# Kac's program in kinetic theory

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.

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# Outlines of the talk

1. Introduction
2. Classical mean field results and Kac's program
3. Main results
4. Quantitative formulations of chaos
5. Proof of the quantitative propagation of chaos
6. Proof of the relaxation time uniformly in $N$
7. Conclusion and open problems

# Plan

# Underlying Physics question

How to derive rigorously the (kinetic) equations for the mesoscopic /statistic dynamics from the description of miscroscopic dynamics (Newton first law of motion for many-particle dynamics) ?

- Grad $\sim$ 1950 : Formal derivation of the nonhomogeneous Boltzmann equation from deterministic dynamic (= "Boltzmann-Grad" limit)
- Lanford 1974 proves rigorously the limit for very short time (shorter than the free mean path) by using Bogoliubov (or BBGKY) hierarchy $\rightarrow$ King, Illner, Pulvirenti, Cercignani
- Neunzert, Braun & Hepp 70s derive the nonlinear Vlasov equation for smooth and bounded potential from Newton first principle (N particles evolve according to Hamiltonian dynamic associated to Coulombian potential) in the "mean-field" limit
- improved by Hauray, Jabin 2007 allowing (too) soft singularity

# Aim of the talk: the Kac's program as a less ambitious goal

- describe the Kac's result in 1956
  - derive the space homogeneous Boltzmann equation as the mean-field limit of a $N$-particle Markov jump (collisional) process
  - but first rigorous mathematic treatment of the deduction of Boltzmann equation from microscopic dynamics!
  - based on the notion of "Kac chaos"
  - begin with simpler models (Vlasov and McKean-Vlasov)
- formulate the "Kac's program" : two open questions in 1956
- and we add two other questions (as intermediate steps)
- give an answer to that four questions (and thus "partially achieve" the Kac's program)

## the results are taken from

- M., Mouhot, Wennberg, "A new approach to quantitative chaos propagation estimates for drift, diffusion and jump processes", arXiv 2011
- M., Mouhot, "Kac's program in kinetic theory", arXiv 2010, 2011
- M. "Introduction aux limites de champs moyen pour les systèmes de particules" (graduate school notes), on my web page
- M. "Programme de Kac sur les limites de champ moyen", EDP-X seminary publication, on my web page
- Hauray, M., "On Kac's chaos and related problems", work in progress
- Carrapataso, "Quantitative and qualitative Kac's chaos on the Boltzmann's sphere", work in progress

# Plan

# Example 1: ODE / Vlasov / empirical measure method

Consider a system of $N$ indistinguishable particles which position $X = (x_1, ..., x_N) \in E^N$, $E = \mathbb{R}^d$, in the phases space evolves (deterministically) according to a system of ODEs

$$(edo) \qquad \dot{x}_i = A_i(X), \quad x_i(0)\,\text{given}, \quad 1 \le i \le N, \quad \text{in } E^N$$

Assume that the interactions term $A_i$ writes

$$A_i(X) = \frac{1}{N} \sum_{j \neq i} a(x_j - x_i) = \frac{1}{N} \sum_{j} a(x_j - x_i) = (a * \mu^N_X)(x_i)$$

where $a$ is smooth, $a(0) = 0$, and the empirical measure $\mu^N_X$ is defined by

$$\forall\, X \qquad \mu^N_X(dz) := \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i}(dz) \in \mathbf{P}(E) = \text{probabilities space.}$$

At the statistical level, consider a density $f := f(t,x) \in \mathbf{P}(E)$ which dynamics is driven by the (mean-field) Vlasov equation

$$(V) \qquad \partial_t f = Q(f) := -\mathrm{div}((a * f)\, f), \quad f(0) = f_0, \quad \text{in } \mathbf{P}(\mathrm{E})$$

## Theorem (Dobrushin 1979)

*For any $f_0 \in \mathbf{P}(E)$ and $X_0 \in E^N$ the solution $X(t)$ of (edo) and the solution $f(t)$ of (V) satisfies*

$$\sup_{[0,T]} W_1(\mu^N_{X(t)}, f(t)) \le e^{LT}\, W_1(\mu^N_{X_0}, f_0).$$

*As a consequence, if $F_0^N \in \mathbf{P}(E^N)$ is the initial density of the particles which all evolve according to (edo), the density $F_t^N$ at time $t \ge 0$ satisfies*

$$(1) \qquad \sup_{[0,T]} \mathcal{D}_0(F_t^N; f_t) \le e^{LT}\, \mathcal{D}_0(F_0^N; f_0)$$

For any $g, h \in \mathbf{P}(E)$ we define the MKW distance $W_p$, $p = 1, 2$, by

$$W_p^p(g,h) := \inf_{\pi \in \Pi(g,h)} \int_{E\times E} d_E^p(X,Y)\, \pi(dX,dY)$$

$$\Pi(g,h) := \{\pi \in \mathbf{P}(E\times E);\ \pi(A\times E) = g(A),\ \pi(E\times B) = h(B)\}$$

For any $G \in \mathbf{P}(E^N)$ and $f \in \mathbf{P}(E)$ we define "the quantification of chaos"

$$\mathcal{D}_0(G; f) := \int_{E^N} W_1(\mu^N_X, f)\, G(dX).$$

# Example 2: SDE / McKean-Vlasov

Stochastic trajectories $X(t) \in E^N$, $E = \mathbb{R}^d$, driven by Brownian SDE plus quadratic and smooth interaction ($(B^i_t)$ independent Brownian motions)

$$(eds) \qquad dx_i = A_i(X)\,dt + dB^i_t, \qquad A_i(X) = (a \star \mu^N_X)(x_i).$$

The associated mean field equation is the McKean-Vlasov equation

$$(McKV) \qquad \partial_t f = Q(f) := \frac{1}{2}\Delta f - \mathrm{div}((a * f)\,f), \qquad f(0) = f_0.$$

### Theorem (Sznitman 1989)

*Consider $f_0 \in \mathbf{P}(E)$, $F^N_0 \in \mathbf{P}_{sym}(E^N)$ and take $X_0 \sim F^N_0$. Then the law $F^N(t)$ of the solution $X(t)$ of (eds) and the solution $f(t)$ of (McKV) satisfy*

$$(2) \qquad \sup_{[0,T]} \mathcal{D}_N(F^N(t); f(t)) \le C_T \left( \mathcal{D}_N(F^N(0); f(0)) + \frac{1}{\sqrt{N}} \right),$$

where for any $G \in \mathbf{P}_{sym}(E^N)$ and $f \in \mathbf{P}(E)$ we define "the quantification of chaos"

$$\mathcal{D}_N(G; f) := W_1(G, f^{\otimes N}).$$

# Example 2: SDE / McKean-Vlasov

**Theorem (Sznitman 1989)**

*Consider* $f_0 \in \mathbf{P}(E)$, $F_0^N \in \mathbf{P}_{sym}(E^N)$ *and take* $X_0 \sim F_0^N$. *Then the law* $F^N(t)$ *of the solution* $X(t)$ *of (eds) and the solution* $f(t)$ *of (McKV) satisfy*

$$(2) \qquad \sup_{[0,T]} \mathcal{D}_N(F^N(t); f(t)) \le C_T \left( \mathcal{D}_N(F^N(0); f(0)) + \frac{1}{\sqrt{N}} \right).$$

For any $G \in \mathbf{P}_{sym}(E^N)$ and $f \in \mathbf{P}(E)$ we define "the quantification of chaos"

$$\mathcal{D}_N(G; f) := W_1(G, f^{\otimes N}),$$

for any $F, G \in \mathbf{P}(E^j)$ we define the MKW distance $W_p$, $p = 1, 2$, by

$$W_p^p(F,G) := \inf_{\pi \in \Pi(F,G)} \int_{E^j \times E^j} d_{E^j}^p(X,Y)\, \pi(dX, dY)$$

$$\Pi(F,G) := \{\pi \in \mathbf{P}(E^j \times E^j);\ \pi(A \times E^j) = F(A),\ \pi(E^j \times B) = G(B)\}$$

$$d_{E^j}^p(X,Y) := \frac{1}{j} \sum_{i=1}^{j} d_E(x_i, y_i)^p$$

# Example 2: SDE / McKean-Vlasov / Coupling method

Stochastic trajectories $X(t) \in E^N$, $E = \mathbb{R}^d$, driven by

$$(eds) \qquad dx_i = (a \star \mu^N_X)(x_i)\, dt \, + \, dB^i_t, \qquad .$$

The associated mean field equation is the McKean-Vlasov equation

$$(McKV) \qquad \partial_t f = Q(f) := \frac{1}{2} \Delta f - \mathrm{div}((a * f)\, f), \qquad f(0) = f_0.$$

**Theorem (Sznitman 1989)**

*Consider $f_0 \in \mathbf{P}(E)$, $F^N_0 \in \mathbf{P}_{sym}(E^N)$ and take $X_0 \sim F^N_0$. Then the law $F^N(t)$ of the solution $X(t)$ of (eds) and the solution $f(t)$ of (McKV) satisfy*

$$(2) \qquad \sup_{[0,T]} \mathcal{D}_N(F^N(t); f(t)) \le C_T \left( \mathcal{D}_N(F^N(0); f(0)) + \frac{1}{\sqrt{N}} \right).$$

Coupling method: consider $Y(t)$ solution to the subsidiary problem:

$$(y_i(0)) \text{ i.i.d. according to } f(0) \quad \text{and} \quad dy_i = (a \star f(t,.))(y_i) + dB^i_t,$$

so that $Y(t) \sim f(t)^{\otimes N}$ and prove that $X(t) \approx Y(t)$ (up to an error term of order $\mathcal{O}\left(\frac{1}{\sqrt{N}}\right)$ thanks to a "quadratic miracle").

$N$-particle system $V = (v_1, ..., v_N)$, $v_i \in E = \mathbb{R}^3$ undergoing random Boltzmann jumps (collisions).
Markov process $(V_t)_{t \geq 0}$ defined step by step as follows:
(i) draw randomly $\forall$ $(v_{i'}, v_{j'})$ collision time $T_{i',j'} \ \sim \ Exp(B(|v_{i'} - v_{j'}|))$; then select the post-collisional velocity $(v_i, v_j)$ such that

$$T_{i,j} = \min_{(i',j')} T_{i',j'}.$$

(ii) draw randomly $\sigma \in S^2$ according to the density law $b(\cos\theta)$ with $\cos\theta = \sigma \cdot (v_i - v_j)/|v_i - v_j|$ and define the post-collisional velocities $(v_i^*, v_j^*)$ thanks to

$$v_i^* = \frac{v_i + v_j}{2} + \frac{|v_j - v_i|}{2}\,\sigma, \qquad v_j^* = \frac{v_i + v_j}{2} - \frac{|v_j - v_i|}{2}\,\sigma.$$

Observe that momentum and energy are conserved

$$v_i^* + v_j^* = v_i + v_j, \qquad |v_i^*|^2 + |v_j^*|^2 = |v_i|^2 + |v_j|^2.$$

Finally, this two bodies collisions jump process satisfies

$$\sum_{i=1} v_i(t) = \text{cst}, \quad \sum_{i=1} |v_i(t)|^2 = \text{cst}.$$

# Example 3: Master equation for Boltzmann-Kac system

Equivalently, after time rescaling, the motion of the $N$-particle system is given through the Master/Kolmogorov equation on the law $F_t^N \in \mathbf{P}_{sym}(E^N)$ which in dual form reads

$$(BKs) \qquad \partial_t \langle F^N, \varphi \rangle = \langle F^N, \Lambda^N \varphi \rangle \qquad \forall\, \varphi \in C_b(E^N), \quad F^N(0) = F_0^N,$$

with

$$(\Lambda^N \varphi)(V) = \frac{1}{N} \sum_{i,j=1}^{N} B(v_i - v_j) \int_{S^2} b(\cos\theta_{ij})\, [\varphi'_{ij} - \varphi]\, d\sigma,$$

where $\varphi = \varphi(V)$, $\varphi'_{ij} = \varphi(V'_{ij})$, $V'_{ij} = (v_1, .., v'_i, .., v'_j, .., v_N)$.

- Maxwell interactions with Grad's cut-off **(MG)**: $B = 1$, $b = 1$;
- Maxwell interactions without cut-off **(M)**: $B = 1$, $b \notin L^1$ ;
- Hard spheres interactions **(HS)**: $B(z) = |z|$, $b = 1$.

# The nonlinear space homogeneous Boltzmann equation

Nonlinear homogeneous Boltzmann equation on $\mathbf{P}(\mathbb{R}^3)$ defined by

$$(Beq) \qquad \partial_t f = Q(f), \quad f(0) = f_0$$

with

$$\langle Q(f), \varphi \rangle := \int_{\mathbb{R}^6 \times S^2} B(v - v_*)\, b(\cos\theta)\, (\phi(v') - \phi(v))\, d\sigma\, f(dv)\, f(dv_*)$$

where again

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\, \sigma.$$

The equation generate a nonlinear semigroup

$$\forall\, f_0 \in P_2(\mathbb{R}^3) \qquad S_t^{NL} f_0 := f_t.$$

# Kac's definition of chaos

A sequence $F^N \in \mathbf{P}_{sym}(E^N)$ is $f$-chaotic, $f \in \mathbf{P}(E)$, iff

$$\forall\, \varphi_1, ..., \varphi_j \in C_b(E) \qquad \int_{E^N} \varphi_1 \otimes ... \otimes \varphi_j \, F^N(dX) \to \prod_{i=1}^{j} \int_E \varphi_i \, f$$

or equivalently

$$\boxed{\begin{array}{ll} \texttt{(def-1)} \qquad \forall\, j \geq 1 & F_j^N \rightharpoonup f^{\otimes j} \ \texttt{weakly in}\ \mathbf{P}(E^j), \\ \qquad\qquad \forall\, j \geq 1 & \mathcal{D}_j(F^N; f) \to 0 \ \texttt{when}\ N \to \infty, \end{array}}$$

where $F_j^N$ stands for the $j$-th marginal of $F^N$ defined by

$$F_j^N := \int_{E^{N-j}} F^N \, dx_{j+1} \dots dx_N = \text{density of the first } j \text{ particles}$$

and for any $G \in \mathbf{P}_{sym}(E^N)$ and $f \in \mathbf{P}(E)$ we define "the quantification of chaos" (for $j \in \{2, ..., N\}$)

$$\mathcal{D}_j(G; f) := W_1(G_j, f^{\otimes j}) \qquad \forall\, j \in \{1, ..., N\}.$$

# Chaos propagation

## Theorem (Kac, McKean, Graham et Méléard)

*Assume* **(MG)**. *Consider* $F_0^N \in \mathbf{P}_{sym}(E^N)$ *and* $F^N(t)$ *the solution to (BKs). Consider* $f_0 \in \mathbf{P}(E)$ *and* $f(t)$ *the solution to (Beq).*
*(a) If* $F_0^N$ *is* $f_0$*-chaotic, then* $F_t^N$ *is* $f(t)$*-chaotic.*
*(b) More precisely, if* $F_0^N = f_0^{\otimes N}$*, then for any* $1 \le \ell \le N$

$$(3) \qquad \sup_{t \in [0,T]} \mathcal{D}_\ell(F^N(t); f(t)) \le \frac{C_{\ell,T}}{N}.$$

**Question 1 by Kac.** "The above proof suffers from the defect that it works only if the restriction on time is independent of the initial distribution. It is therefore inapplicable to the physically significant case of hard spheres because in this case our simple estimates yield a time restriction which depends on the initial distribution. A general proof that Boltzmann's property propagates in time is still lacking"

**Positive answer for (HS)** by Sznitman 1984 thanks to a nonlinear martingale approach, compactness and uniqueness arguments, and by Arkeryd, Caprino, Ianiro 1991 thanks to a BBGKY hierarchy approach

How to deduce the behavior of the typical particle from the behavior of the $N$-particle system ?

Pb 1: Law of large numbers: $\mu^N_{Y(t)} \rightharpoonup f(t)$ or $F^N_1 \rightharpoonup f(t)$ when $N \to \infty$

The density $F^N_1(t)$ of one typical particle of the $N$-particle system behaves as $f(t)$ the solution of the mean-field equation. Mean-field convergence $\approx$ law of large numbers.

Pb 2: propagation of chaos: $F^N_0$ is $f_0$-chaotic implies $F^N_t$ is $f_t$-chaotic?

in the sense that in the large number of particles limit $N \to \infty$:

$$
\begin{aligned}
&(3) : F^N_\ell(t) \rightharpoonup f(t)^{\otimes \ell} \text{ in } \mathbf{P}(E^\ell) && \Leftrightarrow \mathcal{D}_\ell(F^N(t); f(t)) \to 0,\\
&(1) : \hat F^N \rightharpoonup \delta_{f(t)} \text{ in } \mathbf{P}(\mathbf{P}(E)) && \Leftrightarrow \mathcal{D}_0(F^N(t); f(t)) \to 0,\\
&(2) : F^N \approx f^{\otimes N} \text{ in } \mathbf{P}(E^N) && \Leftrightarrow \mathcal{D}_N(F^N(t); f(t)) \to 0
\end{aligned}
$$

- Even when $F^N_0 = f^{\otimes N}_{in}$ we never have $F^N_t = g^{\otimes N}_t$ for a given $N$ (except when there is no interaction between the particles of the $N$-particle system!).
  - we cannot expect independence
  - we may expect recover “independence” at the limit (= chaos)
  - chaos is weaker than independence, chaos $\approx$ asymptotic independence

How to deduce the behavior of the typical particle from the behavior of the $N$-particle system ?

Pb 1: Law of large numbers: $\mu^N_{Y(t)} \rightharpoonup f(t)$ or $F^N_1 \rightharpoonup f(t)$ when $N \to \infty$

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in the sense that in the large number of particles limit $N \to \infty$:

$$
\begin{aligned}
&(3) : F^N_\ell(t) \rightharpoonup f(t)^{\otimes \ell} \text{ in } \mathbf{P}(E^\ell),\\
&(1) : \hat{F}^N \rightharpoonup \delta_{f(t)} \text{ in } \mathbf{P}(\mathbf{P}(E)),\\
&(2) : F^N \approx f^{\otimes N} \text{ in } \mathbf{P}(E^N)
\end{aligned}
$$

- Why are we interested by chaos?
  - chaos is a strong physically relevant information
  - it may help to identify the mean field limit equation (as in Kac's proof). For the Boltzmann model, mean-field limit may only be established when molecular chaos holds at the initial time and is propagated.

Kac claimed that his main motivation was to understand the H-theorem and the time relaxation to the equilibrium for the nonlinear Boltzmann equation from the corresponding properties for the high (and increasing!) dimension linear Boltzmann-Kac system

**Question 2.** Is-it possible to prove something of that kind?

### Theorem (Kac, Janvresse, Carlen, Carvalho, Loss)

*Assume* **(MG)***. Define $\sigma_N$ as the uniform measure on the Boltzmann's sphere $\mathcal{BS}_N :=$ sphere of $E^N$ of radius $\sqrt{N}$. $\exists\, \delta > 0$ s.t. for any $N \geq 1$*

$$\Delta_N := \inf\{ -\langle h, \Lambda^N h\rangle_{L^2},\ \langle h, 1\rangle_{L^2} = 0,\ \|h\|^2_{L^2}\} \geq \delta > 0,$$

*where $\langle \cdot, \cdot\rangle_{L^2}$ and $\|\cdot\|_{L^2}$ stand for the scalar product and the norm in $L^2(\mathcal{BS}_N; d\sigma_n)$. As a consequence, for any $F_0^N = h_0\,\sigma_N \in \mathbf{P}_{sym}(E^N)$, $h_0 \in L^2$, the solution $F^N$ to (BKs) writes $F^N = h(t)\,\sigma_N$ and*

$$(4) \qquad \|h^N(t) - 1\|_{L^2} \leq e^{-\delta\, t}\, \|h_0^N - 1\|_{L^2}.$$

• That result does not answer question 2 because if $F_0^N = h_0^N \, \sigma_N$ is $f_0$-chaotic then $\|h_0^N - 1\|_{L^2} \geq A^N$, with $A > 1$, and we need to wait some time proportional to $N$ in order that (4) implies any convergence to the equilibrium.

• The spectral gap associated to the entropy (which is better adapted to a $N \to \infty$ limit) has been studied recently. Defining

$$\Delta'_N := \inf\Big\{ -\frac{\int_{\mathcal{BS}_N} \log h \, \Lambda^N h \, d\sigma^N}{N H(G|\sigma^N)}, \; G = h \, \sigma^N \Big\}, \; H(G|\sigma^N) \, = \, \frac{1}{N} \int_{\mathcal{BS}_N} h \log h \, d\sigma^N,$$

Villani proved $\Delta'_N \geq 1/N$ and Carlen, Carvalho, Le Roux, Villani proved $\limsup \Delta'_N = 0$. Again, that results does not answer question 2.

• On the other hand, exponential trend to equilibrium for the nonlinear Boltzmann equation (Beq) has been proved by another (direct way), namely for any $f_0 \in \mathbf{P}(E)$ there holds

$$(5) \qquad D(f(t), \gamma) \leq C_{f_0} \, e^{-\lambda \, t}$$

for some distance $D$ on $\mathbf{P}(E)$ and where $\gamma$ is the Maxwell function associated to $f_0$. (cf. Carleman, Grad, Arkeryd, Desvillettes, Carlen, Carvalho, Toscani, Villani, Baranger, Mouhot, ...)

# Questions

- **Question 1′.** May we generalize the result of propagation of chaos to some model mixing derive, diffusion and collisions? May we prove "quantified" version of propagation of chaos (as in (1), (2) and (3)) for realistic physical Boltzmann-Kac model? May we "quantify" the distance to the chaos at time $t$ as a function of the distance to the chaos at time 0?
- **Question 2′.** May we prove convergence of $F^N$ to its equilibrium $\sigma^N$ uniformly in the number of particles $N$?
- **Question 3.** May we prove uniform in time propagation of chaos?
- **Question 4.** What is the relationship between the different distances to the chaos in (1), (2) and (3)?

# Plan

# Quantitative answer to Kac's problem 1

**Theorem ((Th-1) Uniform in time Kac's chaos convergence)**

$$(6) \qquad \sup_{[0,T]} W_1(F_t^N, f(t)^{\otimes N}) \le \Theta_{1,T}(W_1(F_0^N, f_0^{\otimes N})) + \Theta_{2,T}\left(\frac{1}{N}\right)$$

- $T \in (0, +\infty]$
- $E = \mathbb{R}^d$, $d = 3$, $V = (v_1, ...., v_N) \in E^N$
- $f_0 \in \mathbf{P}(E)$ with enough moments bounded,
  $f(t) =$ evolution of one typical particle in the mean-field limit,
  $f_t^{\otimes N}(V) = f_t(v_1) \dots f_t(v_N)$,
- $F_0^N \in \mathbf{P}_{sym}(E^N)$, $F^N(t) =$ evolution of N-particle system $\in \mathbf{P}_{sym}(E^N)$
- $\Theta_i(w) \to 0$ when $w \to 0$,
  with $\Theta_{i,T}(w) = C_i\, w^{\alpha_i}$, $\alpha_i \in (0,1)$, in some situations

## Main features 1: answers question $1'$ and 3

- We prove propagation of chaos with quantitative rates
- Most importantly and new: estimates are uniform in time (we may choose $T = \infty$) for Boltzmann-Kac system (BKs)

  $\Rightarrow N \to \infty$ limit and $t \to \infty$ limit commute!
- We may deal with mixtures of Vlasov, McKean and Boltzmann models at least for smooth and bounded coefficients
- Our theorem applies to the space homogeneous Boltzmann equation in the case of the two important physical collision models:

  - true Maxwell molecules (without Grad's cut-off) cross-section

  - hard spheres cross-section (and hard potential with Grad's cut-off )

  $\Rightarrow$ give quantitative estimates of previous non-constructive convergence result (Sznitman 1984), (Arkeryd et al 1991)

  - Maxwell molecules with Grad's cut-off cross-section

  with optimal rate $\leq C_T/\sqrt{N}$ $\Rightarrow$ recover Kac, McKean, Tanaka, Graham, Méléard, Peyre ...

- For (BKs) associated to **(MG)** and **(M)** interactions, we prove

$$\sup_{[0,\infty)} W_1(F^N(t), f(t)^{\otimes N}) \le C \left( W_1(F_0^N, f_0^{\otimes N})^{\alpha_1} + \frac{1}{N^{\alpha_2}} \right), \quad \alpha_i \in (0,1)$$

- For (BKs) associated to **(MG)** interactions, we are able to prove

$$(7) \qquad \sup_{[0,T]} D(F_t^N; f_t) \le C \left( \frac{1}{N^{\alpha}} + D(F_0^N, f_0) \right), \quad \alpha \in (0,1)$$

for some "distance" $D$ which measures how close to a chaos state "$g \in \mathbf{P}(E)$" is a probability $g^N \in \mathbf{P}_{sym}(E^N)$ and $C, \alpha > 0$.
For smooth $f_0$ and well chosen $F_0^N \in \mathbf{P}_{sym}(\mathcal{BS}_N)$, we have

$$\sup_{[0,\infty)} |H(F^N(t)|\sigma^N) - H(f|\gamma)| \le \frac{C}{N^{\alpha_3}}$$

**Theorem ((Th-2) Convergence to the equilibrium uniformly in $N$)**

$$\sup_N W_1\left(F_t^N, \sigma^N\right) \le \varepsilon_1(t) \xrightarrow[t\to\infty]{} 0$$

*Consider* **(MG)** *interactions,* $f_0$ *with finite Fisher information and assume* $F_0^N = [f_0^{\otimes N}]_{\mathcal{BS}_N}$ *so that* $F_t^N = h^N(t)\sigma^N$ *with* $h^N(t) \in L^1(\mathcal{BS}_N)$*. Then*

$$\sup_N H(F_t|\sigma^N) \le \varepsilon_2(t) \xrightarrow[t\to\infty]{} 0$$

- $E = \mathbb{R}^d$, $d = 3$, $V = (v_1, ...., v_N) \in E^N$
- $F_t^N$ = evolution of N-particle system $\in \mathbf{P}_{sym}(E^N)$,
- we may take $\varepsilon_i(t) = C/t^{a_i}$ with $a_i \in (0,1)$ when we consider **(MG)**.
- remember that

$$H(F|\sigma^N) := \frac{1}{N}\int_{\mathcal{BS}_N} h \log h \, d\sigma^N, \quad \textit{if } F = h\,\sigma^N.$$

I - Very weak uniform in time quantitative chaos propagation

$$(*_1) \qquad \sup_{[0,T]} \|F_2^N(t) - f(t)^{\otimes 2}\|_{\mathcal{F}'} \le \frac{C_T}{N^{1-\varepsilon}} + \Theta_{2,T}(\mathcal{W}_{W_1}(\hat{F}_0^N, \delta_{f_0}))$$

in a weak dual norm $\|\cdot\|_{\mathcal{F}'}$, with $\mathcal{F}$ space of smooth functions $\subset C_b(E^2)$

- the key idea is to compare the dynamics associated to (BKs) and to (Beq) (and not only the two solutions $F_t^N$ and $f_t$ ) but (BKs) defines a linear semigroup $S_t^N$ on $\mathbf{P}(E^N)$ while (Beq) defines a nonlinear semigroup $S_t^{NL}$ on $\mathbf{P}(E)$

- On the one hand, we define $T_t^N$ on $C_b(E^N)$ the dual semigroup of $S_t^N$:

$$\forall\, t \ge 0,\ \forall\, \varphi_0 \in C_b(E^N) \quad T_t^N \varphi_0 := \varphi_t, \quad \partial_t \varphi_t = \Lambda_N \varphi_t,$$

and the generator of $T_t^N$ is $\Lambda_N$ defined on a domain $\mathcal{F}_N \subset C_b(E^N)$.

- On the other hand, we define $T_t^\infty$ on $C_b(\mathbf{P}(E))$ the linear pushforward semigroup of the nonlinear semigroup $S_t^{NL}$ defined by

$$(T_t^\infty \Phi)(\rho) := \Phi(S_t^{NL}\rho), \quad S_t^{NL}\rho = \rho_t, \quad \partial_t \rho_t = Q(\rho_t),$$

and the generator of $T_t^\infty$ is $\Lambda_\infty$ defined on a domain $\mathcal{F}_\infty \subset C_b(\mathbf{P}(E))$.

## I - Very weak uniform in time quantitative chaos propagation

$$(*_1) \qquad \sup_{[0,T]} \|F_2^N(t) - f(t)^{\otimes 2}\|_{\mathcal{F}'} \le \frac{C_T}{N^{1-\varepsilon}} + \Theta_{2,T}(\mathcal{W}_{W_1}(\hat{F}_0^N, \delta_{f_0}))$$

in a weak dual norm $\| \cdot \|_{\mathcal{F}'}$, with $\mathcal{F} \subset C_b(E^2)$

- compare the dynamics associated to (BKs) and to (Beq)
- $T_t^N$ semigroup on $C_b(E^N)$ with generator $\Lambda_N$ and domain $\mathcal{F}_N$
- $T_t^\infty$ semigroup on $C_b(\mathbf{P}(E))$ with generator $\Lambda_\infty$ and domain $\mathcal{F}_\infty$
- introduce $\pi^N$ the projection from $C_b(\mathbf{P}(E))$ onto $C_b(E^N)$ defined by $(\pi^N \Phi)(V) = \Phi(\mu_V^N)$, $\forall\, \Phi \in C_b(\mathbf{P}(E))$
- introduce the "polynomial function" $R_\varphi \in C_b(\mathbf{P}(E))$ for any $\varphi \in C_b(E^j)$ defined by $R_\varphi(\rho) = \langle \rho^{\otimes j}, \varphi \rangle$
- We may (and we have to) estimate the difference

$$(T_t^N \pi^N - \pi^N T_t^\infty)(R_\varphi),$$

where

$$\forall\, t \ge 0 \qquad T_t^N \pi^N - \pi^N T_t^\infty : C_b(\mathbf{P}(E)) \to C_b(E^N)$$

## I - Very weak uniform in time quantitative chaos propagation

$$(*_1) \qquad \sup_{[0,T]} \|F_2^N(t) - f(t)^{\otimes 2}\|_{\mathcal{F}'} \le \frac{C_T}{N^{1-\varepsilon}} + \Theta_{2,T}(\mathcal{W}_{W_1}(\hat F_0^N, \delta_{f_0}))$$

in a weak dual norm $\|\cdot\|_{\mathcal{F}'}$, with $\mathcal{F} \subset UC_b(E^2)$

- $T_t^N$ semigroup on $C_b(E^N)$ with generator $\Lambda_N$ and domain $\mathcal{F}_N$
- $T_t^\infty$ semigroup on $C_b(\mathbf{P}(E))$ with generator $\Lambda_\infty$ and domain $\mathcal{F}_\infty$
- $\pi^N$ the projection from $C_b(\mathbf{P}(E))$ onto $C_b(E^N)$
- $\Phi := R_\varphi \in C_b(\mathbf{P}(E))$ for $\varphi \in C_b(E^j)$
- We estimate the difference thanks to Trotter-Kato formula

$$(T_t^N \pi^N - \pi^N T_t^\infty)\,\Phi = \int_0^t T_{t-s}^N \underbrace{(\Lambda_N \pi^N - \pi^N \Lambda_\infty)}_{consistency} \underbrace{T_s^\infty \Phi}_{stability}\, ds$$

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## I - Very weak uniform in time quantitative chaos propagation

$$(*_1) \qquad \sup_{[0,T]} \|F_2^N(t) - f(t)^{\otimes 2}\|_{\mathcal{F}'} \le \frac{C_T}{N^{1-\varepsilon}} + \Theta_{2,T}(\mathcal{W}_{W_1}(\hat{F}_0^N, \delta_{f_0}))$$

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- consistency result: the difference of generators applied on "smooth" functions is of order $1/N$;

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## I - Very weak uniform in time quantitative chaos propagation

$$(*_1) \qquad \sup_{[0,T]} \|F_2^N(t) - f(t)^{\otimes 2}\|_{\mathcal{F}'} \leq \frac{C_T}{N^{1-\varepsilon}} + \Theta_{2,T}(\mathcal{W}_{W_1}(\hat{F}_0^N, \delta_{f_0}))$$

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- $\pi^N$ the projection from $C_b(\mathbf{P}(E))$ onto $C_b(E^N)$
- $\Phi := R_\varphi \in C_b(\mathbf{P}(E))$ for $\varphi \in C_b(E^j)$
- We estimate the difference thanks to Trotter-Kato formula

$$(T_t^N \pi^N - \pi^N T_t^\infty)\, \Phi = \int_0^t T_{t-s}^N \, \underbrace{(\Lambda_N \pi^N - \pi^N \Lambda_\infty)}_{consistency} \, \underbrace{T_s^\infty \Phi}_{stability} \, ds$$

- consistency: difference of generators applied on smooth functions $= \mathcal{O}(1/N)$;
- stability result (expansion of order $> 1$) for the nonlinear semigroup $S_t^{NL}$
$\Rightarrow \Phi_s := T_s^\infty \Phi$ is a "smooth" function;

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## I - Very weak uniform in time quantitative chaos propagation

$$(*_1) \qquad \sup_{[0,T]} \|F_2^N(t) - f(t)^{\otimes 2}\|_{\mathcal{F}'} \le \frac{C_T}{N^{1-\varepsilon}} + \Theta_{2,T}(\mathcal{W}_{W_1}(\hat{F}_0^N, \delta_{f_0}))$$

in a weak dual norm $\|\cdot\|_{\mathcal{F}'}$, with $\mathcal{F} \subset UC_b(E^2)$

- $T_t^N$ semigroup on $C_b(E^N)$ with generator $\Lambda_N$ and domain $\mathcal{F}_N$
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- $\Phi := R_\varphi \in C_b(\mathbf{P}(E))$ for $\varphi \in C_b(E^j)$
- We estimate the difference thanks to Trotter-Kato formula

$$(T_t^N \pi^N - \pi^N T_t^\infty)\,\Phi = \int_0^t T_{t-s}^N \, \underbrace{(\Lambda_N \pi^N - \pi^N \Lambda_\infty)}_{consistency} \; \underbrace{\Phi_s}_{stability} \; ds$$

- consistency: difference of generators applied on smooth functions $= \mathcal{O}(1/N)$;
- stability for the nonlinear semigroup $S_t^{NL} \Rightarrow \Phi_s$ is a smooth function;
- smooth function $=$ expansion of $\Phi$ up to order $1+a$ in each point of $\mathbf{P}(E)$ seen as an embedded manifold of $\mathcal{F}'$, much more simpler that the "differential calculus" developed in "gradient flow theory"

II - Equivalence of the different "quantifications of chaos" $\Rightarrow$ Uniform in time quantitative chaos propagation for the same "distance", e.g. $D = \mathcal{D}_N$,

$$(*_2) \qquad \sup_{[0,T]} W_1(F_t^N, f_t^{\otimes N}) \le \Theta_{1,T}(W_1(F_0^N, f_0^{\otimes N})) + \Theta_{2,T}\left(\frac{1}{N}\right)$$

- finite dimensional interpolation inequality "all the distance in $\mathbf{P}(E^j)$ are equivalent" $j = 1,\ 2$;
- equivalence between the different notions of "quantification of chaos" $\mathcal{D}_j$, $j \in \{0, 2, ..., N\}$

III - Initial "quantification of chaos" $\Rightarrow$ Uniform in time quantitative chaos

$$(*_3) \qquad \sup_{[0,T]} W_1(F_t^N, f(t)^{\otimes N}) \le \Theta_{3,T}\left(\frac{1}{N}\right)$$

- for any $f_0 \in \mathbf{P}(E)$ "smooth" there exists $F_0^N \in \mathbf{P}_{sym}(\mathcal{BS}_N)$, e.g. $F_0^N = [f_0^{\otimes N}]_{\mathcal{BS}_N}$, s.t. $F_0^N$ is $f_0$-chaotic in a quantified and qualitative way :

$$\mathcal{D}_0(F_0^N, f_0) = \mathcal{W}_{W_1}(\hat{F}_0^N, \delta_{f_0}) \le \frac{C}{N^{\gamma_1}}, \quad |H(F_0^N|\sigma^N) - H(f_0|\gamma)| \le \frac{C}{N^{\gamma_2}}$$

- from here comes the worse error term: $\gamma_2 \le \gamma_1 \approx \frac{1}{d'}$, $d' = \max(d, 2)$

## IV- Entropy convergence to the equilibrium uniformly in $N$

$$(*_4) \qquad \sup_{t\geq 0} |H(F_t^N|\sigma) - H(f_t|\gamma)| \leq \Theta_4\left(\frac{1}{N}\right)$$

- uniform bound on Fisher information

$$I(F^N|\sigma) := \int_{\mathcal{BS}_N} \frac{|\nabla h^N|^2}{h^N}\, d\sigma^N, \quad F^N = h^N\,\sigma^N$$

- use twice the HWI interpolation inequality of Otto-Villani (which is independent of the dimension)

$$H(G^N|\sigma^N) \leq H(F^N|\sigma^N) + \sqrt{I(G^N|\sigma^N)}\, W_2(G^N, F^N)$$

## V - Convergence to the equilibrium uniformly in $N$

$$(*_3') \qquad \sup_N W_1\left(F_t^N, \sigma^N\right) \leq \varepsilon_1(t) \underset{t\to\infty}{\longrightarrow} 0$$

and

$$(*_4') \qquad \sup_N H(F_t^N, \sigma^N) \leq \varepsilon_2(t) \underset{t\to\infty}{\longrightarrow} 0$$

- a triangular inequality

# Plan

# Alternative formulation

To any $F^N \in \mathbf{P}_{sym}(E^N)$ we may associate $\hat F^N \in \mathbf{P}(\mathbf{P}(E))$ by setting

$$\forall\, \Phi \in C_b(\mathbf{P}(E)) \qquad \langle \hat F^N, \Phi\rangle = \int_{E^N} \Phi(\mu^N_X)\, F^N(dX).$$

As a a consequence of Hewitt-Savage theorem:

**Lemma:** $F^N$ is $f$-chaotic iff

(def-1) $\quad F^N_j \rightharpoonup f^{\otimes j}$ weakly in $\mathbf{P}(E^j)$, $\quad \mathcal{D}_j(F^N; f) \to 0$

(def-2) $\quad \hat F^N \rightharpoonup \delta_f$ weakly in $\mathbf{P}(\mathbf{P}(E))$, $\quad \mathcal{D}_0(F^N; f) \to 0$

where for $\alpha, \beta \in \mathbf{P}(\mathbf{P}(E))$ and $D$ a distance on $\mathbf{P}(E)$ we define

$$\mathcal{W}_D(\alpha,\beta) := \inf_{\pi\in\Pi(\alpha,\beta)} \int_{\mathbf{P}(E)\times\mathbf{P}(E)} D(\rho,\eta)\,\pi(d\rho, d\eta).$$

Remark :$\Pi(\hat F^N, \delta_f) = \{\hat F^N \otimes \delta_f\}$

$$\Rightarrow\ \mathcal{W}_{W_1}(\hat F^N, \delta_f) = \int_{E^N} W_1(\mu^N_X, f)\, F^N(dX) = \mathcal{D}_0(F^N; f)$$

# A third formulation

For any $F, G \in \mathbf{P}(E^j)$ we define the MKW distance $W_p$, $p = 1, 2$, by

$$W_p^p(F,G) := \inf_{\pi \in \Pi(F,G)} \int_{E^j \times E^j} d_{E^j}^p(X,Y)\,\pi(dX,dY)$$

with

$$\Pi(F,G) := \{\pi \in \mathbf{P}(E^j \times E^j);\ \pi(A \times E^j) = F(A),\ \pi(E^j \times B) = G(B)\}$$

$$\begin{aligned} d_{E^j}^p(X,Y) &:= \frac{1}{j} \sum_{i=1}^{j} d_E(x_i,y_i)^p \\ &\geq \inf_{\sigma \in \mathfrak{S}_N} \frac{1}{j} \sum_{i=1}^{j} d_E(x_i, y_\sigma(i))^p = W_p(\mu_X^N, \mu_y^N)^p \end{aligned}$$

**Lemma:** $F^N$ is $f$-chaotic if

(def-3) $\quad W_1(F^N, f^{\otimes N}) \to 0$ when $N \to \infty$

Q4: Are these three definitions equivalent ?

# Distances from chaos

**Theorem ( (Th-3) Equivalence of chaos measures)**

$\forall\, M, \forall\, k > 1 \quad \exists\, \alpha_i,\, C > 0$
$\forall\, f \in \mathbf{P}(E), \forall\, F^N \in \mathbf{P}_{sym}(E^N)$ *with* $M_k(F_1^N), M_k(f) \le M$

$$\forall\, j, k \in \{0, 2, ..., N\} \qquad \mathcal{D}_j \le C\,\left(\mathcal{D}_k^{\alpha_1} + \frac{1}{N^{\alpha_2}}\right).$$

Here

$$\mathcal{D}_j := W_1(F_j^N, f^{\otimes j}), \qquad 1 \le j \le N,$$

$$\mathcal{D}_0 := \mathcal{W}_{W_1}(\hat{F}^N, \delta_f).$$

- For $F^N := f^{\otimes N}$ we find $\mathcal{D}_j = 0,\ 1 \le j \le N$,
but $\mathcal{D}_0 \approx \frac{1}{N^{\frac{1}{d'}}},\ d' = d \vee 2,\ \Leftarrow\ \mathcal{W}_{\|.\|^2_{H^{-s}}} = \frac{C_f}{N}$ (quadratic miracle!)

- from here comes the error term (of worse order)

$$\mathcal{D}_0(F_0^N; f_0) \approx \frac{C}{N^{\frac{1}{d'}}}$$

# About the proof

- $W_1(F_j^N, f^{\otimes j}) \le 2\, W_1(F^N, f^{\otimes N})$ for any $1 \le j \le N$
- for the negative Sobolev norm $\|\cdot\|_{H^{-s}}$, $s > d/2$, we prove (quadratic miracle!)

$$\mathcal{W}_{\|\cdot\|^2_{H^{-s}}}(\hat F^N, \delta_f) \lesssim W_1(F_2^N, f^{\otimes 2}) + \|F_1^N - f\|^2_{H^{-s}} + \frac{1}{N}$$

and we conclude by comparing the distance $W_1$ and the norm $\|\cdot\|_{H^{-s}}$ in $E$

- two steps:

$$W_1^\dagger(F^N, f^{\otimes N}) \overset{Def}{:=} \inf_{\pi \in \Pi} \int_{E^N \times E^N} W_1(\mu_X^N, \mu_Y^N)\, \pi(dX, dY) \overset{Lemma(*)}{=} W_1(F^N, f^{\otimes N})$$

and

$$W_1^\dagger(F^N, f^{\otimes N}) \overset{Lemma}{\approx} \mathcal{W}_{W_1}(\hat F^N, \delta_f).$$

(*) Density argument + when $E$ is finite, we define

$$\pi^*(X,Y) := \frac{\pi(\{(X',Y') \sim (X,Y)\})}{\sharp\{d_N(X',Y') = W_1(\mu_X^N, \mu_Y^N)\}} \text{ if } d_N(X,Y) = W_1(\mu_X^N, \mu_Y^N), \quad := 0 \text{ else.}$$

**Theorem (Poincaré Lemma: chaoticity of the $N$-particle steady states)**

*$\sigma^N$ is $\gamma$–chaotic, and more precisely*

$$W_1(\sigma^N_\ell, \gamma^{\otimes \ell}) \le \|\sigma^N_\ell - \gamma^{\otimes \ell}\|_{TV} \le 32\,\frac{\ell}{N-\ell}, \quad 1 \le \ell \le N-4$$

*and*

$$W_1(\sigma^N, \gamma^{\otimes N}) \le \frac{C_\varepsilon}{N^{1/2-\varepsilon}}$$

- $\sigma^N :=$ steady state for the N-particle system
  . $= \mathrm{meas}(S^{dN-1}(\sqrt{N}))^{-1}\,\delta_{S^{dN-1}(\sqrt{N})} \in \mathbf{P}(E^N)$,
- $\gamma(v) := (2\pi)^{-d/2}\,\exp(-|v|^2/2)$,

$\triangleright$ Diaconis-Freedman and adapt Sznitman

## Theorem (Conditioned tensor product)

*For any* $f \in \mathbf{P}_6(E)$ *with finite Fisher information, there exists* $F^N := [f^{\otimes N}]_{\mathcal{BS}_N} \in \mathbf{P}(\mathcal{BS}_N)$ *such that*

- $$W_1(F^N_\ell, f^{\otimes \ell}) \le \frac{C_\ell}{N^{1/2}}, \qquad 1 \le \ell \le N$$
- $$|H(F^N|\sigma^N) - H(f|\gamma)| \le \frac{C}{N^{1/2}}$$
- $$I(F^N|\sigma^N) \le C$$

Mainly use a strong version of the local LCT or Berry-Esseen Theorem

▷ Kac; Carlen-Carvalho-Loss-LeRoux-Villani;Haury-M.;Carrapataso

# Entropy chaos

Definition. $(F^N)$ is entropy $f$-chaotic if

$$F_1^N \rightharpoonup f \quad \text{weakly in } \mathbf{P}(E) \quad \text{and} \quad H(F^N|\sigma^N) \to H(f|\sigma)$$

**Theorem ( (Th-4) On entropy chaos)**

*(i) Assume $(F^N)$ is entropy $f$-chaotic. Then $(F^N)$ is Kac's $f$-chaotic.*
*(ii) Assume $(F^N)$ is Kac's $f$-chaotic and $I(F^N|\sigma^N)$ is uniformly bounded. Then $(F^N)$ is entropy $f$-chaotic.*

For (ii) we use twice the HWI interpolation of Otto-Villani

$$H(G^N|\sigma^N) \le H(F^N|\sigma^N) + \sqrt{I(G^N|\sigma^N)}\, W_2(G^N, F^N)$$

$\mathcal{BS}_N$ is a weak $CD(K,N)$ geodesic space with positive Ricci curvature $K$ and we may apply Theorem 30.22, Optimal Transport, Old & New, C. Villani.

The HWI inequality is an interpolation inequality which is independent of the dimension and is similar to the usual and Hilbert inequality

$$\|g\|_{L^2} \le \|g\|_{H^1}^{1/2} \, \|g\|_{H^{-1}}^{1/2}$$

# Some conclusions about chaos

- The notion of chaos is close (wider/weaker) to the notion of independence in probability theory. If $V$ is a stochastic variable in $E^N$ such that the coordinates are independent variables and have same law $f \in \mathbf{P}(E)$ then $V \sim f^{\otimes N}$. In the case of chaos the tensorization structure is required only asymptotically when $N \to \infty$.

- The seemingly stronger notion of chaos $W_1(F^N, f^{\otimes N}) \to 0$ and $H(F^N) \to H(f)$ (because they involve *all* of variables) are (surprisingly?)
  - equivalent to Kac's definition of chaos for the first one;
  - has a strong link with Kac's definition of chaos for the second one

# Plan

# Quantitative answer to Kac's problem 1

**Theorem ((Th-5) Uniform in time Kac's chaos convergence)**

$$\sup_{t\in[0,T)} \left| \int_{E^k} \left( F_k^N(t) - f_t^{\otimes k} \right) \varphi \, dV \right| \le \theta(N) \underset{N\to\infty}{\longrightarrow} 0.$$

- $T \in (0, +\infty]$,
- $E = \mathbb{R}^d$, $d = 3$, $V = (v_1, ...., v_N) \in E^N$
- $f_0 = f_{in} \in \mathbf{P}(E)$ with enough moments bounded, $f_t =$ evolution of one typical particle in the mean-field limit, $f_t^{\otimes N}(V) = f_t(v_1) \dots f_t(v_N)$,
- $F_0^N$ is $f_{in}$-chaotic, $F_t^N =$ evolution of N-particle system $\in \mathbf{P}_{sym}(E^N)$,
- $\varphi = \varphi_1 \otimes \dots \otimes \varphi_k$, $\varphi_j \in \mathcal{F} \subset C_b(E)$, ex: $\mathcal{F} = W^{1,\infty}$ or $H^s$,
- $N \ge 2\,k$.

# Main features 2

- Our method is strongly inspired by Grünbaum work (1971) where he claimed he proved convergence result for the hard spheres model. But his proof is definitively wrong ! He essentially recovered the non-constructive convergence result for the Maxwell cut-off model by Kac & McKean.
- We follow, complete and improve Grunbaum's program;
- The underlining philosophy is a numerical analyst intuition: based on (A3) consistency estimate and (A4) stability estimate on the limit PDE and refuse any compactness and probability arguments
  - "consistency error" of order $\mathcal{O}(1/N^{1-\varepsilon})$ $\forall\, \varepsilon \in (0,1)$;
  - "stability error" of order $\mathcal{O}(1/N^{1/2})$, $\sim \mathcal{O}(1/N^{1/d})$ or worst because we write the equation in $\mathbf{P}(\mathbf{P}(\mathbb{R}^3))$ and we use some results from the theory of the concentration of measure (at time $t = 0$): the worse error is made at time $t = 0$ (and then it is not deteriorated by the flow);

# Main features 3

- The $\theta$ function splits into

$$\theta(N) = \theta(k,N) = \underbrace{\theta_1(\varphi,N)}_{\mathcal{O}(1/N)} + \underbrace{\theta_2(\varphi,T,N)}_{\mathcal{O}(1/N^{1-\varepsilon})\,\forall\,\varepsilon} + \underbrace{\theta_3(\varphi,T;F_0^N,f_0)}_{\leq\mathcal{O}(1/N^{1/2})},$$

  - $\theta_2$ is the worst term with respect to $\varphi$;
  - $\theta_3$ is the worst term with respect to $N$ dependence;
  - $\theta_3$ is the only term depending on the initial data;

We split

$$
\begin{aligned}
&\left\langle F_t^N - f_t^{\otimes N}, \varphi \otimes 1^{\otimes N-k} \right\rangle = \\
&= \left\langle F_t^N, \varphi \otimes 1^{\otimes N-k} - R_\varphi(\mu_V^N) \right\rangle \qquad (= T_1) \\
&+ \left\langle F_t^N, R_\varphi(\mu_V^N) \right\rangle - \left\langle F_0^N, R_\varphi(S_t^{NL} \mu_V^N) \right\rangle \qquad (= T_2) \\
&+ \left\langle F_0^N, R_\varphi(S_t^{NL} \mu_V^N) \right\rangle - \left\langle f_t^{\otimes k}, \varphi \right\rangle \qquad (= T_3)
\end{aligned}
$$

where $R_\varphi$ is the "polynomial function" on $\mathbf{P}(\mathbb{R}^3)$ defined by

$$
R_\varphi(\rho) = \int_{E^k} \varphi \, \rho(dv_1) \ldots \rho(dv_k)
$$

and $S_t^{NL}$ is the nonlinear semigroup associated to the nonlinear mean-field limit equation by $g_0 \mapsto S_t^{NL} g_0 := g_t$.

$$
\begin{aligned}
|T_1| &= \left| \left\langle F^N_t, \varphi \otimes 1^{\otimes(N-k)}(V) - R_\varphi(\mu^N_V) \right\rangle \right| \\
&= \left| \left\langle F^N_t, \varphi \otimes \widetilde{1^{\otimes(N-k)}}(V) - R_\varphi(\mu^N_V) \right\rangle \right| \\
&\leq \left\langle F^N_t, \frac{2\,k^2}{N} \|\varphi\|_{L^\infty(E^k)} \right\rangle = \frac{2\,k^2}{N} \|\varphi\|_{L^\infty(E^k)} \\
&\leq \frac{2\,k^3}{N} \|\nabla\varphi\|_{L^\infty(E^k)} \, M_1(F^N_1(t)),
\end{aligned}
$$

where we use that $F^N$ is symmetric and a probability and we introduce the symmetrization function associated to $\varphi \otimes 1^{\otimes(N-k)}$ by

$$
\varphi \otimes \widetilde{1^{\otimes(N-k)}}(V) = \frac{1}{\sharp \mathfrak{S}_N} \sum_{\sigma \in \mathfrak{S}_N} \varphi \otimes 1^{\otimes(N-k)}(V_\sigma).
$$

$$
\begin{aligned}
|T_3| &= \left| \left\langle F_0^N, R_\varphi(S_t^{NL}\mu_V^N) \right\rangle - \left\langle (S_t^{NL} f_0)^{\otimes k}, \varphi \right\rangle \right| \\
&= \left| \left\langle F_0^N, R_\varphi(S_t^{NL}\mu_V^N) - R_\varphi(S_t^{NL} f_0) \right\rangle \right| \\
&\le [R_\varphi]_{C^{0,1}} \left\langle F_0^N, W_1(S_t^{NL}\mu_V^N, S_t^{NL} f_0) \right\rangle \\
&\le k \, \|\nabla \varphi\|_{L^\infty(E^k)} \, C_T \left\langle F_0^N, W_1(\mu_V^N, f_0) \right\rangle \\
&\le k \, \|\nabla \varphi\|_{L^\infty(E)} \, C_T \, \mathcal{W}_{W_1}(\hat F_0^N, \delta_{f_0})
\end{aligned}
$$

where

$$[R_\varphi]_{C^{0,1}} := \sup_{W_1(\rho,\eta)\le 1} |R_\varphi(\eta) - R_\varphi(\rho)| = k \, \|\nabla \varphi\|_{L^\infty}$$

and we assume that the nonlinear flow satisfies

$$(A5) \qquad W_1(f_t, g_t) \le C_T \, W_1(f_0, g_0) \quad \forall \, f_0, g_0 \in \mathbf{P}(E)$$

$T_2$ : We write

$$T_2 \;=\; \left\langle F_t^N, R_\varphi(\mu_V^N)\right\rangle - \left\langle F_0^N, R_\varphi(S_t^{NL}\mu_V^N)\right\rangle$$

$T_2$ : We write

$$T_2 = \Big\langle F_t^N, R_\varphi(\mu_V^N) \Big\rangle - \Big\langle F_0^N, R_\varphi(S_t^{NL}\mu_V^N) \Big\rangle$$
$$= \Big\langle F_0^N, T_t^N(R_\varphi \circ \mu_V^N) - (T_t^\infty R_\varphi)(\mu_V^N) \Big\rangle$$

with

- $T_t^N =$ dual semigroup (acting on $C_b(E^N)$) of the N-particle flow $F_0^N \mapsto F_t^N$;
- $T_t^\infty =$ pushforward semigroup (acting on $C_b(\mathbf{P}(E))$) of the nonlinear semigroup $S_t^{NL}$ defined by $(T^\infty \Phi)(\rho) := \Phi(S_t^{NL}\rho)$;

$T_2$ : We write

$$\begin{aligned} T_2 &= \left\langle F_t^N, R_\varphi(\mu_V^N)\right\rangle - \left\langle F_0^N, R_\varphi(S_t^{NL}\mu_V^N)\right\rangle \\ &= \left\langle F_0^N, T_t^N(R_\varphi \circ \mu_V^N) - (T_t^\infty R_\varphi)(\mu_V^N)\right\rangle \\ &= \left\langle F_0^N, (T_t^N \pi_N - \pi_N T_t^\infty)\, R_\varphi\right\rangle \end{aligned}$$

with

- $T_t^N =$ dual semigroup (acting on $C_b(E^N)$) of the N-particle flow $F_0^N \mapsto F_t^N$;
- $T_t^\infty =$ pushforward semigroup (acting on $C_b(\mathbf{P}(E))$) of the nonlinear semigroup $S_t^{NL}$ defined by $(T^\infty \Phi)(\rho) := \Phi(S_t^{NL}\rho)$;
- $\pi_N =$ projection $C(\mathbf{P}(E)) \to C(E^N)$ defined by $(\pi_N \Phi)(V) = \Phi(\mu_V^N)$.

$$\begin{aligned} T_2 &= \left\langle F_0^N, (T_t^N \pi_N - \pi_N T_t^\infty)\, R_\varphi \right\rangle \\ &= \left\langle F_0^N, \int_0^T T_{t-s}^N \, (\Lambda^N \pi_N - \pi_N \Lambda^\infty)\, T_s^\infty \, ds \, R_\varphi \right\rangle \\ &= \int_0^T \left\langle F_{t-s}^N, (\Lambda^N \pi_N - \pi_N \Lambda^\infty)\, (T_s^\infty \, R_\varphi) \right\rangle \, ds \end{aligned}$$

where

- $\Lambda^N$ is the generator associated to $T_t^N$ and $\Lambda^\infty$ is the generator associated to $T_t^\infty$.

Now we have to make some assumptions

- (A1) $F_t^N$ has enough bounded moments;
- (A2) $\Lambda^\infty \Phi(\rho) = \langle Q(\rho), D\Phi(\rho) \rangle$;
- (A3) $(\Lambda^N \pi^N \Phi)(V) = \langle Q(\mu_V^N), D\Phi(\mu_V^N) \rangle + \mathcal{O}([\Phi]_{C^{1,a}}/N)$
- (A4) $S_t^{NL} \in C^{1,a}(\mathbf{P}(E); \mathbf{P}(E))$ "uniformly" in time $t \in [0, T]$

"Differential calculus" on $\mathbf{P}(E)$ :
- see $\mathbf{P}(E)$ as an embedded manifold of $\mathcal{F}'$, $\mathcal{F} \subset UC_b(E)$,
- expansion of $\Phi$ up to order $1+a$ in each point
- much more simpler that the "differential calculus" developed in "gradient flow theory"

$\Phi \in C^{1,a}(\mathbf{P}(E); \mathbb{R})$ if $\Phi \in C(\mathbf{P}(E))$ and $\exists D\Phi : \mathbf{P}(E) \to C(E)$

$$\forall \, \mu, \; \nu \in \mathbf{P}(E) \quad \Big| \Phi(\nu) - \Phi(\mu) - \langle \nu - \mu, D\Phi[\mu] \rangle \Big| \le C \, \| \nu - \mu \|_{TV}^{1+a}.$$

We define

$$[\Phi]_a = \sup_{\mu,\nu \in \mathbf{P}(E)} \frac{\Big| \Phi(\nu) - \Phi(\mu) - \langle \nu - \mu, D\Phi[\mu] \rangle \Big|}{\| \nu - \mu \|_{TV}^{1+a}}.$$

Remark. For any $\varphi \in W^{2,\infty}(E^k)$, $R_\varphi \in C^{1,1}(\mathbf{P}(E))$ and

$$[R_\varphi]_1 \le k^2 \, \|\varphi\|_{W^{2,\infty}(E^k)}.$$

$$\begin{aligned}
T_2 &\le \int_0^T M_0(F^N_{t-s}) \, \|(\Lambda^N \pi_N - \pi_N \Lambda^\infty)\,(T^\infty_s \, R_\varphi)\|_{L^\infty(E^N)} \, ds \\
&\overset{(A3)}{\le} \int_0^T \frac{C}{N} \, [T^\infty_s \, R_\varphi]_{C^{1,a}} \, ds \\
&\le \frac{C}{N} \int_0^T [R_\varphi \circ S^{NL}_t]_{C^{1,a}} \, ds \\
&\le \frac{C}{N} \int_0^T [R_\varphi]_{C^{1,1}} \, [S^{NL}_t]_{C^{1,a}} \, ds \\
&\le \frac{C}{N} \, k^2 \, \|\varphi\|_{W^{2,\infty}} \int_0^T [S^{NL}_t]_{C^{1,a}} \, ds
\end{aligned}$$

## A possible conclusion is :

$$\left\langle F_k^N(t) - f(t)^{\otimes k}, \varphi \right\rangle \le$$
$$\le C_k \left( \frac{\|\nabla\varphi\|_{L^\infty}}{N} + C_T^{(A4)} \frac{\|\varphi\|_{W^{2,\infty}}}{N^a} + C_T^{(A5)} \|\nabla\varphi\|_{L^\infty} \mathcal{W}_{W_1}(\hat{F}_0^N, \delta_{f_0}) \right)$$

and

$$\sup_{[0,T)} \sup_{\|\varphi\|_{W^{2,\infty}} \le 1} \left\langle F_k^N(t) - f(t)^{\otimes k}, \varphi \right\rangle \le$$
$$\le C_k \left( \frac{1}{N} + \frac{C_T^{(A4)}}{N^a} + C_T^{(A5)} \mathcal{W}_{W_1}(\hat{F}_0^N, \delta_{f_0}) \right)$$

with $T = \infty$ if

$$\sup_{t \ge 0} [S_t^{NL}]_{C^{0,1}_{W_1}} + \int_0^\infty [S_t^{NL}]_{C^{1,a}_{TV}} \, dt < \infty.$$

(A2) The nonlinear semigroup $S_t^{NL}$ and operator $Q$ are $C^{0,a}$ for the total variation norm. As a consequence $\forall\,\Phi \in C^{1,a}(\mathbf{P}(E))$, $\forall f_0 \in \mathbf{P}_2(E)$

$$\begin{aligned}
(\Lambda^\infty \Phi)(f_0) &= \frac{d}{dt}(T_t^\infty \Phi)(f_0)|_{t=0} = \frac{d}{dt}\Phi(f_t)|_{t=0} = \lim_{t\to 0} \frac{\Phi(f_t)-\Phi(f_0)}{t} \\
&= \lim_{t\to 0} \left\{ \left\langle \frac{f_t - f_0}{t}, D\Phi[f_0] \right\rangle + \mathcal{O}\left( \frac{\|f_t - f_0\|_{TV}^{1+a}}{t} \right) \right\} \\
&= \left\langle \frac{df_t}{dt}|_{t=0}, D\Phi[f_0] \right\rangle = \langle Q(f_0), D\Phi(f_0) \rangle
\end{aligned}$$

(A3) Consistency: $\forall\,\Phi \in C^{1,a}(\mathbf{P}(E))$, set $\phi = D\Phi[\mu_V^N]$, and compute

$$\begin{aligned}
\Lambda^N(\Phi \circ \mu_V^N) &= \frac{1}{2N} \sum_{i,j=1}^N B(v_i - v_j) \int_{S^2} b\,[\Phi(\mu^N_{V'_{ij}}) - \Phi(\mu^N_V)]\, d\sigma \\
&= \frac{1}{2N} \sum_{i,j} B(v_i - v_j) \int_{S^2} b\, \langle \mu^N_{V'_{ij}} - \mu^N_V, \phi \rangle\, d\sigma \qquad = \langle Q(\mu_V^N), \phi \rangle \\
&+ \frac{1}{2N} \sum_{i,j} B(v_i - v_j) \int_{S^2} \mathcal{O}(\|\mu^N_{V'_{ij}} - \mu^N_V\|_{TV}^{1+a})\, d\sigma \quad = \mathcal{O}(1/N^a)
\end{aligned}$$

(A4) The Boltzmann flow $S_t^{NL}$ is $C^{1,a}$ in total variation norm:
$\forall\, \rho \in \mathbf{P}_k(\mathbb{R}^d)$, $\forall\, t \geq 0$ there exists $\mathcal{L}_t[\rho] \in C(\mathbb{R}^3)$ $\forall\, \eta \in \mathbf{P}_k(\mathbb{R}^d)$

$$\begin{aligned} S_t^{NL}(\eta) &= S_t^{NL}(\rho) + \mathcal{L}_t[\rho](\eta - \rho) + \mathcal{O}(\,\|\eta - \rho\|_{TV}^{1+a}) \\ &= S_t^{NL}(\rho) + \mathcal{L}_t[\rho](\eta - \rho) + \mathcal{O}(e^{-\lambda\, t}\,\|\eta - \rho\|_{TV}^{1+a}) \end{aligned}$$

(A5) The Boltzmann flow $S_t^{NL}$ is $C^{0,1}$ in weak distance (Tanaka, Toscani-Villani, Fournier-Mouhot): $\forall\, \rho, \eta \in \mathbf{P}_k(\mathbb{R}^d)$, $\forall\, t \geq 0$ there holds

$$\begin{aligned} W_1(S_t^{NL}(\eta), S_t^{NL}(\rho)) &\leq C_T\, W_1(\eta, \rho) \\ &\leq \Omega(W_1(\eta, \rho)) \quad \text{uniform in time} \end{aligned}$$

# Plan

## Proof of Theorem 2 part 1 : triangular inequality

• (a) On the one hand, we know from (4) (Kac, Janvresse, Carlen, Loss) that

$$\forall\, N \ge 1 \qquad W_1(F^N(t), \sigma^N) \;\le\; \|h^N \sigma^N - \sigma^N\|_{TV} \;\le\; \|h^N - 1\|_{L^2(\sigma^N)} \le A^N\, e^{-\delta\, t}, \quad A > 1.$$

• (b) On the other hand, Theorem 1 and 2 write (for $N \ge 1$)

$$\sup_{[0,\infty)} W_1(F^N(t), f(t)^{\otimes N}) + W_1(\sigma^N, \gamma^{\otimes N}) \le \theta(N) \underset{N\to\infty}{\longrightarrow} 0.$$

• (c) We recall from (5) that

$$W_1(f(t)^{\otimes N}, \gamma^{\otimes N}) \le \|f_t - \gamma\|_{L^1_1} \le C_{f_0}\, e^{-\lambda\, t}.$$

• (d) Gathering estimates (b) and (c), we get

$$\forall\, N \ge 1 \qquad W_1(F^N(t), \sigma^N(t)) \le \theta(N) + C_{f_0}\, e^{-\lambda\, t}$$

• (e) As a consequence of (a) and (d) we obtain the uniform (with respect to $N$) convergence:

$$W_1(F^N(t), \sigma^N(t)) \le \min\left(2\,\theta(N) + C_{f_0}\, e^{-\lambda\, t}, A^N\, e^{-\delta\, t}\right) \underset{t\to\infty}{\longrightarrow} 0$$

(choose (a) if $\varepsilon\, t \ge N$ and (d) if $\varepsilon\, t \le N$).

# Proof of Theorem 2 part 2 : interpolation inequality

- First, for **(MG)** we show that the relative Fisher information is decreasing :

$$I(F_t^N|\sigma^N) := \frac{1}{N}\int_{S_N} \frac{|\nabla h_t^N|^2}{h_t^N}\, d\sigma^N \le I(F_0^N|\sigma^N)$$

- Next, we use the HWI inequality to get

$$\begin{aligned}\sup_{N\ge1} H(F_t^N|\sigma^N) &\le \sup_{N\ge1} I(F_t^N|\sigma^N)\, W_2(F_t^N,\sigma^N)\\ &\le C\, I(F_0^N|\sigma^N)\, \sup_{N\ge1} W_1(F_t^N,\sigma^N)^{1/2} \to 0 \quad \text{as} \quad t\to\infty;\end{aligned}$$

- First, for **(MG)** we show that the relative Fisher information is decreasing :

$$I(F_t^N|\sigma^N) := \frac{1}{N}\int_{S_N} \frac{|\nabla h_t^N|^2}{h_t^N}\, d\sigma^N \le I(F_0^N|\sigma^N)$$

- and we use also the HWI inequality and Carrapataso's Theorem to get

$$\begin{aligned}
|H(F_t^N|\sigma^N) - H(f_t|\gamma)| &\le |H(F_t^N|\sigma^N) - H([f_t^{\otimes N}]_{\mathcal{BS}_N}|\sigma^N)| \\
&\quad + |H([f_t^{\otimes N}]_{\mathcal{BS}_N}|\sigma^N) - H([f_t^{\otimes N}]_{\mathcal{BS}_N}|\gamma)| \\
&\le \{I(F_t^N|\sigma^N) + I([f_t^{\otimes N}]_{\mathcal{BS}_N}|\sigma^N)\}\, W_2(F_t^N, [f_t^{\otimes N}]_{\mathcal{BS}_N}) \\
&\quad + C(I(f_t|\gamma))/N^{1/2} \\
&\le C(I(f_0|\gamma), I(F_0^N|\sigma^N)) \\
&\quad \left\{\frac{1}{N^{1/2}} + W_2(F_t^N, f_t^{\otimes N}) + W_2(f_t^{\otimes N}, [f_t^{\otimes N}]_{\mathcal{BS}_N})\right\} \\
&\le C/N^\alpha, \quad \alpha \in (0,1)
\end{aligned}$$

# Plan

# Concluding remarks

We have proved a quantified version of chaos propagation which is furthermore uniform in time (for the Boltzmann model)

That result can be seen as a "quantitative version" of BBGKY hierarchy method

First ingredient to estimate the convergence of $T_t^N \pi^N$ to $\pi^N \, T_t^\infty$ as operators acting from $C(\mathbf{P}(E))$ with values in $C(E^N)$ which is a consequence of

- a stability result (expansion of order $> 1$) for the nonlinear semigroup
- consistency result on the associated generators

That requires to develop a "differential calculus" on $\mathbf{P}(E)$ seen as an embedded manifold of $\mathcal{F}'$, $\mathcal{F} \subset UC_b(E)$

Second ingredient equivalent formulations of Kac chaos and interpolation independent of the dimension (HWI)

# Open problems

- $T = +\infty$ with optimal rate $\theta(N) = \mathcal{O}(N^{-1/2})$;

- more general cross-section (true hard or soft potential) and Landau equation;

- Vlasov equation and McKean-Vlasov equation with singular interactions;

- (quantitative) propagation of entropy chaos $\sup_{[0,T]} H(F_t^N | f_t^{\otimes}) \le \theta_H(N)$;

- rate of convergence to equilibrium for the nonlinear PDE from the analysis of the N-particle system dynamic

- for the inelastic Boltzmann equation + diffuse excitation can we deduce from the $N \to \infty$ limit

$$\frac{d}{dt} H(f(t)|g) \le 0$$

where $g$ stands for the unique steady state? Or what is the limit of the entropies sequence as $N \to \infty$