W_1 stability estimate and rate of propagation of chaos for the HS Boltzmann model

S. Mischler

(CEREMADE, U. Paris Dauphine-PSL)

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Joint work with C. Mouhot and J. Norris

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Outlines of the talk

1. Introduction
2. The W_1 stability estimate
3. Short discussion about chaos
4. Uniform in time propagation of chaos

Plan

Aim of the talk

• Establish a W_1 estimate between two solutions of the nonlinear space homogeneous Boltzmann equation for Hard Spheres

$\triangleright$ follow Di Blasio argument, but on a dual problem

$\triangleright$ improve Mouhot-Fournier PDE argument

$\triangleright$ generalize Tanaka, Fournier, Fournier-Perthame coupling argument

$\triangleright$ simplify Norris-Heydecker martingale argument

• Motivation: Deduce a (uniform in time) rate of propagation of chaos for the Kac-Boltzmann Hard Spheres N-particle system

$\triangleright$ recover the same result by Norris, Heydecker (by martingale argument)

$\triangleright$ improve a similar result by M.-Mouhot (by semigroup argument)

The nonlinear HS Boltzmann equation

$$\partial_t f = \int_{\mathbb{R}^d \times S^{d-1}} [\, f' f'_* - f\, f_*]\, B \;\, dv_* d\sigma, \quad f(0,\cdot) = f_0,$$

gives the evolution of the velocities statistical distribution $f = f(t,v) \geq 0$, $t \geq 0$, $v \in \mathbb{R}^d$ under Hard Spheres interactions, so that

$$B = |v - v_*|,$$

where

$$v' = \frac{v+v_*}{2} + \frac{|v_* - v|}{2}\sigma, \qquad v'_* = \frac{v+v_*}{2} - \frac{|v_* - v|}{2}\sigma,$$

and we use the shorthands

$$f(t,v) = f,\ f(t,v_*) = f_*,\ f(t,v') = f',\ f(t,v'_*) = f'_*.$$

Observe that momentum and energy are conserved

$$v' + v'_* = v + v_*, \qquad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2,$$

so that

$$\int f_t\, (1, v, |v|^2)\, dv = (1, 0, d), \quad \forall\, t \geq 0.$$

Constructive W_1 stability estimate

We introduce the notations: the weighted Lebesgue space

$$L^1_k := \{f \in L^1(\mathbb{R}^d);\ |v|^k f \in L^1(\mathbb{R}^d)\}$$

and the Monge-Kantorovich-Wasserstein (MKW) distance

$$\begin{aligned} W_1(f,g) &= \inf\Big\{ \int_{\mathbb{R}^{2d}} 1 \wedge |v-w|\, \pi(dv,dw),\ \pi_1 = f, \pi_2 = g\Big\} \\ &= \sup\Big\{ \int_{\mathbb{R}^d} (f-g)\varphi dv,\ \|\varphi\|_{W^{1,\infty}} \le 1\Big\}. \end{aligned}$$

Theorem

There exists $\kappa > 0$ such that for any $f_0, g_0 \in L^1_3(\mathbb{R}^d)$ the associated solutions $f, g \in C([0,\infty); L^1_2(\mathbb{R}^d))$ to the nonlinear HS Boltzmann equation satisfies

$$\widetilde{W_1}(f_t, g_t) \lesssim e^{\kappa t}\ \widetilde{W_1}(f_0, g_0), \quad \forall\, t \ge 0.$$

▷ biblio for L^1_2 : Arkeryd (1971), DiBlasio (1974), M.-Wennberg (1999), Lu (1999)
▷ biblio: W_1 : Tanaka (1978/79), Fournier-Mouhot (2009), Norris (2016), Heydecker (2019), Fournier-Perthame (arXiv)

We remind the exponential rate of convergence to the normalized gaussian equilibrium $\gamma := (2\pi)^{-d/2} e^{-|v|^2/2}$

Theorem

There exists $\lambda > 0$ such that for any $f_0 \in L^1_3(\mathbb{R}^d)$ the associated solutions $f \in C([0,\infty); L^1_2(\mathbb{R}^d))$ to the nonlinear HS Boltzmann equation satisfies

$$W_1(f_t, \gamma) \lesssim e^{-\lambda t}, \quad \forall\, t \geq 0.$$

$\triangleright$ biblio : Arkeryd (1988), Carlen-Carvalho (1994), Abrahamsson (1999), Toscani-Villani (1999), Villani (2003), Baranger-Mouhot (2005), Mouhot (2006).

Writing

$$\widetilde{W}_1(f_t, g_t) \lesssim \sup_{s \geq 0} \min(e^{\kappa s}\ \widetilde{W}_1(f_0, g_0), e^{-\lambda s}),$$

we deduce

Corollary

$$\widetilde{W}_1(f_t, g_t) \lesssim\ \widetilde{W}_1(f_0, g_0)^{\frac{\lambda}{\kappa+\lambda}}, \quad \forall\, t \geq 0.$$

Kac: Foundations of kinetic theory (1956)

• Go **rigorously** from a microscopic description to a statistical description :

▷ Justify the nonlinear Boltzmann equation at the mesoscopic level

▷ Simplify the huge number of particles microscopic description

• Mean field limit in the sense that each particle interacts with all the other particles with an intensity of order $\mathcal{O}(1/N)$
$\Rightarrow$ statistical description = *law of large numbers limit* of a N-particle system

• at the formal level the identification of the limit is quite clear when one assumes the molecular chaos for the limit model

• main difficulty : propagation of chaos
▷ chaos for ∞ particles = Boltzmann's molecular chaos (stochastic independence)
▷ chaos for $N \to \infty$ particles = Kac's chaos (asymptotic stochastic independence)
▷ propagation of chaos: holds at time $t = 0$ implies holds for any $t > 0$
▷ propagation of chaos is necessary in order to identify the limit as $N \to \infty$

Consider a system of N indistinguishable (exchangeable) particles, each particle being described by its velocity $\mathcal{V}_1^N, ..., \mathcal{V}_N^N \in \mathbb{R}^d$, which undergoes random Boltzmann jumps (collisions): defined step by step as follows:

(i) draw randomly the collision times $T_{i',j'} \sim Exp(B(|\mathcal{V}_{i'} - \mathcal{V}_{j'}|))$ for any $(\mathcal{V}_{i'}, \mathcal{V}_{j'})$; then select the pre-collisional velocity $(\mathcal{V}_i, \mathcal{V}_j)$ such that

$$T_{i,j} = \min_{(i',j')} T_{i',j'}.$$

(ii) pick randomly $\sigma \in S^2$ according to the uniform density law and define the post-collisional velocities $(\mathcal{V}_i^*, \mathcal{V}_j^*)$ thanks to

$$\mathcal{V}_i^* = \frac{\mathcal{V}_i + \mathcal{V}_j}{2} + \frac{|\mathcal{V}_j - \mathcal{V}_i|}{2}\,\sigma, \qquad \mathcal{V}_j^* = \frac{\mathcal{V}_i + \mathcal{V}_j}{2} - \frac{|\mathcal{V}_j - \mathcal{V}_i|}{2}\,\sigma.$$

Observe that momentum and energy are conserved

$$\mathcal{V}_i^* + \mathcal{V}_j^* = \mathcal{V}_i + \mathcal{V}_j, \qquad |\mathcal{V}_i^*|^2 + |\mathcal{V}_j^*|^2 = |\mathcal{V}_i|^2 + |\mathcal{V}_j|^2$$

for each collision, so that

$$\frac{1}{N}\sum_i \mathcal{V}_i(t) = \text{cst} = 0, \quad \frac{1}{N}\sum_i |\mathcal{V}_i(t)|^2 = \text{cst} = d.$$

Alternative formulations

The N-particle random system $\boldsymbol{\mathcal{V}}^N = (\mathcal{V}_1^N, ..., \mathcal{V}_N^N) \in E^N$, $E := \mathbb{R}^d$, evolves according to

$$d\boldsymbol{\mathcal{V}} = \frac{1}{N} \sum_{i,j=1}^{N} \int_{S^{d-1}} (\boldsymbol{\mathcal{V}}'_{ij} - \boldsymbol{\mathcal{V}}) \, |\mathcal{V}_i - \mathcal{V}_j| \, d\mathcal{N}_{i,j}(d\sigma)$$

where $\mathcal{N}$ Poisson measure, $\boldsymbol{\mathcal{V}}'_{ij} = (\mathcal{V}_1, ..., \mathcal{V}'_i, ..., \mathcal{V}'_j, ..., \mathcal{V}_N)$ represents the system after collision of the pair $(\mathcal{V}_i, \mathcal{V}_j)$.

In particular, the law $F^N(t) := \mathcal{L}(\boldsymbol{\mathcal{V}}_t^N)$ satisfies the Master (Liouville or backward Kolmogorov) equation

$$\partial_t F^N = \Lambda^N F^N,$$

where the generator Λ^N writes

$$(\Lambda^N F^N)(V) := \frac{1}{N} \sum_{1 \le i < j \le N} \int_{\mathbb{S}^{d-1}} \left[F^N(V'_{ij}) - F^N(V) \right] \, |v_i - v_j| \, \mathrm{d}\sigma,$$

for any $V = (v_1, \dots, v_N) \in E^N$.

- the normalized MKW distance on $\mathbb{P}(E^j)$ by

$$W_1(F,G) := \inf_{\pi \in \Pi(F,G)} \int_{E^j \times E^j} \Big(\frac{1}{j} \sum_{i=1}^{j} |x_i - x_j| \ \wedge 1 \Big) \, \pi(dX, dY).$$

and a sequence (F^N) is f-Kac's chaotic if

$$W_1(F^N, f^{\otimes N}) \to 0, \quad \text{as} \quad N \to \infty.$$

- a sequence (F^N) is f-entropy chaotic if furthermore

$$H(F^N|\gamma^N) \to H(f|\gamma), \quad \text{as} \quad N \to \infty,$$

where

$$H(F^N|\gamma^N) := \frac{1}{N} \int_{\mathcal{S}^N} F^N \log \frac{dF^N}{d\gamma^N}, \quad H(f|\gamma) := \int_E f \, \log \frac{f}{\gamma},$$

with $\gamma^N =$ uniform probability measure on the Boltzmann sphere $\mathcal{S}^N$

With W_1 the MKW distance on E^N, γ^N the uniform probability measure on the Kac-Boltzmann sphere and γ the normalized gaussian

Theorem (M., Mouhot, Norris, a possible answer to Kac's problems)

For any $f_0 \in \mathbb{P}(E)$ + conditions, there exists a sequence $\mathcal{V}^N(0)$ of initial conditions for the Boltzmann-Kac process for hard spheres such that

$$\sup_{t \geq 0} W_1(F^N(t), f(t)^{\otimes N}) \leq \frac{C}{N^\bullet}$$

$$H(F^N(t)|\gamma^N) \to H(f(t)|\gamma)$$

$$\sup_{N \geq 1} W_1(F^N(t), \gamma^N) \leq \frac{C}{t^\bullet}.$$

▷ biblio: Kac (1956), McKean (1967), Grünbaum (1971), Sznitman (1984), Fontbona-Guérin-Méléard (2009), Fournier (2009), M.-Mouhot (2013), Carrapatoso (2016), Fournier-Hauray (2016), Fournier-M. (2016), Norris (2016), Fournier-Guillin (2017), Cortez-Fontbona (2018), Heydecker (2019), M.-Mouhot-Norris (xxxx)

- Same W_1 estimate between two solutions to the nonlinear Boltzmann equation associated to true (without Grad's angular cut-off) hard potential ?

- Similar W_1 estimate between two solutions to the Boltzmann-Kac N-particle system?

Plan

Previous W_1 stability results for MM and HS

Theorem (Tanaka, Rousset, Fournier-Perthame)

For two solutions to the Boltzmann equation associated to Maxwell molecules

$$\begin{aligned} W_2(f_t, g_t) &\leq W_2(f_0, g_0), \quad \forall\, t \geq 0, \\ W_1(f_t, g_t) &\leq e^{\kappa t}\, W_1(f_0, g_0), \quad \forall\, t \geq 0. \end{aligned}$$

Corollary (Fournier-Mouhot)

There exists $\theta > 0$ such that for two solutions to the Boltzmann equation associated to Hard Spheres

$$W_1(f_t, g_t) \leq W_1(f_0, g_0)^{e^{-\theta t}}, \quad \forall\, t \geq 0, \quad \forall\, f_0, g_0 \in L^1_{\exp}.$$

The idea is to use the splitting

$$\frac{d}{dt} \int \varphi D_t = \iint \dots |v - v_*| \wedge R + \iint \dots (|v - v_*| - R)_+,$$

to roughly estimate the first term as in the Maxwell molecules case and to use exponential weight L^1 estimate for the second case

$$\frac{d}{dt} W_1 \lesssim R W_1 + e^{-R} \leq W_1 (1 + |\log W_1|).$$

Theorem (Di Blasio, Wennberg, M.-Wennberg, Lu)

For two solutions to the Boltzmann equation for HS

$$\|f_t - g_t\|_{L^1_2} \lesssim e^{\kappa t} \, \|f_0 - g_0\|_{L^1_2}, \quad \forall \, t \ge 0, \ \forall \, f_0, g_0 \in L^1_{2\,\log}.$$

Idea of the proof. We define $D := f - g$, $S = f + g$, and we write

$$\frac{d}{dt} \int D\varphi = \iint DS_* [\varphi'_* + \varphi' - \varphi_* - \varphi] \, |v - v_*| d\sigma dv_* dv,$$

with $\varphi := \mathrm{sign}(D) \, \langle v \rangle^2$, $\langle v \rangle^2 := 1 + |v|^2$. We obtain

$$\begin{aligned} \frac{d}{dt} \int |D| \langle v \rangle^2 &\le \iint |D| S_* [\langle v'_* \rangle^2 + \langle v' \rangle^2 + \langle v_* \rangle^2 - \langle v \rangle^2] \, |v - v_*| d\sigma dv_* dv \\ &= \iint |D| S_* [1 + \langle v_* \rangle^2] \, |v - v_*| d\sigma dv_* dv \\ &\lesssim \int S_* \langle v_* \rangle^3 dv_* \int |D| \langle v \rangle^2 dv, \end{aligned}$$

and we conclude thanks to the Gronwall lemma.

For an initial datum $f_0 \in L^1_3$, we have

$$\begin{aligned}
\frac{d}{dt}\int f(1+|v|^3) &= \iiint f\, f_* \left[|v'_*|^3 + |v'|^3 - |v_*|^3 - |v|^3\right] |v-v_*|\, d\sigma dv_* dv \\
&\lesssim \iint f\, f_* \left[|v_*|^2|v| - |v_*|^3\right] |v-v_*|\, dv_* dv \\
&\lesssim \int f_0(1+|v|^2)\, dv \int f(1+|v|^3)\, dv - \int f_0\, dv \int f(1+|v|^4)\, dv
\end{aligned}$$

so that

$$\sup_{t\ge 0} \int f_t(1+|v|^3)\, dv \lesssim \int f_0(1+|v|^3)\, dv.$$

For an initial datum $f_0 \in L^1_{2\log}$, we may prove

$$\int_0^T \int f_t(1+|v|^3)\, dvdt \lesssim (1+T) \int f_0(1+|v|^2(\log|v|^2)_+)\, dv.$$

We write again

$$\frac{d}{dt}\int D_t\varphi\,dv \quad = \quad \int D_t\,[\mathcal{L}_{S_t}\varphi]dv$$

$$\mathcal{L}_{S_t}\varphi \quad := \quad \iint S_{t*}[\varphi'_* + \varphi' - \varphi_* - \varphi]\,|v - v_*|d\sigma dv_*.$$

Introducing the solution φ_s to the backward linear evolution equation

$$\partial_t\varphi_s = \mathcal{L}_{S_{t-s}}\varphi_s, \quad \varphi_{|s=t} = \varphi,$$

we have

$$\int \varphi D_t\,dv = \int \varphi_t D_0\,dv,$$

because

$$\frac{d}{ds}\int \varphi_{t-s}D_s = -\int (\mathcal{L}_{S_s}\varphi_{t-s})D_s + \int \varphi_{t-s}\frac{d}{ds}D_s = 0.$$

As a consequence, this duality trick implies

$$\int \varphi D_t(\mathrm{d}v) \le \widetilde{W}_1(f_0,g_0)\left\|\frac{\varphi_t(v)}{1+|v|^2}\right\|_{W^{1,\infty}}$$

We rather consider the evolution equation

$$\partial_t \phi = \iint \Big[\phi'_* \omega'_* + \phi' \omega' - \phi_* \omega_* - \phi \omega\Big] \frac{|u|}{\omega} \tilde S_{t*} \, d\sigma dv_*,$$

after performing the change of unknown $\phi := \varphi / \omega$ and using the shorthands $u := v - v_*$, $\tilde S_t := S_{T-t}$, for fixed final time $T > 0$.

At the formal level, we may compute (dual from DiBlasio)

$$\begin{aligned}\frac{d}{dt} \|\phi_t\|_{L^\infty} &\le \|\phi_t\|_{L^\infty} \iint \Big[\omega'_* + \omega' + \omega_* - \omega\Big] \frac{|u|}{\omega} \tilde S_t \, d\sigma dv_* \\ &\lesssim \|\phi_t\|_{L^\infty} \int \Big[1 + \omega_*\Big] \frac{|u|}{\omega} \tilde S_t \, dv_* \\ &\lesssim k_t \|\phi_t\|_{L^\infty}, \quad k_t := \int S_{t*} (1 + |v_*|^3) \, dv_*\end{aligned}$$

Gronwall lemma implies

$$(1) \qquad \|\phi_t\|_{L^\infty} \le \|\phi_0\|_{L^\infty} e^{K_t}, \quad K_t := \int_0^t k_s \, ds \le \kappa (t+1).$$

Rigorous proof of L^∞ estimate (G. Toscani has already used a similar trick)

We rather consider truncated evolution equation

$$\partial_t \phi = \iint \Big[\phi'_* \omega'_* + \phi' \omega' - \phi_* \omega_* - \phi \omega \Big] \frac{B_n}{\omega} \tilde S_{t*} \, d\sigma dv_*,$$

with $B_n := |u| \wedge n$, $n \in \mathbb{N}$ fixed. For any $v \in \mathbb{R}^d$, we have

$$\begin{aligned} \partial_t \phi_t(v) &\le k_t \|\phi_t\|_{L^\infty} + \lambda_{nt}(v) \, [\|\phi_t\|_{L^\infty} - \phi_t(v)] \\ &\le k_t \|\phi_t\|_{L^\infty} + \bar\lambda_{nt} \, [\|\phi_t\|_{L^\infty} - \phi_t(v)], \end{aligned}$$

where we define

$$\lambda_{nt}(v) := \int B_n \, \tilde S_{t*} d\sigma dv_*, \quad \bar\lambda_{nt} := \|\lambda_{nt}\|_{L^\infty}, \quad \bar\Lambda_{nt} := \int_0^t \bar\lambda_{ns} \, ds.$$

Using one time the Gronwall lemma implies

$$u_t \le u_0 + \int_0^t (k_s + \bar\lambda_{ns}) u_s \, ds, \quad u_t := \|\phi_t\|_{L^\infty} e^{\bar\Lambda_{nt}}.$$

Using a second time the Gronwall lemma implies

$$u_t \le u_0 \, e^{\,K_t + \bar\Lambda_{nt}},$$

which is nothing but (1) which holds independently of the truncation parameter

We introduce $\psi_t := \nabla \phi_t$ and consider its evolution equation

$$\begin{aligned}\partial_t \psi &= \iint \Big[\psi'_* \omega'_* + \psi' \omega' - \psi_* \omega_* - \psi \omega\Big] \frac{|u|}{\omega} \tilde{S}_{t*}\, d\sigma dv_* \\ &\quad + \iint \Big[\phi'_* \xi'_* + \phi' \xi' - \phi_* \xi_* - \phi \xi\Big] \tilde{S}_{t*}\, d\sigma dv_*,\end{aligned}$$

where

$$\xi(w) := \nabla_v [\omega(w) \frac{|v - v_*|}{\omega(v)}].$$

We compute

$$\xi = \hat{u}, \quad \xi_* = \omega_* (\frac{\hat{u}}{\omega} + 2 \frac{|u| v}{\omega^2}), \quad \xi' = (v' + (v' \cdot \hat{u})\sigma) \frac{|u|}{\omega} + \omega' (\frac{\hat{u}}{\omega} + 2 \frac{|u| v}{\omega^2}),$$

and similarly ξ'_*, so that the 4 terms are bounded by $C \langle v_* \rangle$. Formally, we thus obtain

$$\partial_t \|\psi\|_{L^\infty} \lesssim k_t \|\psi\|_{L^\infty} + \|\phi\|_{L^\infty}.$$

All together, we have established

$$\|\phi_t\|_{W^{1,\infty}} \lesssim e^{\kappa t} \|\phi_0\|_{W^{1,\infty}}.$$

Plan

• for an infinite system of indistinguishable particles: Boltzmann's (molecular) chaos means

$$\mathcal{L}(\mathcal{V}_i, \mathcal{V}_j) = f \otimes f$$

That is the stochastic independence (for a sequence of exchangeable random variables)

• for a system of N indistinguishable particles with $N \to \infty$: Kac's chaos means

$$\mathcal{L}(\mathcal{V}_i^N, \mathcal{V}_j^N) \to f \otimes f \quad \text{as} \quad N \to \infty$$

That is an asymptotically stochastic independence (of the coordinates of a sequence of random vectors with exchangeable coordinates)

• For N fixed particles the states $\mathcal{Z}_1(t)$, ..., $\mathcal{Z}_N(t)$ are **never independent** for positive time $t > 0$ even if the initial states $\mathcal{Z}_1(0), ..., \mathcal{Z}_N(0)$ are assumed to be independent : that is an inherent consequence of the fact that **particles do interact!**

• Equations are written in spaces with increasing dimension $N \to \infty$.
To overcome that difficulty we will work in **fixed spaces** using:
empirical probability measure

$$X \in E^N \mapsto \mu^N_X := \frac{1}{N} \sum_{i=1}^N \delta_{x_i} \in \mathbb{P}(E)$$

and/or marginal densities

$$F^N \in \mathbb{P}_{sym}(E^N) \ \mapsto \ F^N_j := \int_{E^{N-j}} F^N dz_{j+1}...dz_N \in \mathbb{P}_{sym}(E^j)$$

• The nonlinear PDE can be obtained as a *"law of large numbers"* for a **not independent array of exchangeable random variables** in the mean-field limit.

• That is more demanding that the usual LLN. We need to **propagate** some asymptotic independence = Kac's stochastic chaos.

• Kac (1956) defined the notion of chaos for sequences of random vectors. He proved the propagation of chaos for the *"Kac's caricature"* of Boltzmann model. He showed that the stochastic dynamic leaves invariant the Kac's sphere

$$\mathcal{KS}^N := \{V \in \mathbb{R}^N;\ |v_1|^2 + ... + |v_N|^2 = N\},$$

and, for any fixed $N \geq 2$, convergence to the equilibrium (stationary measure)

$$F_t^N = \mathcal{L}(\mathcal{V}_{1t}^N, ..., \mathcal{V}_{Nt}^N) \underset{t\to\infty}{\longrightarrow} \gamma^N = \text{ uniform measure on } \ \mathcal{KS}^N.$$

Kac's Program:

(Pb1) Establish propagation of chaos for more realistic (singular) models

(Pb2) Establish the convergence to the equilibrium as $t \to \infty$ with a uniform chaos rate with respect to the number N of particles

(Pb2′) Establish quantitative chaos estimate (rate) for Kac's chaos

(Pb3) Establish Boltzmann's H-theorem from a microscopic description (seems to be Kac's motivation)

Chaos is the asymptotic independence as $N \to \infty$ for a sequence $(\mathcal{Z}^N)$ of exchangeable random variables with values in E^N

$$\begin{array}{ccc} \mathcal{Z}^N = (\mathcal{Z}_1^N, ..., \mathcal{Z}_N^N) \in E^N & \to & F^N := \mathcal{L}(\mathcal{Z}^N) \in \mathbb{P}_{sym}(E^N) \\ \updownarrow & & \updownarrow \\ \mu^N_{\mathcal{Z}^N} := \frac{1}{N} \sum_{i=1}^N \delta_{\mathcal{Z}_i^N} \in \mathbb{P}(E) & \to & \hat{F}^N := \mathcal{L}(\mu^N_{\mathcal{Z}^N}) \in \mathbb{P}(\mathbb{P}(E)) \end{array}$$

For a random variable $\mathcal{Y}$ taking values in E with law $\mathcal{L}(\mathcal{Y}) = f \in \mathbb{P}(E)$ we say that $(\mathcal{Z}^N)$ is $\mathcal{Y}$-Kac's chaotic if

- $\mathcal{L}(\mathcal{Z}_1^N, ..., \mathcal{Z}_j^N) \rightharpoonup f^{\otimes j}$ weakly in $\mathbb{P}(E^j)$ as $N \to \infty$;
- $\mu^N_{\mathcal{Z}^N} \Rightarrow f$ in law as $N \to \infty$,
  meaning $\mathcal{L}(\mu^N_{\mathcal{Z}^N}) \to \delta_f$ in $\mathbb{P}(\mathbb{P}(E))$ as $N \to \infty$;
- $\mathbb{E}(|\mathcal{X}^N - \mathcal{Y}^N|) \to 0$ as $N \to \infty$ for a sequence $\mathcal{Y}^N$ of i.i.d.r.v with law f

Exchangeable means: $\mathcal{L}(\mathcal{Z}^N_{\sigma(1)}, ..., \mathcal{Z}^N_{\sigma(N)}) = \mathcal{L}(\mathcal{Z}_1^N, ..., \mathcal{Z}_N^N)$ for any permutation σ of the coordinates

Definition of chaos = not about random variables but their laws !

For a given sequence (F^N) in $\mathbb{P}_{sym}(E^N)$ we define

- the marginals $F_j^N \in \mathbb{P}_{sym}(E^j)$ by

$$F_j^N = \int_{E^{N-j}} F^N dz_{j+1} ... dz_N$$

- the projection $\hat{F}^N \in \mathbb{P}(\mathbb{P}(E))$ by

$$\langle \hat{F}^N, \Phi \rangle = \int_{E^N} \Phi(\mu_X^N)\, F^N(dX) \quad \forall\, \Phi \in C_b(\mathbb{P}(E))$$

- the normalized MKW distance on $\mathbb{P}(E^j)$ by

$$W_1(F,G) := \inf_{\pi \in \Pi(F,G)} \int_{E^j \times E^j} \Big(\frac{1}{j} \sum_{i=1}^{j} |x_i - x_j| \ \wedge 1 \Big)\, \pi(dX, dY).$$

- the MKW distance $\mathcal{W}_1$ on $\mathbb{P}(\mathbb{P}(E))$ by

$$\mathcal{W}_1(\alpha, \beta) := \inf_{\pi \in \Pi(\alpha,\beta)} \int_{\mathbb{P}(E)\times\mathbb{P}(E)} W_1(\rho, \eta)\, \pi(d\rho, d\eta).$$

For a given sequence (F^N) in $\mathbb{P}_{sym}(E^N)$ we define

- the marginals $F^N_j \in \mathbb{P}_{sym}(E^j)$,
- the projection $\hat{F}^N \in \mathbb{P}(\mathbb{P}(E))$,
- the normalized MKW distance W_1 on $\mathbb{P}(E^j)$,
- the MKW distance $\mathcal{W}_1$ on $\mathbb{P}(\mathbb{P}(E))$,

and for $f \in \mathbb{P}(E)$ we say that (F^N) is f-Kac's chaotic if (equivalently)

- $\mathcal{D}_j(F^N; f) := W_1(F^N_j, f^{\otimes j}) = \mathbb{E}(|(\mathcal{X}^N_1, ..., \mathcal{X}^N_j) - (\mathcal{X}^N_1, ..., \mathcal{X}^N_j)|) \to 0$
- $\mathcal{D}_\infty(F^N; f) := \mathcal{W}_1(\hat{F}^N, \delta_f) = \mathbb{E}(W_1(\mu^N_{\mathcal{Z}^N}, f) \to 0$

More precisely, for $E = \mathbb{R}^d$

Lemma (Hauray, M.)

For given M and $k > 1$ there exist some constants $\alpha_i, C > 0$ such that $\forall\, f \in \mathbb{P}(E)$, $\forall\, F^N \in \mathbb{P}_{sym}(E^N)$ with $M_k(F^N_1), M_k(f) \le M$

$$\forall\, j, \ell \in \{1, ..., N, \infty\},\ \ell \neq 1 \quad \mathcal{D}_j(F^N; f) \le C \left(\mathcal{D}_\ell(F^N; f)^{\alpha_1} + \frac{1}{N^{\alpha_2}}\right).$$

Stronger chaos: entropy and Fisher's chaos

For $F^N \in \mathbb{P}_{sym}(E^N)$, $E = \mathbb{R}^d$, we define the normalized functionals

$$H(F^N) := \frac{1}{N} \int_{E^N} F^N \log F^N, \quad I(F^N) := \frac{1}{N} \int_{E^N} \frac{|\nabla F^N|^2}{F^N}.$$

Definition

Consider a sequence $F^N \in \mathbb{P}_{sym}(E^N)$ and $f \in \mathbb{P}(E)$

(F^N) is f-entropy chaotic if $F_1^N \rightharpoonup f$ weakly in $\mathbb{P}(E)$ and $H(F^N) \to H(f)$

(F^N) is f-Fisher's chaotic if $F_1^N \rightharpoonup f$ weakly in $\mathbb{P}(E)$ and $I(F^N) \to I(f)$

Theorem (Hauray, M.)

In the list below, each assertion implies the one which follows

(i) (F^N) *is Fisher's chaotic;*

(ii) (F^N) *is Kac's chaotic and* $I(F^N)$ *is bounded;*

(iii) (F^N) *is entropy chaotic;*

(iv) (F_j^N) *converges in* L^1 *for any* $j \geq 1$*;*

(v) (F^N) *is Kac's chaotic.*

Extensions by Carrapatoso, Fournier, Guillin, Hauray, M.

• Kac's chaos, entropic chaos and Fisher's chaos on Kac's spheres and on Boltzmann's spheres

• For a mixture of probability measures = without chaos hypothesis

• Optimal rate of convergence of $\mathcal{D}_\infty(f^{\otimes N}, f) \sim N^{1/d}$ for $f \in \mathbb{P}_q(\mathbb{R}^d)$, $d \geq 2$

Based on many previous works from Funct Analysis, Proba, Stat, Geo, ...

• Mixture: de Finetti (1937), Hewitt-Savage (1955), Robinson-Ruelle (1967)

• Functional and quantified LLN (Glivenko-Cantelli ... Rachev-Rüschendorf ... Barthe-Bordenave)

• local central limit theorem of Berry-Esseen

• HWI inequality of Otto and Villani

• Entropy inequalities: Carlen-Lieb-Loss (2004), Arstein-Ball-Barthe-Naor (2004)

• previous comparison, quantitative and qualitative results on chaos
Kac: Foundations of kinetic theory. (1956)
Sznitman: Topics in propagation of chaos. Saint-Flour -1989 (1991)
Carlen, Carvalho, Le Roux, Loss, Villani: Entropy and chaos ... (2010)

Plan

Theorem (Quantified chaos estimate via semigroup method)

For any $f_0 \in \mathbb{P}(E)$ + *conditions,*

$$\sup_{t\in[0,T]} \|F_k^N - f^{\otimes k}\|_{W^{-2,\infty}} \lesssim \frac{1}{N^{1-\bullet}} + \mathcal{D}_\infty(F_0^N, f_0) \sim \frac{1}{N^{1/d}}$$

$$\sup_{t\geq 0} \|F_k^N - f^{\otimes k}\|_{W^{-2,\infty}} \lesssim \frac{1}{N^{1-\bullet}} + \mathcal{D}_\infty(F_0^N, f_0)^{\bullet} \sim \frac{1}{N^{\bullet}}$$

$\triangleright$ biblio: Kac (1956), McKean (1967), Grünbaum (1971), Sznitman (1984), Fontbona-Guérin-Méléard (2009), Fournier (2009), M.-Mouhot (2013), Carrapatoso (2016), Fournier-Hauray (2016), Fournier-M. (2016), Norris (2016), Fournier-Guillin (2017), Cortez-Fontbona (2018), Heydecker (2019), M.-Mouhot-Norris (xxxx)

Remark. The functional LLN

$$\mathcal{D}_\infty(F_0^N, f_0) \sim \frac{1}{N^{1/d}}$$

is due to Fournier-Guillin (2015).

We split

$$\begin{aligned}\left\langle F^N_{kt} - f_t^{\otimes k}, \varphi \right\rangle &= \left\langle F^N_t - f_t^{\otimes N}, \varphi \otimes 1^{\otimes N-k} \right\rangle = \\ &= \left\langle F^N_t, \varphi \otimes 1^{\otimes N-k} - R_\varphi(\mu^N_V) \right\rangle \qquad (= T_1) \\ &\quad + \left\langle F^N_t, R_\varphi(\mu^N_V) \right\rangle - \left\langle F^N_0, R_\varphi(S^{NL}_t \mu^N_V) \right\rangle \qquad (= T_2) \\ &\quad + \left\langle F^N_0, R_\varphi(S^{NL}_t \mu^N_V) \right\rangle - \left\langle f_t^{\otimes k}, \varphi \right\rangle \qquad (= T_3)\end{aligned}$$

where R_φ is the "polynomial function" on $\mathbb{P}(\mathbb{R}^3)$ defined by

$$R_\varphi(\rho) = \int_{E^k} \varphi \, \rho(dv_1) \dots \rho(dv_k)$$

and S^{NL}_t is the nonlinear semigroup associated to the nonlinear mean-field limit equation by $g_0 \mapsto S^{NL}_t g_0 := g_t$.

$$
\begin{aligned}
|T_1| &= \left|\left\langle F_t^N, \varphi \otimes 1^{\otimes(N-k)}(V) - R_\varphi(\mu_V^N)\right\rangle\right| \\
&= \left|\left\langle F_t^N, \varphi \otimes \widetilde{1^{\otimes(N-k)}}(V) - R_\varphi(\mu_V^N)\right\rangle\right| \\
&\le \left\langle F_t^N, \frac{2\,k^2}{N}\,\|\varphi\|_{L^\infty(E^k)}\right\rangle = \mathcal{O}\Big(\frac{1}{N}\Big)
\end{aligned}
$$

where we use that F^N is symmetric and a probability and we introduce the symmetrization function associated to $\varphi \otimes 1^{\otimes(N-k)}$ by

$$
\varphi \otimes \widetilde{1^{\otimes(N-k)}}(V) = \frac{1}{\sharp \mathfrak{S}_N} \sum_{\sigma \in \mathfrak{S}_N} \varphi \otimes 1^{\otimes(N-k)}(V_\sigma).
$$

$$
\begin{aligned}
|T_3| &= \left|\left\langle F_0^N, R_\varphi(S_t^{NL}\mu_V^N) - R_\varphi(S_t^{NL} f_0)\right\rangle\right| \\
&\le [R_\varphi]_{C^{0,1}} \left\langle F_0^N, W_1(S_t^{NL}\mu_V^N, S_t^{NL} f_0)\right\rangle \\
&\le k\,\|\nabla\varphi\|_{L^\infty(E^k)}\, e^{\kappa t} \left\langle F_0^N, W_1(\mu_V^N, f_0)\right\rangle \\
&\lesssim e^{\kappa t}\,\mathcal{D}_\infty(F_0^N, f_0) \\
&\lesssim \mathcal{D}_\infty(F_0^N, f_0)^{\frac{\lambda}{\kappa+\lambda}}
\end{aligned}
$$

where

$$
[R_\varphi]_{C^{0,1}} := \sup_{W_1(\rho,\eta)\le 1} |R_\varphi(\eta) - R_\varphi(\rho)| = k\,\|\nabla\varphi\|_{L^\infty}
$$

because we have established / we may prove that the nonlinear flow satisfies

$$
\begin{aligned}
&(A5) \qquad W_1(f_t, g_t) \lesssim e^{\kappa t}\, W_1(f_0, g_0) \\
&(A5') \qquad W_1(f_t, g_t) \le W_1(f_0, g_0)^{\frac{\lambda}{\kappa+\lambda}}
\end{aligned}
$$

T_2 : We write

$$T_2 \quad = \quad \left\langle F_t^N, R_\varphi(\mu_V^N) \right\rangle - \left\langle F_0^N, R_\varphi(S_t^{NL} \mu_V^N) \right\rangle$$

T_2 : We write

$$\begin{aligned} T_2 &= \left\langle F_t^N, R_\varphi(\mu_V^N)\right\rangle - \left\langle F_0^N, R_\varphi(S_t^{NL}\mu_V^N)\right\rangle \\ &= \left\langle F_0^N, T_t^N(R_\varphi \circ \mu_V^N) - (T_t^\infty R_\varphi)(\mu_V^N)\right\rangle \end{aligned}$$

with

- $T_t^N =$ dual semigroup (acting on $C_b(E^N)$) of the N-particle flow $F_0^N \mapsto F_t^N$;
- $T_t^\infty =$ pushforward semigroup (acting on $C_b(\mathbb{P}(E))$) of the nonlinear semigroup S_t^{NL} defined by $(T^\infty\Phi)(\rho) := \Phi(S_t^{NL}\rho)$;

T_2 : We write

$$\begin{aligned} T_2 &= \left\langle F_t^N, R_\varphi(\mu_V^N) \right\rangle - \left\langle F_0^N, R_\varphi(S_t^{NL}\mu_V^N) \right\rangle \\ &= \left\langle F_0^N, T_t^N(R_\varphi \circ \mu_V^N) - (T_t^\infty R_\varphi)(\mu_V^N) \right\rangle \\ &= \left\langle F_0^N, (T_t^N \pi_N - \pi_N T_t^\infty)\, R_\varphi \right\rangle \end{aligned}$$

with

- $T_t^N =$ dual semigroup (acting on $C_b(E^N)$) of the N-particle flow $F_0^N \mapsto F_t^N$;
- $T_t^\infty =$ pushforward semigroup (acting on $C_b(\mathbb{P}(E))$) of the nonlinear semigroup S_t^{NL} defined by $(T^\infty \Phi)(\rho) := \Phi(S_t^{NL}\rho)$;
- $\pi_N =$ projection $C(\mathbb{P}(E)) \to C(E^N)$ defined by $(\pi_N \Phi)(V) = \Phi(\mu_V^N)$.

Semigroup method - idea 4 : duality + consistency + stability

$$
\begin{aligned}
T_2 &= \left\langle F_0^N, (T_t^N \pi_N - \pi_N T_t^\infty)\, R_\varphi \right\rangle \\
&= \left\langle F_0^N, \int_0^T T_{t-s}^N\, (\Lambda^N \pi_N - \pi_N \Lambda^\infty)\, T_s^\infty \, ds\, R_\varphi \right\rangle \\
&= \int_0^T \left\langle F_{t-s}^N, (\Lambda^N \pi_N - \pi_N \Lambda^\infty)\, (T_s^\infty\, R_\varphi) \right\rangle \, ds = \mathcal{O}\left(\frac{1}{N^{1-\bullet}}\right)
\end{aligned}
$$

where

- Λ^N is the generator associated to T_t^N and Λ^∞ is the generator associated to T_t^∞.

Now we have to make some assumptions

- (A1) F_t^N has enough bounded moments;
- (A2) $\Lambda^\infty \Phi(\rho) = \langle Q(\rho), D\Phi(\rho)\rangle$;
- (A3) $(\Lambda^N \pi^N \Phi)(V) = \langle Q(\mu_V^N), D\Phi(\mu_V^N)\rangle + \mathcal{O}([\Phi]_{C^{1,a}}/N)$
- (A4) $S_t^{NL} \in C^{1,a}(\mathbb{P}(E); \mathbb{P}(E))$ "uniformly" in time $t \in [0, T]$