Yet another look at Krein-Rutman theorem

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For a positive semigroup $S_t = S_{\mathcal{L}}(t) = e^{t\mathcal{L}}$ with generator $\mathcal{L}$ on a Banach lattice X with positive cone X_+, we ask for

- existence of a first eigenvalue triplet solution $(\lambda, f_1, \phi_1) \in \mathbb{R} \times X \times X'$:

$$f_1 \geq 0,\ \mathcal{L} f_1 = \lambda_1 f_1, \quad \phi_1 \geq 0,\ \mathcal{L}^* \phi_1 = \lambda_1 \phi_1$$

- suitable geometric properties as

  (1) $f_1 > 0$ unique positive eigenvector for $\mathcal{L}$, $N(\mathcal{L} - \lambda_1)^k = \mathrm{vect} f_1$
  and $\phi_1 > 0$ unique positive eigenvector for $\mathcal{L}^*$, $N(\mathcal{L}^* - \lambda_1)^k = \mathrm{vect} \phi_1$

  (1′) $\Sigma_+(\mathcal{L}) - \lambda_1$ is a (discrete) subgroup of $i\mathbb{R}$,
  with $\Sigma_+(\mathcal{L}) := \{\lambda,\ \lambda \in \Sigma_P(\mathcal{L}), \Re e \lambda = \lambda_1\}$

  (2) $\Sigma_+(\mathcal{L}) = \{\lambda_1\}$

- asymptotic attractivity/stability of the principale eigenfunction

$$e^{t\mathcal{L}} f_0 - e^{\lambda_1 t} f_1 \langle \phi_1, f_0 \rangle = o(e^{\lambda_1 t}),$$

with constructive rate.

Framework

• $X = (X, \|\cdot\|, \geq)$ Banach lattice with positive cone $X_+ := \{f \geq 0\}$
Typically $X = L^p$, $X = C_0$ or $X = M^1$ or a weighted such spaces

• $S = (S_t)$ a positive semigroup on X (of linear operators):

- $S_t \in \mathcal{B}(X)$, $S_{t_1} S_{t_2} = S_{t_1+t_2}$, $S_0 = I$,
- strongly or weakly $*$ continuous trajectories,
- $\|S_t\|_{X\to X} \leq M e^{\kappa_1 t}$,
- $S \geq 0$: $S_t f \geq 0$ if $f \geq 0$,
- the generator $\mathcal{L}$ splits as

$$\mathcal{L} = \mathcal{A} + \mathcal{B}, \quad \mathcal{A} \prec \mathcal{B}, \quad S_{\mathcal{B}}(t) = O(e^{\kappa_{\mathcal{B}} t}), \ \kappa_{\mathcal{B}} < \kappa_1$$

with associated Duhamel and iterated Duhamel formulas

$$\begin{aligned} S_{\mathcal{L}} &= S_{\mathcal{B}} + S_{\mathcal{B}} \mathcal{A} * S_{\mathcal{L}} \\ &= S_{\mathcal{B}} + \cdots + (S_{\mathcal{B}} \mathcal{A})^{*N-1} * S_{\mathcal{B}} + (S_{\mathcal{B}} \mathcal{A})^{*N} * S_{\mathcal{L}} \end{aligned}$$

• Examples

$$\begin{aligned} \mathcal{L} f &= \mathrm{div}(a \nabla f) + b \cdot \nabla f + c f \\ &= -a \cdot \nabla f - K f + \int k f_* \, dy_* \end{aligned}$$

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• Examples

$$\begin{aligned} \mathcal{L} f &= \mathrm{div}(a \nabla f) + b \cdot \nabla f + cf - M \chi_R f + M \chi_R f \\ &= -a \cdot \nabla f - Kf + \int k_R^c f_* \, dy_* + \int k_R f_* \, dy_* \end{aligned}$$

Assumptions:

(1) S is a positive semigroup

(2) $\exists \kappa_0 \in \mathbb{R}\ \exists \psi_0 \in X'_+ \backslash \{0\}\ \mathcal{L}^* \psi_0 \ge \kappa_0 \psi_0$

(3) (dissipative case) splitting structure with $\kappa_{\mathcal{B}} < \kappa_0$

(3′) (weakly dissipative case) $\kappa_0 = 0$, $\exists \Theta \in L^1(\mathbb{R}_+)$ such that

$$\|f_t\| \le M\|f_0\| + \int_0^t \Theta(t-s)[f_s]_0\, ds$$

with $[f] = [f]_0 := \langle \psi_0, |f| \rangle$.

Conclusion: $\exists$ a solution (λ, f_1, ϕ_1) to the first eigenvalue triplet problem

Example:

$$\mathcal{L} f = \Delta f + \mathrm{div}(bf) + cf,$$

$b := \nabla |x|^\gamma / \gamma$, $\gamma > 0$, $c \in C_c(\mathbb{R}^d)$

Dissipativity $\exists\, a \in \mathbb{R}$

$$\forall\, f^* \in J_f,\ \langle f^*, \mathcal{B} f\rangle \le a \|f\|^2 \ \Leftrightarrow\ \|S_{\mathcal{B}}(t) f\| \le e^{at} \|f\|,$$

with $J_f := \|f\| (\text{sign} f)$ if $X = L^1$, $J_f := f$ if $X = L^2$

Lyapunov $\exists\, a \in \mathbb{R}$ (or $\mathbb{R}_-$), $\exists\, \psi$, $\exists \psi_0 \lesssim \psi$ (suppψ_0 compact)

$$\mathcal{L}^* \psi \le a \psi + \psi_0$$

Examples

- $\mathcal{L}$ satisfies Lyapunov with $\psi = 1 \Rightarrow \mathcal{B} := \mathcal{L} - M\chi$ is dissipative / $\mathcal{L}$ satsfies (3)
- $\mathcal{L} f := \Delta f + \text{div}(x^{\gamma-1} f)$ satisfies Lyapunov with $\psi = 1 + |x|^2$ when $\gamma \ge 2$

Weak dissipativity with $a = 0$, $X_2 \subset X_1$

$$\forall\, f^* \in J_f,\ \langle f^*, \mathcal{B} f\rangle_{X_2} \le -\|f\|^2_{X_1} \quad \approx \quad \|S_{\mathcal{B}}(t) f\|_{X_1} \le \Theta(t) \|f\|_{X_2}$$

Weak Lyapunov with $a = 0$, $\exists\, \psi_i$, $\psi_0 \lesssim \psi_1 \lesssim \psi_2$

$$\mathcal{L}^* \psi_2 \le -\psi_1 + \psi_0$$

Examples

- weak Lyapunov for $\mathcal{L} \Rightarrow$ weak dissipativity for $\mathcal{B} := \mathcal{L} - \psi_0$ / $\mathcal{L}$ satsfies (3')
- $\mathcal{L} f := \Delta f + \text{div}(x^{\gamma-1} f)$ satisfies weak Lyapunov with same ψ when $\gamma \in (0, 1)$

Assumptions:

(C1) conclusions of the existence part

The consequences of the semigroup positivity assumption

(1′) the weak maximum principle: $(\lambda - \mathcal{L})f \geq 0$, $\lambda > \lambda_1 \Rightarrow f \geq 0$

(1″) Kato's inequalities : $\mathcal{L}|f| \geq (\text{sign} f)\mathcal{L}f$, $\mathcal{L}f_+ \geq (\text{sign}_+ f)\mathcal{L}f$,

(4) the strong maximum principle: $(\lambda - \mathcal{L})f \geq 0$, $f \geq 0$, $f \not\equiv 0 \Rightarrow f > 0$

Conclusions:

- $f_1 > 0$ unique positive eigenvector for $\mathcal{L}$, $N(\mathcal{L} - \lambda_1)^k = \mathrm{vect} f_1$
  and the same for ϕ_1, $\Sigma_+(\mathcal{L}) - \lambda_1$ is a subgroup of $i\mathbb{R}$

- $\Sigma_+(\mathcal{L}) - \lambda_1$ is a **disscrete** subgroup of $i\mathbb{R}$ if $\kappa_\mathcal{B} < \kappa_0$ in the dissipative splitting structure hypothesis (3)

- mean ergodicity

$$\frac{1}{T} \int_0^T \bar{f}_t \, dt \to f_1 \langle f_0, \phi_1 \rangle \text{ as } T \to \infty,$$

under the weakly dissipative splitting structure hypothesis (3′),
with $\bar{S}_\mathcal{L} := e^{-\lambda_1 t} S_\mathcal{L}$, $\bar{f}_t := \bar{S}_t f_0$.

Second Geometry part in the KR theorem

Assumptions:

(C2) conclusions of the existence and first geometric part

(5) reverse Kato's inequality (for eigenvectors)

$$(\mathrm{sign}\bar f)\mathcal{L}f = \mathcal{L}|f| \;\Rightarrow\; f = u|f|,\; u \in S^1$$

(5′) eventual strong positivity

$$\forall\, f_0 \in X_+\backslash\{0\},\; \phi \in X'_+\backslash\{0\},\; \exists\, T, \forall t \ge T,\; \langle f_t, \phi\rangle > 0$$

or in other words when $E = \cup E_k$, E_k compact

$$\forall\, k,\; \exists\, T, \forall t \ge T,\; f_t > 0 \text{ on } E_k \text{ if } f_0 \not\equiv 0 \text{ on } E_k$$

(5′′) reverse strong positivity (for eigenvectors)

$$|S_t f| = S_t|f| \;\Rightarrow\; \exists\, u_t \in S^1,\; f_t = u_t|f_t|$$

(5′′′) splitting structure and $(\lambda_1 + iy - \mathcal{B})^{-N}\mathcal{A} = \mathcal{O}(|y|^{-1-\bullet})$

Conclusions:

- $\Sigma_+ = \{\lambda_1\}$
- ergodicity $\bar f_t \to f_1\langle f_0, \phi_1\rangle$ as $t \to \infty$ (without rate)

Constructive rate part in the KR theorem

Assumptions:

(C1) conclusions of the existence part

(3) (dissipative case) splitting structure with $\kappa_{\mathcal{B}} < \kappa_0$ which implies

$$\|\bar{S}_T f\| \le \gamma_L \|f\| + K[f]_{\psi_0}$$

with $\gamma_L = \gamma_L(T) < 1$ for any $T > 0$ and for some $\psi_0 > 0$

(6) strong Doblin-Harris irreducibility condition and regularity

$$\exists\, T > 0,\ \exists\, g_0 > 0,\ \forall\, f \ge 0,\ S_T f \ge g_0 \langle \psi_0, f \rangle$$
$$\exists\, R_0 > 0,\ \phi_1 \le R_0 \psi_0 \quad \text{(or a weaker estimate)}$$

Conclusion: There exists some constructive constants $C \ge 1$, $a < 0$ such that

$$\|\bar{S}_t f\| \le C e^{at} \|f\|,\ \forall\, t \ge 0,$$

for any $f \in X$ such that $\langle f, \phi_1 \rangle = 0$.

- Similar (but weaker) result in the weak dissipative case

Spectral analysis and KR result

- Perron-Frobenius (dim $< \infty$), Phillips (positive semigroup)
- Krein-Rutman (existence+geo when $\mathrm{int} X_+ \neq \emptyset$, ok $X = C(E)$, E compact)
- German school ($\mathrm{int} X_+ = \emptyset$ ok, but not constructive nor weakly dissipative)

$\to$ Schaefer, Nagel, Greiner, Arendt, ...and many others since $\sim$ 1980
$\to$ more readable book by Bátkai, Kramar Fijavž and Rhandi

- Scher-M ($\simeq$ German school, a bit more constructive)
- P.-L. Lions college de France course (existence, weak compactness argument)
- Alfaro-Gabriel-Kavian and many others ... (KR with more or less explicit rate)

Ergodicity, probabilistic and coupling method approach (conservative case)

- von Neumann, Birkhoff, Markov, Kakutani (existence)
- Doblin, Harris, Meyn, Tweedie (exponential convergence)
- Hairer-Matingly (the same but constructive and simple proof)

Probabilistic and coupling method approach (non conservative)

- Bensaye, Cloez, Gabriel, Marguet (abstract KR via coupling)
- Cañizo-Gabriel-Yoldaș and many others ... (KR with more or less explicit rate)

Discussion - weak dissipative case (but conservative)

Spectral analysis and functional inequalities

- Toscani-Villani, Rochner-Wang (Fokker-Planck operator)
- Kavian-M.-Ndao (idem)

Probabilistic and coupling method approach (non conservative)

- Douc, Fort, Guillin (weak Lypunov condition)
- Cañizo-M. (idem but constructive rate)

What is new?

- KR in the weak dissipative case with possibly no control on strong compactness nor graph norm
- constructive rate
- the necessary assumptions are made clearer at each step

We assume (case $N = 1$ and $X = M^1$) with $\kappa_{\mathcal{B}} < \kappa_0$

$$[f_t] \geq e^{\kappa_0(t-s)}[f_s], \ \forall t > s, \quad f_\tau := S_\tau f_0$$

$$, \qquad \|f_t\| \leq e^{\kappa_{\mathcal{B}} t}\|f_0\| + C_2 \int_0^t e^{\kappa_{\mathcal{B}}(t-s)}[f_s]\, ds$$

Step 1. We define

$$\mathcal{C} := \{f \geq 0, \ [f] = 1, \ \|f\| \leq M\}, \quad \Phi_t(f_0) := \frac{f_t}{[f_t]}.$$

For $f_0 \in \mathcal{C}$ and $\alpha := \kappa_{\mathcal{B}} - \kappa_0 < 0$, we compute for any for $t \leq t_0$,

$$\begin{aligned} \|\Phi_t(f_0)\| &\leq e^{\alpha t}\|f_0\| + C_2 \int_0^t e^{\alpha(t-s)}\, ds \\ &\leq (1 + \alpha t/2)M + C_2 t \leq M \end{aligned}$$

$t_0 > 0$ small and $M > 0$ large. That implies $\Phi_t : \mathcal{C} \to \mathcal{C}$.
From Brouwer-Schauder-Tykonov theorem:

$$\exists\, \xi_t \in \mathcal{C}, \quad \Phi_t(\xi_t) = \xi_t.$$

Step 1. We reformulate

$$\exists\, f_t \in \mathcal{C},\ \exists\, \lambda'_t \in [\kappa_0, \kappa_1], \quad S_t f_t = e^{\lambda'_t} f_t.$$

Step 2. We reformulate again by choising $t = 2^{-n}$:

$$\exists\, f_n \in \mathcal{C},\ \exists\, \lambda'_n \in [\kappa_0, \kappa_1], \quad S_t f_n = e^{\lambda'_n t} f_n, \quad \forall\, t \in \mathbb{D}_m, m \le n,$$

with

$$\mathbb{D}_m := \{t = j 2^{-m}\} = 2^{-m}\mathbb{N} = \text{ part of dyadic real numbers}$$

By compactness, $\exists \lambda_1 \in [\kappa_0, \kappa_1]$, $\exists\, f_1 \in \mathcal{C}$ such that

$$S_t f_{n_k} = e^{\lambda'_{n_k}} f_{n_k}, \quad \lambda_{n_k} \to \lambda_1,\ f_{n_k} \rightharpoonup f_1.$$

We deduce

$$S_t f_1 = e^{\lambda_1 t} f_1, \quad \forall\, t \in \mathbb{D}_m,\ \forall\, m$$

and then

$$S_t f_1 = e^{\lambda_1 t} f_1, \quad \forall\, t \ge 0.$$

We assume (general dissipative case for N and X) with $\kappa_{\mathcal{B}} < \kappa_0$

$$[f_t] \ge e^{\kappa_0 (t-s)} [f_s], \ \forall t > s, \quad f_\tau := S_\tau f_0$$

$$, \qquad \|f_t\| \le C_1 e^{\kappa_{\mathcal{B}} t} \|f_0\| + C_2 \int_0^t e^{\kappa_{\mathcal{B}}(t-s)} [f_s] \, ds, \ C_1 > 0$$

Step 1. With the same notations

$$\|\Phi_{T_0}(f_0)\| \quad \le \quad C_1 e^{\alpha T_0} M + \frac{C_2}{|\alpha|} \le M,$$

for T_0 and $M > 0$ large enough. That implies $\Phi_{T_0} : \mathcal{C} \to \mathcal{C}$.
From Brouwer-Schauder-Tykonov theorem:

$$\exists \, f_{T_0} \in X_+, \ [f_{T_0}] = 1, \ S_{T_0} f_{T_0} = e^{\lambda_1 T_0} f_{T_0}.$$

We cannot make $T_0 \to 0$!!

Step 2. We denote $\bar S_t := S_t e^{-\lambda_1 t}$. We have built a periodic solution

$$\bar S_t f_{T_0} = \bar S_{t-kT_0} f_{T_0}, \quad k := [t/T_0], \quad \forall\, t > 0.$$

For any $t \ge 0$, we deduce

$$\begin{aligned} [\bar S_t f_{T_0}] &\ge e^{(\kappa_0-\lambda_1)(t-kT_0)}[f_{T_0}] \ge e^{(\kappa_0-\lambda_1)T_0} =: r_* > 0, \\ \|\bar S_t f_{T_0}\| &\le C_2 e^{(\kappa_2-\lambda_1)(t-kT_0)}\|f_{T_0}\| \le C_2 e^{(\kappa_2-\lambda_1)T_0)}\|f_{T_0}\| =: R^* < \infty. \end{aligned}$$

The mean u_T satisfies the same estimates:

$$u_T := \frac{1}{T}\int_0^T \bar S_t f_{T_0}\, dt \in \mathcal{G} := \{g \in X_+;\ [g] \ge r_*,\ \|g\| \le R^*\}.$$

By compactness, there exists $f_1 \in \mathcal{G}$ and (T_k) such that $u_{T_k} \rightharpoonup f_1$.
The von Neumann, Birkhoff mean ergodicity trick leads to

$$\begin{aligned} \bar S_t f_1 - f_1 &= \lim_{k\to\infty}\Big\{\frac{1}{T_k}\int_0^{T_k} \bar S_t \bar S_s f_{T_0} ds - \frac{1}{T_k}\int_0^{T_k} \bar S_s f_{T_0} ds\Big\} \\ &= \lim_{k\to\infty}\Big\{\frac{1}{T_k}\int_{T_k}^{T_k+t} \bar S_s f_{T_0} ds - \frac{1}{T_k}\int_0^{t} \bar S_s f_{T_0} ds\Big\} = 0, \end{aligned}$$

because $(\bar S_s f_{T_0})$ is uniformly bounded. We deduce $\mathcal{L} f_1 = \lambda_1 f_1$.

We assume (including weakly dissipative case)

$$[f_t] \geq [f_s],\ \forall t > s, \quad \kappa_0 := 0,$$
$$\|f_t\| \leq M\|f_0\| + \int_0^t \Theta(t-s)[f_s]\, ds,\ M \geq 1$$

For some $g_0 \in X_+$ such that $[g_0] = 1$, we set

$$\mathcal{C} := \{f \geq 0,\ [f] = 1,\ \|f\| \leq R\}, \quad R := \max(2\|\Theta\|_{L^1}, \|g_0\|)$$

and we define the increasing function

$$\lambda(t) := \inf_{f \in \mathcal{C}} [S_t f].$$

We have the alternative

- (case 1) $\sup \lambda \leq 2M$
- (case 2) $\sup \lambda > 2M$

By compactness, there exists $f_0 \in \mathcal{C}$ such that

$$\sup_{t \geq 0} [S_t f_0] \leq 2M.$$

We remind the iterated Duhamel formula

$$S = v + (S_{\mathcal{B}}\mathcal{A})^{(*N)} * S$$

and the associated mean equation

$$U_T = V_T + W_T$$

with

$$U_T := \frac{1}{T}\int_0^T S_t dt, \ v_T := \frac{1}{T}\int_0^T v_t dt, \ W_T := \frac{1}{T}\int_0^T (S_{\mathcal{B}}\mathcal{A})^{(*N)} * S dt.$$

Thanks to Fubini and positivity, we have

$$W_T \leq \int_0^T (S_{\mathcal{B}}\mathcal{A})^{(*N)} dt U_T$$

which implies

$$\|W_T f_0\| \leq \|\Theta\|_{L^1} [U_T f_0]$$

In a simpler way

$$\|V_T f_0\| \le M\|f_0\|.$$

All together, we have ‘

$$\|U_T f_0\| \le M\|f_0\| + \|\Theta\|_{L^1} [U_T f_0] \quad \text{and} \quad 1 \le [S_T f_0] \le 2M.$$

From the first inequality, we deduce that $\|U_T f\|$ is uniformly bounded on $T \in \mathbb{R}_+$.
By compactness, there exists $T_k \to +\infty$ and $f_1 \in X_+$ such that $U_{T_k} f \rightharpoonup f_1$.
Thanks to the second inequality, we have $[f_1] \ge 1$.
From the same and usual mean ergodic trick, for any fixed $s > 0$, we have

$$\begin{aligned} S(s)f_1 - f_1 &= \lim_{k\to\infty} \Big\{ \frac{1}{T_k} \int_0^{T_k} S(s)S(t)f_0 dt - \frac{1}{T_k} \int_0^{T_k} S(t)f_0 \, dt \Big\} \\ &= \lim_{k\to\infty} \Big\{ \frac{1}{T_k} \int_{T_k}^{T_k+s} S(t)f_0 dt - \frac{1}{T_k} \int_0^{s} S(t)f_0 \, dt \Big\} = 0. \end{aligned}$$

That implies that f_1 is a stationary solution, and thus $\lambda_1 = 0$.

Step 1 From the assumption

$$\exists\, T_0',\ \forall\, f \in \mathcal{C},\ [S_t f] \ge [S_{T_0'} f] \ge 2M + \varepsilon,\ \forall\, t \ge T_0'$$

and thus

$$\exists\, T_0,\ \forall\, f \in \mathcal{C},\ \forall\, T \ge T_0,\ [U_T f] \ge 2M.$$

For $f \in \mathcal{C}$, we define

$$\Psi_T f := \frac{U_T f}{[U_T f]}.$$

By the estimate on U_T established when dealing with case 1, we have

$$\|\Psi_T f\| \le \frac{1}{2}\|f\| + \|\Theta\|_{L^1} \le R,$$

so that $\Psi_T : \mathcal{C} \to \mathcal{C}$.

From Brouwer-Schauder-Tykonov theorem: for any $n \ge T_0$

$$\exists\, \lambda_n \in [0, \kappa_1], \exists\, f_n \in \mathcal{C},\ U_n f_n = e^{\lambda_n n} f_n.$$

And ?

We assume

$$(\lambda - \mathcal{L})f \geq 0 \Rightarrow f \geq 0 \text{ (weak maximum principle)}$$
$$\mathcal{L}^*\psi_0 \geq \kappa_0\psi_0$$
$$\text{splitting structure } \mathcal{L} = \mathcal{A} + \mathcal{B},\ \|e^{\mathcal{B}t}\| = \mathcal{O}(e^{\kappa_{\mathcal{B}} t}),\ \kappa_{\mathcal{B}} < \kappa_0$$

We define

$$\begin{aligned} \lambda_1 &:= \inf\{\lambda \in \mathbb{R};\ (\kappa - \mathcal{L}) : X \to X,\ (\kappa - \mathcal{L}) : X_+ \to X_+,\ \forall\, \kappa \geq \lambda\} \\ &= \inf\{\lambda \in \mathbb{R};\ \|(\kappa - \mathcal{L})^{-1}\|_{\mathcal{B}(X)} < \infty,\ \forall\, \kappa \geq \lambda\} \geq \kappa_0 \end{aligned}$$

That implies $\|(\lambda_n - \mathcal{L})^{-1}\|_{\mathcal{B}(X)} \to \infty$ when $\lambda_n \nearrow \lambda_1$ and in particular

$$\exists \hat{f}_n, \varepsilon_n \in X_+,\ \|\hat{f}_n\| = 1,\ \|\varepsilon_n\| \to 0,\ (\lambda_n - \mathcal{L})^{-1}\varepsilon_n = \hat{f}_n.$$

Equivalently (at rank $N = 1$)

$$\hat{f}_n = (\lambda - \mathcal{B})^{-1}\mathcal{A}\hat{f}_n + (\lambda - \mathcal{B})^{-1}\varepsilon_n \in \text{compact}$$

We deduce

$$\hat{f}_n \to f_1,\ \|f_1\| = 1,\ f_1 = (\lambda - \mathcal{B})^{-1}\mathcal{A}\hat{f}_1.$$

Existence - mixing dynamical and resolvent approach

We assume

$S_{\mathcal{L}}$ positive semigroup

$\mathcal{L}^*\psi_0 \geq 0$

splitting structure $\mathcal{L}^*\psi_2 \leq -\psi_1 + \psi_0$

Step 1

$$\begin{aligned} \lambda_1 &:= \inf\{\lambda \in \mathbb{R};\ (\kappa - \mathcal{L}) : X \to X,\ (\kappa - \mathcal{L}) : X_+ \to X_+,\ \forall\, \kappa \geq \lambda\} \\ &= \inf\{\lambda \in \mathbb{R};\ \|(\kappa - \mathcal{L})^{-1}\|_{\mathcal{B}(X_2,X_1)} < \infty,\ \forall\, \kappa \geq \lambda\} \end{aligned}$$

Step 2 Repeat the previous "resolvent approach"