Uniqueness and long time assymptotic for the Keller-Segel equation The parabolic-elliptic case

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Plan

The Keller-Segel (KS) equation writes

$$\begin{aligned} \partial_t f &= \Delta_x f - \nabla_x (f \, \nabla_x c) \quad \text{in} \quad (0,\infty) \times \mathbb{R}^2 \\ \varepsilon \partial_t c &= \Delta_x c + f - \alpha \, c \quad \text{in} \quad (0,\infty) \times \mathbb{R}^2 \end{aligned}$$

time variable $t \geq 0$, position variable $x \in \mathbb{R}^2$,
mass density of cells $f = f(t,x) \geq 0$,
chemo-attractant concentration $c = c(t,x) \in \mathbb{R}$ (or $\in \mathbb{R}_+$ if $\varepsilon > 0$).

We take $\varepsilon = \alpha = 0$ so that

$$-\nabla c = \bar{\mathcal{K}} := \mathcal{K} * f, \quad \mathcal{K} := \nabla \kappa = \frac{1}{2\pi} \frac{z}{|z|^2}, \quad \kappa := \frac{1}{2\pi} \log |z|$$

For the case $\varepsilon, \alpha > 0$ we refer to

- Carrapatoso, Egaña, M., *Uniqueness and long time assymptotic for the Keller-Segel equation - Part II : The parabolic-parabolic case*, work in preparation

The Keller-Segel (KS) equation writes

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and the system rewrites as only one nonlinear parabolic equation

$$\partial_t f = \underbrace{\Delta_x f}_{\textit{spread over}} + \underbrace{\nabla_x (f \, \bar{\mathcal{K}})}_{\textit{aggregate}} \quad \text{in} \quad (0,\infty) \times \mathbb{R}^2$$

Initial datum

The evolution equation is complemented with an initial condition

$$f(0,.) = f_0 \quad \text{in} \quad \mathbb{R}^2.$$

We shall always assume that

$$f_0 \in L^1_2(\mathbb{R}^2) \cap L^1_+(\mathbb{R}^2), \quad f_0 \log f_0 \in L^1(\mathbb{R}^2),$$

and the mass condition (subcritical case)

$$M := \int_{\mathbb{R}^2} f_0(x)\,dx \in (0, 8\pi).$$

Notations: Weighted Lebesgue space $L^p(m)$, $1 \le p \le \infty$, weight m :

$$L^p(m) := \{f \in L^1_{loc}(\mathbb{R}^2);\ \|f\|_{L^p(m)} := \|f\, m\|_{L^p} < \infty\}.$$

$L^p_k = L^p(\langle x\rangle^k)$, $k \ge 0$, polynomial weight function $\langle x\rangle := (1+|x|^2)^{1/2}$

$L^1_+(\mathbb{R}^2) :=$ cone of nonnegative functions of $L^1(\mathbb{R}^2)$

Mass conservation

$$(1) \qquad M(t) := \int_{\mathbb{R}^2} f(t,x)\,dx = \int_{\mathbb{R}^2} f_0(x)\,dx = M,$$

Second moment

$$(2) \qquad M_2(t) := \int_{\mathbb{R}^2} f(t,x)\,|x|^2\,dx = C_1(M)\,t + M_{2,0},$$

with

$$M_{2,0} := \int_{\mathbb{R}^2} f_0(x)\,|x|^2\,dx, \quad C_1(M) := 4M\,\big(1 - \frac{M}{8\pi}\big).$$

Observe that for $M > 8\pi$ there holds $M_2(t) < 0$ in finite time !!

Higher order moment equation

$$\forall\, k \ge 2 \qquad \frac{dM_k}{dt} \le k^2\,M_{k-2} \le k^2\,M^{2/k}\,M_k^{1-2/k}$$

The following free energy-dissipation of the free energy identity (formally) holds

$$\text{(3)} \qquad \mathcal{F}(t) + \int_0^t \mathcal{D}_{\mathcal{F}}(s)\, ds = \mathcal{F}_0 \in \mathbb{R},$$

where the free energy $\mathcal{F}(t) = \mathcal{F}(f(t))$, $\mathcal{F}_0 = \mathcal{F}(f_0)$ is defined by

$$\mathcal{F} = \mathcal{F}(f) := \int_{\mathbb{R}^2} f \log f dx + \frac{1}{2} \int_{\mathbb{R}^2} f\, \bar{\kappa}\, dx,$$

and the dissipation of free energy is defined by

$$\mathcal{D}_{\mathcal{F}} = \mathcal{D}_{\mathcal{F}}(f) := \int_{\mathbb{R}^2} f\, |\nabla(\log f) + \nabla \bar{\kappa}|^2\, dx \geq 0,$$

with again $\bar{\kappa} = \kappa * f$.

Plan

Definition 1. Blanchet, Dolbeault, Perthame 2006

We say that

$$0 \le f \in L^\infty(0,T;L^1(\mathbb{R}^2)) \cap C([0,T);\mathcal{D}'(\mathbb{R}^2)), \quad \forall\, T \in (0,\infty),$$

is a weak solution to the Keller-Segel equation associated to the initial condition f_0 whenever f satisfies (1), (2) and

$$(3') \qquad \mathcal{F}(t) + \int_0^t \mathcal{D}_{\mathcal{F}}(s)\,ds \le \mathcal{F}_0 \quad \forall\, t > 0,$$

as well as the Keller-Segel equation in the distributional sense, namely

$$\int_{\mathbb{R}^2} f_0(x)\,\varphi(0,x)\,dx = \int_0^T \int_{\mathbb{R}^2} f(t,x)\left\{(\nabla_x(\log f) + \bar{\mathcal{K}})\cdot\nabla_x\varphi - \partial_t\varphi\right\} dxdt$$

for any $T > 0$ and $\varphi \in C_c^2([0,T)\times\mathbb{R}^2)$.

Theorem 2. (Existence)

For any initial datum f_0 there exists at least one weak solution

- Blanchet-Dolbeault-Perthame (Electron. JDE 2006)

Theorem 3. (Uniqueness)

For any initial datum f_0 there exists at most one weak solution.

$\triangleright$ Use a trick picked up from the analysis of the two-dimensional Navier-Stokes equation in vortex formulation

- Ben-Artzi (ARMA 1994), Brezis (ARMA 1994)
- Fournier, Hauray, M. (ArXiv 2012)

$\triangleright$ Uniqueness under the additional assumption $f_0 \in L^\infty(\mathbb{R}^2)$

- Carrillo, Lisini, Mainini (ArXiv 2012)

$\triangleright$ Uniqueness of "Maximal free energy solutions" when $M \geq 8\pi$.

Plan

Introduce the rescaled functions g

$$g(t,x) := R(t)^{-2} f(\log R(t), R(t)^{-1}x), \quad R(t) := (1+2t)^{1/2}.$$

It is a solution to the rescaled KS equation

$$\partial_t g = \Delta g + \nabla(g\,x - g\,\mathcal{K} * g) \quad \text{in} \quad (0,\infty)\times\mathbb{R}^2$$

- Mass is conserved, M_2 moment is bounded unifomly in time
- Modified free energy - dissipation of free energy identity

$$(3'') \qquad \frac{d}{dt}\mathcal{E} = -\mathcal{D}_{\mathcal{E}} \qquad \Rightarrow \quad \text{Lyapunov functional}$$

with

$$\mathcal{E} := \int g(1+\log g) + \frac{1}{2}\int g|x|^2 + \frac{1}{2}\int g\,\kappa * g$$

$$\mathcal{D}_{\mathcal{E}} := \int g|\nabla(\log g + \frac{|x|^2}{2} + \kappa * g)|^2$$

We look for

$$G = G_M \in \mathcal{Z}_M := \{g \in L^1_+;\ M_0(g) = M,\ \mathcal{E}(g) < \infty\}$$

such that

Problem 1. G is a solution to the stationary KS PDE

$$\Delta G + \nabla(G\,x + G\,\mathcal{K} * G) = 0 \quad \text{in} \quad \mathbb{R}^2$$

Problem 2. G is a solution to the minimisation problem

$$\mathcal{E}(G) = \min_{g \in \mathcal{Z}_M} \mathcal{E}(g),$$

Problem 3. G makes vanish the rescaled dissipiation of the free energy:

$$\mathcal{D}_{\mathcal{E}}(G) = 0.$$

For a smooth and positive function G

$$\text{Pb } 1 \quad \Leftrightarrow \quad \text{Pb } 3 \quad \Leftarrow \quad \text{Pb } 2$$

Theorem 4. (self-similar problem)

• $\exists!\ G_M$ stationnary solution (PDE, minimisation, vanishing), it satisfies $G_M \in C^\infty(\mathbb{R}^2)$ and $e^{-(1+\varepsilon)|x|^2/2+C_{1,\varepsilon}} \le G_M \le e^{-(1-\varepsilon)|x|^2/2+C_{2,\varepsilon}}$
• $\forall\, f_0$ there holds $\|g(t) - G_M\|_{L^1} \to 0$ without rate as $t \to \infty$;
• $\forall\, f_0 \in L^\infty(G_{M'}^{-1})$ there holds $\|g(t) - G_M\|_{L^1} \le C_{f_0}\, e^{-t}$.

- Blanchet, Dolbeault, Perthame (Electron. JDE 2006)
- Biler, Karch, Laurençot, Nadzieja (M2AS 2006)
- Blanchet, Dolbeault, Escobedo, Fernández (JMAA 2010)
- Calvez, Carrillo (Proc. Amer. Math. Soc. 2012)
- Campos, Dolbeault (arXiv 2012)

Theorem 5. (Universal and optimal self-similar behaviour)

If moreover $M_4(f_0) < \infty$, there holds $\|g(t) - G_M\|_{L^{4/3}} \le C_{f_0}\, e^{-t}$.

- Similar result and method for the homogeneous Boltzmann equation by Mouhot (CMP 2006)

Plan

- A priori estimates
  ⇒ wellknown, since Jäger, Luckhaus (TAMS 1992), Banchet, Dolbeault, Perthame (Electron. JDE 2006)
- A posteriori estimates
  ⇒ renormalization argument “à la DiPerna-Lions”
- Optimal estimates near $t = 0$
  ⇒ Smoothing effect thanks to nonlinear ode argument “à la Nash”
- Uniform in time estimates
  ⇒ in self-similar variables

Theorem 6. (A posterioi estimates)

Any weak solution f is smooth for positive time, namely $f \in C^\infty((0,\infty) \times \mathbb{R}^2)$, satisfies the free energy identity (3) and

$$t^{1/4} \|f(t,.)\|_{L^{4/3}} \to 0 \quad \text{as } t \to 0.$$

Moreover, the rescaled solution g satisfies

$$\sup_{t \geq 0} M_k(g(t)) \leq \max((k-1)^{k/2} M, M_k(f_0)) \quad \forall\, k \geq 2,$$

as well as

$$\sup_{t \geq \varepsilon} \|g(t,.)\|_{W^{2,\infty}} \leq \mathcal{C} \quad \forall\, \varepsilon > 0,$$

for some explicit constant $\mathcal{C}$ which depends on ε, M, $\mathcal{F}_0$ and $M_{2,0}$.

Theorem 6: A posterioi accurate estimates

$\Longrightarrow$ **Theorem 3:** Uniqueness

$\Longrightarrow$ **Theorem 5:** Universal and optimal self-similar behaviour

Plan

The Logarithmic Hardy-Littlewood Sobolev inequality: $\forall f \geq 0$

$$\int_{\mathbb{R}^2} f(x) \log f(x)\, dx + \frac{2}{M} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x)\, f(y)\, \log|x-y|\, dxdy \geq C_2(M),$$

(Beckner, Carlen, Loss) implies that for subcritical mass $M < 8\pi$:

$$\mathcal{H} := \mathcal{H}(f) = \int f \log f \leq C_3(M)\, \mathcal{F} + C_4(M).$$

Together with the classical bound

$$\mathcal{H}^+ := \mathcal{H}^+(f) = \int f (\log f)_+ \leq \mathcal{H} + \frac{1}{4} M_2 + C_5(M),$$

we get

$$\begin{aligned} \mathcal{H}^+(f(t)) + M_2(f(t)) + C_3(M) \int_0^t \mathcal{D}_{\mathcal{F}}(f(s))\, ds \\ \leq C_3(M)\, \mathcal{F}_0 + \frac{5}{4} M_2(t) + C_4(M) + C_5(M), \\ \leq C(M, \mathcal{H}_0, M_{2,0}, T). \end{aligned}$$

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we get

$$\mathcal{H}^+(f), M_2(f) \in L^\infty(0,T), \quad \mathcal{D}_{\mathcal{F}}(f) \in L^1(0,T)$$

Fisher information bound (a priori) Known

We have

$$\frac{1}{2} I(f) \le \mathcal{D}_{\mathcal{F}} + C(M, \mathcal{H}^+) \quad \Rightarrow \quad I(f) \in L^1(0,T).$$

In order to prove the first inequality, we write

$$\mathcal{D}_{\mathcal{F}}(f) = \int f\,|\nabla(\log f + \bar\kappa)|^2 \ge \int f\,|\nabla \log f|^2 + 2\int \nabla f \cdot \nabla \bar\kappa = I(f) - 2\int f^2$$

Using the Cauchy-Schwarz inequality, the Gagliardo-Niremberg type inequality

$$\forall\, p \in [1,\infty), \quad \|f\|_{L^p(\mathbb{R}^2)} \le C_p\, M^{1/p}\, I(f)^{1-1/p}$$

with $p = 3$, and the elementary inequality $f^2 \le 2(f \wedge A)^2 + 2(f - A)_+^2$, we have

$$\begin{aligned}
\int f^2 &\le 2\,A\,M + 2\int f^2\, \mathbf{1}_{f \ge A} \\
&\le 2\,A\,M + 2\Big(\int f\, \mathbf{1}_{f\ge A}\Big)^{1/2}\Big(\int f^3\Big)^{1/2} \\
&\le 2\,A\,M + 2\,\frac{\mathcal{H}^+(f)^{1/2}}{(\log A)^{1/2}}\, C_3^{3/2}\, M^{1/2}\, I(f) \le C(M,\mathcal{H}^+) + \frac{1}{4} I(f)
\end{aligned}$$

L^2 bound (a priori and formal) Known

We easily compute

$$\frac{d}{dt}\int_{\mathbb{R}^2} f^2\,dx + 2\int_{\mathbb{R}^2}|\nabla_x f|^2\,dx = \int_{\mathbb{R}^2} f^3\,dx \le 8\,M\,A^2 + 8\int_{\mathbb{R}^2}(f-A)_+^3\,dx$$

Thanks to the Gagliardo-Niremberg type inequality

$$\forall\, p\in[2,\infty)\quad \|f\|_{L^{p+1}(\mathbb{R}^2)} \le C_p\,\|f\|_{L^1}^{1/(p+1)}\,\|\nabla(f^{p/2})\|_{L^2}^{2/(p+1)}$$

with $p=2$, we deduce

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2} f^2\,dx + 2\int_{\mathbb{R}^2}|\nabla_x f|^2\,dx &\le C_A + 8C_2^3\,\|(f-A)_+\|_{L^1}\,\|\nabla f\|_{L^2}^2\\ &\le C(M,\mathcal{H}^+) + \|\nabla f\|_{L^2}^2.\end{aligned}$$

We conclude with

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2} f^2\,dx + \int_{\mathbb{R}^2}|\nabla_x f|^2\,dx &\le C(M,\mathcal{H}^+)\\ &\le C(M,M_{2,0},\mathcal{H}_0,T) \Rightarrow f\in L^\infty_{loc}(L^2)\end{aligned}$$

From the Gagliardo-Niremberg type inequality

$$\forall\ p \in [1,\infty), \quad \|f\|_{L^p(\mathbb{R}^2)} \le C_p\, M^{1/p}\, I(f)^{1-1/p},$$
$$\forall\ q \in [1,2), \quad \|\nabla f\|_{L^q(\mathbb{R}^2)} \le C_q\, M^{1/q-1/2}\, I(f)^{3/2-1/q}$$

and the bound $I(f) \in L^1(0,T)$, we get

$$f \in L^{p/(p-1)}(0,T; L^p(\mathbb{R}^2)), \quad \forall\ p \in (1,\infty),$$
$$\bar{\mathcal{K}} \in L^{p/(p-1)}(0,T; L^{2p/(2-p)}(\mathbb{R}^2)), \quad \forall\ p \in (1,2),$$
$$\nabla_x \bar{\mathcal{K}} \in L^{p/(p-1)}(0,T; L^p(\mathbb{R}^2)), \quad \forall\ p \in (2,\infty).$$

As a consequence,

$$\partial_t f - \Delta_x f = \bar{\mathcal{K}} \cdot \nabla_x f + f^2,$$

with $f^2, |\nabla_x \bar{\mathcal{K}}|\, f \in L^1((0,T)\times\mathbb{R}^2)$

$\Rightarrow$ weak solutions are renormalized solutions (thanks to DiPerna-Lions renormalization Theorem)

L^2 bound (a posteriori) New

We have

$$\partial_t \beta(f) - \Delta_x \beta(f) + \beta''(f)\,|\nabla f|^2 = \bar{\mathcal{K}} \cdot \nabla_x \beta(f) + \beta'(f) f^2$$

and then

$$\frac{d}{dt}\int_{\mathbb{R}^2} \beta(f)\,dx + \int_{\mathbb{R}^2} \beta''(f)\,|\nabla f|^2 \le \int_{\mathbb{R}^2} [\beta'(f) f^2 - f\,\beta(f)]$$

for any renormalizing function $\beta : \mathbb{R} \to \mathbb{R}$ convex, piecewise C^1 and

$$|\beta(u)| \le C\,(1 + u\,(\log u)_+), \quad (\beta'(u)\,u^2 - \beta(u)\,u)_+ \le C\,(1 + u^2) \quad \forall\, u \in \mathbb{R}.$$

We choose

$$\beta_K(u) := u^2 \text{ if } u \le K, \quad \beta_K(u) := \frac{K}{\log K}\, u \log u \text{ if } u \ge K,$$

and we pass to the limit $K \to \infty$.

Optimal estimate for $t \to 0$ (a posteriori) New

Thanks to Nash inequaliy

$$\|f\|_{L^2}^2 \le C\, M\, \|\nabla f\|_{L^2}$$

and the differential inequality

$$\frac{d}{dt}\|f\|_{L^2}^2 + \|\nabla f\|_{L^2}^2 \le C(M, M_{2,0}, \mathcal{H}_0)$$

we have for some short time $t_0 \in (0, 1]$

$$\|f(t)\|_{L^2}^2 \le C/t \quad \forall\, t \in (0, t_0).$$

Interpolating with

$$\mathcal{H}^+(f(t)) \le C \quad \forall\, t \in (0, t_0),$$

we obtain

$$t^{1/4}\, \|f\|_{L^{4/3}} \le C\, (\log 1/t)^{-1/2} \to 0 \ \text{ as } \ t \to 0.$$

We remark that

$$\mathcal{H}^+(g(t)) + \frac{1}{4} M_2(g(t)) \le C_3(M)\, \mathcal{E}(g(t)) + C_6(M) \le C_3(M)\, \mathcal{E}_0 + C_6(M)$$

and then

$$I(g) \in L^1(0,\infty) + L^\infty(0,\infty).$$

We make the same computation as before and we get

$$\frac{d}{dt}\int g^2 + 2\int |\nabla g|^2 + \int g^2 = \int g^3$$

$$\frac{d}{dt}\int g^2 + \int |\nabla g|^2 + \int g^2 \le C(M,\mathcal{H}^+) \le C(M,\mathcal{H}_0, M_{2,0})$$

Moreover, since $\|g\|_{L^2} \in L^2(0,\infty) + L^\infty(0,\infty)$, we deduce

$$\forall\, \varepsilon > 0,\ \exists\, \mathcal{C}_\varepsilon \quad \sup_{t\ge\varepsilon} \|g(t)\|^2_{L^2} \le \mathcal{C}_\varepsilon.$$

with $\mathcal{C}_\varepsilon$ only depending on ε, M, $\mathcal{H}_0$, $M_{2,0}$ (or equivalently ε, $\mathcal{E}_0$)

Plan

The mild/Duhamel formulation of the Keller-Segel equation writes

$$f_i(t) = e^{t\Delta} f_0 + \int_0^t e^{(t-s)\Delta} \nabla (V_i(s)\, f_i(s))\, ds, \quad V_i = \mathcal{K} * f_i,$$

for two solutions f_1 and f_2 with same initial datum f_0.

The difference $F := f_2 - f_1$ satisfies

$$F(t) = \int_0^t \nabla \cdot e^{(t-s)\Delta} (V_2(s)\, F(s))\, ds + ... = I_1 + I_2,$$

For any $t > 0$, we define

$$Z_i(t) := \sup_{0<s\le t} s^{1/4} \, \|f_i(s)\|_{L^{4/3}}, \quad \Delta(t) := \sup_{0<s\le t} s^{1/4} \, \|F(s)\|_{L^{4/3}}.$$

We recall the regularization estimate of the heat equation

$$\|\nabla(e^{t\Delta} g)\|_{L^{4/3}} \le \frac{C}{t^{3/4}} \, \|g\|_{L^1}$$

and the Hardy-Littlewood-Sobolev inequality

$$\|h\, \mathcal{K} * g\|_{L^1} \le C \, \|h\|_{L^{4/3}} \, \|g\|_{L^{4/3}}$$

We estimate $\Delta(T)$ in the following way:

$$
\begin{aligned}
t^{1/4}\,\|I_1(t)\|_{L^{4/3}} &\leq t^{1/4}\int_0^t \|\nabla\cdot e^{(t-s)\Delta}(V_2(s)\,F(s))\|_{L^{4/3}}\,ds \\
\text{[heat eq regularization]} &\leq t^{1/4}\int_0^t \frac{C}{(t-s)^{3/4}}\,\|V_2(s)\,F(s)\|_{L^1}\,ds \\
\text{[HLS inequality]} &\leq t^{1/4}\int_0^t \frac{C}{(t-s)^{3/4}}\,\|f_2(s)\|_{L^{4/3}}\,\|F(s)\|_{L^{4/3}}\,ds \\
&\leq \int_0^1 \frac{C}{(1-u)^{3/4}}\,\frac{du}{u^{1/2}}\,Z_2(t)\,\Delta(t),
\end{aligned}
$$

We deduce

$$
\begin{aligned}
\Delta(T) &\leq \sup_{[0,T]} t^{1/4}\,\|I_1(t)\|_{L^{4/3}} + \sup_{[0,T]} t^{1/4}\,\|I_2(t)\|_{L^{4/3}} \\
&\leq \underbrace{(Z_1(T)+Z_2(T))}_{\to 0 \text{ as } T\to 0}\,C'\,\Delta(T) \leq \frac{1}{2}\,\Delta(T) \quad \text{for } T \text{ small enough.}
\end{aligned}
$$

$\Rightarrow \Delta(T) \equiv 0$

Plan

Define the linearized Keller-Segel operator

$$\Lambda h := \mathrm{div}_x\big(\nabla h + x\,h + (\mathcal{K} * G)\,h + (\mathcal{K} * h)\,G\big).$$

Define the small (strong confinement/space localisation) Hilbert space $X = L^2(G^{-1/2})$ associated to scalar product

$$(f,g)_X := \int_{\mathbb{R}^2} f\,\bar{g}\,G^{-1}\,dx, \quad \|f\|_X^2 := (f,f)_X.$$

Define (the first eigenfunction) $h_{0,0} := \partial G_M/\partial M$ and

$$X_0^\perp := \{f \in X;\ (f, h_{0,0})_X = 0\}.$$

Define the quadratic form (obtained by linearization ofthe free energy $\mathcal{E}$)

$$Q_1[f] := \int_{\mathbb{R}^2} f^2\,G^{-1}\,dx + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x)\,f(y)\,\kappa(x-y)\,dxdy,$$

equivalent (as a norm) to $\|f\|_X^2$.

Theorem 7. Spectral analysis in the small space X

For any $g \in X_0^\perp$ which belongs to the domain of Λ, there holds

$$\langle \Lambda g, g \rangle \le -Q_1[g].$$

Moreover, there exists $a^* < -1$ and $C > 0$ so that

$$\| e^{t\Lambda} h - e^{-t} \Pi_1 h - \Pi_0 h \|_X \le C\, e^{a^* t} \, \| h - (\Pi_1 + \Pi_0) h \|_X \qquad \forall\, t \ge 0, \ \forall\, h \in X,$$

where Π_0 is the (orthogonal) projection on $\mathrm{Vect}(h_{0,0})$ and Π_1 is the (orthogonal) projection on $\mathrm{Vect}(h_{1,1}, h_{1,2})$ where $h_{1,i} := \partial_{x_i} G$.

- Campos, Dolbeault (arXiv 2012)

$\triangleright$ Nice calssical spectral analysis for self-adjoint operator (Schrödinger type operator, schwarz symmetrization, concentration-compactness, Rayleigh quotient)

Define the large (weaker confinement) Banach $\mathcal{X} := L^{4/3}_k$ with $k > 3/2$ and $a(k) := 1/2 - k$.

Proposition 8. (Spectral analysis in the large space $\mathcal{X}$)

For any $a > \bar{a} := \max(a^*, a(k))$, there exists a constant $C_{k,a}$ such that

$$\|e^{t\Lambda} h - e^{-t}\,\Pi_1 h - \Pi_0 h\|_{\mathcal{X}} \le C\, e^{at}\, \|h - \Pi_1 h - \Pi_0 h\|_{\mathcal{X}} \quad \forall\, t \ge 0,\ \forall\, h \in \mathcal{X},$$

where again
Π_0 = projection on eigenspace $\mathrm{Vect}(h_{0,0})$ associated to eigenvalue 0
Π_1 = projection on eigenspace $\mathrm{Vect}(h_{1,1}, h_{1,2})$ associated to eigenvalue -1

▷ Split $\Lambda = \mathcal{A} + \mathcal{B}$, $\mathcal{A} g := N \chi_R\, g$, $\mathcal{B}$ is dissipative, $\mathcal{A}\,\mathcal{S}_{\mathcal{B}} : \mathcal{X} \to X$.

Apply the extension of functional space for the spectral analysis developed in

- Gualdani, M., Mouhot, *Factorization of Non-Symmetric Operators and Exponential H-Theorem*,

as well as Theorem 7 by Campos-Dolbeault on the spectral analysis of $e^{t\Lambda}$ in the small space X.

The function $h := g - G$ satisfies the NL equation

$$\partial_t h = \Lambda h + \mathrm{div}(h\,\mathcal{K} * h).$$

We introduce the splitting (in order to get the optimal rate)

$$h = h_0 + h_1 + h_2, \quad h_{12} = h_1 + h_2$$

with

$$h_0 := \Pi_0 h = \alpha_0\, h_{0,0}, \quad h_1 = \Pi_1 h,$$

so that the evolution of h_1 and h_2 are given by

$$\partial_t h_1 = -h_1 + \mathcal{Q}_1, \quad \mathcal{Q}_1 := \Pi_1[\mathrm{div}(h\,\mathcal{K} * h)]$$

and

$$\partial_t h_2 = \Lambda h_2 + \mathcal{Q}_2, \quad \mathcal{Q}_2 := \Pi_2[\mathrm{div}(h\,\mathcal{K} * h)].$$

Thanks to the mass conservation, we have

$$0 = \int h = \alpha_0 + \int h_{12}$$

so that

$$\|h_0\|_{\mathcal{X}} = |\alpha_0| \, \|h_{0,0}\|_{\mathcal{X}} \leq C \, \|h_{12}\|_{\mathcal{X}}$$

Multiplying the equation on h_1 by $h_1^* = h_1 \, |h_1|^{-1/3} \, \|h_1\|^{2/3}_{L^{4/3}_k}$, we have

$$\begin{aligned} \frac{d}{dt}\|h_1\|^2_{L^{4/3}_k} &= 2\,\langle -h_1 + \Pi_1[\mathrm{div}(h\,\mathcal{K} * h)], h_1^* \rangle \\ &\leq -2\|h_1\|^2_{L^{4/3}_k} + C\,\|h_1\|_{L^{4/3}_k} \|\mathrm{div}(h\,\mathcal{K} * h)\|_{L^{4/3}_k}. \end{aligned}$$

By an interpolation argument, we have

$$\|\mathrm{div}(h\,\mathcal{K} * h)\|_{L^{4/3}_k} \leq C \, \underbrace{(\|h\|_{W^{2,\infty}} + \|h\|_{L^1_4})^{1-\alpha}}_{\text{bounded for } t \geq \varepsilon} \, \|h\|^{1+\alpha}_{L^{4/3}_k}$$

with $\alpha := 16/121 > 0$.

We define an equivalent norm to $\| \cdot \|_{\mathcal{X}}$ by

$$|||f|||^2 := \eta \, \|f\|_{\mathcal{X}}^2 + \int_0^\infty \|e^{\tau\Lambda} \, e^{\tau} \, f\|_{\mathcal{X}}^2 \, d\tau.$$

For an appropriate choice of $\eta > 0$

$$\frac{d}{dt} |||e^{t\Lambda} f|||^2 \le -2 \, |||e^{t\Lambda} f|||^2 \quad \forall \, t \ge 0, \; \forall \, f \in R(I - \Pi_0 - \Pi_1).$$

With the notations $S_\tau := e^{\tau\Lambda} \, e^{\tau}$ and $\mathcal{Q}_2 := \Pi_2 \mathrm{div}(h \, \mathcal{K} * h)$, we have

$$\begin{aligned} \frac{d}{dt} |||h_2|||^2 &= \eta \, \langle h_2^*, \Lambda h_2 \rangle + \int_0^\infty \langle (S_\tau h_2)^*, S_\tau \Lambda h_2 \rangle \, d\tau \\ &\quad + \eta \, \langle h_2^*, \mathcal{Q}_2 \rangle + \int_0^\infty \langle (S_\tau h_2)^*, \mathcal{Q}_2 \rangle \, d\tau \\ &\le -2 \, |||h_2|||^2 + C \, \|h_2\|_{L^{4/2}_k} \, \|\mathrm{div}(h \, \mathcal{K} * h)\|_{L^{4/2}_k}. \end{aligned}$$

All together, the quantity

$$u(t) := \|h_1\|_{\mathcal{X}}^2 + |||h_2|||^2$$

satisfies the differential inequality

$$u' \le -2u + C \, \|h\|^{2+\alpha} \quad \text{on} \quad (0, \infty)$$

and we easilly conclude recalling that $h = h_0 + h_1 + h_2$ that $u \le C \, e^{-2t}$.