Krein-Rutman theorem
and Kinetic Fokker-Planck equation

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Aleksandr Krein - 8 Esquisses de la jeunesse www.youtube.com/watch?v=sJIG1Cp2n9Y
Bob Rutman www.youtube.com/watch?v=6RTaGQiQfA
Puscifer "Theorem" www.youtube.com/watch?v=bfKV0WmjtT8
and
www.kineticmusicgroup.com
Wanted (Fokker) www.youtube.com/watch?v=nAHHWe4v7Rg
Squarepusher - Maximum Planck www.youtube.com/watch?v=DXrozqhtRMY
(or Baauer - Planck www.youtube.com/watch?v=CHt337NYqTY)
Equation by Camille, Hans Zimmer www.youtube.com/watch?v=bVCAe0qH–A

Outline of the talk

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• framework: a positive semigroup $S_t = S_{\mathcal{L}}(t) = e^{t\mathcal{L}}$ with generator $\mathcal{L}$ on a Banach lattice X with positive cone X_+ in duality with another Banach lattice Y ($Y = X'$ or $X = Y'$).
▷ Concrete situations are $X = C_0$, $X = L^p_m$, $X = M^1_m$, here for KFP: $X := L^2_m$

• (CS1) existence of a solution $(\lambda_1, f_1, \phi_1) \in \mathbb{R} \times X \times Y$ to the first eigentriplet problem :

$$f_1 \geq 0,\ \mathcal{L} f_1 = \lambda_1 f_1, \quad \phi_1 \geq 0,\ \mathcal{L}^* \phi_1 = \lambda_1 \phi_1$$

• suitable geometric properties as

(CS2) $\lambda_1 \in \Sigma_+(\mathcal{L})$ algebraically simple, $f_1 > 0$ unique positive eigenvector for $\mathcal{L}$, $\phi_1 > 0$ unique positive eigenvector for $\mathcal{L}^*$, $\langle f_1, \phi_1 \rangle := 1$

(CS3$_1$) $\Sigma_+(\mathcal{L}) \cap \Sigma_P(\mathcal{L}) - \lambda_1$ is a (discrete) additive subgroup of $i\mathbb{R}$

(CS3$_2$) $\Sigma_+(\mathcal{L}) \cap \Sigma_P(\mathcal{L}) = \{\lambda_1\}$

(CS3$_3$) $\Sigma(\mathcal{L}) \cap \Delta_\kappa = \{\lambda_1\}$, $\kappa < \lambda_1$ (spectral gap)

where
$\Sigma_+(\mathcal{L}) := s(\mathcal{L}) + i\mathbb{R}$ boundary spectrum, $s(\mathcal{L}) := \{\sup \Re e\lambda;\ \lambda \in \Sigma(\mathcal{L})\}$ spectral bound, $\Sigma_P(\mathcal{L})$ point spectrum (set of eigenvalues), $\Delta_\kappa := \{z \in \mathbb{C},\ \Re ez < \kappa\}$

• asymptotic attractivity/stability. The functions $t \mapsto f_1 e^{\lambda_1 t}$ and $t \mapsto \phi_1 e^{\lambda_1 t}$ are particular solutions (with maximal growth) to the primal and the dual evolution problem. Is this first function attractive

$$e^{t\mathcal{L}} f_0 - e^{\lambda_1 t} f_1 \langle \phi_1, f_0 \rangle = o(e^{\lambda_1 t})?$$

Introducing $\widetilde{S}_t := S_{\mathcal{L}}(t) e^{-\lambda_1 t}$ and assuming $\langle \phi_1, f_0 \rangle = 0$, we wonder if

• (CE1) mean ergodicity when

$$\frac{1}{T} \int_0^T \widetilde{S}_t f_0 dt \to 0;$$

• (CE2) ergodicity when

$$\widetilde{S}_t f_0 \to 0;$$

• (CE3) quantitative asymptotical stability when

$$\|\widetilde{S}_t f_0\|_0 \le \Theta(t) \|f_0\|_1, \quad \Theta(t) \to 0,$$

with non constructive rate or constructive rate with $X_1 = X \subset X_0$, Θ exponential (geometric, dissipative) or sub-exponential (subgeometric, weak dissipative)

Discussion about KR - dissipative case

Spectral analysis

- Perron-Frobenius 1907-09 (dim $< \infty$), Phillips 60' (positive semigroup)
- Krein-Rutman 1948 (exist+geo when $\text{int}X_+ \neq \emptyset \leftrightarrow X = C(E)$, E compact)
- Greiner 1984, Webb 1984, Bürger 1988 ($\text{int}X_+ = \emptyset$, not constructive)

$\rightarrow$ Schaefer (and German school), Voigt, Gearhart, Prüss, Nagel, Arendt, ...
$\rightarrow$ more readable book by Bátkai, Kramar Fijavž and Rhandi

- Scher-M 2016 ($\simeq$ German school, a bit more constructive)
- Lions' college de France course 2020, (existence, weak compactness argument)

Ergodicity, probabilistic and coupling method approach (conservative case)

- von Neumann, Birkhoff, Markov, Kakutani (existence) $\sim$ 30'
- Doblin 40', Harris 50', Meyn-Tweedie 90', Hairer-Matingly 2011 (convergence)

Probabilistic and coupling method approach (non conservative)

- Collet-Martinez-Méléart-San Martin 2011-13, Champagnat-Villemonais 2016
- Bensaye, Cloez, Gabriel, Marguet (abstract KR via coupling) 2019-22

Spectral analysis and functional inequalities

- Toscani-Villani 2000, Rochner-Wang 2001 (Fokker-Planck operator)
- Kavian-M.-Ndao 2021 (idem)

Entropy, probabilistic and coupling method approach (non conservative)

- M.-Michel-Perthame 2005 (GRE)
- Douc, Fort, Guillin 2009 (weak Lypunov condition)
- Cañizo-M. 2023 (idem but constructive rate)

Two goals

First goal: to revisit Krein-Rutman theory

• more general than the initial Krein-Rutmann theorem ($\mathrm{int} X_+ = \emptyset$ is possible) / less abstract than usual semigroup school approach

• more intuitive = series of a priori estimates / the necessary assumptions are made clearer at each step

What is new?

• KR in the weak dissipative case (no spectral gap!!)

• only weak compactness arguments are needed

• (towards) constructive rate

Proof based on

• simple Banach lattice tools (no ideal, no quasi interior point, no Calkin algebra)

• simple spectral analysis tools (no essential spectrum/growth bound), ergodicity and probabilistic (coupling method) tools

• additional strong positivity assumptions

Second goal: apply / illustrate on KFP model

We consider the KFP equation

$$\begin{cases} \partial_t f = \mathcal{L} f, & (0,\infty) \times \mathcal{O}, \\ \gamma_- f = \mathscr{R} \gamma_+ f = \alpha \mathcal{D} \gamma_+ f + \beta \Gamma \gamma_+ f, & (0,\infty) \times \Sigma_-, \end{cases}$$

where $\mathcal{O} := \Omega \times \mathbb{R}^d$, $\Omega \subset \mathbb{R}^d$ bounded domain, Σ_- incoming boundary $\subset \Sigma = \partial\Omega \times \mathbb{R}^d$, with general Kinetic Fokker-Planck operator

$$\mathcal{L} f := -v \cdot \nabla_x f + \Delta_v f + b \cdot \nabla_v f + c f,$$

and general reflection operator $\mathscr{R}$ associated to the specular reflection operator

$$\Gamma_x(g(x,\cdot))(v) = g(x, \mathcal{V}_x v), \quad \mathcal{V}_x v = v - 2n(x)(n(x) \cdot v),$$

and the diffusive operator

$$\mathcal{D}_x(g(x,\cdot))(v) = c_{\mathscr{M}} \mathscr{M}(v) \widetilde{g}(x), \quad \widetilde{g}(x) = \int_{\Sigma_+^x} g(x,w)\, n(x) \cdot w \, dw,$$

for a normalized Maxwellian $\mathscr{M}$ and some accomodation coefficients $\alpha, \beta \in [0,1]$, $\alpha + \beta \le \zeta \le 1$ $\Rightarrow$ leak is possible.

Theorem - KR for the KFP equation

The conclusions (CS1), (CS2), ($CS3_3$) and (CE3) with exponential rate but non constructive constants hold

Discussion about KFP

Existence in $\mathbb{R}^d$ and in a domain (with additional NL terms)

- Degond 1986, Dreseler 1987, Bouchut 1993, Carrillo-Soler 1993-97
- Carrillo 1998, M. 2000-2010

Regularity

- Golse-Lions-Perthame-Sentis 1988, Lions-Perthame 1992, M.-Weng 2017
- Hörmander 1967, Bouchut 1995, Hérau 2007, Hwang-Ju-Velázquez 2014
- De Giorgi 1956, DiBenedetto-Gianazza-Vespri 1986-2012
- Hérau+Pravda-Starov 2011, Golse-Imbert-Mouhot-Vasseur 2019, Zhu 2021

Longtime behaviour and Hypocoercivity

- Bonilla-Carrillo-Soler-Vasquez 1995-98
- Nier-Helfer-Hérau 2004-05, Neuman-Mouhot 2006, Villani 2009

KR for the KFP operator with zero inflow boundary condition

- Lelièvre-Maurin-Monmarché 2022, Guillin-Nectoux-Wu arXiv 2022

What is new?

- reflection at the boundary condition for a non conservative model

Outline of the talk

We assume that $\mathcal{L}$ enjoys the following properties

(H1) $\mathcal{R}_\mathcal{L}(\alpha) := (\alpha - \mathcal{L})^{-1} \in \mathscr{B}(X) \cap \mathscr{B}(X_+)$ for any $\alpha \ge \kappa_1$

We may define

$$\mathcal{I} := \{\kappa \in \mathbb{R};\, \forall\, \alpha \ge \kappa,\, \mathcal{R}_\mathcal{L}(\alpha) \in \mathscr{B}(X) \cap \mathscr{B}(X_+)\} \neq \emptyset.$$

(H2) $\exists\, \kappa_0 \in \mathbb{R},\ \kappa_0 \notin \mathcal{I}$.

We may define

$$\lambda_1 := \inf \mathcal{I} \in [\kappa_0, \kappa_1].$$

(H3) If (λ_n) in $\mathbb{R}$, (g_n) in X_+ and (ε_n) in X_+ satisfy

$$\lambda_n \searrow \lambda_1,\ \ \|g_n\| = 1,\ \varepsilon_n \to 0,\ \ \lambda_n g_n - \mathcal{L} g_n = \varepsilon_n,$$

then (for a subsequence) $g_n \rightharpoonup f_1,\ f_1 \neq 0$.

Theorem ($\sim$Lions)

Under conditions (H1)-(H2)-(H3) there exists a solution (λ_1, f_1) to the first eigenvalue problem

• Because $\lambda_n \searrow \lambda_1$ we must have $\|\mathcal{R}_\mathcal{L}(\lambda_n)\|_{\mathscr{B}(X)} \to \infty$.
On the contrary $\lambda_1 \notin \Sigma(\mathcal{L})$.
That implies that there exists $\eta > 0$ such that $B(\lambda_1, \eta) \cap \Sigma(\mathcal{L}) = \emptyset$ and

$$\mathcal{R}(\alpha) = \mathcal{R}(\lambda_1) \sum_{k=0}^{\infty} (\lambda_1 - \alpha)^k \mathcal{R}(\lambda_1)^k \geq 0,$$

for any $\lambda_1 - \eta < \alpha \leq \lambda_1$.
$]\lambda_1 - \eta, \lambda_1] \subset \mathcal{I}$ and that is in contradiction with definition of λ_1.

• That means there exists $\varepsilon_n \to 0$ and $\|g_n\| = 1$ such that $g_n = \mathcal{R}_\mathcal{L}(\lambda_n)\varepsilon_n$.
Because $\mathcal{R}_\mathcal{L}(\lambda_n) \in \mathscr{B}(X_+)$ we may assume $g_n \geq 0$, $\varepsilon_n \geq 0$.

• We just apply (H3) and conclude

(H1′) $S = S_{\mathcal{L}}$ is a positive semigroup with growth bound $\omega(S_{\mathcal{L}}) \le \kappa_1$

(H2′) $\exists\, \kappa_0 < \kappa_1$, $\exists \psi_0 \gneqq 0$ such that $\mathcal{L}^* \psi_0 \ge \kappa_0 \psi_0$ or $\exists f_0 \gneqq 0$ such that $\mathcal{L} f_0 \ge \kappa_0 f_0$

(H3′) $\mathcal{L} = \mathcal{A} + \mathcal{B}$ with $\mathcal{A} \in \mathscr{B}(X)$, $\exists\, \kappa_{\mathcal{B}} < \kappa_0$, $\mathcal{R}_{\mathcal{B}}(\alpha) \in \mathscr{B}(X)$ uniformly in $\alpha \ge \kappa_{\mathcal{B}}$ and $\exists\, N \ge 1$ such that $\mathcal{W}(\alpha) := (\mathcal{R}_{\mathcal{B}}(\alpha)\mathcal{A})^N$ satisfies

$$\mathcal{W}(\alpha) : \mathcal{X}_0 \to \mathcal{X}_1 \ \text{ is positive and uniformly bounded in } \ \alpha \ge \kappa_0$$

with $\mathcal{X}_1 \subset X \subset \mathcal{X}_0$ and assuming that for any $R_1 \ge R_0 > 0$ the set (ring)

$$\mathcal{C} := \{ g \in X_+; \ \|g\|_{\mathcal{X}_0} \ge R_0, \ \|g\|_{\mathcal{X}_1} \le R_1 \}$$

is relatively sequentially compact for the weak topology $\sigma(X, Y)$ and $0 \notin \overline{\mathcal{C}}$.

(H3″) same condition on the (almost) dual operator $\mathcal{W}^\sharp(\alpha) := (\mathcal{R}_{\mathcal{B}^*}(\alpha)\mathcal{A}^*)^N$

Theorem

Under conditions (H1′)-(H2′)-(H3′)-(H3″) there exists a solution (λ_1, f_1, ϕ_1) to the first eigentriplet problem

- (H3′) and (H3″) are automatically verified when $\mathcal{W}(\alpha) \in \mathscr{K}(X)$
- (H3′) holds if $\mathcal{W} : L^p_m \to L^q_\omega$ weakly compact in L^p_m
- (H3′) holds if $\mathcal{W} : M^1_{m_0} \to M^1_{m_1}$ weakly $*$ compact in $M^1_{m_0}$

• (H1′) $\Rightarrow$ (H1) $0 \le S_{\mathcal{L}} = \mathcal{O}(e^{\kappa t})$, $\forall\, \kappa > \kappa_1$, is equivalent to the Kato inequality $(\mathrm{sign} f)\mathcal{L} f \le \mathcal{L}|f|$ and $\kappa - \mathcal{L} \in \mathscr{B}(X) \cap \mathscr{B}(X_+)$, $\forall\, \kappa > \kappa_1$.

• (H2′) $\Rightarrow$ (H2)

• (H3′) $\Rightarrow$ (H3) For a sequence of approximation solutions

$$g_n \ge 0,\ \|g_n\| = 1,\ \lambda_n g_n - \mathcal{B} g_n - \mathcal{A} g_n = \lambda_n g_n - \mathcal{L} g_n = \varepsilon_n \to 0,$$

we write

$$g_n = \mathcal{R}_{\mathcal{B}}(\lambda_n)\varepsilon_n + \mathcal{R}_{\mathcal{B}}(\lambda_n)\mathcal{A} g_n$$

and iterating

$$\begin{aligned} g_n &= \{\mathcal{R}_{\mathcal{B}}(\lambda_n) + \cdots + (\mathcal{R}_{\mathcal{B}}(\lambda_n)\mathcal{A})^{N-1}\mathcal{R}_{\mathcal{B}}(\lambda_n)\}\varepsilon_n + \mathcal{W}(\lambda_n) g_n \\ &=: v_n + w_n. \end{aligned}$$

By assumptions $v_n \to 0$ so that $w_n \in \mathcal{C} = \mathcal{C}_{1/2,R}$, $\mathcal{X}_0 = X$, $\mathcal{X}_1 \subset X$.
As a consequence, $w_{n_k} \rightharpoonup f_1 \neq 0$, $v_{n_k} \to 0$, so that $g_{n_k} \rightharpoonup f_1$ and $f_1 \ge 0$.
Passing to the limit in the first equation, we also get $\lambda_1 f_1 - \mathcal{L} f_1 = 0$.

- (CS2) is a consequence of the weak maximum principle (H1$'$) (or (H1$''$) + (H1$'''$)) and the strong maximum principle (H4), where :
  - (H1$''$) weak maximum principle: $(\lambda - \mathcal{L})f \ge 0$, $\lambda > \lambda_1 \Rightarrow f \ge 0$
  - (H1$'''$) Kato's inequalities : $\mathcal{L}|f| \ge (\mathrm{sign} f)\mathcal{L}f$, $\mathcal{L}f_+ \ge (\mathrm{sign}_+ f)\mathcal{L}f$
  - (H4) strong maximum principle: $(\lambda - \mathcal{L})f \ge 0$, $f \ge 0$, $f \not\equiv 0 \Rightarrow f > 0$
- (CS3$_1$) is a consequence of (CS2), (H4) (and (H3$'$) for the additional discrete property)
- (CS3$_2$) is a consequence of (CS2), (H4) and $(\mathrm{H5}_i)_{i=1,2}$ with
  - (H5$_1$) inverse Kato's inequalities : $\mathcal{L}|f| = (\mathrm{sign} f)\mathcal{L}f$ implies $f = u|f|$
  - (H5$_2$) aperiodicity condition :

$$\forall f \in X_+\backslash\{0\}, \forall \phi \in Y_+\backslash\{0\},\ \exists\, T > 0,\, \forall \tau \ge T \quad \langle S_\tau f, \phi\rangle > 0.$$

- (CS3$_3$) is a consequence of (CS2), (H4) and (H5$_3$) with
  - (H5$_3$) Voigt's quasi-compactness :

$$S_T = V_T + K_T, \quad V_t = \mathcal{O}(e^{\kappa_{\mathcal{B}} t}), \quad K_T \in \mathscr{K}(X)$$

• (CE1) (weak convergence) is a consequence of (CS1) and (CS2) when $X \subset L^1_{loc}$

• (CE2) is a consequence of (CS1), (CS2), ($CS3_2$) and the trajectories $(\widetilde{S}_t f)_{t \ge 0}$ are relatively compact

• (CE3) (exponential rate but without constructive constants) is a consequence of (CS2), (H4) and ($H5_3$) thanks to Voigt-Greiner-Webb-Bürger theorem

• (CE3) (exponential or sub-exponential rate with constructive constants) is consequence of (CS1) together with a Lyapunov condition and a Doblin-Harris condition. Typically :

$$\|\bar{S}_T f\| \le \gamma_L \|f\| + K \langle |f|, \psi_0 \rangle$$

with $\gamma_L = \gamma_L(T) < 1$ for any $T > 0$ and for some $\psi_0 > 0$ and

$$\exists\, T > 0,\ \exists\, g_0 > 0,\ \forall\, f \ge 0,\ S_T f \ge g_0 \langle \psi_0, f \rangle$$
$$\exists\, R_0 > 0,\ \phi_1 \le R_0 \psi_0$$

Outline of the talk

We consider the Kinetic Fokker-Planck operator

$$\begin{cases} \mathcal{L}f = -v \cdot \nabla_x f + \Delta_v f + b \cdot \nabla_v f + cf \quad \text{in} \quad \mathcal{O}, \\ \gamma_- f = \alpha \mathcal{D} \gamma_+ f + \beta \Gamma \gamma_+ f \quad \text{on} \quad \Sigma_-, \end{cases}$$

with $\Sigma_\pm := \{(x,v) \in \partial\Omega \times \mathbb{R}^d,\ \pm v \cdot n(x) > 0\}$. We assume furthermore

$$c, \mathrm{div}_v b, b/\langle v \rangle \in L^\infty(\mathcal{O}), \quad \limsup_{(x,v)\to\infty} b \cdot \hat{v} > 0,$$

with notations $\hat{v} := v/\langle v \rangle$, $\langle v \rangle^2 = 1 + |v|^2$. For simplicity, we assume here

$$\varpi := c + \frac{|v|^2 + d}{2} - \frac{1}{2}\mathrm{div}_v b - b \cdot v \to -\infty$$

and we denote

$$\begin{aligned} \kappa_1' &:= \Big\| \Big(c + \frac{\Delta m^2}{2m^2} - \frac{1}{2}\mathrm{div}_v b - b \cdot \frac{\nabla m}{m} \Big)_+ \Big\|_{L^\infty} \\ \kappa_1'' &:= \Big\| \big(c - \mathrm{div}_v b \big)_+ \Big\|_{L^\infty}, \quad m := \mathscr{M}^{-1/2} \end{aligned}$$

We finaly assume that

$$n(x) = -\nabla \delta(x) \in W^{1,\infty}(\Omega), \quad \delta(x) := \mathrm{dist}(x, \partial\Omega).$$

Consider a solution f to $(\lambda - \mathcal{L})f = \mathscr{F}$ in $\mathcal{O}$, $\gamma_- f = \mathscr{R}\gamma_+ f +$ on Σ_-.

- Multiplying by $f\, m^2$, we get

$$\int (\lambda - \varpi) f^2 m^2 + \int |\nabla_v f|^2 m^2 + \int_\Sigma (\gamma f)^2 n(x) \cdot v m^2 = \int \mathscr{F} f m^2,$$

so that

$$\int (\lambda - \varpi) f^2 m^2 + \int |\nabla_v f|^2 m^2 \le \|\mathscr{F}\|_{L^2_m} \|f\|_{L^2_m},$$

with $\inf(\lambda - \varpi) > 0$ when $\lambda > \kappa_1'$. $\Rightarrow$ existence

- Multiplying by $f\, n(x) \cdot \hat{v}/\langle v \rangle$ and using Cauchy-Schwarz, we get

$$\Big(\int_\Sigma |\gamma f| |n(x) \cdot v| \Big)^2 \lesssim \int_\Sigma f^2 m^2 (n(x) \cdot \hat{v})^2 \lesssim \|f\|^2_{L^2_m} + \|\mathscr{F}\|^2_{L^2_m}.$$

$\Rightarrow$ trace condition is meaningful

- Multiplying by $S'(f)$, $S(f) = |f|$ or $S(f) = f_+$, when $\mathscr{F} = 0$, we get

$$0 \ge \int_{\mathcal{O}} S(f)(\lambda - c + \mathrm{div} b) + \int_\Sigma S(\gamma f) \nu \cdot v \varphi.$$

with $\inf(\lambda - c + \mathrm{div} b) > 0$ when $\lambda > \kappa_1''$. $\Rightarrow$ uniqueness and positivity

- The three estimates together $\Rightarrow \mathcal{R}_{\mathcal{L}}(\lambda) \in \mathscr{B}(X) \cap \mathscr{B}(X_+)$ when $\lambda > \kappa_1 = \max(\kappa_1', \kappa_1'')$

Additional a priori estimates

• Multiplying by $f\psi$, with $\psi = \delta(x)^{1/2} n(x) \cdot v$ Lions-Perthame type multiplicator, we get

$$\int \frac{(n(x)\cdot \hat v)^2}{\delta(x)^{1/2}} f^2 \lesssim \|f\|^2_{L^2_m} + \|\mathscr{F}\|^2_{L^2_m}.$$

• Together with the first estimate, we get

$$\int_{\mathcal{O}} f^2 \{ m_1^2 + \frac{(n(x)\cdot \hat v)^2}{\delta(x)^{1/2}} \} + |\nabla_v f|^2 m^2 \lesssim \|\mathscr{F}\|^2_{L^2_m},$$

for $\lambda > \kappa_1'$, $m_1 := m\langle \varpi_- \rangle^{1/2} >> m$.

• Defining

$$\mathcal{U}_\varepsilon := \{(x,v) \in \mathcal{O};\ \delta(x) > \varepsilon,\ |v| \le \varepsilon^{-1}\},$$

there exists $\Lambda(\varepsilon) \to \infty$ when $\varepsilon \to 0$, such that

$$\Lambda(\varepsilon) \int_{\mathcal{U}_\varepsilon^c} f^2 m^2 \le \|f\|^2_{L^2_m} + \|\mathscr{F}\|^2_{L^2_m}$$

We define

$$\mathcal{A}f := M\chi f, \quad \mathbf{1}_{B_R} \le \chi(v) \le \mathbf{1}_{B_{2R}} \quad \mathcal{B} := \mathcal{L} - \mathcal{A}.$$

• For any $\kappa \in \mathbb{R}$, we may fix $M, R > 0$ large enough such that

$$\sup_{z \in \Delta_\kappa} \|\mathcal{R}_\mathcal{B}(z)\|_{\mathscr{B}(L^2_m)} < \infty,$$

what comes from the very first a priori estimate.

• Moreover in that situation $\mathcal{R}_\mathcal{B}(\lambda) \in \mathscr{K}(L^2_m)$ for any fixed $\lambda > \kappa$. We consider

$$(\lambda - \mathcal{B})g_n = G_n, \quad \|G_n\|_{L^2_m} = 1.$$

From the previous a priori estimates, we have

$$\|g_n\|_{L^2_{m_1}} + \|\nabla_v g_n\|_{L^2_m} + \sup_{\varepsilon > 0} \Lambda(\varepsilon)\|g_n\|_{L^2(\mathcal{U}^c_\varepsilon)} \lesssim 1.$$

Using the averaging lemma for the kinetic equation

$$v \cdot \nabla_x g_n = G_n + \Delta_v g_n + b \cdot \nabla_v g_n + (c - \lambda - M\chi)g_n \;\text{ bdd }\; L^2_{x,loc}H^{-1}_{v,loc},$$

we deduce

$$g_n = g_n *_v \rho_\alpha + (g_n - g_n *_v \rho_\alpha) \in \text{ compact } L^2_{loc}.$$

Additional local regularity and local positivity estimates

Consider the KFP equation

$$\partial_t g + v \cdot \nabla_x g - \Delta_v g + \langle v \rangle^2 g = \mathcal{G} \text{ in } I \times \mathcal{U},$$

with $I = \emptyset$, $I \subset \mathbb{R}_+$, $\mathcal{U} = \mathbb{R}^d \times \mathbb{R}^d$, $\mathcal{U} \subset\subset \mathcal{O}$.

• From Hörmander, Hérau & Pravda-Starov, we have

$$\|D_x^{2/3} g\|_{L^2} + \|D_v^2 g\|_{L^2} \lesssim \|\mathcal{G}\|_{L^2} + \|g\|_{L^2}.$$

• From De Girogi, ..., DiBenedetto, ..., Golse-Imbert-Mouhot-Vasseur, we have

$$\begin{aligned}
&\|g\|_{L^{p_1}(Q_{r_1})} \lesssim \|g\|_{L^2(Q_{r_0})} + \|\mathcal{G}\|_{L^2(Q_{r_0})} \\
&\|g\|_{L^{p_{j+1}}(Q_{r_1})} \lesssim \|g\|_{L^2(Q_{r_0})} + \|\mathcal{G}\|_{L^{p_j}(Q_{r_0})}, \quad 1 \le j \le k-1 \\
&\|g\|_{L^\infty(Q_{r_1})} \lesssim \|g\|_{L^2(Q_{r_0})} + \|\mathcal{G}\|_{L^{p_k}(Q_{r_0})} \\
&\|g\|_{C^\alpha(Q_{r_1})} \lesssim \|g\|_{L^2(Q_{r_0})} + \|\mathcal{G}\|_{L^\infty(Q_{r_0})}
\end{aligned}$$

with $Q_r :=$ cone ..., but also $Q_r := \mathcal{U}_r$, $r_1 > r_0 > 0$.

• From Harnack inequality in GIMV or using the energy estimates + regularity estimates in GIMV + barrier function (Villani), for any $\mathcal{U} \subset\subset \mathcal{O}$, we have

$$g \ge \varepsilon_{\mathcal{U}} > 0 \quad \text{on} \quad \mathcal{U},$$

when $\mathcal{G} = 0$, $g \ge 0$, $\|g\|_{L^2_m} = 1$.

For $0 \le h_0 \in C^2_c(\mathcal{O})$, $\|h_0\|_{L^2_m} = 1$, we define $f_0 \in L^2_m \cap L^2 H^1$ the solution to

$$(\kappa_1 - \mathcal{L}) f_0 = h_0 \text{ on } \mathcal{O}, \quad \gamma_- f = \mathscr{R} \gamma_+ f \text{ on } \Sigma_-.$$

• $\|f_0\|_{L^2_m} \ge 1/C_1 > 0$ because

$$\begin{aligned} 1 &= \int h_0^2 m^2 = \int (\kappa_1 - \mathcal{L}) f_0 h_0 m^2 \\ &\le \int f_0 (\kappa_1 - \mathcal{L}^*)(h_0 m^2) \le C_1 \|f_0\|_{L^2_m}. \end{aligned}$$

• From the additional estimate

$$\|f_0\|_{L^2 H^1_m} + \|f_0 \frac{\hat{v} \cdot \nu}{\delta^{1/4}}\|_{L^2} \le C_2,$$

we deduce

$$\int_{\mathcal{U}} f_0^2 m^2 \ge (2C_1)^{-1}, \quad \mathrm{supp} h_0 \subset \mathcal{U},$$

• From Harnack inequality, we deduce $f_0 \ge \varepsilon_{\mathcal{U}} \mathbf{1}_{\mathcal{U}} \ge 1/C_0 h_0$ for some $\varepsilon_{\mathcal{U}}, C_0 > 0$ and next

$$\mathcal{L} f_0 = \kappa_1 f_0 - h_0 \ge \kappa_1 f_0 - \|h_0\|_{L^\infty} \mathbf{1}_{\mathcal{U}} \ge (\kappa_1 - \|h_0\|_{L^\infty} C_0) f_0,$$

• (H1)-(H2)-(H3) implies $\exists\,(\lambda_1, f_1, \phi_1)$ solution to the eigentriplet problem (CS1)

• Harnack inequality implies strong maximum principle (H4), which in turn implies uniqueness, strict positivity and algebraic simplicity (CS2)

• Reverse Kato inequality condition ($H5_1$) implies triviality of eigenvalues in the boundary spectrum $\Sigma_+(\mathcal{L}) \cap \Sigma_P(\mathcal{L}) = \{\lambda_1\}$ ($CS3_2$)

• For free, ergodicity holds for the L^1 weak convergence and, working slightly more, for the L^1 strong convergence (using GRE techniques)

Convenient splitting/representation formula for the semigroup

We introduce the splitting

$$\mathcal{A}g := M\Upsilon_\varepsilon g, \quad \Upsilon_\varepsilon g := \chi_\varepsilon g, \quad \mathcal{B} := \mathcal{L} - \mathcal{A},$$

with $\chi_\varepsilon \in C^2_c(\mathcal{O})$, $\mathbf{1}_{\mathcal{U}_{2\varepsilon}} \le \chi_\varepsilon \le \mathbf{1}_{\mathcal{U}_\varepsilon}$ (truncation in both x and v).
We write the Duhamel and iterated Duhamel formulas (with $N := k+2$)

$$\begin{aligned} S_\mathcal{L} &= S_\mathcal{B} + S_\mathcal{B}\mathcal{A} * S_\mathcal{L} \\ &= S_\mathcal{B} + \dots + (S_\mathcal{B}\mathcal{A})^{*N-1} * S_\mathcal{B} + (S_\mathcal{B}\mathcal{A})^{*N} * S_\mathcal{L} \\ &= U + W * S_\mathcal{L} \end{aligned}$$

For $T > 0$ large and $0 < \tau < T$ small, we next define a modified convolution operator

$$\begin{cases} (a *_\tau b)(t) := \int_\tau^{t-\tau} a(t-s)b(s)\,ds \ \text{ if } \ t \in [\tau, T-\tau] \\ (a *_\tau b)(t) := 0 \ \text{ if } \ t \in [\tau, T-\tau]^c, \end{cases}$$

(with these notations $*_0 = *$) and the new splitting

$$S_\mathcal{L} = U + K_1^c + K_2^c + K,$$

with

$$K := \Upsilon_\nu W_\tau *_\tau S_\mathcal{L}, \quad K_1^c := W * S_\mathcal{L} - W_\tau *_\tau S_\mathcal{L}, \quad K_2^c := (1 - \Upsilon_\nu) W_\tau *_\tau S_\mathcal{L}$$

where $W_\tau := (S_\mathcal{B}\mathcal{A})^{*_\tau N}$.

Let us fix $\kappa < \kappa_{\mathcal{B}} < \kappa_0 \le \lambda_1$ and choose $\mathcal{A}$ in such a way that $S_{\mathcal{B}}(t) = \mathcal{O}(e^{\kappa_{\mathcal{B}} t})$.

• We choose T large, $\tau > 0$ small and $\nu > 0$ small such that

$$\|U\| \le \frac{1}{3} e^{\kappa T}, \quad \|K_1^c\| \le \frac{1}{3} e^{\kappa T}, \quad \|K_2^c\| \le \frac{1}{3} e^{\kappa T}.$$

For the third estimate we must first observe that

$$W_\tau *_\tau S_{\mathcal{L}} : L^2_m \to L^{p_1}_{m_{p_1}}$$

with $m_{p_1} := \mathcal{M}^{1/p_1 - 1}$ so that (u_n^2) is weakly compact in L^1 if (u_n) is a bounded sequence of $L^{p_1}_{m_{p_1}}$ as a consequence of the first De Giorgi-GIMV estimate.

• From the series of De Giorgi-GIMV estimates, we have

$$K = \Upsilon_\nu W_\tau *_\tau S_{\mathcal{L}} : L^2_m \to C^\alpha(\mathcal{U}_\nu), \quad \mathcal{U}_\nu \subset\subset \mathcal{O},$$

so that $K \in \mathscr{K}(L^2_m)$.

• We conclude to the exponential rate but without constructive constants (CE3) thanks to Voigt-Greiner-Webb-Bürger theorem

- The same for a variable wall temperature $0 \le \vartheta \in W^{1,\infty}(\partial\Omega)$, $\vartheta^{-1} \in L^\infty(\partial\Omega)$
- Constructive constants for the exponential rate of convergence
- L^p convergence for any $1 \le p \le \infty$
- The same for a relaxation type equation (a kernel term instead of the Δ_v term)