^* w \le -\xi + b \, \mathbf{1}_{U_0},$$

for $\xi \simeq |x|^{s+\gamma-2} \, w$, the same computation as in the potential case leads to

$$\int |\nabla h|^2 GW^* - \frac{1}{2} \int h^2 \, (\mathcal{L}^* W) \, G \ge \frac{1}{4} \int |\nabla h|^2 G \xi$$

for some $\lambda > 0$ large enough. We immediately deduce our first differential inequality

$$\frac{d}{dt} \int f^2 W G^{-1} \le -\frac{1}{4} \int f^2 \xi G^{-1}.$$

* modified norm $\simeq$ “hypodissipativity trick”

- L^2 estimate from mass conservation. We split

$$\mathcal{L} = \mathcal{A} + \mathcal{B}, \qquad \mathcal{A} := M\chi(x/R), \quad \mathbf{1}_{B(0,1)} \le \chi \in \mathcal{D}(\mathbb{R}^d),$$

we use iterated Duhamel formula

$$S_{\mathcal{L}} = S_{\mathcal{B}} + ... + (S_{\mathcal{B}}\mathcal{A})^{(*n)} * S_{\mathcal{L}}$$

and we have to prove $S_{\mathcal{B}} : L^p(m_2) \to L^p(m_1)$ with decay $\Theta \in L^1_t$, $m_1 << m_2$, and $(S_{\mathcal{B}}\mathcal{A})^{(*n)}$ has some smoothing property (by Nash technique), namely $(S_{\mathcal{B}}\mathcal{A})^{(*n)} : L^1(m_1) \to L^2$.

- L^p decay from L^2 decay. We use the iterated Duhamel formula

$$S_{\mathcal{L}} = S_{\mathcal{B}} + ... + S_{\mathcal{L}} * (\mathcal{A}S_{\mathcal{B}})^{(*n)}$$

and $(\mathcal{A}S_{\mathcal{B}})^{(*n)} : L^p(m_2) \to L^2(G^{-1})$ if $p < 2$.
We use the iterated Duhamel formula

$$S_{\mathcal{L}} = S_{\mathcal{B}} + ... + (S_{\mathcal{B}}\mathcal{A})^{(*n)} * S_{\mathcal{L}} * (\mathcal{A}S_{\mathcal{B}})$$

and $(S_{\mathcal{B}}\mathcal{A})^{(*n)} : L^2(G^{-1}) \to L^p$ if $p > 2$.

Outline of the talk

We consider a Markov semigroup $S_t = S_{\mathcal{L}}(t)$ defined on $X := M^1(E)$, meaning $S_t \geq 0$ and $S^* 1 = 1$. We furthermore assume

(H1) Subgeometric Lyapunov condition. There are two weight functions $m_0, m_1 : E = \mathbb{R}^d \to [1, \infty)$, $m_1 \geq m_0$, $m_0(x) \to \infty$ as $x \to \infty$, and two real constants $b, R > 0$ such that

$$\mathcal{L}^* m_1 \leq -m_0 + b \, \mathbf{1}_{B_R}.$$

Theorem 2 Douc, Fort, Guillin ? deterministic proof by Cañizo, M.

Any Feller-Markov semigroup (S_t) which fulfills the above Lyapunov condition has at least one invariant borelian measure $G \in M^1(m_0)$.

Remark.

- $m_0 = m_1$: geometric Lyapunov condition = spectral gap (the result is true, the proof is simpler)
- Feller-Markov semigroup acts on $C_0(E)$ and $S_t := (S_{\mathcal{L}^*}(t))^*$.

We introduce the splitting

$$\mathcal{A} := b\mathbf{1}_{B_R}, \quad \mathcal{B} := \mathcal{L} - \mathcal{A}.$$

We observe that $S_\mathcal{B}$ is a submarkovian semigroup and

$$0 \le S_\mathcal{B} \in L^\infty_t(\mathcal{B}(M^1(m_i))); \quad \int_0^\infty \|S_\mathcal{B}(t) f_0\|_{M^1(m_0)}\, dt \le \|f_0\|_{M^1(m_1)}.$$

We write the Duhamel formula

$$S_\mathcal{L} = S_\mathcal{B} + S_\mathcal{B} * \mathcal{A} S_\mathcal{L},$$

and we consider the associated Cezaro means

$$U_T := \frac{1}{T}\int_0^T S_\mathcal{L}\, dt, \quad V_T := \frac{1}{T}\int_0^T S_\mathcal{B}\, dt, \quad W_T := \frac{1}{T}\int_0^T S_\mathcal{B} * \mathcal{A} S_\mathcal{L}\, dt.$$

We define $X := M^1(m_0)$, $0 \le f_0 \in X$, $\langle f_0 \rangle = 1$, and we observe that

$$\| V_T \|_{X \to X} := \frac{1}{T} \Big\| \int_0^T S_\mathcal{B} \, dt \Big\|_{X \to X} \le 1$$

On the other hand, by Fubini and positivity

$$\begin{aligned}
\| W_T f_0 \|_{M^1(m_0)} &= \Big\| \frac{1}{T} \int_0^T S_\mathcal{B}(\tau) \int_0^{T-\tau} \mathcal{A} \, S_\mathcal{L}(s) \, f_0 \, d\tau ds \Big\|_{M^1(m_0)} \\
&\le \frac{1}{T} \int_0^\infty \Big\| S_\mathcal{B}(\tau) \int_0^T \mathcal{A} \, S_\mathcal{L}(s) \, ds f_0 \Big\|_{M^1(m_0)} d\tau \\
&\le \frac{1}{T} \Big\| \int_0^T \mathcal{A} \, S_\mathcal{L}(s) \, ds f_0 \Big\|_{M^1(m_1)} \le C_\mathcal{A} \| f_0 \|_{M^1(m_0)},
\end{aligned}$$

We deduce $U_{T_k} f_0 \to G$ weakly and G satisfies $\mathcal{L} G = 0$ because for any $s > 0$:

$$\begin{aligned}
S_\mathcal{L}(s) G - G &= \lim_{k \to \infty} \Big\{ \frac{1}{T_k} \int_0^{T_k} S_\mathcal{L}(s) S_\mathcal{L}(t) f_0 dt - \frac{1}{T_k} \int_0^{T_k} S_\mathcal{L}(t) f_0 \, dt \Big\} \\
&= \lim_{k \to \infty} \frac{1}{T_k} \Big\{ \int_{T_k}^{T_k + s} S_\mathcal{L}(\tau) f_0 d\tau - \int_0^s S_\mathcal{L}(t) f_0 \, dt \Big\} = 0.
\end{aligned}$$

Let us denoted by G the steady state for the Fokker-Planck equation provided by Theorem 2 under the general assumption $x \cdot \mathbf{E} \sim |x|^\gamma$, $\gamma \in (0,1)$.

• Thanks to a bootstrap argument: G is smooth, or at least a bit smoother than E, and in any cases $G \in W^{1,p}(\mathbb{R}^d)$ for any $p \in [1,\infty)$.

• From Step 4 in the proof of Theorem 1 (2), we get

$$G \le e^{-\kappa_1 |x|^\gamma}, \ \kappa_1 > 0.$$

• Because of the strong maximum principle, we have $G > 0$. More accurately, using a comparison to a subsolution technique , we have

$$G \ge e^{-\kappa_1 |x|^\gamma}, \ \kappa_2 > 0.$$

Outline of the talk

We consider a Markov semigroup $S_t = S_{\mathcal{L}}(t)$ defined on $X := L^1(\mathbb{R}^d)$, meaning $S_t \ge 0$ and $S_t^* 1 = 1$. We furthermore assume

(H1) Subgeometric Lyapunov condition. There are two weight functions $m_0, m_1 : \mathbb{R}^d \to [1, \infty)$, $m_1 \ge m_0$, $m_0(x) \to \infty$ as $x \to \infty$, and two real constants $b, R > 0$ such that

$$\mathcal{L}^* m_1 \le -m_0 + b\,\mathbf{1}_{B_R}.$$

(H2) Doeblin-Harris condition. $\exists\, T > 0\ \forall\, R > 0\ \exists\, \nu \ge 0, \not\equiv 0$, such that

$$S_T g \ge \nu \int_{B_R} g, \quad \forall\, g \in X_+.$$

(H3) There are two other weight functions $m_2, m_3 : \mathbb{R}^d \to [1, \infty)$, $m_3 \ge m_2 \ge m_1$ such that

$$\mathcal{L}^* m_i \le -m_0 + b\,\mathbf{1}_{B_R}$$

and $m_2 \le m_0^\theta m_3^{1-\theta}$ with $\theta \in (1/2, 1]$.

Theorem 3 Douc, Fort, Guillin, Hairer, deterministic proof by Cañizo, M.

Consider a Markov semigroup S on $X := L^1(m_2)$ which satisfies (H1), (H2), (H3). There holds

$$\|S_t f_0\|_{L^1} \lesssim \Theta(t) \|f_0\|_{L^1(m_2)}, \quad \forall\, t \ge 0,\ \forall\, f_0 \in X,\ \langle f_0 \rangle = 0,$$

for the function Θ given by

$$\Theta(t) := \inf_{\lambda > 0} \left\{ \, e^{-\varepsilon_\lambda t} + \xi_\lambda \, \right\},$$

where

$$m_1 \le \frac{1}{2\varepsilon_\lambda} m_0 + \eta_\lambda m_2,\ \forall\, \lambda, \quad \varepsilon_\lambda, \eta_\lambda \to 0 \text{ as } \lambda \to \infty.$$

• For the Fokker-Planck equation, assumption (H2) can be proved in a similar way (maybe a bit more tricky) as for the lower bound in Theorem 1 (1).

• The assumption (H3) is not necessary: m_1 satsisfies a Lyaponov condition implies that $\phi(m_1)$ satsisfies a Lyaponov condition for any $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ concave.

• The probabilistic proof use Martingale argument, renewal theory and (if possible?) constants are not easily tractable.

• In the probabilistic result, one writes $m_0 = \xi(m_1)$, $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ concave, and

$$\tilde{\Theta}(t) := \frac{C}{\xi(H^{-1}(t))}, \quad H(u) := \int_1^u \frac{ds}{\xi(s)}.$$

- If $\xi(s) = s$ then $\tilde{\Theta}(t) = e^{-\lambda t}$;

- If $m_1 = \langle x \rangle^k$, $m_0 := \langle x \rangle^{k+\gamma-2}$ then $\tilde{\Theta}(t) = t^{1-\frac{k}{2-\gamma}} >> \Theta(t)$;

- If $m_1 = e^{\kappa \langle x \rangle^s}$, $m_0 := \langle x \rangle^{s+\gamma-2} e^{\kappa \langle x \rangle^s}$ then $\tilde{\Theta}(t) \simeq e^{-\lambda t^{\frac{s}{2-\gamma}}} \simeq \Theta(t)$.

Rk 1. Assuming just that (S_t) is a Markov semigroup, we have

$$|S_t f| = |S_t f_+ - S_t f_-| \le |S_t f_+| + |S_t f_-| = S_t |f|.$$

Integrating, we deduce that (S_t) is a L^1 contraction

$$\int |S_t f| \le \int S_t |f| = \int |f| S_t^* 1 = \int |f|.$$

Rk 2. We assume furthermore the strong Doeblin-Harris condition:

$$(H2') \qquad \exists\, T, \exists\, \nu, \quad S_T g \ge \nu \int_{\mathbb{R}^d} g, \quad \forall\, g \in X_+.$$

For $f \in L^1$, $\langle f \rangle = 0$, we have

$$S_T f_\pm \ge \nu \int_{\mathbb{R}^d} f_\pm = \frac{\nu}{2} \int_{\mathbb{R}^d} |f| =: \eta.$$

We may adapt the proof in Rk 1 in the following way

$$\begin{aligned} |S_T f| &= |S_T f_+ - \eta - (S_T f_- - \eta)| \\ &\le |S_T f_+ - \eta| + |S_T f_- - \eta| = S_T |f| - 2\eta. \end{aligned}$$

Integrating, we deduce that (S_T) is a strict contraction

$$\|S_T f\|_{L^1} \le \|f\|_{L^1} - 2\|\eta\|_{L^1} = (1 - \langle \nu \rangle)\, \|f\|_{L^1}$$

Rk 3. Assuming (H2), we have similarly

$$\int |S_T f| \le \theta \int |f| \quad \text{if} \quad \int |f| m_0 \le \frac{m_0(R)}{4} \int |f|,$$

with

$$\theta := 1 - \langle \nu \rangle / 2 \in (0,1).$$

Indeed, we mainly observe that

$$\begin{aligned} S_T f_\pm &\ge \nu \int_{\mathbb{R}^d} f_\pm - \nu \int_{B_R^c} f_\pm \\ &\ge \frac{\nu}{2} \int_{\mathbb{R}^d} |f| - \nu \int_{B_R^c} |f| \\ &\ge \frac{\nu}{2} \int_{\mathbb{R}^d} |f| - \frac{\nu}{m_0(R)} \int_{\mathbb{R}^d} |f| m_0 \\ &\ge \frac{\nu}{2} \int_{\mathbb{R}^d} |f| - \frac{\nu}{4} \int_{\mathbb{R}^d} |f| \\ &= \frac{\nu}{4} \int_{\mathbb{R}^d} |f|, \end{aligned}$$

and we then follow the same proof as when we have assumed (H2′).

We fix $f_0 \in L^1(m_3)$, we denote $f_{\mathcal{B}t} := S_{\mathcal{B}}(t) f_0$.
From (H1) and (H3), we have

$$\frac{d}{dt} \|f_{\mathcal{B}t}\|_{m_3} \le -\| f_{\mathcal{B}t}\|_{m_0} \le 0$$
$$\frac{d}{dt} \|f_{\mathcal{B}t}\|_{m_2} \le -\| f_{\mathcal{B}t}\|_{m_0} \le -\| f_{\mathcal{B}t}\|_{m_2}^{1/\theta} \| f_0\|_{m_3}^{1-1/\theta} \le 0$$

so that $t \mapsto \|f_{\mathcal{B}t} \|_{m_2} \lesssim \langle t\rangle^{-\frac{\theta}{1-\theta}} \| f_0\|_{m_3} \in L^1(\mathbb{R}_+)$.

Using the splitting

$$S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{B}} * \mathcal{A} S_{\mathcal{L}}$$

and the L^1 contraction, we deduce

$$\|S_{\mathcal{L}}(t) f_0\|_{m_2} \le M_2 \, \|f_0\|_{m_2}.$$

We set $t_{n+1} \simeq t_n + T$, $A := m_0(R)/4 \geq 2b$ and we have the following alternative:

• Or
$$\exists\, t \in [t_n, t_n + T), \quad \int |f_t| m_0 \leq A \int |f_t|$$

and assuming $t := t_n$ (to make the discussion simpler) we get from the variant of Doeblin-Harris contraction argument (using (H2) assumption)

$$\int |f_{t_{n+1}}| \leq \theta \int |f_{t_n}|$$

• Or
$$\forall\, t \in [t_n, t_n + T), \quad \int |f_t| m_0 \geq A \int |f_t|,$$

and we simply compute (thanks to assumption (H2) and (H3))

$$\begin{aligned} \frac{d}{dt} \int |f| m_1 &\leq b \int |f| - \int |f| m_0 \\ &\leq -\frac{1}{2} \int |f| m_0 \leq -\varepsilon_\lambda \int |f| m_1 + \varepsilon_\lambda \eta_\lambda C \int |f_0| m_2. \end{aligned}$$

We deduce

$$\int |f_{t_{n+1}}| m_1 \leq e^{-\varepsilon_\lambda T} \int |f_{t_n}| m_1 + (1 - e^{-\varepsilon_\lambda T})\, \eta_\lambda C \int |f_0| m_2.$$

We define

$$\|f\|_\beta := \|f\|_{L^1} + \beta \|f\|^*_{m_1}, \quad \beta > 0.$$

In both cases and for $\beta > 0$ small enough*, we have

$$\|f_{t_{n+1}}\|_\beta \le e^{-\varepsilon_\lambda T} \, \|f_{t_n}\|_\beta + (1 - e^{-\varepsilon_\lambda T}) \, \eta_\lambda C \int |f_0| m_2.$$

After iteration, we deduce

$$\begin{aligned} \|f_{t_n}\|_\beta &\le e^{-\varepsilon_\lambda t_n} \, \|f_0\|_\beta + (1 - e^{-\varepsilon_\lambda t_n}) \, \eta_\lambda C \int |f_0| m_2. \\ &\le [e^{-\varepsilon_\lambda t_n} + \eta_\lambda] \; C_\beta \|f_0\|_{L^1(m_2)}. \end{aligned}$$

* modified norm $\simeq$ "hypodissipativity trick"

Outline of the talk

• Fractional Fokker-Planck equation with weak confinement.
L. Lafleche (phD U. Paris-Dauphine & Ecole polytechnique) by (generalized) weakly Poincaré inequality.

• Kinetic Fokker-Planck equation with weak confinement.
C. Cao (phD U. Paris-Dauphine) by twisting H^1 norm technique (Villani) and micro-macro decomposition (Hérau, Dolbeault-Mouhot-Schmeiser).

• Age structured equation: Cañizo,Yoldas by using Theorem 3 above.

• Relaxation equation with weak confinement.
Cañizo, Cao, ... by using Theorem 3 above.

• Free transport equation with Maxwellian reflexion (in general domain)
A. Bernou (phD U. Paris-Dauphine & Sorbonne U.) using coupling method (I have not spoken about in this talk)

• What about the inelastic Boltzmann equation with very weak confinement force with possible application to the stability of Saturn's rings ??