Semigroup methods for evolution PDE

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Outline of the talk

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Evolution equation and semigroup

We consider an evolution equation

$$\partial_t f = \mathcal{L} f, \quad f(0) = f_0,$$

and the associated semigroup of operators $S_\mathcal{L}(t)$ defined through the relation $S_\mathcal{L}(t) f_0 := f(t)$ on a Banach space X. Our purpose is then to explain when and how we can show that the semigroup splits as

$$S_\mathcal{L}(t) = S_0(t) + S_1(t),$$

where

$$\begin{cases} S_1(t) \text{ ranges in a finite dimensional non trivial subspace of } X \\ \text{and } \|S_0(t)\| = o(\|S_1(t)\|) \text{ as } t \to \infty. \end{cases}$$

Better, we would like to identify some cases where, if possible in a quantitative/constructive way,

$$\lim_{t\to\infty} \|e^{-s(\Lambda)t} S(t) - P\| = 0,$$

for some projector $P \in \mathcal{B}(X)$ (with rank $P = 1$ if possible!) and real number (spectral bound) $s(\Lambda) \in \mathbb{R}$.

Framework

- X Banach space, possibly
  - a Hilbert space (or not),
  - a Banach lattice with positive cone $X_+ := \{f \geq 0\}$ (or not).

Typically $X = L^p$, $X = C_0$ or $X = M^1$ or a weighted such spaces

- $S = (S_t)$ a positive semigroup on X (of linear operators):
  - $S_t \in \mathcal{B}(X)$, $S_{t_1} S_{t_2} = S_{t_1+t_2}$, $S_0 = I$,
  - strongly or weakly $*$ continuous trajectories,
  - $\|S_t\|_{X\to X} \leq M e^{\kappa_1 t}$, $M \geq 1$, $\kappa_1 \in \mathbb{R}$,
  - the generator $\mathcal{L}$ splits as

$$\mathcal{L} = \mathcal{A} + \mathcal{B}, \quad \mathcal{A} \prec \mathcal{B}, \quad S_{\mathcal{B}}(t) = O(e^{\kappa_{\mathcal{B}} t}), \ \kappa_{\mathcal{B}} < \kappa_1$$

- Examples: A general elliptic/Fokker-Planck operator, the growth-fragmentation operator and the age structured operator

$$\begin{aligned}
\mathcal{L}f &= \mathrm{div}(a\nabla f) + b \cdot \nabla f + cf, \quad (\text{for FP: } c = \mathrm{div}\ \mathrm{b}) \\
&= -a \cdot \nabla f - Kf + \int k f_* \, dy_* \\
&= -\partial_x f - Kf + \delta_0 \int_0^\infty K(y) f(y)\, dy
\end{aligned}$$

Framework

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- Examples: A general elliptic/Fokker-Planck operator, the growth-fragmentation operator and the age structured operator

$$\begin{aligned}
\mathcal{L}f &= \operatorname{div}(a\nabla f) + b \cdot \nabla f + cf - M\chi_R f + M\chi_R f \\
&= -a \cdot \nabla f - Kf + \int k_R^c f_* \, dy_* + \int k_R f_* \, dy_* \\
&= - - \partial_x f - Kf + \delta_0 \int_0^\infty K(y) f(y) \, dy
\end{aligned}$$

Spectral analysis and semigroup analysis

• describe spectrum set $\Sigma(\mathcal{L})$, set of its eigenvalues and associated eigenspaces

• *spectral mapping theorem*

$$\Sigma(e^{t\mathcal{L}})\backslash\{0\} = e^{t\Sigma(\mathcal{L})}, \qquad \forall\, t \ge 0$$

• *Extension of the spectral analysis to other spaces:* enlargement/shrinkage

• *Weyl's theorem* on compact perturbation and discrete spectrum or partial (but principal) spectral mapping theorem

$$\Sigma(e^{t\mathcal{L}})\backslash B(0, e^{at}) = e^{t\Sigma(\mathcal{L})\cap\Delta_a}, \qquad \forall\, t \ge 0,\ \forall\, a > a^*,$$

for some abscissa $a^* \in \mathbb{R}$, where $\Delta_a := \{\xi \in \mathbb{C};\ \Re e\xi > a\}$ the half-plane $\forall a \in \mathbb{R}$ and deduce the asymptotical behaviour of trajectories

• *Small perturbation theorem*

• *Self-adjointeness, spectral gap, related coercivity estimates and beyond: hypocoercivity estimates*

• *Krein-Rutman Theorem for positive semigroup*

• *Doblin-Harris Theorem for Markov/stochastic semigroup*

Outline of the talk

Dissipative and hypodissipative generator

Consider a semigroup $S_{\mathcal{B}}$ with generator $\mathcal{B}$ in a Banach space X with norm $\|\cdot\|$.
We say that $\mathcal{B} - a$ is dissipative if

$$\forall f \in D(\mathcal{B}),\ \forall f^* \in J_f, \quad \Re e\langle f^*, (\mathcal{B} - a)f\rangle \le 0 \tag{1}$$

or equivalently

$$\Re e\langle f^*, \mathcal{B} f\rangle \le a\|f\|^2,$$

where J_f is the dual set

$$J_f := \{\varphi \in X';\ \langle \varphi, f\rangle = \|f\|_X^2 = \|\varphi\|_{X'}^2\}.$$

By Hahn-Banach separation theorem $J_f \neq \emptyset$.
When X is an Hilbert space then $J_f = \{f\}$, we say that $\mathcal{B} - a$ is coercive.
When $X = L^p$, $1 \le p < \infty$, then $J_f := \{cf|f|^{p-2}\}$.
We say that $\mathcal{B} - a$ is hypodissipative if (1) holds for any $f^* \in J_{f,|||\cdot|||}$, with

$$J_{f,|||\cdot|||} := \{\varphi \in X';\ \langle \varphi, f\rangle = |||f|||^2 = |||\varphi|||_{X'}^2\},$$

where $|||\cdot|||$ stands for an equivalent norm in X.

Hypodissipative and growth/decay estimate : Hille-Yosida, Lumer-Phillips

Consider a dissipative semigroup $S_{\mathcal{L}}$ with generator $\mathcal{L}$ in a Banach space X. For $a \in \mathbb{R}$, $M \geq 1$, there is equivalence between
(a) $\mathcal{L} - a$ is hypodissipative, and the norm of dissipativity satisfies

$$\forall f \in X \qquad \|f\| \leq |||f||| \leq M \|f\|; \tag{2}$$

(b) the semigroup $S_{\mathcal{L}}$ satisfies the growth estimate

$$\|S_{\mathcal{L}}(t)\|_{\mathcal{B}(X)} \leq M e^{a t}, \quad \forall t \geq 0. \tag{3}$$

We define $\omega(S) := \inf\{a \in \mathbb{R};\ (3) \text{ holds}\}$ the growth bound.

Proof of (a) $\Rightarrow$ (b) for a equivalent regular norm such that the square norm function $\Phi(f) := |||f|||^2/2$ satisfies

$$\Phi : X \to \mathbb{R}_+ \text{ G-differentiable and } \quad J_{f,|||\cdot|||} = \{\Phi'(f)\}, \quad \forall f \in X.$$

We compute

$$\frac{d}{dt}|||f|||^2 = \Re e\langle \Phi'(f), \mathcal{L}f\rangle \leq a|||f|||^2,$$

and we use the Gronwall lemma.

The reverse implication (b) $\Rightarrow$ (a)

By assumption

$$\|S(t)\|_{\mathcal{B}(X)} \le M\, e^{\alpha\, t}, \quad \Re e\, \langle f^*, \mathcal{L} f\rangle \le b\, \|f\|^2 \ \ \forall\, f \in D(\mathcal{L}),$$

with $M \ge 1$, $a^* \le \alpha < a < b \in \mathbb{R}$, and where $J_{f,\|\cdot\|} = \{f^*\}$. We define the new norm

$$|\!|\!| f |\!|\!|^2 := \eta\, \|f\|^2 + \int_0^\infty \|S(\tau)\, e^{-a\tau}\, f\|^2\, d\tau.$$

With $f_t := S(t) f$, we compute

$$\frac{1}{2}\frac{d}{dt} |\!|\!| f_t |\!|\!|^2 \le a |\!|\!| f_t |\!|\!|^2,$$

by choosing $\eta > 0$ small enough, and

$$\frac{1}{2}\frac{d}{dt} |\!|\!| f_t |\!|\!|^2 = \Re e\, \langle (f_t)^{**}, \mathcal{L} f_t\rangle$$

with

$$g^{**} := \eta\, g^* + \int_0^\infty S_{\mathcal{L}}(\tau)^* (S_{\mathcal{L}}(\tau) g)^*\, d\tau \in X', \quad \forall\, g \in X.$$

Hypocoercivity $\simeq$ twisted norm

Duhamel formulas

Consider $S_{\mathcal{L}}$ a semigroup with generator $\mathcal{L}$ enjoying the splitting structure

$$\mathcal{L} = \mathcal{A} + \mathcal{B}, \quad \mathcal{B} \text{ generator of } S_{\mathcal{B}}, \ \mathcal{A} \prec \mathcal{B}.$$

Typically $\mathcal{A} \in \mathcal{B}(X)$. The following Duhamel formulas

$$S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{B}}\mathcal{A} * S_{\mathcal{L}} = S_{\mathcal{B}} + S_{\mathcal{L}} * \mathcal{A}S_{\mathcal{B}}$$

hold, as well as the iterated Duhamel formulas (or "stopped" Dyson-Phillips series: the Dyson-Phillips series corresponds to the choice $N = \infty$)

$$\begin{aligned} S_{\mathcal{L}} &= S_{\mathcal{B}} + \cdots + (S_{\mathcal{B}}\mathcal{A})^{*(N-1)} * S_{\mathcal{B}} + (S_{\mathcal{B}}\mathcal{A})^{*N} * S_{\mathcal{L}} \\ &= S_{\mathcal{B}} + \cdots + S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{*(N-1)} + S_{\mathcal{L}} * (\mathcal{A}S_{\mathcal{B}})^{*N}. \end{aligned}$$

Here we define $V * U$ by

$$t \mapsto (V * U)(t) := \int_0^t V(t-s)U(s)ds \ \in \ L^1_{loc}(\mathbb{R}_+; \mathcal{B}(X_1; X_3)),$$

for $U \in L^1_{loc}(\mathbb{R}_+; \mathcal{B}(X_1; X_2))$ and $V \in L^1_{loc}(\mathbb{R}_+; \mathcal{B}(X_2; X_3))$.

Enlargement and shrinkage of the functional space for semigroup growth

Th 1. Assume

$$\mathcal{L} = \mathcal{A} + \mathcal{B}, \ \ L = A + B, \ \ A = \mathcal{A}_{|E}, \ \ B = \mathcal{B}_{|E}, \ \ E \subset \mathcal{E}$$

For any $a > a^*$

- **(i)** $(B - a)$ is hypodissipative on E, $(\mathcal{B} - a)$ is hypodissipative on $\mathcal{E}$;
- **(ii)** $A \in \mathcal{B}(E)$, $\mathcal{A} \in \mathcal{B}(\mathcal{E})$;
- **(iii)** there is $n \geq 1$ and $C_a > 0$ such that

$$\left\|(S_{\mathcal{B}}\mathcal{A})^{(*n)}(t)\right\|_{\mathcal{E}\to E} + \left\|(\mathcal{A}S_{\mathcal{B}})^{(*n)}(t)\right\|_{\mathcal{E}\to E} \leq C_a\, e^{at}.$$

Then there is equivalence between

$$\forall\, t \geq 0, \quad \left\|S_{\mathcal{L}}(t)\right\|_{\mathcal{E}\to\mathcal{E}} \leq C_{\mathcal{L},a}\, e^{a\,t}$$

and

$$\forall\, t \geq 0, \quad \left\|S_L(t)\right\|_{E\to E} \leq C_{L,a}\, e^{a\,t}.$$

$\triangleright$ Bobylev (Boltzmann), Gallay-Wayne (harmonic Fokker-Planck), Gualdani-M.-Mouhot (abstract and applications)

Proof of the change of functional space : as an immediate consequence of the iterated Duhamel formula

$S_L = \mathcal{O}(e^{at})$ implies $S_\mathcal{L} = \mathcal{O}(e^{at})$:

$$S_\mathcal{L} = \underbrace{S_\mathcal{B} + \cdots + S_\mathcal{B} * (\mathcal{A} S_\mathcal{B})^{*(N-1)}}_{\mathcal{E}\to\mathcal{E}} + \underbrace{S_L}_{E\to E\subset\mathcal{E}} * \underbrace{(\mathcal{A} S_\mathcal{B})^{*N}}_{\mathcal{E}\to E}.$$

$S_\mathcal{L} = \mathcal{O}(e^{at})$ implies $S_L = \mathcal{O}(e^{at})$:

$$S_\mathcal{L} = \underbrace{S_\mathcal{B} + \cdots + (S_\mathcal{B} \mathcal{A})^{*(N-1)} * S_\mathcal{B}}_{E\to E} + \underbrace{(S_\mathcal{B} \mathcal{A})^{*N}}_{\mathcal{E}\to E} * \underbrace{S_\mathcal{L}}_{E\subset\mathcal{E}\to\mathcal{E}}$$

because $e_a * e_a = t e_a \le e_{a'}$ for any $a' > a > a^*$, with $e_a(t) := e^{at}$

Example 1 : the Fokker-Planck equation

We consider the Fokker-Planck equation

$$\partial_t f = \mathcal{L} f = \Delta f + \mathrm{div}(E f)$$

on $f = f(t, x)$, $t \geq 0$, $x \in \mathbb{R}^d$, with force confinement

$$E = \nabla \frac{\langle x \rangle^\gamma}{\gamma} = x \langle x \rangle^{\gamma - 2}, \quad \gamma > 0.$$

Th 1'. For any $k \geq 0$ and $p \in [1, \infty]$, there exists a constant $M \geq 1$ such that

$$\sup_{t \geq 0} \|f_t\|_{L^p_k} \leq M \|f_0\|_{L^p_k}$$

with

$$\|f\|_{L^p_k} := \|f \langle x \rangle^k\|_{L^p}, \quad \langle x \rangle^2 := 1 + |x|^2.$$

▷ Toscani-Villani, Röckner-Wang, Kavian-M.-Ndao

Elements of proof

We observe that

$$\frac{d}{dt} \int f dx = 0,$$

so that mass is conserved !

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Elements of proof

Similarly

$$\frac{d}{dt} \int |f| dx \le 0,$$

so that

$$S_{\mathcal{L}} : L^1 \to L^1, \quad \text{uniformly bounded.}$$

The idea is to use the shrinkage result taking advantage of the splitting structure

$$\partial_t f = \mathcal{L} f = \underbrace{\partial_{xx} f + \partial_x (x^{\gamma-1} f) - M\chi_R f}_{=:\mathcal{B} f} + \underbrace{M \chi_R f}_{=:\mathcal{A} f}$$

L^1_k estimate for $S_{\mathcal{L}}$ when $\gamma \geq 2$

$\mathcal{L}$ satisfies the (strong for $\gamma \geq 2$, weak for $\gamma < 2$) Lyapunov condition

$$\mathcal{L}^* \langle x \rangle^k \lesssim -\langle x \rangle^{k+\gamma-2} + \mathbf{1}_{B_R},$$

because

$$\partial_{xx} x^k - x^{\gamma-1} \partial_x x^k \sim -k x^{k+\gamma-2}.$$

When $\gamma \geq 2$, we may proceed in a very simple way :

$$\begin{aligned} \frac{d}{dt} \int f \langle x \rangle^k &\lesssim -\int f \langle x \rangle^k + \int f \\ &\lesssim -\int f \langle x \rangle^k + \int f_0, \end{aligned}$$

and thanks to the Gronwall lemma we conclude directly

$$S_{\mathcal{L}} : L^1_k \to L^1_k \quad \text{uniformly bounded.}$$

L^1_k estimate for $S_\mathcal{L}$ (general case)

We write

$$f_t = S_\mathcal{B}(t) f_0 + (S_\mathcal{B}\mathcal{A} * S_\mathcal{L})(t) f_0$$

and we next compute

$$\begin{aligned}
\|f_t\|_{L^1_k} &\le \|S_\mathcal{B}(t) f_0\|_{L^1_k} + \int_0^t \|S_\mathcal{B}(t-s)\mathcal{A}S_\mathcal{L}(s) f_0\|_{L^1_k}\, ds \\
&\le \|f_0\|_{L^1_k} + \int_0^t \Theta(t-s)\|\mathcal{A}S_\mathcal{L}(s) f_0\|_{L^1_m}\, ds \\
&\lesssim \|f_0\|_{L^1_k} + \int_0^t \Theta(t-s)\|S_\mathcal{L}(s) f_0\|_{L^1}\, ds \\
&\le \|f_0\|_{L^1_k} + \int_0^t \Theta(t-s)\|f_0\|_{L^1}\, ds \\
&\le (1 + \|\Theta\|_{L^1})\|f_0\|_{L^1_k}.
\end{aligned}$$

We have to prove

$$S_\mathcal{B}(t) : L^1_k \to L^1_k \quad \text{uniformly bounded}$$

$$S_\mathcal{B}(t) : L^1_m \to L^1_k \quad \text{with rate } t \mapsto \Theta(t) \in L^1 \text{ for } m > k \text{ (large enough)}$$

L^1_k estimate for $S_\mathcal{B}$

$\mathcal{B}$ satisfies the (weak) dissipativity condition

$$\mathcal{B}^* \langle x \rangle^k \lesssim -\langle x \rangle^{k+\gamma-2} \le 0.$$

A solution f to the evolution equation $\partial_t f = \mathcal{B} f$ satisfies

$$\frac{d}{dt} \int f \langle x \rangle^k \le - \int f \langle x \rangle^{k+\gamma-2} \le 0,$$

so that first

$$S_\mathcal{B} : L^1_k \to L^1_k, \ L^1_m \to L^1_m, \quad \text{uniformly bounded } \ \forall\, m \ge k.$$

Observing that

$$\langle x \rangle^k \le A^{2-\gamma} \langle x \rangle^{k+\gamma-2} + A^{k-m} \langle x \rangle^m, \quad \forall\, A > 0,$$

we compute

$$\frac{d}{dt} \int f \langle x \rangle^k + A^{\gamma-2} \int f \langle x \rangle^k \le A^{k-m+\gamma-2} \int f \langle x \rangle^m,$$

and next

$$\frac{d}{dt} \Big(e^{tA^{\gamma-2}} \int f \langle x \rangle^k \Big) \le e^{tA^{\gamma-2}} A^{k-m+\gamma-2} \int f_0 \langle x \rangle^m.$$

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So that first

$$S_\mathcal{B} : L^1_k \to L^1_k, \; L^1_m \to L^1_m, \quad \text{uniformly bounded } \; \forall \, m \ge k.$$

A solution f to the evolution equation $\partial_t f = \mathcal{B} f$ satisfies

$$\frac{d}{dt} \Big(e^{tA^{\gamma-2}} \int f \langle x \rangle^k \Big) \le e^{tA^{\gamma-2}} A^{k-m+\gamma-2} \int f_0 \langle x \rangle^m.$$

Integrating in time (using the Gronwall lemma), we deduce

$$\begin{aligned} \int f \langle x \rangle^k &\le e^{-tA^{\gamma-2}} \int f_0 \langle x \rangle^k + A^{k-m} \int f_0 \langle x \rangle^m, \quad \forall \, A > 0, \\ &\le \inf_{A>0} \Big(e^{-tA^{\gamma-2}} + A^{k-m} \Big) \int f_0 \langle x \rangle^m \\ &=: \Theta(t) \int f_0 \langle x \rangle^m \end{aligned}$$

We find

$$\Theta(t) \le t^{-2} + (t/\ln t^2)^{\frac{k-m}{2-\gamma}}$$

by making the choice $A := (t/\ln t^2)^{\frac{1}{2-\gamma}}$. We have $\Theta \in L^1$ when $m > k + 2 - \gamma$.

L^p_k estimate for $S_{\mathcal{B}}$ (and next $S_{\mathcal{L}}$) in the case $\gamma \geq 2$ and $p = 2$

We use Nash trick and Nash inequality

$$\|f\|_{L^2}^{1+2/d} \leq C_d \, \|f\|_{L^1}^{2/d} \, \|\nabla f\|_{L^2}$$

for a solution f to the evolution equation $\partial_t f = \mathcal{B} f$. Taking advantage of the available L^1 estimate (for M, R large enough)

$$\|f_t\|_{L^1} \lesssim e^{-t},$$

we may compute

$$\begin{aligned} \frac{d}{dt} \, \|f\|_{L^2}^2 \quad &\lesssim \quad - \, \|\nabla f\|_{L^2}^2 - 2\|f\|_{L^2}^2 \\ &\lesssim \quad - \, \frac{\|f\|_{L^2}^{2(1+\alpha)}}{\|f\|_{L^1}^{2\alpha}} - 2\|f\|_{L^2}^2, \end{aligned}$$

with $\alpha := 2/d > 0$, so that

$$\frac{d}{dt} \, \left(\|f\|_{L^2}^2 e^{2t}\right) \quad \lesssim \quad - \, \frac{\left(\|f\|_{L^2}^2 e^{2t}\right)^{1+\alpha}}{\|f_0\|_{L^1}^{2\alpha}}.$$

Nonlinear ODE

We recall that the solution to the ODE

$$u' \leq -K\, u^{1+\alpha},$$

satisfies

$$u(t) \leq \frac{1}{(\alpha K t)^{1/\alpha}}.$$

The proof is elementary. We write equivalently

$$\frac{du}{u^{1+\alpha}} \leq -K dt$$

and after integration in time, we get

$$u^{-\alpha}(t) \geq \alpha\, K\, t + u_0^{\alpha} \geq \alpha\, K\, t.$$

Using that result with the choice $\alpha = 2/d$ and $K = C \|f_0\|_{L^1}^{-4/d}$, we deduce

$$\|f\|_{L^2}^2 e^{2t} \lesssim \frac{\|f_0\|_{L^1}^2}{t^{d/2}}$$

and finally

$$\|f\|_{L^2} \lesssim \frac{e^{-t}}{t^{d/4}} \|f_0\|_{L^1}$$

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and finally

$$\|f\|_{L^2} \lesssim \frac{e^{-t}}{t^{d/4}} \|f_0\|_{L^1}$$

We have established

$$S_{\mathcal{B}}(t) : L^1 \to L^2 \text{ with rate } \Theta := \frac{e^{-t}}{t^{d/4}} \in L^1, \quad \text{if } \ d \le 3.$$

In general, we have

$$S_{\mathcal{B}}(t) : L^1 \to L^p \text{ with rate } \Theta := \frac{e^{-t}}{t^{d/2}},$$

and whatever is $p \in [1, \infty]$, $d \ge 1$, $k \ge 0$, we may prove

$$(\mathcal{A} S_{\mathcal{B}})^{*N}(t) : L^1 \to L^p_k \text{ with rate } \Theta \in L^1, \quad \text{for } \ N \ge 1 \ \text{ large enough.}$$

Dissipativity / Lyapunov / weak dissipativity / weak Lyapunov

Dissipativity $\exists\, a \in \mathbb{R}$

$$\langle f^*, \mathcal{B} f\rangle \le a \|f\|^2 \;\Leftrightarrow\; \|S_\mathcal{B}(t) f\| \le e^{at}\|f\|$$

Hypo-dissipativity $\exists\, a \in \mathbb{R}$

$$\langle f^*, \mathcal{B} f\rangle \le a |||f|||^2 \;\Leftrightarrow\; \|S_\mathcal{B}(t) f\| \le M e^{at}\|f\|$$

• $\mathcal{B} - a$ dissipative implies $\mathcal{L} - (a + \|\mathcal{A}\|)$ dissipative and we may sometime show $\mathcal{L} - \kappa$ hypodissipative with $\kappa \in [a, a + \|\mathcal{A}\|)$.

Lyapunov condition $\exists\, a \in \mathbb{R}$ (or $\mathbb{R}_-$), $\exists\, \psi \ge 1$, $\exists \psi_c \lesssim \psi$ (suppψ_c compact)

$$\mathcal{L}^* \psi \le a\psi + \psi_c$$

• For positive semigroup in L^1 we have Kato's inequality: $(\mathrm{sign} f)\mathcal{L} f \le \mathcal{L}|f|$. Lyapunov condition then implies $\mathcal{B} - a$ is dissipative with $\mathcal{B} := \mathcal{L} - \psi_c$.
When $\psi = 1$, we may compute

$$\begin{aligned}
\langle f^*, \mathcal{B} f\rangle &= \langle f^*, \mathcal{L} f\rangle - \langle f^*, \psi_c f\rangle \\
&\le \langle 1, \mathcal{L}|f|\rangle - \langle 1, \psi_c |f|\rangle \\
&= \langle \mathcal{L}^* 1 - \psi_c, |f|\rangle \\
&\le a\langle 1, |f|\rangle = a\|f\|_{L^1}.
\end{aligned}$$

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Dissipativity $\exists\, a \in \mathbb{R}$

$$\langle f^*, \mathcal{B} f\rangle \le a\|f\|^2 \;\Leftrightarrow\; \|S_\mathcal{B}(t) f\| \le e^{at}\|f\|$$

Hypo-dissipativity $\exists\, a \in \mathbb{R}$

$$\langle f^*, \mathcal{B} f\rangle \le a|\!|\!|f|\!|\!|^2 \;\Leftrightarrow\; \|S_\mathcal{B}(t) f\| \le M e^{at}\|f\|$$

Lyapunov condition $\exists\, a \in \mathbb{R}$ (or $\mathbb{R}_-$), $\exists\, \psi \ge 1$, $\exists \psi_c \lesssim \psi$ (suppψ_c compact)

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Weakly dissipativity $a = 0$, $X_1 \subset X_0$

$$\langle f^*, \mathcal{B} f\rangle_{X_1} \le -\|f\|_{X_0} \;\Leftrightarrow\; \text{not clear}$$

but

$$\langle f^*, \mathcal{B} f\rangle_{X_1} \le -\|f\|_{X_0}, \quad \langle f^*, \mathcal{B} f\rangle_{X_2} \le 0, \quad X_2 \subset X_1 \subset X_0$$

imply

$$\|S_\mathcal{B}(t) f\|_{X_i} \le \|f\|_{X_i}, \; i = 1,2, \quad \|S_\mathcal{B}(t) f\|_{X_0} \le \Theta(t)\|f\|_{X_2}.$$

Weak Lyapunov with $a = 0$, $\exists\, \psi_i$, $\psi_c \lesssim \psi_0 \lesssim \psi_1$

$$\mathcal{L}^*\psi_1 \le -\psi_0 + \psi_c$$

- weak Lyapunov condition for $\mathcal{L} \Rightarrow$ weak dissipative property for $\mathcal{B}$

Outline of the talk

Let's start with a picture

Weyl's theorem - characterization

Th 2.

(0) $\mathcal{L} = \mathcal{A} + \mathcal{B}$, where $\mathcal{A}$ is $\mathcal{B}^{\zeta'}$-bounded with $0 \le \zeta' < 1$,

(1) $\|S_\mathcal{B} * (\mathcal{A}S_\mathcal{B})^{(*\ell)}\|_{X\to X} \le C_\ell \, e^{at}$, $\forall\, a > a^*$, $\forall\, \ell \ge 0$,

(2) $\int_0^\infty \|(\mathcal{A}S_\mathcal{B})^{(*n)}\|_{X\to D(\mathcal{B}^\zeta)} \, e^{-at} \, dt < \infty$, $\forall\, a > a^*$, with $\zeta > \zeta'$,

(3) $\int_0^\infty \|(\mathcal{A}S_\mathcal{B})^{(*m)}\|_{X\to Y} \, e^{-at} \, dt < \infty$, $\forall\, a > a^*$, with $Y \subset X$ compact,

is equivalent to

(4) there exist $\xi_1, ..., \xi_J \in \bar\Delta_a$, there exist $\Pi_1, ..., \Pi_J$ some finite rank projectors, there exists $T_j \in \mathcal{B}(R\Pi_j)$ such that $\mathcal{L}\Pi_j = \Pi_j\mathcal{L} = T_j\Pi_j$, $\Sigma(T_j) = \{\xi_j\}$, in particular

$$\Sigma(\mathcal{L}) \cap \bar\Delta_a = \{\xi_1, ..., \xi_J\} \subset \Sigma_d(\Sigma)$$

and there exists a constant C_a such that

$$\|S_\mathcal{L}(t) - \sum_{j=1}^J e^{tT_j}\Pi_j\|_{X\to X} \le C_a \, e^{at}, \quad \forall\, a > a^*$$

▷ Weyl (1910), Ribarič-Vidav (1969), Vidav (1974), Voigt (1980), M.-Scher (2016)

• It can be seen as a condition under which a *"spectral mapping theorem for the principal part of the spectrum holds"*

• Issue : constants are not constructive !!

Resolvent and semigroup

We define

$$\mathcal{R}_\mathcal{L}(\lambda) := (\lambda - \mathcal{L})^{-1},$$

when $\lambda - \mathcal{L} : D(\mathcal{L}) \to X$ is one-to-one.
In that case, we write $\lambda \in \rho(\mathcal{L}) \subset \mathbb{C}$ the resolvent set.
We have $\rho(\mathcal{L}) \supset \Delta_{\omega(S_\mathcal{L})} \neq \emptyset$, $\Delta_a := \{z \in \mathbb{C};\ \Re e z > a\}$ and

$$\mathcal{R}_\mathcal{L}(\lambda) = \int_0^\infty S_\mathcal{L}(t) e^{-\lambda t}\, dt, \quad \forall\, \lambda \in \Delta_{\omega(\mathcal{L})}. \tag{4}$$

The counterpart of the Duhamel formulas are

$$\mathcal{R}_\mathcal{L} = \mathcal{R}_\mathcal{B} + \mathcal{R}_\mathcal{B}\mathcal{A}\mathcal{R}_\mathcal{L} = \mathcal{R}_\mathcal{B} + \mathcal{R}_\mathcal{L}\mathcal{A}\mathcal{R}_\mathcal{L}$$

and some counterpart of the iterated Duhamel formulas is e.g.

$$\mathcal{R}_\mathcal{L} = \mathcal{R}_\mathcal{B} + \dots + (\mathcal{R}_\mathcal{B}\mathcal{A})^{(N-1)}\mathcal{R}_\mathcal{B} + (\mathcal{R}_\mathcal{B}\mathcal{A})^N \mathcal{R}_\mathcal{L}.$$

Inversing the Laplace transform (4), we get

$$\begin{aligned} S_\mathcal{L}(t) &= \frac{i}{2\pi} \int_{\uparrow_a} e^{zt} \mathcal{R}_\mathcal{L}(z) dz \\ &= S_\mathcal{B} + \dots + (S_\mathcal{B}\mathcal{A})^{*(N-1)} * S_\mathcal{B} + \frac{i}{2\pi} \int_{\uparrow_a} e^{zt} (\mathcal{R}_\mathcal{B}(z)\mathcal{A})^N \mathcal{R}_\mathcal{L}(z) dz \end{aligned}$$

Resolvent and spectrum

- We define the spectrum set $\Sigma(\mathcal{L}) := \mathbb{C}\backslash\rho(\mathcal{L})$.
- We define the point spectrum set (the set of eigenvalues)

$$\Sigma_P(\mathcal{L}) := \{\lambda \in \mathbb{C};\ \exists f \in X\backslash\{0\}\ \mathcal{L}f = \lambda f\}.$$

- We say that $\lambda \in \Sigma(\mathcal{L})$ is isolated if $\exists r > 0$, $\Sigma(\mathcal{L}) \cap B(\lambda, r) = \{\lambda\}$.
- For $\lambda \in \Sigma_P(\mathcal{L})$, we define $M_\lambda := \lim_{n\to\infty} N(\lambda - \mathcal{L})^n$ the almost algebraic eigenspace and $m_{aa} := \dim M(\mathcal{L} - \lambda) \in \{1..., \infty\}$ the "almost algebraic multiplicity".
- If it exists, the algebraic eigenspace $\mathcal{E}_\lambda$ associated to $\lambda \in \Sigma_P(\mathcal{L})$ satisfies
  - there exists a projection Π which commutes with $\mathcal{L}$ and satisfies $\Pi X = \mathcal{E}_\lambda$,
  - $\mathcal{L}_{|\mathcal{E}_\lambda} \in \mathcal{B}(\mathcal{E}_\lambda)$, $\Sigma_P(\mathcal{L}_{|\mathcal{E}_\lambda}) = \Sigma(\mathcal{L}_{|\mathcal{E}_\lambda}) = \{\lambda\}$ and $\lambda \notin \Sigma_P(\mathcal{L}_{|X_0})$ with $X_0 := (I - \Pi)X$.
- We define the discrete spectrum set $\Sigma_d(\mathcal{L})$ as the set of $\lambda \in \Sigma_P(\mathcal{L})$ which is isolated and which algebraic multiplicity $\dim \mathcal{E}_\lambda$ is finite.

We have

$$\Sigma_d(\mathcal{L}) \subset \Sigma_P(\mathcal{L}) \subset \Sigma(\mathcal{L}), \quad M_\lambda \subset \mathcal{E}_\lambda \ \text{ if } \ \lambda \in \Sigma_P(\mathcal{L})$$

and

$$\Pi = \frac{i}{2\pi} \int_{|z-\lambda|=r/2} \mathcal{R}_\mathcal{L}(z)\, dz \ \text{ if } \ \lambda \in \Sigma_d(\mathcal{L}).$$

Sketch of the proof of Weyl + spectral mapping theorem

We split the semigroup into invariant linear sub-manifolds (eigenspaces)

$$S_{\mathcal{L}} = \Pi\, S_{\mathcal{L}} + \Pi^\perp\, S_{\mathcal{L}}\, \Pi^\perp,$$

with $\Pi^\perp := I - \Pi$, $\Sigma(\mathcal{L}\Pi^\perp) \cap \Delta_{a^*} = \emptyset$ and write the (iterated) Duhamel formula

$$S_{\mathcal{L}} = \sum_{\ell=0}^{N-1} S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)} + S_{\mathcal{L}} * (\mathcal{A}S_{\mathcal{B}})^{(*N)}$$

Using the inverse Laplace formula for $b > \omega(\mathcal{L}) \geq s(\mathcal{L}) = \sup \Re e \Sigma(\mathcal{L})$ and the fact that $\Pi^\perp R_{\mathcal{L}}(z)$ is analytic in Δ_{a^*}, we get

$$\begin{aligned} \{\Pi^\perp\, S_{\mathcal{L}}\} * (\mathcal{A}S_{\mathcal{B}})^{(*N)} &= \frac{i}{2\pi} \int_{b-i\infty}^{b+i\infty} e^{zt}\, \Pi^\perp\, R_{\mathcal{L}}(z) (\mathcal{A}R_{\mathcal{B}}(z))^N\, dz \\ &= \lim_{M\to\infty} \frac{i}{2\pi} \int_{a-iM}^{a+iM} e^{zt}\, \Pi^\perp\, R_{\mathcal{L}}(z) (\mathcal{A}R_{\mathcal{B}}(z))^N\, dz \end{aligned}$$

These three identities together

$$\begin{aligned} S_{\mathcal{L}} &= \Pi\, S_{\mathcal{L}} + \Pi^\perp \{ \sum_{\ell=0}^{N-1} S_{\mathcal{B}} * (\mathcal{A}S_{\mathcal{B}})^{(*\ell)} \}\, \Pi^\perp \\ &+ \frac{i}{2\pi} \int_{\uparrow_a} e^{zt}\, \Pi^\perp\, R_{\mathcal{L}}(z) (\mathcal{A}R_{\mathcal{B}}(z))^N\, dz \quad = \mathcal{O}(e^{at})? \end{aligned}$$

The key estimate on the last term

We clearly have

$$\sup_{z=a+iy,\, y\in[-M,M]} \|\Pi^\perp R_{\mathcal{L}}(z)(\mathcal{A}R_{\mathcal{B}}(z))^N\| \le C < \infty \quad \text{(not constructive!)}$$

and it is then enough to get the bound

$$\|R_{\mathcal{L}}(z)(\mathcal{A}R_{\mathcal{B}}(z))^N\| \le C/|y|^2, \quad \forall\, z = a + iy,\ |y| \ge M,\ a > a_*$$

We assume (in order to make the proof simpler) that $\zeta = 1$ in estimate (2), namely

$$\|(\mathcal{A}S_{\mathcal{B}})^{(*n)}\|_{X \to X_1} = \mathcal{O}(e^{at}) \quad \forall\, t \ge 0,$$

with $X_1 := D(\mathcal{L}) = D(\mathcal{B})$, which implies

$$\|(\mathcal{A}R_{\mathcal{B}}(z))^n\|_{X \to X_1} \le C_a \quad \forall\, z = a + iy,\ a > a_*.$$

We also assume (for the same reason) that $\zeta' = 0$, so that

$$\mathcal{A} \in \mathcal{B}(X) \ \text{ and } \ R_{\mathcal{B}}(z) = \frac{1}{z}(R_{\mathcal{B}}(z)\mathcal{B} - I) \in \mathcal{L}(X_1, X)$$

imply

$$\|\mathcal{A}R_{\mathcal{B}}(z)\|_{X_1 \to X} \le C_a/|z| \quad \forall\, z = a + iy,\ a > a_*.$$

The two estimates together imply

$$(*) \qquad \|(\mathcal{A}R_{\mathcal{B}}(z))^{n+1}\|_{X \to X} \le C_a/|z| \quad \forall\, z = a + iy,\ a > a_*.$$

The key estimate on the last term - 2nd step

We write

$$R_{\mathcal{L}}(I-\mathcal{V}) = \mathcal{U}$$

with

$$\mathcal{U} := \sum_{\ell=0}^{n} R_{\mathcal{B}}(\mathcal{A}R_{\mathcal{B}})^{\ell}, \quad \mathcal{V} := (\mathcal{A}R_{\mathcal{B}})^{n+1}$$

For M large enough

$$(**) \qquad \|\mathcal{V}(z)\| \le 1/2 \quad \forall\, z = a+iy,\ |y| \ge M,$$

and we may write the Neuman series

$$R_{\mathcal{L}}(z) = \underbrace{\mathcal{U}(z)}_{\text{bounded}} \underbrace{\sum_{j=0}^{\infty} \mathcal{V}(z)^j}_{\text{bounded}}$$

For $N = 2(n+1)$, we finally get from $(*)$ and $(**)$

$$\|R_{\mathcal{L}}(z)(\mathcal{A}R_{\mathcal{B}}(z))^N\| \le C/\langle y\rangle^2, \quad \forall\, z = a+iy,\, |y| \ge M$$

The key argument for the first term

We write again

$$R_{\mathcal{L}}(I-\mathcal{V})=\mathcal{U}$$

with

$$\mathcal{U}:=\sum_{\ell=0}^{n} R_{\mathcal{B}}(\mathcal{A}R_{\mathcal{B}})^{\ell}, \quad \mathcal{V}:=(\mathcal{A}R_{\mathcal{B}})^{n+1}$$

Because

- $I-\mathcal{V}$ is holomorphic on Δ_{a^*},
- it is a compact perturbation of the identity
- it satisfies $I-\mathcal{V}(z)\to I$ when $\Re e z\to\infty$,

one may use the theory of *degenerate-meromorphic functions* of Ribarič and Vidav (1969), and conclude that $\mathcal{V}(z)$ is invertible outside of a discrete set $\mathcal{D}$ of Δ_{a^*}.

That implies that $\Sigma(\mathcal{L})\cap\Delta_{a^*}=\mathcal{D}$ is a discrete set of Δ_*.

On the other hand, thanks to the Fredholm alternative, one deduces that the eigenspace associated to each spectral value $\lambda\in\mathcal{D}$ is non zero and finite dimensional, so that $\lambda\in\Sigma_d(\mathcal{L})$.

We define

$$\Pi=\frac{i}{2\pi}\int_{\uparrow_a}\mathcal{R}_{\mathcal{L}}(z)\,dz, \quad \text{with } \uparrow_a\cap\Sigma(\mathcal{L})=\emptyset.$$

Outline of the talk

Let's start again with a picture

The KR theorem issue

For a positive semigroup $S_t = S_{\mathcal{L}}(t) = e^{t\mathcal{L}}$ with generator $\mathcal{L}$ on a Banach lattice X with positive cone X_+, we ask for

- existence of a first eigenvalue triplet solution $(\lambda, f_1, \phi_1) \in \mathbb{R} \times X \times X'$:

$$f_1 \ge 0,\ \mathcal{L} f_1 = \lambda_1 f_1, \quad \phi_1 \ge 0,\ \mathcal{L}^* \phi_1 = \lambda_1 \phi_1$$

- suitable geometric properties as

  (1) $f_1 > 0$ unique positive eigenvector for $\mathcal{L}$, $N(\mathcal{L} - \lambda_1)^k = \mathrm{vect} f_1$
  and $\phi_1 > 0$ unique positive eigenvector for $\mathcal{L}^*$, $N(\mathcal{L}^* - \lambda_1)^k = \mathrm{vect} \phi_1$

  (1') $\Sigma_+(\mathcal{L}) - \lambda_1$ is a (discrete) subgroup of $i\mathbb{R}$,
  with $\Sigma_+(\mathcal{L}) := \{\lambda,\ \lambda \in \Sigma_P(\mathcal{L}), \Re e \lambda = \lambda_1\}$

  (2) $\Sigma_+(\mathcal{L}) = \{\lambda_1\}$

- asymptotic attractivity/stability of the principal eigenfunction

$$e^{t\mathcal{L}} f_0 - e^{\lambda_1 t} f_1 \langle \phi_1, f_0 \rangle = o(e^{\lambda_1 t}),$$

with constructive rate.

Krein-Rutmann for positive operator

Th 3. Consider a semigroup generator $\mathcal{L}$ on a Banach lattice such that
(1) $\mathcal{L}$ such as in Weyl's Theorem for some $a^* \in \mathbb{R}$;
(2) $\exists b > a^*$ and $\psi \in D(\mathcal{L}^*) \cap X'_+ \backslash \{0\}$ such that $\mathcal{L}^* \psi \ge b\,\psi$;
(3) $S_\mathcal{L}$ is positive (and $\mathcal{L}$ satisfies Kato's inequalities);
(4) $-\mathcal{L}$ satisfies a strong maximum principle.

Defining $\lambda_1 := s(\mathcal{L})$, there holds

$$a^* < \lambda_1 = \omega(\mathcal{L}) \quad \text{and} \quad \lambda_1 \in \Sigma_d(\mathcal{L}),$$

and there exists $0 < f_1 \in D(\mathcal{L})$ and $0 < \phi_1 \in D(\mathcal{L}^*)$ such that

$$\mathcal{L} f_1 = \lambda_1\, f_1, \quad \mathcal{L}^* \phi_1 = \lambda_1\, \phi_1, \quad R\Pi_{\mathcal{L},\lambda_1} = \text{Vect}(f_1),$$

and then

$$\Pi_{\mathcal{L},\lambda_1} f = \langle f, \phi_1 \rangle\, f_1 \quad \forall\, f \in X.$$

Moreover, there exist $\alpha \in (a^*, \lambda_1)$ and $C > 0$ such that for any $f_0 \in X$

$$\|S_\mathcal{L}(t) f_0 - e^{\lambda_1 t}\, \Pi_{\mathcal{L},\lambda_1} f_0\|_X \le C\, e^{\alpha t}\, \|f_0 - \Pi_{\mathcal{L},\lambda_1} f_0\|_X \qquad \forall\, t \ge 0.$$

$\triangleright$ In M. & Scher, that is mainly a consequence of Weyl + spectral mapping theorem by establishing furthemore that

$$\Sigma(\mathcal{L}) \cap \Delta_{a^*} = \{\lambda_1\}, \quad \lambda_1 \in \mathbb{R}.$$

Existence part in the KR theorem

Th 3′ Assumptions:

(1) S is a positive semigroup

(2) $\exists \kappa_0 \in \mathbb{R}\ \exists \psi_0 \in X'_+ \backslash \{0\}\ \mathcal{L}^* \psi_0 \ge \kappa_0 \psi_0$

(3) (dissipative case) splitting structure with $\kappa_\mathcal{B} < \kappa_0$

(3′) (weakly dissipative case) $\kappa_0 = 0$, $\exists \Theta \in L^1(\mathbb{R}_+)$ such that

$$\|f_t\| \le M\|f_0\| + \int_0^t \Theta(t-s)[f_s]\, ds, \quad f_t := S_t f_0,$$

with $[f] := \langle \psi_0, |f| \rangle$ and $X \subset \mathcal{X}$ (weakly) compact, with $\|f\|_\mathcal{X} := [f]$.

Conclusion: $\exists$ a solution (λ, f_1, ϕ_1) to the first eigenvalue triplet problem

Example: (1) The Fokker-Planck operator

$$\mathcal{L}f = \Delta f + \mathrm{div}(Ef) + cf, \quad E := \nabla |x|^\gamma / \gamma,\ \gamma > 0, \quad c \in C_c(\mathbb{R}^d).$$

(2) The condition (3′) is natural under a splitting structure

$$S_\mathcal{L} = S_\mathcal{B} + \cdots + (S_\mathcal{B}\mathcal{A})^{*(N-1)} * S_\mathcal{B} + (S_\mathcal{B}\mathcal{A})^{*N} * S_\mathcal{L},$$

with $\mathcal{A}$ bounded, $\mathcal{B}$ weakly dissipative, $(S_\mathcal{B}\mathcal{A})^{*N} : \mathcal{X} \to X$.

Existence - 1st proof $\sim$ Collet-Martínez-Méléard-San Martín?

We assume (case $N = 1$ and $X = M^1$) with $\kappa_{\mathcal{B}} < \kappa_0$

$$[f_t] \ge e^{\kappa_0 (t-s)} [f_s], \ \forall t > s, \quad f_\tau := S_\tau f_0$$

$$, \qquad \|f_t\| \le e^{\kappa_{\mathcal{B}} t} \|f_0\| + C_2 \int_0^t e^{\kappa_{\mathcal{B}}(t-s)} [f_s] \, ds \ \Leftrightarrow \ C_1 = 1$$

Step 1. We define

$$\mathcal{C} := \{ f \ge 0, \ [f] = 1, \ \|f\| \le M \}, \quad \Phi_t(f_0) := \frac{f_t}{[f_t]}.$$

For $f_0 \in \mathcal{C}$ and $\alpha := \kappa_{\mathcal{B}} - \kappa_0 < 0$, we compute for $t \le t_0$,

$$\begin{aligned} \|\Phi_t(f_0)\| &\le e^{\alpha t} \|f_0\| + C_2 \int_0^t e^{\alpha (t-s)} \, ds \\ &\le (1 + \alpha t/2) M + C_2 t \le M \end{aligned}$$

$t_0 > 0$ small and $M > 0$ large. That implies $\Phi_t : \mathcal{C} \to \mathcal{C}$.

From the Schauder/Tykonov theorem:

$$\exists \, \xi_t \in \mathcal{C}, \quad \Phi_t(\xi_t) = \xi_t.$$

Existence - 1st proof (continuation)

Step 1. We reformulate

$$\exists\, f_t \in \mathcal{C},\ \exists\, \lambda'_t \in [\kappa_0,\kappa_1], \quad S_t f_t = e^{\lambda'_t t} f_t.$$

Step 2. We reformulate again by choising $t = 2^{-n}$:

$$\exists\, f_n \in \mathcal{C},\ \exists\, \lambda'_n \in [\kappa_0,\kappa_1], \quad S_t f_n = e^{\lambda'_n t} f_n, \quad \forall\, t \in \mathbb{D}_m, m \le n,$$

with

$$\mathbb{D}_m := \{t = j2^{-m}\} = 2^{-m}\mathbb{N} = \text{ part of dyadic real numbers}$$

By compactness, $\exists \lambda_1 \in [\kappa_0,\kappa_1]$, $\exists\, f_1 \in \mathcal{C}$ such that

$$S_t f_{n_k} = e^{\lambda'_{n_k} t} f_{n_k}, \quad \lambda_{n_k} \to \lambda_1,\ f_{n_k} \rightharpoonup f_1.$$

We deduce

$$S_t f_1 = e^{\lambda_1 t} f_1, \quad \forall\, t \in \mathbb{D}_m,\ \forall\, m$$

and then

$$S_t f_1 = e^{\lambda_1 t} f_1, \quad \forall\, t \ge 0.$$

Existence - 2nd proof $\sim$ Cañizo-M

We assume (general dissipative case for N and X) with $\kappa_{\mathcal{B}} < \kappa_0$

$$[f_t] \geq e^{\kappa_0 (t-s)} [f_s], \ \forall t > s, \quad f_\tau := S_\tau f_0$$

$$, \qquad \|f_t\| \leq C_1 e^{\kappa_{\mathcal{B}} t} \|f_0\| + C_2 \int_0^t e^{\kappa_{\mathcal{B}}(t-s)} [f_s] \, ds, \ C_1 > 1$$

Step 1. With the same notations

$$\|\Phi_{T_0}(f_0)\| \quad \leq \quad C_1 e^{\alpha T_0} M + \frac{C_2}{|\alpha|} \leq M,$$

for T_0 and $M > 0$ large enough. That implies $\Phi_{T_0} : \mathcal{C} \to \mathcal{C}$.

From the Schauder/Tykonov theorem:

$$\exists \, f_{T_0} \in X_+, \ [f_{T_0}] = 1, \ S_{T_0} f_{T_0} = e^{\lambda_1 T_0} f_{T_0}.$$

We cannot make $T_0 \to 0$!!

Existence - 2nd proof (continuation)

Step 2. We denote $\bar S_t := S_t e^{-\lambda_1 t}$. We have built a periodic solution

$$\bar S_t f_{T_0} = \bar S_{t-kT_0} f_{T_0}, \quad k := [t/T_0], \quad \forall\, t > 0.$$

For any $t \ge 0$, we deduce

$$\begin{aligned} [\bar S_t f_{T_0}] &\ge e^{(\kappa_0-\lambda_1)(t-kT_0)}[f_{T_0}] \ge e^{(\kappa_0-\lambda_1)T_0} =: r_* > 0, \\ \|\bar S_t f_{T_0}\| &\le C_2 e^{(\kappa_2-\lambda_1)(t-kT_0)}\|f_{T_0}\| \le C_2 e^{(\kappa_2-\lambda_1)T_0)}\|f_{T_0}\| =: R^* < \infty. \end{aligned}$$

The mean u_T satisfies the same estimates:

$$u_T := \frac{1}{T}\int_0^T \bar S_t f_{T_0}\, dt \in \mathcal{G} := \{g \in X_+;\ [g] \ge r_*,\ \|g\| \le R^*\}.$$

By compactness, there exists $f_1 \in \mathcal{G}$ and (T_k) such that $u_{T_k} \rightharpoonup f_1$.
The von Neumann, Birkhoff mean ergodicity trick leads to

$$\begin{aligned} \bar S_t f_1 - f_1 &= \lim_{k\to\infty}\Big\{\frac{1}{T_k}\int_0^{T_k} \bar S_t \bar S_s f_{T_0}\, ds - \frac{1}{T_k}\int_0^{T_k} \bar S_s f_{T_0}\, ds\Big\} \\ &= \lim_{k\to\infty}\Big\{\frac{1}{T_k}\int_{T_k}^{T_k+t} \bar S_s f_{T_0}\, ds - \frac{1}{T_k}\int_0^{t} \bar S_s f_{T_0}\, ds\Big\} = 0, \end{aligned}$$

because $(\bar S_s f_{T_0})$ is uniformly bounded. We deduce $\mathcal{L} f_1 = \lambda_1 f_1$.

Existence - third proof (dynamical approach)

We assume (including weakly dissipative case)

$$[f_t] \geq [f_s], \ \forall t > s, \quad \kappa_0 := 0,$$
$$\|f_t\| \leq M\|f_0\| + \int_0^t \Theta(t-s)[f_s]\, ds, \ \ M \geq 1$$

For some $g_0 \in X_+$ such that $[g_0] = 1$, we set

$$\mathcal{C} := \{f \geq 0, \ [f] = 1, \ \|f\| \leq R\}, \quad R := \max(2\|\Theta\|_{L^1}, \|g_0\|)$$

and we define the increasing function

$$\lambda(t) := \inf_{f \in \mathcal{C}} [S_t f].$$

We have the alternative

- (case 1) $\sup \lambda \leq 2M$
- (case 2) $\sup \lambda > 2M$

Existence - third proof (case 1)

By compactness, there exists $f_0 \in \mathcal{C}$ such that

$$\sup_{t\ge 0}[S_t f_0] \le 2M.$$

We remind the iterated Duhamel formula

$$S = v + (S_{\mathcal{B}}\mathcal{A})^{(*N)} * S$$

and the associated mean equation

$$U_T = V_T + W_T$$

with

$$U_T := \frac{1}{T}\int_0^T S_t dt,\;\; v_T := \frac{1}{T}\int_0^T v_t dt,\;\; W_T := \frac{1}{T}\int_0^T (S_{\mathcal{B}}\mathcal{A})^{(*N)} * S dt.$$

Thanks to Fubini and positivity, we have

$$W_T \le \int_0^T (S_{\mathcal{B}}\mathcal{A})^{(*N)} dt U_T$$

which implies

$$\|W_T f_0\| \le \|\Theta\|_{L^1}[U_T f_0]$$

In a simpler way

$$\|V_T f_0\| \le M\|f_0\|.$$

All together, we have '

$$\|U_T f_0\| \le M\|f_0\| + \|\Theta\|_{L^1} [U_T f_0] \quad \text{and} \quad 1 \le [S_T f_0] \le 2M.$$

From the first inequality, we deduce that $\|U_T f_0\|$ is uniformly bounded on $T \in \mathbb{R}_+$.
By compactness, there exists $T_k \to +\infty$ and $f_1 \in X_+$ such that $U_{T_k} f_0 \rightharpoonup f_1$.
Thanks to the second inequality, we have $[f_1] \ge 1$.
From the same and usual mean ergodic trick, for any fixed $s > 0$, we have

$$\begin{aligned} S(s)f_1 - f_1 &= \lim_{k\to\infty} \Big\{ \frac{1}{T_k} \int_0^{T_k} S(s)S(t)f_0 dt - \frac{1}{T_k} \int_0^{T_k} S(t)f_0 \, dt \Big\} \\ &= \lim_{k\to\infty} \Big\{ \frac{1}{T_k} \int_{T_k}^{T_k+s} S(t)f_0 dt - \frac{1}{T_k} \int_0^{s} S(t)f_0 \, dt \Big\} = 0. \end{aligned}$$

That implies that f_1 is a stationary solution, and thus $\lambda_1 = 0$.

Existence - third proof (case 2 - step 1)

Step 1 From the assumption

$$\exists\, T_0 > 0, \quad \forall\, f \in \mathcal{C}, \quad [S_{T_0} f] \geq 2M.$$

For $f \in \mathcal{C}$, we define

$$\Phi_{T_0} f := \frac{S_{T_0} f}{[S_{T_0} f]},$$

so that $\Phi_{T_0} f \geq 0$ and $[\Phi_{T_0} f] = 1$. Because of the above assumption and the Lyapunov like estimate, we have

$$\|\Phi_{T_0} f\| \leq \frac{1}{2}\|f\| + \|\Theta\|_{L^1} \leq R.$$

We have established $\Phi_{T_0} : \mathcal{C} \to \mathcal{C}$ and from the Schauder/Tykonov theorem, there exists $f_{T_0} \in \mathcal{C}$ such that $\Phi_{T_0} f_{T_0} = f_{T_0}$. In other words : we have built a pair of "almost eigenvalue and eigenfunction"

$$f_{T_0} \geq 0, \quad [f_{T_0}] = 1, \quad S_{T_0} f_{T_0} = e^{\lambda_1 T_0} f_{T_0},$$

with $e^{\lambda_1 T_0} = [S_{T_0} f]$ and thus $\lambda_1 \in [0, \kappa_1]$.

Step 2 We conclude as in the 2nd proof !

Outline of the talk

Hypothesis

We consider a Markov semigroup $S_t = S_{\mathcal{L}}(t)$ defined on $X := L^1(\mathbb{R}^d)$, meaning $S_t \geq 0$ and $S_t^* 1 = 1$. We furthermore assume

(H1) Subgeometric Lyapunov condition. There are two weight functions $m_0, m_1 : \mathbb{R}^d \to [1, \infty)$, $m_1 \geq m_0$, $m_0(x) \to \infty$ as $x \to \infty$, and two real constants $b, R > 0$ such that

$$\mathcal{L}^* m_1 \leq -m_0 + b\,\mathbf{1}_{B_R}.$$

(H2) Doeblin-Harris condition. $\exists\, T > 0\ \forall\, R > 0\ \exists\, \nu \geq 0, \not\equiv 0$, such that

$$S_T g \geq \nu \int_{B_R} g, \quad \forall\, g \in X_+.$$

(H3) There are two other weight functions $m_2, m_3 : \mathbb{R}^d \to [1, \infty)$, $m_3 \geq m_2 \geq m_1$ such that

$$\mathcal{L}^* m_3 \leq -m_2 + b\,\mathbf{1}_{B_R}$$

and $m_2 \leq m_0^\theta m_3^{1-\theta}$ with $\theta \in (1/2, 1]$.

Conclusion

Theorem 4

Consider a Markov semigroup S on $X := L^1(m_3)$ which satisfies (H1), (H2), (H3). There holds

$$\|S_t f_0\|_{L^1} \lesssim \Theta(t) \|f_0\|_{L^1(m_3)}, \quad \forall\, t \geq 0,\ \forall\, f_0 \in X,\ \langle f_0 \rangle = 0,$$

for the function Θ given by

$$\Theta(t) := \inf_{\lambda > 0} \left\{ e^{-\varepsilon_\lambda t} + \xi_\lambda \right\},$$

where

$$m_1 \leq \frac{1}{2\varepsilon_\lambda} m_0 + \eta_\lambda m_3,\ \forall\, \lambda, \quad \varepsilon_\lambda, \eta_\lambda \to 0 \text{ as } \lambda \to \infty.$$

Comments

• The assumption (H3) is not necessary: m_1 satsisfies a Lyaponov condition implies that $\phi(m_1)$ satsisfies a Lyaponov condition for any $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ concave.

• The probabilistic proof use Martingale argument, renewal theory and (if possible?) constants are not easily tractable.

• In the probabilistic approach, one writes $m_0 = \xi(m_1)$, $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ concave, and

$$\tilde\Theta(t) := \frac{C}{\xi(H^{-1}(t))}, \quad H(u) := \int_1^u \frac{ds}{\xi(s)}.$$

- If $\xi(s) = s$ then $\tilde\Theta(t) = e^{-\lambda t}$;

- If $m_1 = \langle x\rangle^k$, $m_0 := \langle x\rangle^{k+\gamma-2}$ then $\tilde\Theta(t) = t^{1-\frac{k}{2-\gamma}} >> \Theta(t)$;

- If $m_1 = e^{\kappa\langle x\rangle^s}$, $m_0 := \langle x\rangle^{s+\gamma-2} e^{\kappa\langle x\rangle^s}$ then $\tilde\Theta(t) \simeq e^{-\lambda t^{\frac{s}{2-\gamma}}} \simeq \Theta(t)$.

A second version of the subgeometric Doeblin-Harris theorem

Consider a Markov semigroup $S_t = S_\mathcal{L}(t)$ defined on $X := L^1_m(\mathbb{R}^D)$, meaning $S_t \ge 0$ and $S_t^* 1 = 1$. We furthermore assume

(H1) Subgeometric Lyapunov condition. There is a weight function $m : \mathbb{R}^D \to [1,\infty)$, $m \nearrow \infty$, an increasing concave function $\varphi : [1,\infty) \to [1,\infty)$, $\varphi \nearrow \infty$, and three real constants $b, R, \delta > 0$ such that

$$\mathcal{L}^* m \le -\delta \varphi(m) + b\,\mathbf{1}_{B_R}, \quad B_R := \{y \in \mathbb{R}^D;\ V(y) \le R\}.$$

(H2) Doeblin-Harris irreducibility condition. $\exists\, T > 0\ \forall\, R > 0\ \exists\, \nu \ge 0, \not\equiv 0$, such that

$$S_T g \ge \nu \int_{B_R} g, \quad \forall\, g \in X_+.$$

Theorem 4′

For any $f_0 \in X$, $\langle f_0 \rangle = 0$, there holds

$$\forall\, t \ge 0, \quad \|S_t f_0\| \lesssim \frac{1}{H^{-1}(t)} \|f_0\|_m, \quad H(u) := \int_1^u \frac{ds}{\varphi(u)}.$$

In particular,

$$\frac{1}{H^{-1}(t)} = e^{-t} \text{ if } \varphi(u) = u, \quad \frac{1}{H^{-1}(t)} = t^{-1/a} \text{ if } \varphi(u) = u^{1-a}.$$

several proofs :

Theorem 4 (geometric case)

- Doeblin
- Harris, Proceedings 1956
- Meyn, Tweedie, AAP 1992, 1993, 1994
- Hairer, Mattingly, Proceedings 2011
- Cañizo-M. (semigroup approach)

Theorem 4 (subgeometric case)

- Douc, Fort, Guillin, SPA 2009
- Hairer, unpublished lecture notes, 2016
- Cañizo-M. (semigroup approach)

Doeblin-Harris irreducibility/strong positivity condition implies coupling weak generator Lyapunov implies weak semigroup Lyapunov

Lemma 5

The Harris condition (H2) implies the coupling condition:
(H2′) $\exists \gamma_H \in (0,1)$, $A > 0$,

$$\|f\|_m \le A\|f\|, \ \langle f\rangle = 0 \implies \|S_T f\| \le \gamma_H \|f\|.$$

proof : splitting $\mathbb{R}^D = \mathcal{C}_R \cup \mathcal{C}_R^c$

Lemma 6

The generator Lyapunov condition (H1) implies the semigroup Lyapunov condition:
(H1′) $\forall\, t > 0$, $\exists K_t \ge 0$,

$$\|S_t f_0\|_m + t\|S_t f_0\|_{\varphi(m)} \le \|f_0\|_m + K_t\|f_0\|,$$

proof : integration in time

About Lemma 5 : contraction and strict contraction

Rk 1. Assuming just that (S_t) is a Markov semigroup, we have

$$|S_t f| = |S_t f_+ - S_t f_-| \leq |S_t f_+| + |S_t f_-| = S_t |f|.$$

Integrating, we deduce that (S_t) is a L^1 contraction

$$\int |S_t f| \leq \int S_t |f| = \int |f| S_t^* 1 = \int |f|.$$

Rk 2. We assume furthermore the strong Doeblin-Harris condition:

$$\text{(strong H2)} \qquad \exists\, T, \exists\, \nu, \quad S_T g \geq \nu \int_{\mathbb{R}^D} g, \quad \forall\, g \in X_+.$$

For $f \in L^1$, $\langle f \rangle = 0$, we have

$$S_T f_\pm \geq \nu \int_{\mathbb{R}^D} f_\pm = \frac{\nu}{2} \int_{\mathbb{R}^D} |f| =: \eta.$$

We may adapt the proof in Rk 1 in the following way

$$\begin{aligned} |S_T f| &= |S_T f_+ - \eta - (S_T f_- - \eta)| \\ &\leq |S_T f_+ - \eta| + |S_T f_- - \eta| = S_T |f| - 2\eta. \end{aligned}$$

Integrating, we deduce that (S_T) is a strict contraction

$$\|S_T f\|_{L^1} \leq \|f\|_{L^1} - 2\|\eta\|_{L^1} = (1 - \langle \nu \rangle)\, \|f\|_{L^1}$$

Proof of Lemma 5: the Harris condition (H2) implies the coupling condition (H2′)

Rk 3. Assuming (H2), we have similarly

$$\int |S_T f| \le \gamma_H \int |f| \quad \text{if} \quad \int |f| m \le \frac{m(R)}{4} \int |f|,$$

with

$$\gamma_H := 1 - \langle \nu \rangle / 2 \in (0,1).$$

Indeed, we mainly observe that

$$\begin{aligned} S_T f_\pm &\ge \nu \int_{\mathbb{R}^D} f_\pm - \nu \int_{B_R^c} f_\pm \\ &\ge \frac{\nu}{2} \int_{\mathbb{R}^D} |f| - \nu \int_{B_R^c} |f| \\ &\ge \frac{\nu}{2} \int_{\mathbb{R}^D} |f| - \frac{\nu}{m(R)} \int_{\mathbb{R}^D} |f| m \\ &\ge \frac{\nu}{2} \int_{\mathbb{R}^D} |f| - \frac{\nu}{4} \int_{\mathbb{R}^D} |f| \\ &= \frac{\nu}{4} \int_{\mathbb{R}^D} |f|, \end{aligned}$$

and we then follow the same proof as when we have assumed (strong H2).

Strict contraction for time discrete semigroup $S := S_T$

S satisfies a Lyapunov operator condition (H1$''$) if $\exists\, \gamma_L \in (0,1)$, $K \ge 0$

$$\|Sf\|_m + \gamma_L \|Sf\|_{\varphi(m)} \le \|f\|_m + K\|f\|, \quad \forall\, f$$

S satisfies a coupling operator condition (H2$''$) if $\exists \gamma_H \in (0,1)$, $A > 0$,

$$\|f\|_m \le A\|f\|,\ \langle f \rangle = 0 \implies \|Sf\| \le \gamma_H \|f\|.$$

Lemma 7

If $A > K/\gamma_L$ there exists $\alpha > 0$ and an equivalent norm $|||\cdot|||_m$ to $\|\cdot\|_m$ such that

$$|||Sf|||_m + \alpha \|Sf\|_{\varphi(m)} \le |||f|||_m, \quad \forall\, f,\ \langle f \rangle = 0.$$

Proof: a hypocoercivity trick and an alternative.
We introduce the equivalent norm for convenient choice of $\beta, \gamma > 0$

$$|||f|||_m := \|f\| + \beta \|f\|_{\varphi(m)} + \gamma\, \|f\|_m{}^*$$

If $\|f\|_{\varphi(m)} \le A\|f\|$, we use the coupling condition (H2$''$)
If $\|f\|_{\varphi(m)} \ge A\|f\|$, we use the Lyapunov condition (H1$''$)

* modified norm $\simeq$ "hypodissipativity trick"

Subgeometric convergence for time discrete semigroup $S := S_T$

We assume that S satisfies (H1'') and (H2'') for two pairs $m_i, \varphi_i, K_i, \gamma_{Li}$ and A_i, K_i, γ_{Hi} with $A_i > K_i/\gamma_{Hi}$, $m_1 \le m_2$, $\varphi_1(m_1) \le \varphi_2(m_2)$, as well as the interpolation condition

$$(H3) \qquad \lambda m_1 \le \varphi_1(m_1) + \xi(\lambda) m_2, \forall\, \lambda > 0,$$

with $\xi : \mathbb{R}_+ \to \mathbb{R}_+$, $\xi(\lambda)/\lambda \to 0$ as $\lambda \to 0$. That means $\varphi_1(m_1) << m_1 << m_2$.

Lemma 8

Under the above conditions, for any f, $\langle f \rangle = 0$,

$$\|S^n f\| \lesssim \tilde\Theta(n) \|f\|_{m_2}, \quad \forall\, n,$$

with

$$\tilde\Theta(n) = \frac{\Theta(n/2)}{n}, \quad \Theta(t) := F^{-1}(t), \quad F(\lambda) := \int_\lambda^1 \frac{ds}{\xi^*(\theta s)}$$

Proof:

$$|||Sf|||_{m_i} + \alpha \|Sf\|_{\varphi_i(m_i)} \le |||f|||_{m_i}$$

implies

$$|||Sf|||_{m_1} + \alpha\lambda \|Sf\|_{m_1} \le |||f|||_{m_1} + \alpha\xi(\lambda) |||f|||_{m_2}, \quad \forall\, \lambda > 0,$$

and next

$$|||S^{n+1} f|||_{m_1} \le (1 - \theta\lambda_n) |||S^n f|||_{m_1} + \alpha\xi(\lambda_n) |||f|||_{m_2}$$

Outline of the talk

Example 2 : the age structured equation