# Hypocoercivity and geometrical constraints /space confinement

**S. Mischler**

*(Université Paris-Dauphine - PSL)*

Isaac Newton Institute seminar during the “Frontiers in kinetic theory: connecting microscopic to macroscopic scales” semester
March 1st, 2022

# Outline of the talk

1. Introduction
   - Villani's program
   - second step: quantitative hypocoercivity estimates
2. Relaxation equation with confinement
   - The relaxation operator in the torus
   - The relaxation operator with confinement force
   - The relaxation operator in bounded domain
3. Linearized Boltzmann equation with confinement
   - Linearized Boltzmann equation in the torus
   - Linearized Boltzmann equation in bounded domain
   - Linearized Boltzmann equation with force confinement
4. Perspectives

# Outline of the talk

1. Introduction
   - Villani's program
   - second step: quantitative hypocoercivity estimates
2. Relaxation equation with confinement
   - The relaxation operator in the torus
   - The relaxation operator with confinement force
   - The relaxation operator in bounded domain
3. Linearized Boltzmann equation with confinement
   - Linearized Boltzmann equation in the torus
   - Linearized Boltzmann equation in bounded domain
   - Linearized Boltzmann equation with force confinement
4. Perspectives

1. Find a constructive method for bounding below the spectral gap in $L^2(M^{-1})$, the space of self-adjointness, say for the Boltzmann operator with hard spheres.

3. Find a constructive argument to overcome the degeneracy in the space variable, to get an exponential decay for the linear semigroup associated with the linearized spatially inhomogeneous Boltzmann equation; something similar to hypo-ellipticity techniques.

2. Find a constructive argument to go from a spectral gap in $L^2(M^{-1})$ to a spectral gap in $L^1$, with all the subtleties associated with spectral theory of non-self-adjoint operators in infinite dimension ...

4. Combine the whole things with a perturbative and linearization analysis to get a constructive exponential decay for the nonlinear equation close to equilibrium.

$\Rightarrow$ constructive constants are fundamental for connecting microscopic to macroscopic scales

# Space inhomogeneous Boltzmann equation (or related models)

Consider a kinetic equation on the density of particles of a gas

$$\partial_t F + v \cdot \nabla_x F + \ldots = Q(F)$$
$$F(0,.) = F_0$$

where $F = F(t,x,v) \geq 0$, time $t \geq 0$, velocity $v \in \mathbb{R}^3$, position $x \in \Omega$,

$\Omega = \mathbb{T}^3$ (torus)

$\Omega = \mathbb{R}^3$ + confinement force field

$\Omega \subset \mathbb{R}^3$ + boundary reflection conditions

$Q$ = linear relaxation or Fokker-Planck collisions operator : 1 conservation (of mass)

$Q$ = nonlinear (quadratic) Boltzmann (or Landau) collisions operator
: $d+2$ conservations (of mass, momentum and energy)

**Theorem** (expected)

There exists a unique stationary solution $F_\infty(x,v) = M(v) = (2\pi)^{-3/2} e^{-|v|^2/2}$ and for any $F_0$ the (unique?) solution $F_t$ satisfies

$$F_t \to F_\infty, \quad t \to \infty.$$

What about a constructive rate = quantitative and constructive H-Theorem ?

## Splitting of the proof into 4 constructive steps :

We introduce the linearized Boltzmann operator

$$\mathcal{L} := \mathcal{T} + \mathcal{S}, \quad \mathcal{T} := -v \cdot \nabla_x, \quad \mathcal{S} := Q(\cdot, M) + Q(M, \cdot)$$

and the projections

$$\begin{aligned} \pi f &:= \text{ microscopic projection on } N(\mathcal{S}) \\ \Pi f &:= \text{ macroscopic projection on } N(\mathcal{L}) \end{aligned}$$

- coercivity in $v$ of $\mathcal{S}$: there exist some Hilbert spaces $\mathfrak{h}$ and $\mathfrak{h}_*$

$$(-\mathcal{S}h, h)_{\mathfrak{h}} \geq \lambda \, \|\pi^\perp h\|^2_{\mathfrak{h}_*}, \quad \pi^\perp = I - \pi$$

- hypocoercivity in $(x, v)$ of $\mathcal{L}$: there exists a Hilbert space $\mathcal{H} = L^2$ or $H^k$ and an equivalent Hilbert norm such that

$$(\!( -\mathcal{L}h, h )\!)_{\mathcal{H}} \geq \kappa \, |\!|\!| \, \Pi^\perp h \, |\!|\!|^2_{\mathcal{H}}, \quad \Pi^\perp = I - \Pi$$

- there exists a Banach algebra $\mathcal{X}$ such that

$$\|S_{\mathcal{L}}(t) f_0 - \Pi f_0\|_{\mathcal{X}} \leq C \, e^{at} \|f_0 - \Pi f_0\|_{\mathcal{X}}, \quad \forall \, t \geq 0.$$

- In a conditional bounded regime or a close to the equilibrium regime:

$$\|F_t - F_\infty\| \leq C \, e^{at}, \quad \forall \, t \geq 0.$$

## "$(x,v)$ coercivity" estimate issue $\Rightarrow$ hypocoercivity answer

In a Hilbert space $\mathcal{H} \supset \mathcal{H}_x \otimes \mathcal{H}_v$, consider an operator

$$\mathcal{L} = \mathcal{S} + \mathcal{T}, \quad \mathcal{S}^* = \mathcal{S} \le 0, \quad \mathcal{T}^* = -\mathcal{T}.$$

- Microscopic conservation. $\mathcal{S}$ acts on the $v$ variable space $\mathcal{H}_v$ and is coercive:

$$(-\mathcal{S}f, f)_{\mathfrak{h}} \gtrsim \|f^\perp\|^2_{\mathfrak{h}_*}, \quad f^\perp = f - \pi f,$$

for a finite dimensional range projector $\pi$ in $\mathfrak{h} = \mathcal{H}_v$. We have

$$f \in N(\mathcal{S}) \Leftrightarrow \ (\mathcal{S}f, f) = 0 \Leftrightarrow \ f = \pi f$$

- Macroscopic conservation. The main issue is

$$N(\mathcal{L}) = N(\mathcal{S}) \cap N(\mathcal{T}) \ \neq \ N(\mathcal{S}) \quad \text{in} \quad \mathcal{H}!!$$

In $\mathcal{H}$ the operator $\mathcal{S}$ is degenerately / partially coercive: for the initial Hilbert norm, we get the same degenerate / partial positivity of the Dirichlet form

$$D[f] := (-\mathcal{L}f, f) = (-\mathcal{S}f, f) \gtrsim \|\pi^\perp f\|^2_{\mathcal{H}_*} \neq \|\Pi^\perp f\|^2_{\mathcal{H}_*}, \quad \forall f.$$

That information is not strong enough in order to control the longtime behavior of the dynamic of the associated semigroup !! We need to control $\pi f \in \mathcal{H}_x$ !

# What is the $L^2$-hypocoercivity about - the twisted norm approach

▷ Find a new Hilbert norm by twisting

$$|||f|||^2 := \|f\|^2 + 2(Af, Bf)$$

such that the new Dirichlet form is coercive for $f$ such that $\Pi f = 0$:

$$\begin{aligned} D[f] &:= ((-\mathcal{L}f, f)) \\ &= (-\mathcal{L}f, f) + (-A\mathcal{L}f, Bf) + (Af, -B\mathcal{L}f) \\ &\gtrsim \|f^\perp\|^2 + \|\pi f\|^2. \end{aligned}$$

▷ We destroy the nice symmetric / skew symmetric structure and we have also to be very careful with the "remainder terms".
▷ That functional inequality approach is equivalent (and more precise if constructive) to the other more dynamical approach (called "Lyapunov" or "energy" approach).

**Theorem.** (for strong coercive operators in both variables, in particular $\mathfrak{h}_* \subset \mathfrak{h}$)

There exist some new but equivalent Hilbert norm $|||\cdot|||$ and a (constructive) constant $\lambda > 0$ such that the associated Dirichlet form satisfies

$$D[f] \ge \lambda\, |||f|||^2, \quad \forall f,\ \Pi f = 0$$

▷ It implies $|||e^{\mathcal{L}t} f||| \le e^{-\lambda t} |||f|||$ and then $\|e^{\mathcal{L}t} f\| \le C e^{-\lambda t} \|f\|$, $\forall f,\ \Pi f = 0$.

From the hypocoercivity estimate

$$D[f] \geq \|f^\perp\|^2 + \|\pi f\|^2, \quad \text{if} \quad \Pi f = 0,$$

we are able to establish

$$\mathcal{L} f = 0 \quad \Rightarrow \quad f = \Pi f.$$

We have more

$$\mathcal{L} f \simeq 0 \quad \Rightarrow \quad f \simeq \Pi f.$$

# About the Boltzmann equation

General regime and conditional bounded regime

- DiPerna-Lions renormalized solutions ($\sim$ 1990)
- Constructive entropy approach: Desvillettes-Villani (2001-2005)
- Exponential convergence: Mouhot + Baranger, Strain, Neumann, Gualdani-M., Carrapatoso-M. (since 2006)

Close to the equilibrium regime:

- Non constructive spectral analysis approach : Ukai (1974), Arkeryd, Esposito, Pulvirenti (1987), Wennberg (1995)
- Energy (in high order Sobolev space) approach [2002-...] : Guo and Guo' school
- Micro-Macro approach : Shizuta, Kawashima (1984), Liu, Yu (2004), Yang, Guo, Duan, ...

# About Hypocoercivity estimates:

More about close to the equilibrium regime and hypocoercivity

- Constructive estimate and hypoellipticty : Hérau, Nier, Helffer, Eckmann, Hairer (2003-2005), Villani (2009)
- Constructive hypocoercivity estimates without hypoellipticty [2006-...]: Hérau, Villani, Mouhot, Neumann, Dolbeault, Schmeiser, Guo, . . .
- Carrapatoso, Dolbeault, Hérau, M., Mouhot, *Weighted Korn and Poincaré-Korn Inequalities in the Euclidean Space and Associated Operators*, ARMA (2022)
- Bernou, Carrapatoso, M., Tristani, *Hypocoercivity for kinetic linear equations in bounded domains with general Maxwell boundary condition*, Annales IHP (?)
- Carrapatoso, Dolbeault, Hérau, M., Mouhot, Schmeiser, *Special macroscopic modes and hypocoercivity*, arXiv (2021)

- relaxation equation in the torus - Hérau
- relaxation equation with confinement force - Dolbeault-Mouhot-Schmeiser
- relaxation equation in a bounded domain $\sim$ Guo (but Villani's formalism)
- linearized Boltzmann equation in the torus
  - Mouhot-Neumann by $H^1$-hypocoercivity $\neq L^2$-hypocoercivity
- linearized Boltzmann in a bounded domain
  - Guo, Briant-Guo, . . . , Bernou, Carrapatoso, M., Tristani
- linearized Boltzmann with confinement force
  - Duhan, Duhan-Li, . . . , Carrapatoso, Dolbeault, Hérau, M., Mouhot, Schmeiser

# Outline of the talk

We consider the "simplest" relaxation kinetic operator

$$\mathcal{L} := \mathcal{S} + \mathcal{T}$$

where $\mathcal{S}$ is the "simplest" relaxation operator

$$\mathcal{S}f := \rho_f M - f =: f^\perp, \quad \rho_f := \langle f \rangle := \int f dv$$

and $\mathcal{T}$ is the transport operator

$$\mathcal{T}f := -v \cdot \nabla_x f + \dots .$$

We may assume

$$\begin{aligned} &\text{(case 1)} \qquad \dots = 0, \quad \Omega := \mathbb{T}^d, \quad (\phi := 0); \\ &\text{(case 2)} \qquad \dots = \nabla_x \phi \cdot \nabla_v f, \quad \Omega := \mathbb{R}^d, \quad e.g.\ \phi \sim |x|^\gamma,\ \gamma \ge 1; \\ &\text{(case 3)} \qquad \dots = 0, \quad \Omega \subset \mathbb{R}^d \ + \text{ reflection}, \quad (\phi := 0). \end{aligned}$$

• By definition

$$\mathcal{S}f = 0 \Leftrightarrow \; f - \rho_f M(v) = 0,$$

so that

$$N(\mathcal{S}) = \{\rho(x)M\}, \quad \pi f := \rho_f(x)M(v)$$

• We remind that

$$N(\mathcal{L}) = N(\mathcal{S}) \cap N(\mathcal{T}),$$

so that

$$f \in N(\mathcal{L}) \Leftrightarrow \; \rho_f(x)M \in N(\mathcal{T}).$$

We compute

$$v \cdot \nabla_x(\rho_f(x)M(v)) = 0 \;\Rightarrow\; \nabla_x \rho_f = 0.$$

By periodicity, we deduce $\rho_f(x) = \langle \rho_f \rangle$.
As a conclusion:

$$N(\mathcal{L}) = \mathrm{vect} M, \quad \Pi f = \langle \rho_f \rangle M(v)$$

and the only macroscopic law of conservation is the mass conservation.

We introduce the twisted Hilbert norm

$$|||f|||^2 := \|f\|^2_{\mathcal{H}} - 2\eta(\nabla_x \Delta^{-1}\rho, m)$$

with $1 >> \eta > 0$ and then the Dirichlet form

$$\begin{aligned} D(f) &= ((-\mathcal{L}f, f)) \\ &= (-\mathcal{L}f, f) + \eta(\nabla_x \Delta^{-1}\rho_f, m[\mathcal{L}f]) + \eta(\nabla_x \Delta^{-1}\rho[\mathcal{L}f], m_f). \end{aligned}$$

Here

$$\begin{aligned} \rho &:= \rho_f = \rho[f] = \langle f \rangle = \int f dv, \\ m &:= m_f = m[f] = \langle f\, v \rangle = \int f v dv. \end{aligned}$$

**Theorem 1**

For a convenient choice of $1 >> \eta > 0$ there holds (with explicit constant)

$$D(f) \gtrsim |||f|||^2 \simeq \|f\|^2_{\mathcal{H}}, \quad \forall f, \ \Pi f = 0,$$

with

$$\Pi f = \langle \rho_f \rangle M(v).$$

$\Delta^{-1}$ := solution to the Poisson equation with periodic condition.
We split $D = D_0 + D_1 + D_2$.

- We have

$$D_0 := (-\mathcal{L}f, f)_{L^2(M^{-1})} = \|f^\perp\|^2_{\mathcal{H}}$$

- We compute

$$\begin{aligned} m[\mathcal{L}f] &= \langle v\mathcal{T}\pi f\rangle + \langle v\mathcal{L}f^\perp\rangle \\ &= -\nabla_x\rho_f + \nabla_x\langle v\otimes v\, f^\perp\rangle + \langle vf^\perp\rangle, \end{aligned}$$

so that

$$\begin{aligned} D_1 &:= \eta(\nabla_x\Delta^{-1}\rho_f, m[\mathcal{L}f]) \\ &:= \eta\big(\nabla_x\Delta^{-1}\rho_f, -\nabla_x\rho_f + \nabla_x\langle v\otimes v\, f^\perp\rangle + \langle vf^\perp\rangle\big) \\ &\gtrsim \eta\|\rho_f\|^2_{L^2} - \eta\|\rho_f\|_{L^2}\,\|f^\perp\|_{\mathcal{H}}. \end{aligned}$$

with $\|\rho_f\|^2_{L^2} = \|\pi f\|^2_{L^2}$ !

- Similarly

$$\rho[\mathcal{L}f] = \langle\mathcal{T}\pi f\rangle + \langle\mathcal{L}f^\perp\rangle = -\nabla_x\langle vf^\perp\rangle,$$

so that

$$D_2 := \eta(\nabla_x\Delta^{-1}\rho[\mathcal{L}f], m_f) = -\eta(\nabla_x\Delta^{-1}\nabla_x\langle vf^\perp\rangle, m_f) \gtrsim -\eta\|f^\perp\|^2_{\mathcal{H}}$$

We have used the splitting

$$f = \pi f + f^{\perp},$$

the cancellation

$$\mathcal{S}\pi f = 0,$$

the identities

$$(\nabla \Delta^{-1}\rho, -\nabla\rho) = \|\rho\|_{L^2}^2 = \|\pi f\|_{L^2}^2,$$

and

$$\|f\|^2 = \|\rho\|^2 + \|f^{\perp}\|^2,$$

the two estimates

$$\begin{aligned} \Delta^{-1} &: H^{-1} \to H^1, \\ \Delta^{-1} &: L^2 \to H^2. \end{aligned}$$

and the Young inequality

$$\begin{aligned} D &\gtrsim A^2 + \eta B^2 - \eta AB - \eta A^2 \\ &\gtrsim (1 - \eta - \tfrac{1}{2})A^2 + \eta(1 - \tfrac{\eta}{2})B^2 \end{aligned}$$

We rather define $\Delta^{-1} := \Delta_\phi^{*-1}$, $\Delta_\phi^*$ stands for the modified Laplacian operator

$$\Delta_\phi^* u \quad := \quad \Delta u - \nabla\phi \cdot \nabla u = e^{\phi}\nabla(e^{-\phi}\nabla u),$$

and the twisted $L^2$ scalar product

$$((f,g)) \quad = \quad (f,g)_{\mathcal{H}} - \eta(\nabla\Delta_\phi^{*-1}(\rho_f e^{\phi}), m_g)_{L^2} - \eta(m_f, \nabla\Delta_\phi^{*-1}(\rho_g e^{\phi}))_{L^2}.$$

We compute

$$\begin{aligned} m[-\mathcal{L}f] &= m[-\mathcal{T}\pi f] + \dots \\ &= m[v \cdot \nabla_x \rho_f M - \nabla\phi \cdot \nabla_\phi \rho_f M] + \dots \\ &= m[Mv \cdot (\nabla_x \rho_f + \nabla\phi\rho_f)] + \dots \\ &= \nabla_x \rho_f + \nabla\phi\rho_f + \dots = e^{-\phi}\nabla(\rho_f e^{\phi}) + \dots \end{aligned}$$

We deduce that the leader term in $D_1$ is

$$\begin{aligned} D_{1,1} &:= -\eta(\nabla\Delta_\phi^{*-1}(\rho_f e^{\phi}), m[-\mathcal{T}\pi f])_{L^2} \\ &= -\eta(\nabla\Delta_\phi^{*-1}(\rho_f e^{\phi}), e^{-\phi}\nabla(\rho_f e^{\phi}))_{L^2} \\ &= \eta(e^{\phi}\nabla(e^{-\phi}\nabla\Delta_\phi^{*-1}(\rho_f e^{\phi})), \rho_f)_{L^2} \\ &= \eta\, \|\rho_f\|^2_{L^2(e^{\phi})}. \end{aligned}$$

We complement the "simplest" relaxation kinetic operator with the reflection condition at the boundary

$$f_- = \mathcal{C} f_+ \quad \text{on} \quad \Sigma_-, \quad f_\pm = f_{|\Sigma_\pm},$$

where

$$\Sigma_\pm := \{(x,v) \in \Sigma := \partial\Omega \times \mathbb{R}^d, \ \pm n(x) \cdot v > 0\}$$

and $n(x)$ stands for the outward unit normal vector at boundary point $x \in \partial\Omega$.

The reflection operator $\mathcal{C}$ splits as

$$\mathcal{C} g = (1-\alpha)\mathcal{R} g + \alpha \mathcal{D} g,$$

with accomodation coefficient $\alpha \in [0,1]$, $\mathcal{R}$ the specular reflection operator

$$\mathcal{R} g(x,v) := g(x, R_x v), \quad R_x v := v - 2(v \cdot n(x)) n(x),$$

and $\mathcal{D}$ the diffusion reflection operator

$$\mathcal{D} g := c\, M(v)\, \widetilde{g}, \quad \widetilde{g}(x) := \int_{n(x)\cdot w > 0} g(x,w) n(x) \cdot w \, dw,$$

where $c$ such that $c\widetilde{M} = 1$.

Same defintion of the twisted Hilbert norm, with now $u := \Delta^{-1}\rho_f$ solution to the Poisson equation with Neumann boundary condition (mass is conserved !).

- Because of the dissipation property of the diffusion reflection operator

$$D_0 := (-\mathcal{L}f, f) \geq \lambda\|f^\perp\|^2 + \frac{1}{2}\|\sqrt{\alpha(2-\alpha)}\mathcal{D}^\perp f_+\|^2_{\partial\mathcal{H}_+}$$

with $\mathcal{D}^\perp = I - \mathcal{D}$, $\partial\mathcal{H}_+ := L^2(\Sigma_+, n(x)\cdot v dv d\sigma_x)$.

- We compute (with $\neq$ integration by part)

$$\begin{aligned} \eta^{-1}D_1 &:= \big(\nabla_x u, -v\nabla_x\langle vf\rangle\big) + \dots \\ &= \big(\partial_{ij}u, \langle v_i v_j f\rangle\big) + \int_\Sigma (\nabla u\cdot v) f n\cdot v + \dots \\ &= \big(\partial_{ij}u, \delta_{ij}\rho_f\big) + \int_{\Sigma_+}(\nabla u\cdot v)\alpha\mathcal{D}^\perp f_+\, n\cdot v + \big(\partial_{ij}u, \langle v_i v_j f^\perp\rangle\big) + \dots, \end{aligned}$$

where we have used the identity (reformulation if the reflection condition)

$$\begin{aligned} \int_\Sigma \psi f\, n\cdot v &= \int_{\Sigma_+}\psi\alpha\mathcal{D}^\perp f_+\, n\cdot v + \int_{\Sigma_+}\{\psi - \psi\circ R_x\}(1-\alpha)\mathcal{D}^\perp f_+\, n\cdot v \\ &\quad + \int_{\Sigma_+}\{\psi - \psi\circ R_x\}\mathcal{D}f_+\, n(x)\cdot v, \end{aligned}$$

with $\psi := \nabla u\cdot v$, so that $\psi - \psi\circ R_x = 0$.

Same defintion of the twisted Hilbert norm, with now $u := \Delta^{-1}\rho_f$ solution to the Poisson equation with Neumann boundary condition (mass is conserved !).

• Because of the dissipation property of the diffusion reflection operator

$$D_0 := (-\mathcal{L}f, f) \geq \lambda \|f^\perp\|^2 + \frac{1}{2}\|\sqrt{\alpha(2-\alpha)}\mathcal{D}^\perp f_+\|^2_{\partial\mathcal{H}_+}$$

with $\mathcal{D}^\perp = I - \mathcal{D}$, $\partial\mathcal{H}_+ := L^2(\Sigma_+, n(x)\cdot v dv d\sigma_x)$.

• We compute (with $\neq$ integration by part)

$$\begin{aligned} \eta^{-1} D_1 &= \big(\partial_{ij} u, \delta_{ij}\rho_f\big) + \int_{\Sigma_+} (\nabla u \cdot v)\alpha\mathcal{D}^\perp f_+ \, n \cdot v + \dots \\ &= \|\rho_f\|^2_{L^2} - \mathcal{O}\big(\|u\|_{H^1(\partial\Omega)}\|\alpha\mathcal{D}^\perp f_+\|_{\partial\mathcal{H}_+}\big) + \dots \\ &= \|\rho_f\|^2_{L^2} - \mathcal{O}\big(\|\rho_f\|_{L^2}\|\alpha\mathcal{D}^\perp f_+\|_{\partial\mathcal{H}_+}\big) + \dots, \end{aligned}$$

by Cauchy-Schwarz inequality and elliptic regularity estimate.

# Outline of the talk

Consider the equation

$$\partial_t f + v \cdot \nabla_x f = \mathcal{S} f, \quad (0,\infty) \times \mathbb{T}^d \times \mathbb{R}^d,$$

with linearized Boltzmann collisional operator $\mathcal{S}$, and thus

$$\mathcal{L} := \mathcal{T} + \mathcal{S}, \quad \mathcal{T} := -v \cdot \nabla_x \ \& \ \mathbb{T}^d\text{-periodicity}.$$

- The microscopic null space is

$$f \in N(\mathcal{S}) \ \Leftrightarrow \ (\mathcal{S} f, f) = 0 \ \Leftrightarrow \ f = \pi f$$

with

$$\pi f := \rho_f M(v) + m_f v M(v) + e_f \mathfrak{E}(v) M(v), \quad \mathfrak{E}(v) := \tfrac{1}{\sqrt{2d}}(|v|^2 - d)$$

- The naive macroscopic conservation are

$$\frac{d}{dt} \int f(1, v_i, |v|^2) dv dx = 0$$

and the naive macroscopic projector is

$$\Pi f := \langle \rho_f \rangle M(v) + \langle m_f \rangle v M(v) + \langle e_f \rangle \mathfrak{E}(v) M(v)$$

- The macroscopic null space is

$$f \in N(\mathcal{L}) \;\Leftrightarrow\; \mathcal{S}f = \mathcal{T}f = 0 \;\Leftrightarrow\; \mathcal{T}\pi f = 0$$

The last equation writes

$$\begin{cases} \partial_t \rho_f = -\nabla_x \cdot m_f \\ \partial_t m_f = -\nabla_x \rho_f - \sqrt{\frac{2}{d}}\, \nabla_x e_f \\ \partial_t e_f = -\sqrt{2/d}\, \nabla_x \cdot m_f \\ \frac{1}{\sqrt{2d}} (\partial_t e_f) I = \nabla^s_x m_f \\ 0 = \nabla_x e_f \end{cases}$$

where $(\nabla^s_x m)_{ij} = (\partial_i m_j + \partial_j m_i)/2$ is the symmetric gradient. The Schwarz Lemma

$$\partial^2_{x_i x_j} m_k = \partial_{x_i} (\nabla^s m)_{j,k} + \partial_{x_j} (\nabla^s m)_{i,k} - \partial_{x_k} (\nabla^s m)_{i,j}$$

and differential calculus yield

$$\rho_f \sim a - x \cdot b' + |x|^2 c'', \quad m_f \sim Ax + b - xc', \quad e_f = c$$

with $b = b(t), c = c(t) \in \mathbb{R}$ and $a \in \mathbb{R}$, $A \in M^a = \{A^* = -A\}$ independent of time. The periodicity condition and $\Pi f = 0$ imply $a = b = c = A = 0$. That proves

$$\Pi f := \langle \rho_f \rangle M(v) + \langle m_f \rangle v M(v) + \langle e_f \rangle \mathfrak{E}(v) M(v)$$

The appropriate twisted $L^2$ norm is

$$\begin{aligned} |||f|||^2 \quad &:= \quad \|f\|_{L^2}^2 - 2\eta_1(\nabla_x u[e_f], M_p[f]) \\ &\quad -2\eta_2(\nabla_x^s U[m_f], M_q[f]) - 2\eta_3(\nabla_x u[\rho_f], m_f) \end{aligned}$$

with $1 >> \eta_1 >> \eta_2 >> \eta_3 > 0$, $u = u(E)$ and $U = U(M)$ are given by

$$-\Delta u = E \quad \text{in } \mathbb{T}^d,$$

$$-\text{div}(\nabla^s U) = M \quad \text{in } \mathbb{T}^d$$

and $M_r[f] = \langle rf \rangle$, $p := v(|v|^2 - 5)/2$, $q := v \otimes v - I$.

The four main contributions in the associated Dirichlet form are

$$\begin{aligned} D[f] \quad &\gtrsim \quad \|f^\perp\|^2 + \eta_1(\nabla u[e_f], \nabla e_f) + \eta_2(\nabla_x^s U[m_f], \nabla m_f) \\ &\quad +\eta_3(\nabla_x u[\rho_f], \nabla \rho_f) - \dots \\ &\gtrsim \quad \|f^\perp\|^2 + \eta_1\|e_f\|^2 + \eta_2\|m_f\|^2 + \eta_3\|\rho_f\|^2 - \dots \end{aligned}$$

In order to solve the system

$$-\mathrm{div}(\nabla^s U) = M \quad \text{in } \mathbb{T}^d$$

and to prove it is elliptic, we introduce the bilinear form

$$\begin{aligned} a(U,V) &:= (-\mathrm{div}(\nabla^s U), V) = (\nabla^s U, \nabla V) \\ &= (\nabla^s U, \nabla^s V), \end{aligned}$$

which is continuous in $H^1(\mathbb{T}^d)$. It is also coercive thanks to Korn and Poincaré inequalities

$$\begin{aligned} a(U,U) &= \|\nabla^s U\|^2 \\ &\gtrsim \|\nabla U\|^2 \gtrsim \|U\|^2_{H^1}, \end{aligned}$$

when $\langle U \rangle = 0$.

**Theorem 2**

For a convenient choice of $1 >> \eta_1 >> \eta_2 >> \eta_3 > 0$, there holds

$$(\!(-\mathcal{L}h, h)\!) \geq |\!|\!|\Pi^\perp f|\!|\!|^2 \simeq \|\Pi^\perp f\|^2_{\mathcal{H}}$$

with

$$\Pi f = \langle \rho_f \rangle M(v) + \langle m_f \rangle v M(v) + \langle e_f \rangle \mathfrak{E}(v) M(v)$$

Consider the equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \mathcal{S} f, & (0,\infty) \times \mathcal{O}, \\ f_- = \mathcal{C} f_+ = (1-\alpha)\mathcal{R} f_+ + \alpha \mathcal{D} f_+, & (0,\infty) \times \Sigma_-, \end{cases}$$

with linearized Boltzmann collisional operator $\mathcal{S}$, accomodation coefficient $\alpha \in [0,1]$, specular reflection operator $\mathcal{R}$ and diffusion reflection operator $\mathcal{D}$.

Same microscopic conservations and macroscopic mass is conserved

$$\frac{d}{dt} \int f dx dv = 0, \quad \forall\, \alpha \in [0,1].$$

It is the macroscopic conservation law when $\alpha > 0$. When $\alpha = 0$, energy is conserved

$$\frac{d}{dt} \int f |v|^2 dx dv = 0$$

as well as the *total angular momentum*

$$\frac{d}{dt} \int (A\, x \cdot v)\, f dx dv = 0$$

associated to rotation deplacements preserving $\Omega$:

$$A \in \mathcal{A}_\Omega := \{A^* = -A;\ Ax \cdot n(x) = 0\ \forall\, x \in \partial\Omega\}.$$

The appropriate modified $L^2$ norm is

$$\begin{aligned} |||f|||^2 \quad &:= \quad \|f\|^2_{L^2} - 2\eta_1(\nabla_x u[e_f], M_p[f]) \\ &\quad -2\eta_2(\nabla^s_x U[m_f], M_q[f]) - 2\eta_3(\nabla_x u_N[\rho_f], m_f) \end{aligned}$$

with $1 >> \eta_1 >> \eta_2 >> \eta_3 > 0$, $u = u(E)$, $U = U(M)$ and $u_N = u_N[\rho]$ are given by

$$\begin{cases} -\Delta u = E \quad \text{in } \Omega, \\ (2-\alpha)\frac{\partial u}{\partial n} + \alpha u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

$$\begin{cases} -\mathrm{div}(\nabla^s U) = M \quad \text{in } \Omega, \\ U \cdot n = 0 \quad \text{on } \partial\Omega, \\ (2-\alpha)[\nabla^s U - (\nabla^s U : n \otimes n)n] + \alpha U = 0 \quad \text{on } \partial\Omega, \end{cases}$$

$$\begin{cases} -\Delta u_N = \rho \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \end{cases}$$

and $M_r[f] = \langle rf \rangle$, $p := v(|v|^2 - 5)/2$, $q := v \otimes v - I$.

The macrocopic projector is

$$\begin{aligned} \Pi f \quad &:= \quad \langle \rho_f \rangle M(v) \quad \text{if } \alpha > 0, \\ \Pi f \quad &:= \quad \langle \rho_f \rangle M(v) + (P_{\mathcal{A}_\Omega} \langle \nabla^a m_f \rangle) x \cdot v M(v) + \langle e_f \rangle \mathfrak{E}(v) M(v) \quad \text{if } \alpha = 0, \end{aligned}$$

Consider the equation

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = \mathcal{S} f, \quad (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d,$$

with the linearized Boltzmann collisional operator $\mathcal{S}$ (with same microscopic conservations).

• mass is conserved $\Rightarrow$ mass mode

$$F := \mathcal{M}, \quad \mathcal{M} := e^{-|v|^2/2} e^{-\phi}$$

is a stationary state.

• Hamiltonian energy

$$\mathcal{H} := \tfrac{1}{2}|v|^2 + \phi(x)$$

is conserved $\Rightarrow$ energy mode $F := \mathcal{H}\mathcal{M}$ is a stationary state.

• roatations $A$ compatible with $\phi$ if

$$A \in \mathcal{A}_\phi := \{ \, A \in M^a; \; \nabla(x) \cdot \nabla\phi(x) = 0, \; \forall \, x\}$$

$\Rightarrow$ rotation mode $F := (Ax \cdot v)\mathcal{H}\mathcal{M}$ is a stationary state if $A \in \mathcal{A}_\phi$.

# There are possibly other non stationary special modes

Define

$$E_\phi := span\{\nabla\phi(x) - x\}$$

with dimension $d_\phi \in \{0, \dots, d\}$.

• If $1 \le d_\phi \le d-1$ and $i$ such that $\partial_{x_i}\phi = x_i$ then

$$(x_i \cot t - v_i \sin t)\mathcal{M}, \quad (x_i \sin t + v_i \cos t)\mathcal{M},$$

are harmonic directional modes (particular oscillating solutions).

• If $d_\phi = 0 \Leftrightarrow \phi(x) = |x|^2/2$, there are additional harmonic pulsation modes

$$[(|x|^2 - |v|^2)\cos(2t) - 2x\cdot v\sin(2t)]\mathcal{M}.$$

• We find

$$\Pi f = \langle\rho\rangle M + \langle\langle\mathcal{H}f\rangle\rangle\mathcal{H}M + P_\phi\langle\nabla^a m\rangle x\cdot v\mathcal{M},$$

when we additionally assume if $d_\phi \in \{1, \dots, d-1\}$

$$\langle\langle x_i f\rangle\rangle = \langle\langle x_i f\rangle\rangle = 0$$

and if $d_\phi = 0$

$$\langle\langle 2x\cdot vf\rangle\rangle = \langle\langle(|x|^2 - |v|^2)f\rangle\rangle = 0$$

The appropriate modified $L^2$ norm is not

$$\begin{aligned} |||f|||^2 \quad := \quad & \|f\|^2_{L^2} - 2\eta_1(\Delta_\phi^{-1}\nabla_x e_f, M_p[f]) \\ & -2\eta_2(\Delta_\phi^{-1}\nabla_x^s m_f, M_q[f]) - 2\eta_3(\Delta_\phi^{-1}\nabla_x \rho_f, m_f) \end{aligned}$$

with $1 >> \eta_1 >> \eta_2 >> \eta_3 > 0$ and

$$\Delta_\phi u := \Delta u - \nabla\phi \cdot \nabla u - u.$$

We need to control additional macroscopic quantities $b = b(t), c = c(t) \in \mathbb{R}$ and $A \in M^a$ defined by

$$\rho_f \sim -x \cdot b' + |x|^2 c'' + \phi c, \quad m_f \sim Ax + b - xc', \quad e_f = c$$

which appear when considering the hyperbolic system $\mathcal{T}\pi f = 0$, or more precisely and worst $\mathcal{T}\pi f = \mathcal{O}(\|f^\perp\|)$. We also need a Korn inequality

$$\|u\| \lesssim \|\Delta_\phi^{-1/2}\nabla^s u\|$$

in order to control $m$.

We rather define

$$\begin{aligned}\mathcal{F}(t) \quad := \quad & \|f\|^2_{L^2} - \eta_1(\Delta_\phi^{-1}\nabla_x e, M_p[f]) - \eta_2(\Delta_\phi^{-1}\nabla^s_x m_s, M_q[f]) \\ & -\eta_3(\Delta_\phi^{-1}\nabla_x w_s, m_s) + \eta_3(\Delta_\phi^{-1}\nabla_x \partial_t w_s, w_s) \\ & -\eta_5 \left\langle (X - Y \cdot \nabla_x \phi), \nabla_x \phi \cdot A\, x \right\rangle - \eta_6 \left\langle b, b' \right\rangle - \eta_6 \left\langle c', c'' \right\rangle,\end{aligned}$$

with convenient $1 >> \eta_1 >> \eta_2 >> \eta_3 >> \eta_4 >> \eta_5 >> \eta_6 > 0$, where

$$\begin{aligned}\rho_s \quad & \sim \quad \rho - \langle \nabla \rho \rangle x - \langle \Delta \rho \rangle |x|^2 \\ m_s \quad & \sim \quad m - \langle \nabla^a m \rangle x - \langle \nabla \cdot m \rangle x - \langle m \rangle \\ w_s \quad & \sim \quad \rho_s - \langle e \rangle (\phi - \langle \Delta \phi \rangle |x|^2) \\ X \quad & \sim \quad (2\phi + \nabla \phi \cdot x - d) c + |x|^2 c'' - x \cdot b' \\ Y \quad & \sim \quad \langle x \phi \rangle c + \langle |x|^2 x \rangle c'' - \langle x \otimes x \rangle b'\end{aligned}$$

and $\rho$, $m$, $e$, $A$, $b$ and $c$ are defined by

$$\rho = \langle f \rangle, \quad m = \langle v f \rangle, \quad e = \langle \mathfrak{E} f \rangle$$
$$A = \langle \nabla^a m \rangle, \quad b = \langle m \rangle, \quad c = \langle e \rangle$$

We prove

$$\mathcal{F}' \lesssim -\mathcal{F} \sim -\|f\|^2 \quad \text{when} \quad \Pi f = 0.$$

# Outline of the talk

1. Introduction
   - Villani's program
   - second step: quantitative hypocoercivity estimates
2. Relaxation equation with confinement
   - The relaxation operator in the torus
   - The relaxation operator with confinement force
   - The relaxation operator in bounded domain
3. Linearized Boltzmann equation with confinement
   - Linearized Boltzmann equation in the torus
   - Linearized Boltzmann equation in bounded domain
   - Linearized Boltzmann equation with force confinement
4. Perspectives

- Same semigroup decay in $L^\infty$ ? in any cases :
  - linearized Boltzmann
  - linearized Landau
  - with force confinement
  - in a bounded domain
- The nonlinear problem ?
- Uniform estimate in the grazing collisions limit (Boltzmann $\rightarrow$ Landau) ?
- Uniform estimate in the fluid limit (Boltzmann $\rightarrow$ Navier-Stokes) ?