Chaos and Statistical solutions

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Boltzmann equation: mathematics, modeling and simulations
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Outlines of the talk

Plan

Aim of the talk

The aim of the talk is to present some remarks about

- "quantitative chaos"
- the statistical solutions of a BBGKY hierarchy
- and their relations with some recent result about "quantitative and uniform in time propagation of chaos".

Underlying problem: How to derive rigorously mesoscopic/statistic dynamics (Boltzmann and Vlasov equations) from microscopic dynamics (Newton first law of motion) ?

the results are taken from

- M., Mouhot, Wennberg, “A new approach to quantitative chaos propagation estimates for drift, diffusion and jump processes”, arxiv 2011
- M., Mouhot, “Quantitative uniform in time chaos propagation for Boltzmann collision processes”, arxiv 2010
- M. “Introduction aux limites de champs moyen pour les systèmes de particules” (graduate school notes)
- M. "Programme de Kac sur les limites de champ moyen", EDP-X seminary publication
- Hauray, M., Mouhot, work in progress

Very short historical introduction

- Newton - Philosophiæ Naturalis Principia Mathematica (XVII century)
- Maxwell and Boltzmann - Boltzmann equation (XIX century)
- Hilbert's sixth problem (ICM 1900 Paris):
- Grad $\sim$ 1950 : Formal derivation of the nonhomogeneous Boltzmann equation from deterministic dynamic (= "Boltzmann-Grad" limit)
- Kac (1959) : space homogeneous Kac-Boltzmann equation as the mean-field limit of a N-particle Markov jump process
- Lanford (1973) : Rigorous proof of the "Boltzmann-Grad" limit for very short time. Idea: use Bogoliubov (or BBGKY) hierarchy
- Sznitman (1984) : Kac's program for hard spheres Boltzmann model
- Hauray, Jabin (2007) : Rigorous derivation of the Vlasov equation for (not too) singular interaction potential

Main open problems in the "mean-field limit theory"

- Derive the Vlasov-Poisson equation from Newton first principle (N particles evolve according to Hamiltonian dynamic associated to Coulombian potential) in the "mean-field" limit
- Derive the nonhomogeneous Boltzmann equation from Newton first principle (N particles evolve according to deterministic Hamiltonian dynamic) in the "Boltzmann-Grad" limit for large time
- Achieve the Kac's program

Derive the (space homogeneous) Boltzmann equation from a jump (collisional) process. First rigorous mathematic treatment of the deduction of Boltzmann equation from microscopic dynamics.

Kac introduced the notion of chaos

Kac stressed two open questions

- Hard spheres model: "The above proof suffers from the defect that it works only if the restriction on time is independent of the initial distribution. It is therefore inapplicable to the physically significant case of hard spheres because in this case our simple estimates yield a time restriction which depends on the initial distribution. A general proof that Boltzmann's property propagates in time is still lacking"

$\rightarrow$ proved by Sznitman 1984 (nonlinear martingale approach, compactness and uniqueness arguments)

- Uniform spectral gap: Deduce spectral gap/exponential trend to equilibrium for the nonlinear Boltzmann eq from the spectral gap for the family of Master eqs

$\rightarrow$ proved by a direct way by Mouhot 2006 (using : linearized L^2 spectral gap Grad 63; L^1 moments Povzner 1965, quantitative H-theorem: Carlen, Carvalho 1992)

Bibliography

- Equations: Maxwell 1867, Boltzmann 1871, Vlasov 1938
- Hierarchy: Bogoliubov 1946, Kirkwood, Born, Green
- Kac program 1951-1975: Wild, Kac, McKean, Tanaka, Grünbaum
- Chaos to Maxwell function: Mehler 1866, Poincaré Lemma, Borel 1925, Sznitman 1989
- Grad limit: Grad 1958, Lanford, King, Illner, Pulvirenti, Cercignani 1994
- Mean field Vlasov limit: Neunzert, Wick 1971, Braun, Hepp 1977, Dobrushin 1979, Spohn, 1991, Hauray, Jabin 2007
- Probability approach: Sznitman 1989, Méléard, Graham, Fournier, Guérin, Malrieu, Villani, Bolley, Guillin
- Uniform spectral gap: Kac, Janvresse 2001, Carlen, Carvalho, Loss, Maslen, Villani, Lieb, Gernimo, Le Roux

More Bibliography on Boltzmann

Propagation of chaos for Maxwell molecules

- M. Kac, *Foundation of kinetic theory* (1956), *Some probabilistic aspects of the Boltzmann equation (1973)*
- H.P. McKean, *An exponential formula for solving Boltmann's equation for a Maxwellian gas* (1967)
- H. Tanaka, *Propagation of chaos for certain Markov processes of jump type with nonlinear generators*, Proc. Japan Acad (1969)
- R. Peyre, *Some ideas about quantitative convergence of collision models to their mean eld limit*, JSP (2009)

Propagation of chaos for hard spheres cross-section

- A.F. Grunbaum, *Propagation of chaos for the Boltzmann equation*, ARMA (1971)
- A.-S. Sznitman, *Equations de type de Boltzmann, spatialement homogènes*, ZWVG (1984)

Plan

Kac's definition of chaos

E = a locally compact polish space ($E = \mathbb{R}^d$)
$\mathbf{P}(E)$ = the space of probability measures
$\mathbf{P}_{sym}(E^N)$ = probabilities which are invariant under indexes permutations.

A sequence $F^N \in \mathbf{P}_{sym}(E^N)$ is f-chaotic, $f \in \mathbf{P}(E)$, iff

$$\forall\, \varphi_1, ..., \varphi_j \in C_b(E) \qquad \int_{E^N} \varphi_1 \otimes ... \otimes \varphi_j \, F^N(dX) \to \prod_{i=1}^{j} \int_E \varphi_i \, f$$

or equivalently

$$\boxed{\text{(def-1)} \qquad \forall\, j \ge 1 \qquad F_j^N \rightharpoonup f^{\otimes j} \text{ weakly in } \mathbf{P}(E^j),}$$

where F_j^N stands for the j-th marginal of F^N defined by

$$F_j^N := \int_{E^{N-j}} F^N \, dx_{j+1} \dots dx_N.$$

Alternative formulation

To any $F^N \in \mathbf{P}_{sym}(E^N)$ we may associate $\hat F^N \in \mathbf{P}(\mathbf{P}(E))$ by setting

$$\forall\, \Phi \in C_b(\mathbf{P}(E)) \qquad \langle \hat F^N, \Phi \rangle = \int_{E^N} \Phi(\mu^N_X)\, F^N(dX),$$

where the empirical measure μ^N_X is defined by

$$X = (x_1, ..., x_N) \in E^N \;\mapsto\; \mu^N_X := \frac{1}{N} \sum_{i=1}^N \delta_{x_i} \in \mathbf{P}(E).$$

Lemma: F^N is f-chaotic iff

(def-2) $\quad \hat F^N \rightharpoonup \delta_f$ weakly in $\mathbf{P}(\mathbf{P}(E))$

It is (for instance) a consequence of Hewitt-Savage theorem:

$\Updownarrow$
- $(\pi_j)_{j \ge 1} \in \mathbf{P}_{sym}(E^j)$ compatible, i.e. $\pi_{j+1}|_{E^j} = \pi_j$
- $\hat\pi \in \mathbf{P}(\mathbf{P}(E))$,

by setting $\pi_j := \displaystyle\int_{\mathbf{P}(E)} \rho^{\otimes j}\, \hat\pi(d\rho)$.

A third formulation

For any $F, G \in \mathbf{P}(E^j)$ we define the MKW distance W_p, $p = 1, 2$, by

$$W_p^p(F,G) := \inf_{\pi \in \Pi(F,G)} \int_{E^j \times E^j} d_j^p(X,Y)\, \pi(dX, dY)$$

with

$$\Pi(F,G) := \{\pi \in \mathbf{P}(E^j \times E^j);\ \pi(A \times E^j) = F(A),\ \pi(E^j \times B) = G(B)\}$$

$$\begin{aligned} d_j^p(X,Y) &:= \frac{1}{j} \sum_{i=1}^{j} d_E(x_i, y_i)^p \\ &\geq \inf_{\sigma \in \mathfrak{S}_N} \frac{1}{j} \sum_{i=1}^{j} d_E(x_i, y_\sigma(i))^p = W_p(\mu_X^N, \mu_y^N)^p \end{aligned}$$

Lemma: F^N is f-chaotic if

(def-3) $\qquad W_1(F^N, f^{\otimes N}) \to 0$ when $N \to \infty$

Are these three definitions equivalent ?

A positive answer

Theorem ((I-1) Equivalence of chaos measures)

$\forall\, M, \forall\, k > 1 \quad \exists\, \gamma_i,\ C > 0$
$\forall\, f \in \mathbf{P}(E), \forall\, F^N \in \mathbf{P}_{sym}(E^N)$ *with* $M_k(F_1^N), M_k(f) \le M$

$$\forall\, j,k \in \{0,2,...,N\} \qquad \mathcal{D}_j \le C\,\left(\mathcal{D}_k^{\gamma_1} + \frac{1}{N^{\gamma_2}}\right).$$

Here

$$\mathcal{D}_j := W_1(F_j^N, f^{\otimes j}), \qquad 1 \le j \le N,$$
$$\mathcal{D}_0 := \mathcal{W}_{W_1}(\hat F^N, \delta_f)$$

where for $\alpha, \beta \in \mathbf{P}(\mathbf{P}(E))$ and D a distance on $\mathbf{P}(E)$ we define

$$\mathcal{W}_D(\alpha,\beta) := \inf_{\pi \in \Pi(\alpha,\beta)} \int_{\mathbf{P}(E)\times\mathbf{P}(E)} D(\rho,\eta)\,\pi(d\rho,d\eta).$$

Remark 1: $\Pi(\hat F^N, \delta_f) = \{\hat F^N \otimes \delta_f\} \;\Rightarrow\; \mathcal{W}_D(\hat F^N, \delta_f) = \int_{E^N} D(\mu^N_X, f)\,F^N(dX)$.
Remark 2: For $F^N := f^{\otimes N}$ we find $\mathcal{D}_j = 0,\ 1 \le j \le N$,
but $\mathcal{D}_{N+1} \approx \frac{1}{N^{\frac{1}{d'}}},\ d' = d \vee 2,\ \Leftarrow\ \mathcal{W}_{\|.\|^2_{H^{-s}}} = \frac{C_f}{N}$ (quadratic miracle!)

About the proof

• $W_1(F_j^N, f^{\otimes j}) \le 2\, W_1(F^N, f^{\otimes N})$ for any $1 \le j \le N$

• for the negative Sobolev norm $\|\cdot\|_{H^{-s}}$, $s > d/2$, we prove (quadratic miracle again!)

$$\mathcal{W}_{\|\cdot\|^2_{H^{-s}}}(\hat F^N, \delta_f) \lessapprox W_1(F_2^N, f^{\otimes 2}) + \|F_1^N - f\|^2_{H^{-s}} + \frac{1}{N}$$

and we conclude by comparing the distance W_1 and the norm $\|\cdot\|_{H^{-s}}$ in E

• two steps:

$$W_1^\dagger(F^N, f^{\otimes N}) \overset{Def}{:=} \inf_{\pi\in\Pi} \int_{E^N\times E^N} W_1(\mu^N_X, \mu^N_Y)\, \pi(dX, dY) \overset{Lemma(*)}{=} W_1(F^N, f^{\otimes N})$$

and

$$W_1^\dagger(F^N, f^{\otimes N}) \overset{Lemma}{\approx} \mathcal{W}_{W_1}(\hat F^N, \delta_f).$$

(*) Density argument + when E is finite, we define

$$\pi^*(X,Y) := \frac{\pi(\{(X',Y') \sim (X,Y)\})}{\sharp\{d_N(X',Y') = W_1(\mu^N_X, \mu^N_Y)\}} \text{ if } d_N(X,Y) = W_1(\mu^N_X, \mu^N_Y), \quad := 0 \text{ else.}$$

Entropic chaos - a definition

Definition: $F^N \in \mathbf{P}_{sym}(E^N)$ is entropic f-chaotic, $f \in \mathbf{P}(E)$, if

- F^N is (weakly) f-chaotic (in the sense of Kac)
- $H(F^N) \to H(f)$ when $N \to \infty$

Here the entropy $H(G)$ of $G \in \mathbf{P}_{sym}(E^j)$ is defined by

$$H(G) \overset{Def}{:=} \frac{1}{j} \int_{E^j} G \log G.$$

Notice that if F^N is f-chaotic, then

$$H(f) \leq \liminf H(F^N).$$

Entropic chaos - Another definition by

[CCLLV] Carlen, Carvalho, Loss, Le Roux, Villani, Kinet. Relat. Models (2010)

Definition′: Assume $E = \mathbb{R}$ and define $\sigma^N :=$ uniform probability on $S^{N-1}(\sqrt{N})$ the sphere of $\mathbb{R}^N$ of radius $\sqrt{N}$.
We say that $F^N \in \mathbf{P}_{sym}(E^N)$ with $\operatorname{supp} F^N \subset S^{N-1}(\sqrt{N})$ is entropic f-chaotic, $f \in \mathbf{P}(E)$, if

- F^N is (weakly) f-chaotic (in the sense of Kac)
- $H(F^N|\sigma^N) \to H(f|\gamma)$ when $N \to \infty$

Here the relative entropy $H(g|G)$ of $g, G \in \mathbf{P}_{sym}(E^j)$ is defined by

$$H(g|G) \overset{Def}{:=} \frac{1}{j} \int_{E^j} \frac{g}{G} \log \frac{g}{G}\, G$$

where g/G stands for the Radon-Nykodym derivative of g with respect to G, and γ is the normalized Gaussian

$$\gamma(dx) = \gamma(x)\, dx := \frac{e^{-x^2/2}}{\sqrt{2\pi}}\, dx.$$

A sufficient condition of entropic chaos in E^N

Theorem ((I-2) Fisher bound condition for entropic chaos)

Consider (F^N) a sequence of $\mathbf{P}_{sym}(E^N)$. Then
(i) *F^N is weakly f-chaotic;*
(ii) *$I(F^N)$ is bounded;*
(iii) *F_1^N is bounded in $\mathbf{P}_k(E)$, $k > 2$;*

$$\Rightarrow \qquad F^N \text{ is entropic } f\text{-chaotic}: \quad H(F^N) \to H(f).$$

Here the Fisher information $I(G)$ of $G \in \mathbf{P}_{sym}(E^j)$ is defined by

$$I(G) \overset{Def}{:=} \frac{1}{j} \int_{E^j} \frac{|\nabla G|^2}{G}$$

Proof. Use the HWI inequality

$$H(F^N|\gamma^{\otimes N}) \le H(f^{\otimes N}|\gamma^{\otimes N}) + W_2(F^N, f^{\otimes N}) \sqrt{I(F^N|\gamma^{\otimes N})}$$

with

$$I(F^N|\gamma^{\otimes N}) = \frac{1}{N} \int_{E^N} \left|\nabla \log \frac{F^N}{\gamma^{\otimes N}}\right|^2 \gamma^{\otimes N} \le C \quad \text{and} \quad W_2(F^N, f^{\otimes N}) \to 0.$$

A sufficient condition of entropic chaos in $S^{N-1}(\sqrt{N})$

Theorem ((I-3) Fisher bound condition for entropic chaos)

Consider (F^N) *a sequence of* $\mathbf{P}_{sym}(E^N)$, $E = \mathbb{R}$, *with* $supp\, F^N \subset S^{N-1}(\sqrt{N})$.
Then
(i) F^N *is weakly* f*-chaotic;*
(ii) $I(F^N|\sigma^N)$ *is bounded;*
(iii) F_1^N *is bounded in* $\mathbf{P}_4(E)$*;*

$$\Rightarrow \qquad F^N \text{ is entropic } f\text{-chaotic, i.e.} \quad H(F^N|\sigma^N) \to H(f|\gamma).$$

Same proof. Remark that the Ricci curvature of $S^{N-1}(\sqrt{N})$ is $K = (N-1)/N \geq 0$ and use HWI inequality in weak $CD(K, N)$ geodesic space (Theorem 30.22, Optimal Transport, Old & New, C. Villani)

Partial answer to Open Problem 11 in [CCLLV]

Theorem ((I-4) relative entropic chaos)

Consider (F^N) *a sequence of* $\mathbf{P}_{sym}(E^N)$ *and* $f \in \mathbf{P}(E)$, $E = \mathbb{R}^d$. *Then*
(i) F^N *is weakly* f*-chaotic;*
(ii) $I(F^N)$ *is bounded;*
(iii) F_1^N, f *bounded in* $\mathbf{P}_k(E)$, $k > 2$*;*
(iii) $I(f) < \infty$, $D^2(-\log f) \geq K \in \mathbb{R}$, $|\nabla \log f| \leq C \langle v \rangle^{k/2}$*;*

$\Rightarrow$ F^N *is relative entropic* f*-chaotic, i.e.* $H(F^N|f^{\otimes N}) \to 0$.

Similar proof. Use the HWI inequality

$$H(F^N|f^{\otimes N}) \leq H(f^{\otimes N}|f^{\otimes N}) + W_2(F^N, f^{\otimes N}) \sqrt{I(F^N|f^{\otimes N})} + (K_-)\, W_2(F^N, f^{\otimes N})^2$$

so that

$$\limsup_{N \to \infty} H(F^N|f^{\otimes N}) \leq 0.$$

Some conclusions about chaos

- The notion of chaos is close (wider) to the notion of independence in probability theory. If V is a stochastic variable in E^N such that the coordinates are independent variables and have same law $f \in \mathbf{P}(E)$ then $V \sim f^{\otimes N}$. In the case of chaos the tensorization structure is required only asymptotically when $N \to \infty$.
- The seemingly stronger notion of chaos $W_1(F^N | f^{\otimes N}) \to 0$ and $H(F^N) \to H(f)$ (because they involve *all* of variables) are (surprisingly?)
  - equivalent to Kac's definition of chaos for the first one;
  - has a strong link with Kac's definition of chaos for the second one.

Plan

N-particle system / Mean-field limit

The N-particle system is described by

- $Y(t) \in E^N$ deterministic/stochatstic trajectories $\leftrightarrow$ $\mu^N_{Y(t)} \in \mathbf{P}(E)$;
- $F^N(t,\cdot) \in \mathbf{P}_{sym}(E^N)$ the law of Y,
  $\mathbf{P}_{sym}(E^N) \approx$ undistinguishable particles;
  $\partial_t F^N = \Omega^N F^N$ Liouville or Kolmogorov equation
  $\leftrightarrow$ $\hat{F}^N(t) = \pi^N_P F^N \in \mathbf{P}(\mathbf{P}(E))$ law of $\mu^N_{Y(t)}$.
  $\leftrightarrow$ $F^N_k(t) \in \mathbf{P}(E^k) \, \forall \, k \le N$

At the statistical (mean-field) limit the system is described by

- $f(t,\cdot) \in \mathbf{P}(E)$ the probability density of particles,
  $\partial_t f = Q(f)$ nonlinear PDE equation

How to deduce the behavior of the typical particle from the behavior of the N-particle system ?

Pb 1: Law of large numbers: $\mu^N_{Y(t)} \rightharpoonup f(t)$ or $F^N_1 \rightharpoonup f(t)$ when $N \to \infty$

The density $F^N_1(t)$ of one typical particle of the N-particle system behaves as $f(t)$ the solution of the mean-field equation. Mean-field convergence $\approx$ law of large numbers.

Pb 2: propagation of chaos: F^N_0 is f_0-chaotic implies F^N_t is f_t-chaotic?

in the sense that in the large number of particles limit $N \to \infty$:

$$F^N_k(t) \rightharpoonup f(t)^{\otimes k}, \quad \hat{F}^N \rightharpoonup \delta_{f(t)} \quad \text{or} \quad F^N \approx f^{\otimes N}$$

- Even when $F^N_0 = f_{in}^{\otimes N}$ we never have $F^N_t = g_t^{\otimes N}$ for a given N (except when there is no interaction between the particles of the N-particle system!).
  - we cannot expect independence
  - we may expect recover "independence" at the limit (= chaos)
- Why are we interested by chaos?
  - chaos is a strong physically relevant information
  - it may help to identify the mean field limit equation (as in Kac's proof). For the Boltzmann model, mean-field limit may only be established when molecular chaos holds at the initial time and is propagated.

Example 1: ODE / Vlasov / empirical measure method

Deterministic trajectories $X(t) \in E^N$, $E = \mathbb{R}^d$, driven by ODE with smooth coefficients

$$\dot{x}_i = A_i(X) = A(x_i, \mu^{N-1}_{X_i}) = A(x_i, \mu^N_X), \quad 1 \le i \le N, \quad X_i = X \backslash \{x_i\}$$

Its law F^N satisfies the Master/Liouville equation

$$\partial_t F^N = \Omega^N F^N := -\sum_i \mathrm{div}_{x_i}(A(x_i, \mu^N_X)\, F^N)$$

We aim to prove that its (mean-field) limit ($N \to \infty$) satisfies Vlasov equation

$$(*) \qquad \partial_t f = Q(f) := -\mathrm{div}(A(x, f)\, f)$$

We prove: $\mu^N_{X(t)}$ is a solution of $(*)$ for any $X(0)$ and for any other solution $f(t)$

$$W_1(\mu^N_{X(t)}, f(t)) \le C_T\, W_1(\mu^N_{X(0)}, f(0)).$$

We deduce the propagation of chaos estimate

$$\mathcal{W}_{W_1}(\hat{F}^N(t), \delta_{f(t)}) \le C_T\, \mathcal{W}_{W_1}(\hat{F}^N(0), \delta_{f(0)}) \approx \frac{1}{N^{\frac{1}{d'}}},\; d' = d \vee 2$$

Example 2: SDE / McKean-Vlasov / Coupling method

Stochastic trajectories $X(t) \in E^N$, $E = \mathbb{R}^d$, driven by Brownian SDE plus quadratic and smooth interaction ((B^i_t) independent Brownian motions)

$$dx_i = A_i(X)\, dt + dB^i_t, \qquad A_i(X) = (a \star \mu^N_X)(x_i).$$

Its law $F^N \in \mathbf{P}_{sym}(E^N)$ satisfies the Master/Kolomogorov equation

$$\partial_t F^N = \Omega^N F^N := -\sum_i^N \Delta_i F^N - \sum_{i=1}^N \mathrm{div}_i(A_i(X)\, F^N) \quad (0,\infty) \times E^N$$

and the associated mean field equation is the McKean-Vlasov equation

$$\partial_t f = Q(f) := \frac{1}{2}\, \Delta f - \mathrm{div}(A(x,f)\, f) \quad (0,\infty) \times E.$$

For a given solution $f(t)$, consider $Y(t)$ solution to the subsidiary problem:

$$(y_i(0)) \text{ i.i.d. according to } f(0) \ \text{ and } \ dy_i = (a \star f(t,.))(y_i) + dB^i_t,$$

so that $Y(t) \sim f(t)^{\otimes N}$, we prove

$$W_1(F^N(t), f^{\otimes N}(t)) \le C_T \left(W_1(F^N(0), f^{\otimes N}(0)) + \frac{1}{\sqrt{N}} \right)$$

Notice that

$$W_1(F^N(t), f^{\otimes N}(t)) = \inf_{(X_t,Y_t);X_t\sim F^N(t),Y_t\sim f(t)^{\otimes N}} u_{X_t,Y_t}$$

with

$$u_{X_t,Y_t} = \mathbf{E}\underbrace{\Big(\frac{1}{N}\sum_{j=1}^{N}|x_j(t)-x_j(t)|\Big)}_{=:distance\ d_N\ in\ E^N}$$

Write a differential inequality on $u(t) := u_{X_t,Y_t}$

$$\dot{u} \le C\,u + \mathcal{A}(t)$$

with

$$\mathcal{A}(t)^2 := \frac{1}{N}\sum_{i=1}^{N}\int_{E^N}\Big[(a*(\mu_Y^N - f_t)(y_i)\Big]^2 f_t^{\otimes N}(dY) \approx \frac{C}{N} \quad (\textit{quadratic miracle}!)$$

N-particle system $V = (v_1, ..., v_N)$, $v_i \in E = \mathbb{R}^3$ undergoing random Boltzmann jumps (collisions).
Markov process $(V_t)_{t \geq 0}$ defined step by step as follows:
(i) draw randomly $\forall\ (v_{i'}, v_{j'})$ collision time $T_{i',j'} \ \sim \ Exp(B(|v_{i'} - v_{j'}|))$; then select the post-collisional velocity (v_i, v_j) such that

$$T_{i,j} = \min_{(i',j')} T_{i',j'}.$$

(ii) draw randomly $\sigma \in S^2$ according to the density law $b(\cos\theta)$ with $\cos\theta = \sigma \cdot (v_i - v_j)/|v_i - v_j|$ and define the post-collisional velocities (v_i^*, v_j^*) thanks to

$$v_i^* = \frac{v_i + v_j}{2} + \frac{|v_j - v_i|}{2}\,\sigma, \qquad v_j^* = \frac{v_i + v_j}{2} - \frac{|v_j - v_i|}{2}\,\sigma.$$

Observe that momentum and energy are conserved

$$v_i^* + v_j^* = v_i + v_j, \qquad |v_i^*|^2 + |v_j^*|^2 = |v_i|^2 + |v_j|^2.$$

Finally, this two bodies collisions jump process satisfies

$$\sum_{i=1} v_i(t) = \text{cst}, \quad \sum_{i=1} |v_i(t)|^2 = \text{cst}.$$

Example 3: Master equation for Boltzmann-Kac system

Equivalently, after time rescaling, the motion of the N-particle system is given through the Master/Kolmogorov equation on the law $F_t^N \in \mathbf{P}(E^N)$ which in dual form reads

$$\partial_t \langle F_t, \varphi \rangle = \langle F_t^N, G^N \varphi \rangle \qquad \forall \, \varphi \in C_b(E^N)$$

with $G^N = (\Omega^N)^*$ given by

$$(G^N \varphi)(V) = \frac{1}{N} \sum_{i,j=1}^{N} B(v_i - v_j) \int_{S^2} b(\cos \theta_{ij}) \, [\varphi'_{ij} - \varphi] \, d\sigma,$$

where $\varphi = \varphi(V)$, $\varphi'_{ij} = \varphi(V'_{ij})$, $V'_{ij} = (v_1, .., v'_i, .., v'_j, .., v_N)$.

- Maxwell interactions with cut-off: $B = 1$, $b = 1$;
- Maxwell interactions without cut-off: $B = 1$, $b \notin L^1$;
- Hard spheres interactions: $B(z) = |z|$, $b = 1$.

The nonlinear Boltzmann equation

Nonlinear homogeneous Boltzmann equation on $\mathbf{P}(\mathbb{R}^3)$ defined by

$$\partial_t f_t = Q(f_t), \quad f_0 \in P_2(\mathbb{R}^3)$$

with

$$\langle Q(f), \varphi \rangle := \int_{\mathbb{R}^6 \times S^2} B(v - v_*)\, b(\cos\theta)\, (\phi(v') - \phi(v))\, d\sigma\, f(dv)\, f(dv_*)$$

where again

$$v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\, \sigma.$$

The equation generate a nonlinear semigroup

$$\forall\, f_0 \in P_2(\mathbb{R}^3) \qquad S_t^{NL} f_0 := f_t.$$

Quantitative answer to Kac's problem 1

Theorem ((II-1) Uniform in time Kac's chaos convergence)

$$\sup_{t\in[0,T)} \left| \int_{E^k} \left(F_k^N(t) - f_t^{\otimes k} \right) \varphi \, dV \right| \leq \theta(N) \underset{N\to\infty}{\longrightarrow} 0.$$

- $T \in (0, +\infty]$,
- $E = \mathbb{R}^d$, $d = 3$, $V = (v_1, ..., v_N) \in E^N$
- $f_0 = f_{in} \in \mathbf{P}(E)$ with enough moments bounded,
  $f_t =$ evolution of one typical particle in the mean-field limit,
  $f_t^{\otimes N}(V) = f_t(v_1) \dots f_t(v_N)$,
- F_0^N is f_{in}-chaotic, $F_t^N =$ evolution of N-particle system $\in \mathbf{P}_{sym}(E^N)$,
- $\varphi = \varphi_1 \otimes ... \otimes \varphi_k$, $\varphi_j \in \mathcal{F} \subset C_b(E)$, ex: $\mathcal{F} = W^{1,\infty}$ or H^s,
- $N \geq 2\,k$.

Main features 1

- We prove propagation of chaos with quantitative rates
- Most importantly and new: estimates are uniform in time for the Boltzmann equation (and the McKean-Vlasov)

  $\Rightarrow N \to \infty$ limit and $t \to \infty$ limit commute!
- We may deal with mixtures of Vlasov, McKean and Boltzmann models at least for smooth and bounded coefficients
- Our theorem applies to the space homogeneous Boltzmann equation in the case of the two important physical collision models:

  - true Maxwell molecules (without Grad's cut-off) cross-section

  - hard spheres cross-section (and hard potential with Grad's cut-off)

  $\Rightarrow$ give quantitative estimates of previous non-constructive convergence result (Sznitman 1984)

  - Maxwell molecules with Grad's cut-off cross-section

  with optimal rate $\leq C_T/\sqrt{N} \Rightarrow$ recover Kac, McKean, Tanaka, Graham, Méléard, Peyre ...

Main features 2

- Our method is strongly inspired by Grünbaum work (1971) where he claimed he proved convergence result for the hard spheres model. But his proof is definitively wrong ! He essentially recovered the non-constructive convergence result for the Maxwell cut-off model by Kac & McKean.
- We follow, complete and improve Grunbaum's program;
- The underlining philosophy is a numerical analyst intuition: based on (A3) consistency estimate and (A4) stability estimate on the limit PDE and refuse any compactness and probability arguments
  - "consistency error" of order $\mathcal{O}(1/N^{1-\varepsilon})\ \forall\, \varepsilon \in (0,1)$;
  - "stability error" of order $\mathcal{O}(1/N^{1/2})$, $\sim \mathcal{O}(1/N^{1/d})$ or worst because we write the equation in $\mathbf{P}(\mathbf{P}(\mathbb{R}^3))$ and we use some results from the theory of the concentration of measure (at time $t=0$): the worse error is made at time $t=0$ (and then it is not deteriorated by the flow);

Main features 3

- The θ function splits into

$$\theta(N) = \theta(k,N) = \underbrace{\theta_1(\varphi,N)}_{\mathcal{O}(1/N)} + \underbrace{\theta_2(\varphi,T,N)}_{\mathcal{O}(1/N^{1-\varepsilon})\,\forall\,\varepsilon} + \underbrace{\theta_3(\varphi,T;F_0^N,f_0)}_{\le \mathcal{O}(1/N^{1/2})},$$

  - θ_2 is the worst term with respect to φ;
  - θ_3 is the worst term with respect to N dependence;
  - θ_3 is the only term depending on the initial data;

- We are not able to prove that

$$\sup_{[0,T]} D(F_t^N; f_t) \le C \left(\frac{1}{N^\alpha} + D(F_0^N, f_0) \right)$$

for some "distance" D which measures how close to a chaos state "$g \in \mathbf{P}(E)$" is a probability $g^N \in \mathbf{P}_{sym}(E^N)$ and $C, \alpha > 0$, but we prove

$$\sup_{[0,T]} W_1(F^N(t), f(t)^{\otimes N}) \le C_T \left(W_1(F^N(0), f(0)^{\otimes N})^{\gamma_1} + \frac{1}{N^{\gamma_2}} \right)$$

Theorem ((II-2) Chaoticity of the N-particle steady states)

$$\left| \int_{E^k} \left(\sigma_k^N - \gamma^{\otimes k} \right) \varphi \, dV \right| \le \theta(N) \underset{N \to \infty}{\longrightarrow} 0.$$

- $E = \mathbb{R}^d$, $d = 3$, $V = (v_1, ..., v_N) \in E^N$
- $\sigma^N :=$ steady state for the N-particle system
  . $= \text{meas}(S^{dN-1}(\sqrt{N}))^{-1} \, \delta_{S^{dN-1}(\sqrt{N})} \in \mathbf{P}(E^N)$,
- $\gamma(v) := (2\pi)^{-d/2} \, \exp(-|v|^2/2)$,
- $\varphi = \varphi_1 \otimes ... \otimes \varphi_k$, $\varphi_j \in H^s$,
- $N \ge 2\, k$,
- $F_0^N = [f_{in}^{\otimes N}]_{S^{dN-1}(\sqrt{N})} =$ conditioned product measure.

In other words,

$$\sigma^N \text{ is } \gamma\text{-chaotic}$$

Theorem ((II-3) Convergence to the equilibrium uniformly in N)

$$\sup_N W_1\left(F^N(t),\sigma^N\right) \le \varepsilon(t) \underset{t\to\infty}{\longrightarrow} 0$$

- $E = \mathbb{R}^d$, $d = 3$, $V = (v_1, ..., v_N) \in E^N$
- $F_0^N = [f_{in}^{\otimes N}]_{S^{dN-1}(\sqrt{N})}$,
- F_t^N = evolution of N-particle system $\in \mathbf{P}_{sym}(E^N)$,

Plan

• (1) On the one hand, we know (from Kac, and then Carlen, Loss, ...) that

$$\forall N \geq 1 \qquad W(F_1^N(t), \sigma^N(t)) \leq \|F^N \sigma^N - \sigma^N\|_{TV} \leq \|F^N - 1\|_{L^2(\sigma^N)} \leq A^N e^{-\lambda_N t}, \quad A > 1.$$

• (2) On the other hand, Theorem II-1 and II-2 write (for $N \geq 2$)

$$\sup_{[0,\infty)} W_1(F^N(t), f(t)^{\otimes N}) + W_1(\sigma^N, \gamma^{\otimes N}) \leq \theta(N) \underset{N\to\infty}{\longrightarrow} 0.$$

• (3) We know (from Carlen, Carvalho, and then Villani, Mouhot 90'-2006) that

$$W_1(f(t)^{\otimes N}, \gamma^{\otimes N}) \leq \|f_t - \gamma\|_{L^1_1} \leq C_{f_0}\, e^{-\lambda\, t}.$$

• (4) Gathering estimates (2) and (3), we get

$$\forall N \geq 2 \qquad W_1(F^N(t), \sigma^N(t)) \leq \theta(N) + C_{f_0}\, e^{-\lambda\, t}$$

• (5) As a consequence of (1) and (4) we obtain the uniform (with respect to N) convergence:

$$W_1(F^N(t), \sigma^N(t)) \leq \min\left(2\,\theta(N) + C_{f_0}\, e^{-\lambda\, t}, C_{N,F_0^N}\, e^{-\lambda_N\, t}\right) \underset{t\to\infty}{\longrightarrow} 0$$

(choose (1) if $\varepsilon\, t \geq N$ and (4) if $\varepsilon\, t \leq N$).

We split

$$
\begin{aligned}
&\left\langle F_t^N - f_t^{\otimes N}, \varphi \otimes 1^{\otimes N-k} \right\rangle = \\
&= \left\langle F_t^N, \varphi \otimes 1^{\otimes N-k} - R_\varphi(\mu_V^N) \right\rangle \qquad (= T_1) \\
&+ \left\langle F_t^N, R_\varphi(\mu_V^N) \right\rangle - \left\langle F_0^N, R_\varphi(S_t^{NL} \mu_V^N) \right\rangle \qquad (= T_2) \\
&+ \left\langle F_0^N, R_\varphi(S_t^{NL} \mu_V^N) \right\rangle - \left\langle f_t^{\otimes k}, \varphi \right\rangle \qquad (= T_3)
\end{aligned}
$$

where R_φ is the "polynomial function" on $\mathbf{P}(\mathbb{R}^3)$ defined by

$$R_\varphi(\rho) = \int_{E^k} \varphi \, \rho(dv_1) \dots \rho(dv_k)$$

and S_t^{NL} is the nonlinear semigroup associated to the nonlinear mean-field limit equation by $g_0 \mapsto S_t^{NL} g_0 := g_t$.

$$
\begin{aligned}
|T_1| &= \left|\left\langle F_t^N, \varphi \otimes 1^{\otimes(N-k)}(V) - R_\varphi(\mu_V^N)\right\rangle\right| \\
&= \left|\left\langle F_t^N, \widetilde{\varphi \otimes 1^{\otimes(N-k)}}(V) - R_\varphi(\mu_V^N)\right\rangle\right| \\
&\leq \left\langle F_t^N, \frac{2\,k^2}{N}\,\|\varphi\|_{L^\infty(E^k)}\right\rangle = \frac{2\,k^2}{N}\,\|\varphi\|_{L^\infty(E^k)} \\
&\leq \frac{2\,k^3}{N}\,\|\nabla\varphi\|_{L^\infty(E^k)}\,M_1(F_1^N(t)),
\end{aligned}
$$

where we use that F^N is symmetric and a probability and we introduce the symmetrization function associated to $\varphi \otimes 1^{\otimes(N-k)}$ by

$$
\widetilde{\varphi \otimes 1^{\otimes(N-k)}}(V) = \frac{1}{\sharp\mathfrak{S}_N} \sum_{\sigma\in\mathfrak{S}_N} \varphi \otimes 1^{\otimes(N-k)}(V_\sigma).
$$

$$
\begin{aligned}
|T_3| &= \left| \left\langle F_0^N, R_\varphi(S_t^{NL}\mu_V^N) \right\rangle - \left\langle (S_t^{NL} f_0)^{\otimes k}, \varphi \right\rangle \right| \\
&= \left| \left\langle F_0^N, R_\varphi(S_t^{NL}\mu_V^N) - R_\varphi(S_t^{NL} f_0) \right\rangle \right| \\
&\le [R_\varphi]_{C^{0,1}} \left\langle F_0^N, W_1(S_t^{NL}\mu_V^N, S_t^{NL} f_0) \right\rangle \\
&\le k \, \|\nabla \varphi\|_{L^\infty(E^k)} \, C_T \left\langle F_0^N, W_1(\mu_V^N, f_0) \right\rangle \\
&\le k \, \|\nabla \varphi\|_{L^\infty(E)} \, C_T \, \mathcal{W}_{W_1}(\hat F_0^N, \delta_{f_0})
\end{aligned}
$$

where

$$
[R_\varphi]_{C^{0,1}} := \sup_{W_1(\rho,\eta)\le 1} |R_\varphi(\eta) - R_\varphi(\rho)| = k \, \|\nabla \varphi\|_{L^\infty}
$$

and we assume that the nonlinear flow satisfies

$$
(A5) \qquad W_1(f_t, g_t) \le C_T \, W_1(f_0, g_0) \quad \forall \, f_0, g_0 \in \mathbf{P}(E)
$$

T_2 : We write

$$T_2 \quad = \quad \left\langle F_t^N, R_\varphi(\mu_V^N) \right\rangle - \left\langle F_0^N, R_\varphi(S_t^{NL} \mu_V^N) \right\rangle$$

T_2 : We write

$$\begin{aligned} T_2 &= \left\langle F_t^N, R_\varphi(\mu_V^N) \right\rangle - \left\langle F_0^N, R_\varphi(S_t^{NL}\mu_V^N) \right\rangle \\ &= \left\langle F_0^N, T_t^N(R_\varphi \circ \mu_V^N) - (T_t^\infty R_\varphi)(\mu_V^N) \right\rangle \end{aligned}$$

with

- T_t^N = dual semigroup (acting on $C_b(E^N)$) of the N-particle flow $F_0^N \mapsto F_t^N$;
- T_t^∞ = pushforward semigroup (acting on $C_b(\mathbf{P}(E))$) of the nonlinear semigroup S_t^{NL} defined by $(T^\infty \Phi)(\rho) := \Phi(S_t^{NL}\rho)$;

T_2 : We write

$$
\begin{aligned}
T_2 &= \Big\langle F_t^N, R_\varphi(\mu_V^N) \Big\rangle - \Big\langle F_0^N, R_\varphi(S_t^{NL}\mu_V^N) \Big\rangle \\
&= \Big\langle F_0^N, T_t^N(R_\varphi \circ \mu_V^N) - (T_t^\infty R_\varphi)(\mu_V^N) \Big\rangle \\
&= \Big\langle F_0^N, (T_t^N \pi_N - \pi_N T_t^\infty)\, R_\varphi \Big\rangle
\end{aligned}
$$

with

- $T_t^N =$ dual semigroup (acting on $C_b(E^N)$) of the N-particle flow $F_0^N \mapsto F_t^N$;
- $T_t^\infty =$ pushforward semigroup (acting on $C_b(\mathbf{P}(E))$) of the nonlinear semigroup S_t^{NL} defined by $(T^\infty \Phi)(\rho) := \Phi(S_t^{NL}\rho)$;
- $\pi_N =$ projection $C(\mathbf{P}(E)) \to C(E^N)$ defined by $(\pi_N \Phi)(V) = \Phi(\mu_V^N)$.

$$
\begin{aligned}
T_2 &= \left\langle F_0^N, (T_t^N \pi_N - \pi_N T_t^\infty)\, R_\varphi \right\rangle \\
&= \left\langle F_0^N, \int_0^T T_{t-s}^N \, (G^N \pi_N - \pi_N G^\infty)\, T_s^\infty \, ds \, R_\varphi \right\rangle \\
&= \int_0^T \left\langle F_{t-s}^N, (G^N \pi_N - \pi_N G^\infty)\, (T_s^\infty \, R_\varphi) \right\rangle \, ds
\end{aligned}
$$

where

- G^N is the generator associated to T_t^N and G^∞ is the generator associated to T_t^∞.

Now we have to make some assumptions

- (A1) F_t^N has enough bounded moments;
- (A2) $G^\infty \Phi(\rho) = \langle Q(\rho), D\Phi(\rho)\rangle$;
- (A3) $(G^N \pi^N \Phi)(V) = \langle Q(\mu_V^N), D\Phi(\mu_V^N)\rangle + \mathcal{O}([\Phi]_{C^{1,a}}/N)$
- (A4) $S_t^{NL} \in C^{1,a}(\mathbf{P}(E); \mathbf{P}(E))$.

A parentesis: the $C^{1,a}$ space, $a \in (0,1]$

$\Phi \in C^{1,a}(\mathbf{P}(E);\mathbb{R})$ if $\Phi \in C(\mathbf{P}(E))$ and $\exists D\Phi : \mathbf{P}(E) \to C(E)$

$$\forall\, \mu,\ \nu \in \mathbf{P}(E) \quad \Big|\Phi(\nu) - \Phi(\mu) - \langle \nu - \mu, D\Phi[\mu]\rangle\Big| \le C\, \|\nu - \mu\|_{TV}^{1+a}.$$

We define

$$[\Phi]_a = \sup_{\mu,\nu\in\mathbf{P}(E)} \frac{\Big|\Phi(\nu) - \Phi(\mu) - \langle \nu - \mu, D\Phi[\mu]\rangle\Big|}{\|\nu - \mu\|_{TV}^{1+a}}.$$

Remark. For any $\varphi \in W^{2,\infty}(E^k)$, $R_\varphi \in C^{1,1}(\mathbf{P}(E))$ and

$$[R_\varphi]_1 \le k^2\, \|\varphi\|_{W^{2,\infty}(E^k)}.$$

$$
\begin{aligned}
T_2 &\le \int_0^T M_0(F^N_{t-s}) \, \|(G^N \pi_N - \pi_N G^\infty)\,(T^\infty_s \, R_\varphi)\|_{L^\infty(E^N)} \, ds \\
&\overset{(A3)}{\le} \int_0^T \frac{C}{N} \, [T^\infty_s \, R_\varphi]_{C^{1,a}} \, ds \\
&\le \frac{C}{N} \int_0^T [R_\varphi \circ S^{NL}_t]_{C^{1,a}} \, ds \\
&\le \frac{C}{N} \int_0^T [R_\varphi]_{C^{1,1}} \, [S^{NL}_t]_{C^{1,a}} \, ds \\
&\le \frac{C}{N} \, k^2 \, \|\varphi\|_{W^{2,\infty}} \int_0^T [S^{NL}_t]_{C^{1,a}} \, ds
\end{aligned}
$$

$$\left\langle F_k^N(t) - f(t)^{\otimes N}, \varphi \right\rangle \leq$$
$$\leq C_k \left(\frac{\|\nabla\varphi\|_{L^\infty}}{N} + C_T^{(A4)} \frac{\|\varphi\|_{W^{2,\infty}}}{N^a} + C_T^{(A5)} \|\nabla\varphi\|_{L^\infty} \mathcal{W}_{W_1}(\hat{F}_0^N, \delta_{f_0}) \right)$$

and

$$\sup_{[0,T)} \sup_{\|\varphi\|_{W^{2,\infty}} \leq 1} \left\langle F_k^N(t) - f(t)^{\otimes N}, \varphi \right\rangle \leq$$
$$\leq C_k \left(\frac{1}{N} + \frac{C_T^{(A4)}}{N^a} + C_T^{(A5)} \mathcal{W}_{W_1}(\hat{F}_0^N, \delta_{f_0}) \right)$$

with $T = \infty$ if

$$\sup_{t \geq 0} [S_t^{NL}]_{C^{0,1}_{W_1}} + \int_0^\infty [S_t^{NL}]_{C^{1,a}_{TV}} \, dt < \infty.$$

(A2) The nonlinear semigroup S_t^{NL} and operator Q are $C^{0,a}$ for the total variation norm. As a consequence $\forall\,\Phi \in C^{1,a}(\mathbf{P}(E))$, $\forall f_0 \in \mathbf{P}_2(E)$

$$\begin{aligned}
(G^\infty \Phi)(f_0) &= \frac{d}{dt}(T_t^\infty \Phi)(f_0)|_{t=0} = \frac{d}{dt}\Phi(f_t)|_{t=0} = \lim_{t\to 0} \frac{\Phi(f_t)-\Phi(f_0)}{t} \\
&= \lim_{t\to 0} \left\{ \left\langle \frac{f_t - f_0}{t}, D\Phi[f_0] \right\rangle + \mathcal{O}\left(\frac{\|f_t - f_0\|_{TV}^{1+a}}{t} \right) \right\} \\
&= \left\langle \frac{df_t}{dt}|_{t=0}, D\Phi[f_0] \right\rangle = \langle Q(f_0), D\Phi(f_0)\rangle
\end{aligned}$$

(A3) Consistency: $\forall\,\Phi \in C^{1,a}(\mathbf{P}(E))$, set $\phi = D\Phi[\mu_V^N]$, and compute

$$\begin{aligned}
G^N(\Phi \circ \mu_V^N) &= \frac{1}{2N} \sum_{i,j=1}^N B(v_i - v_j) \int_{S^2} b\,[\Phi(\mu^N_{V'_{ij}}) - \Phi(\mu^N_V)]\, d\sigma \\
&= \frac{1}{2N} \sum_{i,j} B(v_i - v_j) \int_{S^2} b\,\langle \mu^N_{V'_{ij}} - \mu^N_V, \phi\rangle\, d\sigma \qquad = \langle Q(\mu^N_V), \phi\rangle \\
&+ \frac{1}{2N} \sum_{i,j} B(v_i - v_j) \int_{S^2} \mathcal{O}(\|\mu^N_{V'_{ij}} - \mu^N_V\|_{TV}^{1+a})\, d\sigma \quad = \mathcal{O}(1/N^a)
\end{aligned}$$

(A4) The Boltzmann flow S_t^{NL} is $C^{1,a}$ in total variation norm:
$\forall\,\rho \in \mathbf{P}_k(\mathbb{R}^d)$, $\forall\,t \geq 0$ there exists $\mathcal{L}_t[\rho] \in C(\mathbb{R}^3)$ $\forall\,\eta \in \mathbf{P}_k(\mathbb{R}^d)$

$$\begin{aligned} S_t^{NL}(\eta) &= S_t^{NL}(\rho) + \mathcal{L}_t[\rho](\eta-\rho) + \mathcal{O}(\,\|\eta-\rho\|_{TV}^{1+a}) \\ &= S_t^{NL}(\rho) + \mathcal{L}_t[\rho](\eta-\rho) + \mathcal{O}(e^{-\lambda\,t}\,\|\eta-\rho\|_{TV}^{1+a}) \end{aligned}$$

(A5) The Boltzmann flow S_t^{NL} is $C^{0,1}$ in weak distance (Tanaka, Toscani-Villani, Fournier-Mouhot): $\forall\,\rho,\eta \in \mathbf{P}_k(\mathbb{R}^d)$, $\forall\,t \geq 0$ there holds

$$\begin{aligned} W_1(S_t^{NL}(\eta), S_t^{NL}(\rho)) &\leq C_T\, W_1(\eta,\rho) \\ &\leq \Omega(W_1(\eta,\rho)) \quad \text{uniform in time} \end{aligned}$$

Plan

1. Introduction
2. Quantitative formulations of chaos
3. Quantitative propagation of chaos
4. Outlines of the proofs
5. Statistical solutions
6. Conclusion and open problems

Naive idea: 1-marginal

$$\partial_t F^N = \Omega_N F^N$$

implies

$$\partial_t F^N_1 = (\Omega_N F^N)_1 = \Omega_{N,2} F^N_2 \quad \to \quad \partial_t \pi_1 = \Omega_2^\infty \pi_2 \quad \text{and ? ...}$$

We carry on the idea by taking ℓ-th marginal

- Start from a N-particle system

$$\partial_t F^N = \Omega_N F^N \quad \text{or} \quad F^N(t) = e^{t\,\Omega_N} f_0^N = \sum_{k=0}^{\infty} \frac{t^k}{k!} A_N^k f_0^{\otimes N}$$

- Write the equation for the ℓ-th marginal distribution

$$\partial_t F^N_\ell = \Omega_{N,\ell+1} F^N_{\ell+1} \quad \text{or} \quad F^N_\ell(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} (A_N^k f_0^{\otimes N})_\ell$$

Unclosed equation when $\ell < N$.

- Kac's argument: take $\varphi \in C_b(E^\ell)$ and write the dual identity

$$\langle F_\ell^N(t), \varphi \rangle = \sum_{k=0}^{\infty} \frac{t^k}{k!} \langle f_0^{\otimes N}, \Omega_N^k(\varphi \otimes 1^{\otimes N-\ell}) \rangle$$

- Pass now to the limit $N \to \infty$

$$\langle \pi_\ell(t), \varphi \rangle = \sum_{k=0}^{\infty} \frac{t^k}{k!} \langle f_0^{\otimes k+\ell}, \varphi_k \rangle, \quad \varphi_k \in C(E^{k+\ell}).$$

For $\varphi, \psi \in C(E)$ Kac proves

$$(\varphi \otimes \psi)_k = \sum_{i=0}^{k} \frac{k!}{i!\,(k-i)!} \varphi_i \otimes \psi_{k-i}$$

so that we may recognize

$$\langle \pi_2(t), \varphi \otimes \psi \rangle = \sum_{i \le k} \frac{t^k}{i!\,(k-i)!} \langle f_0^{\otimes k+2}, \varphi_i \otimes \psi_{k-i} \rangle = \langle \pi_1(t), \varphi \rangle \, \langle \pi_1(t), \psi \rangle.$$

- We come back to the family of equations on the marginals of order ℓ

$$\partial_t F^N_\ell = \Omega_{N,\ell+1}\, F^N_{\ell+1},$$

which (again) are unclosed equations if $\ell < N$.

- We pass to the limit $N \to \infty$

$$(*) \qquad \partial_t \pi_\ell = \Omega^\infty_{\ell+1}\, \pi_{\ell+1}.$$

under some hypothesis

  - (A1′) F^N has bounded moments
  - (A3′) $G_{N,\ell+1}(\varphi) \;\to\; G^\infty_{\ell+1}\varphi \approx \langle Q^*(\varphi), DR_\varphi\rangle$

We obtain a family of solutions $(\pi_\ell)_{\ell\ge1}$ to an infinite hierarchy of equations

- We remark that $\bar\pi_\ell(t) = f(t)^{\otimes \ell}$ is a solution of $(*)$

Theorem ((III-1) Uniqueness of BBGKY hierachy)

Assume (A2) and (A4).
For a given initial datum $\hat\pi_0 = (\pi_{0,\ell})_\ell$, there is equivalence between

- $(\pi_\ell(t))_{\ell \ge 1}$ *is a solution to* $(*)$
- $\hat\pi(t) \in \mathbf{P}(\mathbf{P}(E))$ *(linked by Hewitt-Savage theorem) is a solution to*

$$(**) \qquad \partial_t \hat\pi = \Omega^\infty \hat\pi.$$

- $\hat\pi(t) = \bar\pi(t)$ *defined by*

$$\langle \bar\pi(t), R_\varphi \rangle := \langle \hat\pi_0, T^\infty_t R_\varphi \rangle$$

As a consequence: $\hat\pi_0 = \delta_{f_0} \Leftrightarrow \pi_{0,\ell} = f_0^{\otimes \ell}$
implies that the solution of $(**) \Leftrightarrow (*)$ *is* $\hat\pi_t = \delta_{f_t} \Leftrightarrow \pi_\ell(t) = f(t)^{\otimes \ell}$

Proof inspired from Arkeryd, Caprino, Ianiro (1991)

Theorem ((III-2) Propagation of chaos)

Assume (A1′), (A2), (A3′) and (A4).
If F_0^N is f_0-chaotic then $F_\ell^N(t) \to \pi_\ell(t) = f(t)^{\otimes \ell}$: $F^N(t)$ is $f(t)$-chaotic.

Plan

Concluding remarks

We have proved a quantified version of chaos propagation which is furthermore uniform in time (for the Boltzmann model)

That result can be seen as a "quantitative version" of BBGKY method

The key point is to estimate the convergence of $T_t^N \pi^N$ to $\pi^N \, T_t^\infty$ as operators acting from $C(\mathbf{P}(E))$ with values in $C(E^N)$ which a consequence of

- a stability result (expansion of order > 1) for he nonlinear semigroup
- consistency result on the associated generators

That requires to develop a "differential calculus" on $\mathbf{P}(E)$ seen as an embedded manifold of $\mathcal{F}'$, $\mathcal{F} \subset UC_b(E)$

Open problems

- $T = +\infty$ with optimal rate $\theta(N) = \mathcal{O}(N^{-1/2})$;

- more general cross-section (true hard or soft potential) and Landau equation;

- Vlasov equation and McKean-Vlasov equation with singular interactions;

- (quantitative) propagation of entropy chaos $\sup_{[0,T]} H(F_t^N | f_t^{\otimes}) \le \theta_H(N)$;

- quantification of the chaos for the equilibrium state (elastic or inelastic Boltzmann model)

- rate of convergence to equilibrium for the nonlinear PDE from the analysis of the N-particle system dynamic

- for the inelastic Boltzmann equation + diffuse excitation can we deduce from the $N \to \infty$ limit

$$\frac{d}{dt} H(f(t)|g) \le 0$$

where g stands for the unique steady state?