Uniform propagation of chaos for McKean-Vlasov equations

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Abstract

These notes sum up the ideas presented in a talk for the workshop "Mean Field Limits" organised by Stéphane Mischler. They deal with the property of propagation of chaos for the McKean-Vlasov equations. These equations appear as an idealized model for the evolution of the distribution of gaz. The main goal in the following lines is to explain how it is possible to get uniform (in time) bounds for the convergence of the particle system.

Contents

Introduction

The McKean-Vlasov equation with initial condition u_0 is given by

$$\begin{cases} \partial_t u = \operatorname{div}\big(u\nabla(\sigma^2 \log u + V + W * u)\big), \\ u(0,\cdot) = u_0(\cdot), \end{cases} \tag{1}$$

where

- $\sigma \in (0 + \infty)$,
- the confinement potential $V\colon \mathbb{R}^d \to \mathbb{R}$ goes to infinity as $|x| \to \infty$ (or is null),
- the interaction potential $W\colon \mathbb{R}^d \to \mathbb{R}$ is even and convex,

- the symbol $*$ stands for the convolution on $\mathbb{R}^d$.

Let us associate to Equation (1) the stochastic process $(\bar{X}_t)_{t\geqslant 0}$ solution of

$$\begin{cases} d\bar{X}_t = \sqrt{2\sigma^2}dB_t - \nabla V(\bar{X}_t)\,dt - \nabla W * \mu_t(\bar{X}_t)\,dt, \\ \mathcal{L}(\bar{X}_t) = \mu_t, \\ \bar{X}_0 \sim u_0(x)\,dx. \end{cases} \tag{2}$$

Such a process exists and its law at time t admits a density $u(t,\cdot)$ which is solution of Equation (1). Moreover, one can derive uniform (in time) bounds for the moments of $\bar{X}$ provided its initial condition is sufficiently intergable (see [27, 24] when the interaction term is bounded and [8] and the refeerence therein for polynomial W).

Remark 1. *If $W(x) = |x|^2/2$, then,*

$$\nabla W * \mu_t(\bar{X}_t) = \bar{X}_t - \mathbb{E}(\bar{X}_t).$$

In other words, the process is attracted by its averaged position at time t.

This process is no easy to deal with since its driving coefficients depend on its law at time t. Let us introduce the associated particle system:

$$\begin{cases} dX_t^{i,N} = \sqrt{2\sigma^2}dB_t^i - \nabla V(X_t^{i,N})\,dt - \nabla W * \Pi_t^N(X_t^{i,N})\,dt, \\ X_0^{i,N} = X_0^i, \end{cases} \tag{3}$$

where Π_t^N is the empirical measure of the system:

$$\Pi_t^N(dx) = \frac{1}{N}\sum_{j=1}^N \delta_{X_t^{j,N}}(dx). \tag{4}$$

In Equation (3), $(B^i)_{i\geqslant 1}$ are independent Brownian motions, $(X_0^i)_{i\geqslant 1}$ are independent random variables with law $u_0(x)\,dx$ (and the two sequences are independent). Obviously, one has

$$\nabla W * \Pi_t^N(X_t^{i,N}) = \frac{1}{N}\sum_{j=1}^N \nabla W(X_t^{i,N} - X_t^{j,N}).$$

Intuition of the propagation of chaos phenomenon. As the size N of the particle system goes to ∞,

- two particles are less and less correlated,
- the empical measure is closer and closer to the law of a single particle,
- a particle among N behaves more and more like a nonlinear one.

The coupling. One has to construct on the same probability space (*i.e* with the same randomness) the particle system and N independent nonlinear processes (correlated with it) : $\bar{X}^i$ and $X^{i,N}$ have the same initial condition and are driven by the same Brownian motion:

$$\begin{cases} d\bar{X}_t^i = \sqrt{2\sigma^2}dB_t^i - \nabla V(\bar{X}_t^i)\,dt - \nabla W * \mu_t(\bar{X}_t^i)\,dt, \\ \mathcal{L}(\bar{X}_t^i) = \mu_t, \\ \bar{X}_0^i = X_0^i. \end{cases}$$

Remark 2. *As it is explained in the sequel, the comparison between $\bar{X}^i$ and $X^{i,N}$ does not depend on σ but on the dissipativity of the drift term. It is a challenging issue to use the diffusion term to compensate some degeneracy of the drift (see Section 3).*

These notes are organised as follows. Section 1 deals with the ideal case when the confinement potential is strong enough to ensure the stability of the system. In Sections 2 and 3, V is null and one has to use the interaction potential in a more clever way. If V is a double-well potential, uniform propagation of chaos does not hold any more (see Section 4). One can also study the long time behavior of both the particle system and the nonlinear process. Some results in these directions are presented in Section 5. At last, Section 6 deals with generalizations and comments.

1 The simplest case

Everything is simple if the interaction potential is convex (the particles attract each other) and the confinement potential is strictly convex. This ideal situation has been studied in [21].

Theorem 3. *Assume that*

$$(x-y)\cdot(\nabla W(x)-\nabla W(y))\geqslant 0,$$

and that there exists $\beta>0$ such that

$$(x-y)\cdot(\nabla V(x)-\nabla V(y))\geqslant \beta|x-y|^2.$$

Then there exists a K such that, for every $N\geqslant 1$,

$$\sup_{t\in\mathbb{R}}\mathbb{E}\left(\left|X_t^{1,N}-\bar{X}_t^1\right|^2\right)\leqslant\frac{K}{N}, \tag{5}$$

and

$$\mathbb{E}\left(\sup_{0\leqslant t\leqslant T}\left|X_t^{1,N}-\bar{X}_t^1\right|^2\right)\leqslant K\frac{T}{N}.$$

Proof of Theorem 3. For $i=1,...,N$,

$$\begin{aligned}X_t^{i,N}-\bar{X}_t^i &= X_s^{i,N}-\bar{X}_s^i-\int_s^t\big(\nabla V(X_r^{i,N})-\nabla V(\bar{X}_r^i)\big)\,dr\\ &\quad -\frac{1}{N}\sum_{j=1}^N\int_s^t\big(\nabla W(X_r^{i,N}-X_r^{j,N})-\nabla W*\mu_r(\bar{X}_r^i)\big)\,dr.\end{aligned}$$

By Itô's formula,

$$\begin{aligned}\sum_{i=1}^N\left|X_t^{i,N}-\bar{X}_t^i\right|^2 &= \sum_{i=1}^N\left|X_s^{i,N}-\bar{X}_s^i\right|^2\\ &\quad -2\sum_{i=1}^N\int_s^t(X_r^{i,N}-\bar{X}_r^i)\cdot\big(\nabla V(X_r^{i,N})-\nabla V(\bar{X}_r^i)\big)\,dr\\ &\quad -\frac{2}{N}\sum_{i,j=1}^N\int_s^t\rho_{ij}^{(1)}(r)\,dr \end{aligned}\tag{6}$$

where

$$\rho_{ij}^{(1)}(r)=\big(X_r^{i,N}-\bar{X}_r^i\big)\cdot\big[\nabla W(X_r^{i,N}-X_r^{j,N})-\nabla W*\mu_r(\bar{X}_r^i)\big].$$

One can decompose $\rho_{ij}^{(1)}(r) = \rho_{ij}^{(2)}(r) + \rho_{ij}^{(3)}(r)$ with

$$\begin{aligned}
\rho_{ij}^{(2)}(r) &= \left(X_r^{i,N} - \bar{X}_r^i\right) \cdot \left[\nabla W(X_r^{i,N} - X_r^{j,N}) - \nabla W(\bar{X}_r^i - \bar{X}_r^j)\right] \\
\rho_{ij}^{(3)}(r) &= \left(X_r^{i,N} - \bar{X}_r^i\right) \cdot \left[\nabla W(\bar{X}_r^i - \bar{X}_r^j) - \nabla W * \mu_r(\bar{X}_r^i)\right].
\end{aligned}$$

The vector field ∇W is odd and satisfies

$$(\nabla W(x) - \nabla W(y)) \cdot (x - y) \geqslant 0$$

then, by definition of $\rho_{ij}^{(2)}(r)$,

$$\rho_{ij}^{(2)}(r) + \rho_{ji}^{(2)}(r) = \left[(X_r^{i,N} - X_r^{j,N}) - (\bar{X}_r^i - \bar{X}_r^j)\right] \cdot \left[\nabla W(X_r^{i,N} - X_r^{j,N}) - \nabla W(\bar{X}_r^i - \bar{X}_r^j)\right] \geqslant 0.$$

It has been shown that

$$\sum_{i,j=1}^{N} \rho_{ij}^{(2)}(r) = \sum_{1 \leqslant i < j \leqslant N} \left(\rho_{ij}^{(2)}(r) + \rho_{ji}^{(2)}(r)\right) \geqslant 0.$$

On the other hand, Cauchy-Schwarz inequality leads to

$$\begin{aligned}
-\mathbb{E}\left[\sum_{j=1}^{N} \rho_{ij}^{(3)}(r)\right] &= -\mathbb{E}\left[(X_r^{i,N} - \bar{X}_r^i) \cdot \left(\sum_{j=1}^{N} \left(\nabla W(\bar{X}_r^i - \bar{X}_r^j) - \nabla W * \mu_r(\bar{X}_r^i)\right)\right)\right] \\
&\leqslant \left(\mathbb{E}\left[\left|X_r^{i,N} - \bar{X}_r^i\right|^2\right]\right)^{\frac{1}{2}} (\theta_i(r))^{\frac{1}{2}},
\end{aligned}$$

where

$$\theta_i(r) = \mathbb{E}\left(\left|\sum_{j=1}^{N} \left[\nabla W(\bar{X}_r^i - \bar{X}_r^j) - \nabla W * \mu_r(\bar{X}_r^i)\right]\right|^2\right).$$

Then, we get

$$\theta_i(r) = \sum_{j=1}^{N} \mathbb{E}\left(|\xi_j(r)|^2\right) + 2 \sum_{1 \leqslant j < k \leqslant N} \mathbb{E}(\xi_j(r) \cdot \xi_k(r)),$$

with the obvious notation

$$\xi_j(r) = \nabla W(\bar{X}_r^i - \bar{X}_r^j) - \nabla W * \mu_r(\bar{X}_r^i).$$

If j is not equal to k, one of them is not equal to i and then,

$$\mathbb{E}(\xi_j(r) \cdot \xi_k(r)) = 0 \quad \text{if } j \neq k$$

since the random variables $(\bar{X}_r^j)_j$ are independent copies of $\bar{X}_r^1$ with law μ_r. At last,

$$\begin{aligned}
\mathbb{E}\left(|\xi_j(r)|^2\right) &= \mathbb{E}\left(\left|\nabla W(\bar{X}_r^i - \bar{X}_r^j) - \nabla W * \mu_r(\bar{X}_r^i)\right|^2\right) \\
&\leqslant \mathbb{E}\left(\left|\nabla W(\bar{X}_r^i - \bar{X}_r^j)\right|^2\right) \\
&\leqslant K\,\mathbb{E}\left(\left|\bar{X}_r^i\right|^{2p}\right) + K\,\mathbb{E}\left(\left|\bar{X}_r^j\right|^{2p}\right) \\
&\leqslant 2K\, M_{2p}.
\end{aligned}$$

We have established that

$$-\mathbb{E}\left(\sum_{j=1}^{N} \rho_{ij}^{(3)}(r)\right) \leqslant \sqrt{KN} \gamma(r)^{1/2}$$

where γ is defined by

$$\gamma(t) = \mathbb{E}\Big[(X_t^{i,N} - \bar{X}_t^i)^2\Big].$$

Let take the expectation of (6). Using the exchangeability of the marginals of the particle system, we get that

$$\gamma(t) \leqslant \gamma(s) - 2\beta \int_s^t \gamma(r)\,dr + \frac{2\lambda K}{\sqrt{N}} \int_s^t \gamma(r)^{1/2}\,dr. \tag{7}$$

This means that

$$\gamma'(t) \leqslant -2\beta\,\gamma(t) + \frac{2\lambda K}{\sqrt{N}}\gamma(t)^{1/2}.$$

Gronwall's lemma implies (since $\alpha(0)$ is 0) that

$$\gamma(t)^{1/2} \leqslant \frac{\lambda K}{\beta\sqrt{N}}\Big[1 - e^{-\beta t}\Big]$$

which is (5).The second estimate in Theorem 3 follows classically from (5) (see [1]). □

2 Without confinement

Assume now that V is null. This model has been investigated initially by [1, 2] in dimension one and then in [22]. Is the interaction potential sufficient to stick the particle system on the independent nonlinear processes? The answer is essentially yes.

Taking the expectation in (2), one has that

$$\mathbb{E}(\bar{X}_t) = \mathbb{E}(\bar{X}_0) - \int_0^t \mathbb{E}(\nabla W(\bar{X}_s - \tilde{X}_s))\,ds$$

where $\tilde{X}_s$ is an independent copy of $\bar{X}_s$. Since W is even, ∇W is odd and $t \mapsto \mathbb{E}(\bar{X}_t)$ is constant. Assume for example that $\mathbb{E}(\bar{X}_0) = 0$. The empirical mean of the particle system does not share this property:

$$\frac{1}{N}\sum_{i=1}^N X_t^{i,N} = \frac{1}{N}\sum_{i=1}^N X_0^i + \frac{\sqrt{2}}{N}\sum_{i=1}^N B_t^i.$$

In particular the variance of this random variable goes to infinity as $t \to 0$. The idea is to modify slightly the particle system in such a way that its empirical mean is stable as $t \to \infty$: one has to remove the empirical mean to each position:

$$Y_t^{i,N} = X_t^{i,N} - \frac{1}{N}\sum_{j=1}^N X_t^{j,N}$$

The process $(Y_\cdot^{i,N})_{1\leqslant i\leqslant N}$ is still a diffusion process but on the hyperplane

$$\mathcal{M}_N = \left\{ x \in (\mathbb{R}^d)^N \,:\, \sum_{i=1}^N x_i = 0 \right\}.$$

It is solution of the following stochastic differential equation on $(\mathbb{R}^d)^N$:

$$dY_t^N = \sqrt{2}d\tilde{B}_t - \boldsymbol{\nabla}\mathcal{V}(Y_t^N)\,dt$$

where $\tilde{B}$ is a Brownian motion on $\mathcal{M}_N$ and $\mathcal{V} : (\mathbb{R}^d)^N \to \mathbb{R}$ is given by

$$\mathcal{V}(y) = \frac{1}{N}\sum_{1\leqslant i,j\leqslant N} W(y_i - y_j)$$

for any $y = (y_1, y_2, \ldots, y_N) \in (\mathbb{R}^d)^N$. The following lemma ensures that the potential $\mathcal{V}$ is convex on $(\mathbb{R}^d)^N$ but it is strictly convex on $\mathcal{M}_N$.

Lemma 4. *If there exists $\lambda > 0$ such that* $\operatorname{Hess} W(x) \geqslant \lambda I_d$*, then, for any $y \in (\mathbb{R}^d)^N$,*

1. *the matrix* $\mathbf{Hess}\, \mathcal{V}(y)$ *admits* 0 *as an eigenvalue with multiplicity d and the associated eigenvectors are of the following form $\boldsymbol{v} = (v, \ldots, v)$ where v belongs to $\mathbb{R}^d$,*

2. *the others eigenvalues are greater or equal than λ.*

In particular, if $y, \tilde{y} \in \mathcal{M}_N$, then

$$(y - \tilde{y}) \cdot (\boldsymbol{\nabla}\mathcal{V}(y) - \boldsymbol{\nabla}\mathcal{V}(\tilde{y})) \geqslant \lambda |y - \tilde{y}|^2.$$

Let us state the propagation of chaos estimate.

Theorem 5. *If there exists $\lambda > 0$ such that*

$$\operatorname{Hess} W(x) \geqslant \lambda I_d, \tag{8}$$

then there exists a constant K such that, for every $N \geqslant 1$,

$$\sup_{t \geqslant 0} \mathbb{E}\left(\left| Y_t^{i,N} - \bar{X}_t^i \right|^2 \right) \leqslant \frac{K}{N}.$$

Sketch of proof. Let us define

$$\bar{Y}_t^{i,N} = \bar{X}_t^i - \frac{1}{N} \sum_{j=1}^{N} \bar{X}_t^j$$

Since

$$\begin{aligned}
\mathbb{E}\left(\left| Y_t^{i,N} - \bar{X}_t^i \right|^2 \right) &\leqslant 2\mathbb{E}\left(\left| Y_t^{i,N} - \bar{Y}_t^{i,N} \right|^2 \right) + 2\mathbb{E}\left(\left| \bar{Y}_t^{i,N} - \bar{X}_t^i \right|^2 \right) \\
&\leqslant 2\mathbb{E}\left(\left| Y_t^{i,N} - \bar{Y}_t^{i,N} \right|^2 \right) + \frac{C}{N^2},
\end{aligned}$$

one has to focus on $\left| Y_t^{i,N} - \bar{Y}_t^{i,N} \right|^2$. To deal with $\rho_{ij}^{(2)}(s)$, one has to gather the crossing terms (as in the previous case):

$$\sum_{1 \leqslant i,j \leqslant N} \rho_{ij}^{(2)}(s) = \frac{1}{2} \sum_{1 \leqslant i,j \leqslant N} \left[\rho_{ij}^{(2)}(s) + \rho_{ji}^{(2)}(s) \right],$$

and

$$\begin{aligned}
\rho_{ij}^{(1)}(s) + \rho_{ji}^{(1)}(s) &= \left[\nabla W(Y_s^{i,N} - Y_s^{j,N}) - \nabla W(\bar{Y}_s^{i,N} - \bar{Y}_s^{j,N}) \right] \cdot \left[\left(Y_s^{i,N} - Y_s^{j,N} \right) - \left(\bar{Y}_s^{i,N} - \bar{Y}_s^{j,N} \right) \right] \\
&\geqslant \lambda \left| \left(Y_s^{i,N} - \bar{Y}_s^{i,N} \right) - \left(Y_s^{j,N} - \bar{Y}_s^{j,N} \right) \right|^2.
\end{aligned}$$

Since the vectors Y^N and $\bar{Y}^N$ are on $\mathcal{M}_N$, the sum of their coordinates is equal to 0 and then, we get by a straightforward computation:

$$\begin{aligned}
\sum_{1 \leqslant i,j \leqslant N} \rho_{ij}^{(1)}(s) &\geqslant \frac{\lambda}{2} \sum_{1 \leqslant i,j \leqslant N} \left| \left(Y_s^{i,N} - \bar{Y}_s^{i,N} \right) - \left(Y_s^{j,N} - \bar{Y}_s^{j,N} \right) \right|^2 \\
&= \lambda N \sum_{i=1}^{N} \left| Y_s^{i,N} - \bar{Y}_s^{i,N} \right|^2.
\end{aligned}$$

The end of the proof is the one of Theorem 3. $\square$

3 Degenerated interaction potential

A motivating example of McKean-Vlasov equations is derived in [3] as a simple variant of Boltzmann equations. In this case, the interaction potential is given by $W(x) = |x|^3$. Thus it is important to study the case of potential that are degenerately convex in 0.

We say that condition $\mathbf{C}(A, \alpha)$ holds if there exist $A, \alpha > 0$ such that for any $0 < \varepsilon < 1$,

$$\forall x, y \in \mathbb{R}^d, \qquad (x - y) \cdot (\nabla W(x) - \nabla W(y)) \geq A\varepsilon^\alpha(|x - y|^2 - \varepsilon^2).$$

Remark 6. *This condition holds for $x \mapsto |x|^{2+\alpha}$.*

We can now prove the following propagation of chaos estimate for the projected particle system which is uniform in time but degenerate in N (see [8]).

Theorem 7. *Assume that W satisfies $\mathbf{C}(A, \alpha)$. Then there exists $K > 0$ such that*

$$\sup_{t\geq 0} \mathbb{E}\left(|Y_t^{i,N} - \bar{X}_t^i|^2\right) \leq \frac{K}{N^{\frac{1}{1+\alpha}}}. \tag{9}$$

Proof. Once again, the single modification is the control of $\rho_{ij}^{(2)}(s)$. Condition $\mathbf{C}(A, \alpha)$ yields

$$\sum_{i,j=1}^N \rho_{ij}^{(2)}(s) \geq A\varepsilon^\alpha \left(N \sum_{i=1}^N |Y_s^{i,N} - \bar{Y}_s^{i,N}|^2 - \varepsilon^2 N^2/2 \right),$$

so that the differential inequality satisfied by $\gamma(t)$ becomes

$$\gamma'(s) \leq -2A\varepsilon^\alpha(\gamma(s) - \varepsilon^2) + \frac{c}{\sqrt{N}}\sqrt{\gamma(s)}.$$

Since the moments of Y^N and $\bar{Y}^N$ are bounded (uniformly in time) we know that $\gamma(s) \leqslant K$ for some $K > 1$. We may choose $\varepsilon = \sqrt{\gamma(s)}/2\sqrt{K} < 1$ and get

$$\gamma'(s) \leq -J\,\gamma(s)^{1+\alpha/2} + \frac{c}{\sqrt{N}}\sqrt{\gamma(s)}$$

with $J = \frac{2A}{(2\sqrt{K})^\alpha}\left(1 - \frac{1}{4K}\right)$. Define $\beta(s) = \sqrt{\gamma(s)}$. Then

$$\beta'(s) + (J/2)\beta^{1+\alpha}(s) \leq \frac{c}{2\sqrt{N}}$$

so that

$$\beta(s) \leq C/N^{1/2(1+\alpha)}$$

for any s such that $\beta'(s) \geq 0$. Since $\beta(0) = 0$ it easily follows that $\beta(s) \leq C/N^{1/2(1+\alpha)}$ everywhere, hence the result. □

4 Non convex confinement

If V is no longer convex, as for example $x \mapsto |x|^4/4 - |x|^2/2$, the method above does not provide uniform estimates for the propagation of chaos. One can only get something like

$$\mathbb{E}\left(\left|X_t^{1,N} - \bar{X}_t^1\right|^2\right) \leqslant \frac{Ke^{Kt}}{N}.$$

In fact uniform estimates do not hold (see Remark 8 in the next section).

5 Long time behavior

A related issue is the long time behavior of the particle system and the nonlinear process (or equivalently the McKean-Vlasov equation). Roughly speaking, "good" properties of ergodicity and uniform propagation of chaos hold simultaneously.

5.1 The particle system

The particle system (up to the projection trick when V is null) admits a unique invariant probability measure on $(\mathbb{R}^d)^N$:

$$\mu_N(dx) = \frac{1}{Z_N} \exp\left(-\frac{1}{\sigma^2} \sum_{i=1}^N V(x_i) - \frac{1}{2N\sigma^2} \sum_{i=1}^N W(x_i - x_j) \right).$$

In the first two cases (V strictly convex or V null and W strictly convex), the law of the particle system at time t goes to μ_N exponentially fast with a rate that does not depend on N in terms of relative entropy (see [21, 22]).

If V is null and W is degenerate (as $x \mapsto |x|^{\alpha+2}$) then one can only derive an algebraic rate of convergence for the Wasserstein distance which does not depend on the size of the particle system (see [8]).

If V is not convex, this is no longer true (the rate of convergence goes to 0 exponentially fast with N). A study of the long time behavior of the particle system (as a metastable process) is performed in [28].

5.2 The McKean-Vlasov equation

It is a little bit complicated for the nonlinear process. The fruitful idea is to introduce the free energy $\eta(u)$ of a probability density given by

$$\eta(u) = \sigma^2 \int u \log u + \int V u + \frac{1}{2} \iint W(x-y) u(x) u(y)\, dx dy.$$

If u is solution of (1) then $t \mapsto \eta(u)$ is decreasing:

$$\begin{aligned} \frac{d}{dt}\eta(u) &= \int \partial_t u (1 + \sigma^2 \log u + V + W * u) \\ &= -\int u \left|\nabla(\sigma^2 \log u + V + W * u)\right|^2 \leqslant 0, \end{aligned}$$

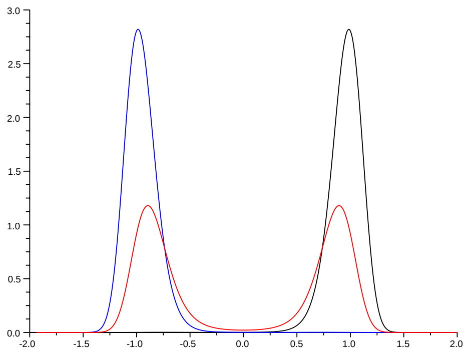
and the invariant measures for the McKean-Vlasov equation are the critical points of η. In other words, they are solutions of the implicit relation:

$$\bar{\mu}(dx) = \frac{1}{Z(\bar{\mu})} e^{-\frac{1}{\sigma^2}(V(x) + W * \bar{\mu}(x))}\, dx, \quad \text{with} \quad Z(\bar{\mu}) = \int e^{-\frac{1}{\sigma^2}(V(y) + W * \bar{\mu}(y))}\, dy.$$

If V and W are convex (and non zero), the nonlinear process $\bar{X}$ admits a unique invariant measure. If V is null, the uniqueness holds up to a translation since the evolution in (1) preserves the mean. If W is strictly convex, the exponential convergence still holds. If it is degenerate, one can use the diffusion term to get exponential rate of convergence (but with a rate that depends on the initial distribution). All these situations are investigated in [6, 7].

If V is a symmetric double well potential, as for example,

$$V(x) = \frac{x^4}{4} - \frac{x^2}{2},$$



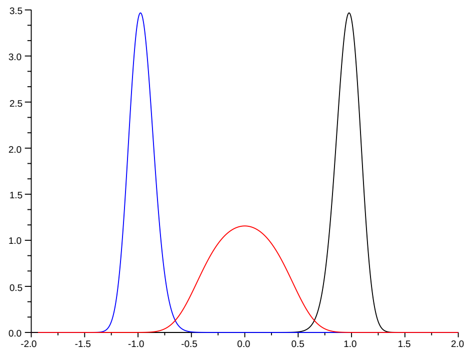


Figure 1: Invariant measures for Equation (1) with $d = 1$, $V(x) = x^4/4 - x^2/2$, $W(x) = \alpha x^2/2$ and $\sigma^2 = 0.04$. On the left, $\alpha = 0.2$, on the right, $\alpha = 1.2$.

the nonlinear process $\bar{X}$ may have exactly three invariant measures (see [11, 18, 19, 17]) if σ is sufficently small: two of them are localised around the minima of V and the third one is symmetric. Moreover the symmetric measure is concentrated around 0 if $W''(0) + V''(0) \geqslant 0$ and around two points x_+ and $-x_+$ otherwise as σ goes to zero (see Figure 5.2).

For any (reasonable) initial condition, the solution of (1) goes to one of these three invariant measures. It is rather obvious that the free energy of the symmetric one is greater than the one of the others. Moreover, symmetry is preserved by (1): all the symmetric densities belong to the attraction domain of the symmetric invariant measure. Is this attraction domain greater?

Remark 8. *If $\bar{X}$ has several invariant measures, the propagation of chaos cannot be uniform in time.*

6 Comments

6.1 Physical motivations

The McKean-Vlasov equation can be seen as an idealized model for the evolution of particles colliding inelastically (see [3] and the references therein). In this framework, the interaction potential really looks like $x \mapsto |x|^4$: it is the motivation for the degenerate case presented in Section3.

Others modelling can been found in Cappasso,etc

6.2 Uniform (or not) propagation of chaos

The propagation of chaos has been widely studied from McKean (see [23]). The Saint Flour course by Sznitman [27] and the CIME course by Méléard [24] are enlightening references on the topic.

In [1, 2], the model of Section 2 is under study but the propagation of chaos is not uniform in time (without the trick of the projection). Let us mention several models that are closely related to these ones as

- reflected McKean-Vlasov equations [12],
- competitive particles [16],
- McKean-Vlasov equations on the torus [10],
- particles intaracting through theirs ranks [25, 20, 9],

Particle systems can be associated to other nonlinear equations as the Boltzmann equation (see [26]) or the Landau equation (see [15, 14]).

Uniform estimates for the propagation of chaos property also hold for genetic type algorithms (see the review [13] and the references therein).

6.3 Concentration of the empirical measure of the particle system

A natural question in this framework is the following: how the empirical measure of the particle system Π_t^N is close to the law $\bar\mu_t$ of the nonlinear process? One can expect for example that

$$W_2(\Pi_t^N, \bar\mu_t) \xrightarrow[N\to\infty]{a.s.} 0.$$

It is possible to get uniform Gaussian estimates for this convergence (see [5, 4]).

6.4 Non gradient case and non convex interaction

This study has nothing to do with the fact that the drift coefficient are the gradients (see [29] and the thesis of Angela Ganz). One can consider a particle system solution of

$$\begin{cases} dX_t^{i,N} = \sqrt{2}dB_t^i + A(X_t^{i,N})\,dt + B * \Pi_t^N(X_t^{i,N})\,dt, \\ X_0^{i,N} = X_0^i, \end{cases}$$

assuming (for example) that

$$(x-y)\cdot(A(x)-A(y)) \leqslant -\beta|x-y|^2 \quad \text{and} \quad (x-y)\cdot(B(x)-B(y)) \leqslant 0.$$

Similarly, a lack of convexity of W can be managed by the convexity of V.

6.5 Non constant diffusion coefficients

If the particle system is given by

$$\begin{cases} dX_t^{i,N} = \sigma(X_t^{i,N})dB_t^i - \nabla V(X_t^{i,N})\,dt - \nabla W * \Pi_t^N(X_t^{i,N})\,dt, \\ X_0^{i,N} = X_0^i, \end{cases}$$

all the results above hold providing that σ and its derivative is small enough with respect to the dissipative coefficients (see [8]).

6.6 Optimality for the degenerate case

Is the estimate (9) optimal? The proof has nothing to do with the diffusion part of the particle system. Does the Brownian shaking is able to improve this bound? This is the case for long time behavior of the nonlinear process (see [6])...

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