Preserving hypocoercivity when enlarging the space setting for linear PDE and applications

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Outline of the talk

Plan

1. Introduction
   - The “functional space enlarging” issue
   - List of related works
2. Motivation, applications
   - The two mains outcomes
   - Larger class of initial data
   - Nonlinear stability
   - Perturbation argument
3. Abstract theorem for space homogeneous equation
   - Statement of the theorem
   - Proof of the theorem
4. Abstract theorem for space inhomogeneous equation
5. Miscellaneous
   - Extensions / Open problems

The "functional space enlarging" issue

Consider E Hilbert space, L a generator

Favorite example:

space homogeneous Fokker-Planck equation

$$E = L^2_v(\mathbb{R}^d, G^{-1}\,dv),\ G = (2\pi)^{-d/2}\,e^{-|v|^2/2}$$

$$L = \mathrm{div}(\nabla + v) = \mathrm{div}(G\nabla(\cdot/G))$$

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Spectral analysis of L

- localization of the spectrum $\Sigma(L)$
- eigenspace + eigenspace projector
- growth estimate on semigroup e^{tL}

(here spectral mapping theorem $\Sigma(e^{tL}) = e^{t\Sigma(L)}$ holds)

Main question:

prove the same "spectral properties"
on a larger Banach space $\mathcal{E} \supset E$?

Aim of the talk

Consider E Hilbert space, L a generator on E

Spectral analysis of L
- localization of the spectrum $\Sigma(L)$
- eigenspace + eigenspace projector
- growth estimate on semigroup e^{tL}
- $=$ L is hypocoercive

positive answer:

$\mathcal{L}$ generator on $\mathcal{E} \supset E$ s.t. $\mathcal{L}|_E = L$ (the same pde operator on a larger space)

$\Rightarrow$ $\mathcal{L}$ satisfies the "same spectral properties as L"

$\Rightarrow$ $\mathcal{L}$ is hypocoercive

- make precise that statement
- holds for many PDE
- the proof is very simple
- the result has interesting applications
- the result is based on "explicit argument" (no compactness)

(Hypo)dissipative and (hypo)coercive operators

$\Lambda - a$ dissipative in $X = (X, \|.\|_X)$ if

$$\|e^{t\Lambda}\|_{B(X)} \le e^{at} \qquad \Rightarrow \Sigma(\mathcal{B}) \cap \Delta_a = \emptyset, \text{bdd resolvent}$$

$$\text{or} \qquad \Re e \langle (\Lambda - a)\, f, \phi \rangle \le 0 \quad \forall f \in D(\Lambda) \, \forall \phi \in F(f)$$

with $\phi \in F(f)$ if $\langle f, \phi \rangle = \|f\|_X^2 = \|\phi\|_{X^*}^2$

$\Lambda - a$ hypodissipative in X if

$$\|e^{t\Lambda}\|_{B(X)} \le C_a \, e^{at}, \quad C_a > 1$$

$$\text{or} \qquad \Lambda - a \text{ is dissipative in } (X, |\cdot|_X), \ |\cdot|_X \approx \|\cdot\|_X$$

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Λ (hypo)coercive in X if $\Delta_a \cap \Sigma(\Lambda) =$ discrete for some $a \in \mathbb{R}$

$$\|e^{t\Lambda} - e^{t\Lambda}\, \Pi_{\Lambda,a}\|_{B(X)} \le C_a\, e^{at}, \ C_a = 1 \text{ or } C_a > 1$$

$$\text{or} \qquad \Lambda - a \text{ is (hypo)dissipative on an invariant set } X_0, \ \text{codim} X_0 < \infty$$

(then $X_0 = R(I - \Pi_{\Lambda,a})$)

L is coercive

Half complex plane $\Delta_a := \{z \in \mathbb{C};\ \Re e\, z > a\}$

Localization of the spectrum/discrete spectrum

$\Sigma(L) \cap \Delta_a = \{\xi_1, ..., \xi_k\}$
$\xi_1, \ldots, \xi_k :=$ discrete eigenvalues
$\Pi_{L,\xi_j} :=$ eigenspace projector on the finite dimensional eigenspace

growth estimate on the semigroup e^{tL}

$\Pi_{L,a} := \Pi_{L,\xi_1} + ... + \Pi_{L,\xi_k}$
$\|e^{tL} - e^{tL}\Pi_{L,a}\|_{B(E)} \le C_a\, e^{a\,t}\ \forall\, t \ge 0$

Favorite example:

$\Sigma(L) = \{-(j-1)/2;\ j \ge 1\}$
$\Rightarrow \Sigma(L) \cap \Delta_a = \{0\}$, for some $a < 0$
$\Pi_{L,a} f = \langle f \rangle\, G,\ \ \langle f \rangle := \int_{\mathbb{R}^d} f\, dv = (f, G)_E$
$\|e^{tL} f_0 - \langle f_0 \rangle\, G\|_E \le \|f_0\|_E\, e^{a\,t}\ \forall\, t \ge 0$

A positive answer (rough version)

Consider
- $\mathcal{E}$ Banach space $\supset E$
- $\mathcal{L}$ generator of a C_0-semigroup s.t. $\mathcal{L}_{|E} = L$

If $\mathcal{L}$ decomposes as $\mathcal{L} = \mathcal{A} + \mathcal{B}$
- $\mathcal{A} : \mathcal{E} \to E$ ("regularizing" term)
- $\mathcal{B} - a$ is dissipative (good spectral localization term)

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$\Rightarrow$ $\mathcal{L}$ inherits the spectral properties of L

| $\Sigma(\mathcal{L}) \cap \Delta_a = \{\xi_1, ..., \xi_k\}$, with ξ_j discrete eigenvalue
| $\Pi_{\mathcal{L},\xi_j}|_E = \Pi_{L,\xi_j}$ = spectral projector

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$\Rightarrow$ $e^{t\,\mathcal{L}}$ inherits the growth estimate of e^{tL}

$\forall\, t \geq 0,\ \forall\, a' > a \qquad \|e^{t\mathcal{L}} - e^{tL}\Pi_{\mathcal{L},a}\|_{\mathcal{E}\to\mathcal{E}} \leq C_{a'}\, e^{a'\,t}$

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$\left| \; \forall\, t \ge 0, \forall\, a' > a \quad \|e^{t\mathcal{L}} - e^{tL}\Pi_{\mathcal{L},a}\|_{\mathcal{E}\to\mathcal{E}} \le C_{a'}\, e^{a'\,t} \right.$

$\Rightarrow$ $\mathcal{L}$ is hypocoercive $\approx$ partial spectral mapping theorem

$\Sigma(e^{\mathcal{L}\,t}) \cap \Delta_{e^{a'\,t}} = e^{\Sigma(L)\,t} \cap \Delta_{e^{a'\,t}}$

Favorite example: $\left| \begin{array}{ll} \mathcal{E} := L^2(g^{-1}), & g(v) = (1+|v|^2)^{-k},\ k > d/2 \\ \mathcal{A} f := M\,\chi_R\, f, & 0 \le \chi_R \in \mathcal{D}(\mathbb{R}^d),\ \chi_R \equiv 1 \text{ on } B(0,R) \\ \mathcal{B} f := \mathcal{L} f - M\,\chi_R\, f & \end{array} \right.$

Main difficulties / ideas

Difficulties

- L and $\mathcal{L}$ may be $\neq$ symmetric
- $\mathcal{E}$ may be $\neq$ Hilbert space
- constructive estimates

Ideas

- $\mathcal{L} = \mathcal{A} + \mathcal{B} \approx$ smooth + well known
- $\mathcal{R}_{\mathcal{L}} = \mathcal{R}_{\mathcal{B}} - \mathcal{R}_L \, \mathcal{A} \, \mathcal{R}_{\mathcal{B}}$

or more generally $\mathcal{R}_{\mathcal{L}} = \mathcal{R}_{\mathcal{B}} - ... + (-1)^n \mathcal{R}_L \, (\mathcal{A} \, \mathcal{R}_{\mathcal{B}})^n$

- use that expression of $\mathcal{R}_{\mathcal{L}}$ to get informations on $\Sigma(\mathcal{L})$, ..., $e^{t \, \mathcal{L}}$

List 1 of related works

[1) C. Mouhot, *Rate of convergence to equilibrium for the spatially homogeneous Boltzmann equation with hard potentials*, CMP (2006)

[2) S.M., C. Mouhot, *Stability, convergence to self-similarity and elastic limit for the Botlzmann equation for inelastic hard spheres*, CMP (2009)

[3) S.M., C. Mouhot, *Stability, convergence to the steady state and elastic limit for the Boltzmann equation for diffusively excited granular media*, DCDS (2009)

[4) M.P. Gualdani, S. M., C. Mouhot, *Factorization for non-symmetric operators and exponential H-Theorem*, ArXiv 2010

[5) A. Arnold, I. M. Gamba, M. P. Gualdani, S. M., C. Mouhot, C. Sparber, *The Wigner-Fokker-Planck equation: Stationary states and large time behavior*, ArXiv 2010

[6) M.J. Caceres, J.A. Cañizo, S. M., *Rate of convergence to an asymptotic profile for the self-similar fragmentation and growth-fragmentation equations*, ArXiv 2010

List 2 of related works to be complemented!

- Lax 1954
- Bobylev ????
- Gallay-Wayne 2002

Plan

Conditionally (up to time uniform strong estimate) exponential H-Theorem

• $(f_t)_{t\geq 0}$ solution to the inhomogeneous Boltzmann equation for hard spheres interactions in the torus with strong estimate

$$\sup_{t\geq 0}\left(\|f_t\|_{H^k}+\|f_t\|_{L^1(1+|v|^s)}\right)\leq C_{s,k}<\infty.$$

Conditionally (up to time uniform strong estimate) exponential H-Theorem

• $(f_t)_{t\ge 0}$ solution to the inhomogeneous Boltzmann equation for hard spheres interactions in the torus with strong estimate

$$\sup_{t\ge 0} \left(\|f_t\|_{H^k} + \|f_t\|_{L^1(1+|v|^s)} \right) \le C_{s,k} < \infty.$$

• Desvillettes, Villani proved [Invent. Math. 2005]: for any $s \ge s_0$, $k \ge k_0$

$$\forall\, t \ge 0 \qquad \int_{\mathbb{T}\times\mathbb{R}^d} f_t \log \frac{f_t}{G_1(v)}\, dvdx \le C_{s,k}\, (1+t)^{-\tau_{s,k}}$$

with $C_{s,k} < \infty$, $\tau_{s,k} \to \infty$ when $s, k \to \infty$.

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with $C_{s,k}<\infty$, $\tau_{s,k}\to\infty$ when $s,k\to\infty$.

Theorem

$\exists\, s_1, k_1$ s.t. for any $a>\lambda_2$ exists C_a

$$\forall\, t\geq 0 \qquad \int_{\mathbb{T}\times\mathbb{R}^d} f_t \log\frac{f_t}{G_1(v)}\,dvdx\leq C_a e^{-\frac{a}{2}\,t},$$

with $\lambda_2<0$ (2^{nd} eigenvalue of the linearized Boltzmann eq. in $L^2(G_1^{-1})$).

Elastic and inelastic inhomogeneous Boltzmann equation for hard spheres interactions in the torus: global existence for weakly inhomogeneous initial data

Theorem

For any $F_0 \in L^1_3(\mathbb{R}^d)$ there exists $e_0 \in (0,1)$ and $\varepsilon_0 > 0$ such that if $f_0 \in W^{k,1}_x(\mathbb{T}^d; L^1_3(\mathbb{R}^d))$ satisfies $\|f_0 - F_0\| \le \varepsilon_0$ and if $e \in [e_0, 1]$ then

- *there exists a unique global mild solution $f(t,x,v)$ starting from f_0;*
- *$f(t) \to G_1$ when $t \to \infty$ (with rate) when $e = 1$;*
- *$f(t) \to \bar{G}_e$ when $t \to \infty$ (with rate) when $e < 1$ (diffuse forcing).*

The case $e = 1$ has been treated by non constructive arguments by Arkeryd-Esposito-Pulvirenti (CMP 1987), Wennberg (Nonlinear Anal. 1993) and for the space homogeneous analogous by Arkeryd (ARMA 1988), Wennberg (Ser. Adv. Math. Appl. Sci. 1992)

Motivation 1: Linear evolution equation

Drastically increase the class of initial data such that the long trend convergence to the steady state holds

Starting point: spectral property of the operator L in the usual space $L^2(G_1^{-1})$ associated to the equilibrium G_1

- NL Boltzmann eq. [1] : $L^2(\mathbb{R}^d, e^{|v|^2}) \Rightarrow L^p(\mathbb{R}^d, e^{|v|^s})$
- Fokker-Planck eq. [4,5]: $L^2(\mathbb{R}^d, e^{|v|^2}) \Rightarrow$ or L^p (polynomial weight)
- Fragmentation eq. [6]: $L^2(\mathbb{R}_+, \phi G^{-1}) \Rightarrow$ with $p \in [1,2]$, $s \in (0,2]$

Variants of example 1 ([4], [5], [6]):

- $L f := \mathrm{div}(\nabla f + (\nabla\phi + U) f)$ with $\mathrm{div} U = \nabla\phi \cdot U = 0$
- $L f := \int [k(v,v')f(v') - k(v',v) f(v)] \, dv'$
- $L f := \int_v^\infty k(v',v) f(v') \, dv' - (K(v) - \lambda) f(v) - \partial_v(a(v) f), \ v > 0$
- $L f := -v \cdot \nabla_x f + ...$

Operators and their decomposition

General rule 1 for FP/Boltzmann type operator

$$L := \textit{order} \ \leq 1 + \textit{order}\ 2$$

$$L := \textit{compact} + \textit{explicit}$$

$$\mathcal{L} := \underbrace{\textit{smooth}/\textit{order} \leq 1}_{\mathcal{A}} + \underbrace{\textit{small} + \textit{explicit}/\textit{order}\ 2}_{\mathcal{B}}$$

General rule 2 for non space homogeneous operator

$$L_v := \mathcal{A}_v + \mathcal{B}_v$$

$$\mathcal{L}_{x,v} := \underbrace{\mathcal{A}_v}_{\mathcal{A}_{x,v}} + \underbrace{\mathcal{B}_v + T_x}_{\mathcal{B}_{x,v}}, \qquad T_x = v \cdot \nabla_x \text{ or } \Delta_x$$

Our favorite example: Fokker-Planck operator

Operator L:
space homogeneous Fokker-Planck equation
$E = L^2_v(\mathbb{R}^d, G^{-1}\,dv),\ G = (2\pi)^{-d/2}\,e^{-|v|^2/2}$
$L = \mathrm{div}(\nabla + v) = \mathrm{div}(G\nabla(\cdot/G))$

spectral properties
$\Sigma(L) = \{-(j-1)/2;\ j \geq 1\}$
$\Rightarrow \Sigma(L) \cap \Delta_a = \{0\}$, for some $a < 0$
$\Pi_{L,a}\, f = \langle f\rangle\, G,\ \ \langle f\rangle := \int_{\mathbb{R}^d} f\,dv = (f, G)_E$
$\|e^{tL} f_0 - \langle f_0\rangle\, G\|_E \leq \|f_0\|_E\, e^{a\,t}\ \forall\, t \geq 0$

Larger space:
$\mathcal{E} := L^2(g^{-1}),\qquad g(v) = (1+|v|^2)^{-k},\ k > d/2$
or $g(v) = e^{-|v|^s},\ s \in (0,2]$

Decomposition:

$$\mathcal{L} := \underbrace{M\,\chi_R}_{\mathcal{A}} + \underbrace{\mathrm{div}(\nabla + v) - M\,\chi_R}_{\mathcal{B}}$$

with $0 \leq \chi_R \in \mathcal{D}(\mathbb{R}^d),\ \chi_R \equiv 1$ on $B(0,R)$

The key estimate for our favorite example

$$
\begin{aligned}
(\mathcal{L} f, f)_{L^2(g^{-1})} &= \int \operatorname{div}\left(G \nabla \left(\frac{f}{g} \frac{g}{G} \right) \right) \frac{f}{g} \, dv \\
&= \int \operatorname{div}\left(g \nabla \left(\frac{f}{g} \right) \right) \frac{f}{g} \, dv \\
&\quad + \int \operatorname{div}\left(\frac{f}{g} G \nabla \left(\frac{g}{G} \right) \right) \frac{f}{g} \, dv \\
&= - \int \left| \nabla \left(\frac{f}{g} \right) \right|^2 g \, dv \\
&\quad + \frac{1}{2} \int \operatorname{div}\left(G \nabla \left(\frac{g}{G} \right) \right) \left(\frac{f}{g} \right)^2 dv \\
\Big[\dots \Big] \underset{|v| \to \infty}{\longrightarrow} a(k) < 0 \quad &\le \int \left[\frac{1}{2g} \operatorname{div}\left(G \nabla \left(\frac{g}{G} \right) \right) \right] f^2 g^{-1} \, dv \\
\forall\, a < 0 \ \exists\, k, R, M \quad &\le a \int f^2 g^{-1} \, dv + \int M \chi_R f^2 g^{-1} \, dv
\end{aligned}
$$

Conclusion for our favorite example

Theorem

For any $\alpha > d/2$, $a \in (d/4 - \alpha/2, 0)$ *and* $f_0 \in L^2(\langle v \rangle^\alpha)$

$$\| e^{\mathcal{L}\,t} f_0 - e^{t\,L}\,\Pi_{\mathcal{L},a}\,f_0 \|_{L^2(\langle v \rangle^\alpha)} \le C_{\alpha,a}\,e^{a\,t}\,\|f_0\|_{L^2(\langle v \rangle^\alpha)}$$

For any $\alpha > 0$, $a \in (-\alpha, 0)$ *and* $f_0 \in L^1(\langle v \rangle^\alpha)$

$$\| e^{\mathcal{L}\,t} f_0 - e^{t\,L}\,\Pi_{\mathcal{L},a}\,f_0 \|_{L^1(\langle v \rangle^\alpha)} \le C_{\alpha,a}\,e^{a\,t}\,\|f_0\|_{L^1(\langle v \rangle^\alpha)}$$

with $\langle v \rangle^\alpha = (1 + |v|^2)^{\alpha/2}$.

See Appendix A in Gallay-Wayne (ARMA 2002) for that result (maybe sharper) in L^2 obtained thanks to explicit computations

Example 2: space homogeneous linear like Boltzmann equation

Operator L:

$$
\begin{array}{|l}
Lf := \int k(v,v')f(v')\,dv' - K(v)\,f(v) - \partial_v(a(v)\,f(v)) \\
L\,G = 0,\ L^*\psi = 0,\ E := L^2(G^{-1}\psi) \\
\Sigma(L)\cap\Delta_a = \{0\} \text{ for some } -\min\nu_0 < a < 0 \\
\text{eigenspace } E_1 \text{ associated to 0 is } \mathbb{R}\,G \\
\text{spectral gap: } E = E_0 \oplus E_1 \\
\qquad \forall f \in E_0 \quad (Lf,f)_E \le a\,\|f\|^2
\end{array}
$$

Decomposition of $\mathcal{L}$:

$$
\mathcal{L}f = \overbrace{\int k^s(v,v')f(v')\,dv'}^{=:\mathcal{A}\,f}
+ \underbrace{\int k^r(v,v')f(v')\,dv' - K(v)\,f(v) - \partial_v(a(v)\,f(v))}_{=:\mathcal{B}\,f}
$$

$\mathcal{E} := L^p(g^{-1}\psi)$, $\mathcal{A} : \mathcal{E} \to E$ bounded, $\Sigma(\mathcal{B})\cap\Delta_a = \emptyset$

Motivation 2: Nonlinear evolution equation

Difficulty:

- Nonlinear equation well posed in L^1
- Spectral properties/rate of decay for the linearized equation around the steady state in the natural space of linearization $L^2(G^{-1})$
- $L^2(G^{-1})$ is too small for a well posedness theory (for large initial data)

Idea: | Spectral properties/rate of decay for the linearized equation around the steady state in L^1

Example 3: space homogeneous Boltzmann equation cf. C. Mouhot (CMP 2006)

Linearized Boltzmann operator L in $E = L^2(G^{-1}\,dv)$

$$Lf := 2\,Q(f,G) = L^{+,*}f - \nu_0\,f,$$
$$L^{+,*}f = L^+f - L^*f,\; L^+f := 2\,Q^+(f,G),\; L^*f = \Phi * f$$
$\nu_0(v) = \Phi * G \approx \langle v \rangle$, $Q^{+,*}(\cdot,G)$ is compact relatively to ν_0
$\Sigma(L) \cap \Delta_a = \{0\}$ for some $-\min \nu_0 < a < 0$
eigenspace E_1 associated to 0 of dimension $d+2$
spectral gap: $E = E_0 \oplus E_1 \qquad \forall\, f \in E_0 \quad (Lf,f)_E \le a\,\|f\|^2$

Decomposition of $\mathcal{L}$ in $\mathcal{E} := L^1(m^{-1})$, $m = \langle v \rangle^k$, $k > 2$,
or $m = e^{|v|^s}$, $s \in (0,2)$

$$\mathcal{L}f = \underbrace{\mathcal{L}^*f + \mathcal{L}^{+,s}f}_{=:\mathcal{A}\,f} + \underbrace{\mathcal{L}^{+,r}f - \nu_0\,f}_{=:\mathcal{B}\,f}$$

where $\mathcal{L}^{+,s}$ is a smooth part of $\mathcal{L}^+$ and $\mathcal{L}^{+,r}$ is a (small) remainder part
$\mathcal{A} : \mathcal{E} \to E$ bounded
$\Sigma(\mathcal{B}) \cap \Delta_a = \emptyset$

Motivation 3: Perturbation argument

We prove in [2], [3], [5]: uniqueness and linearized/nonlinear stability of the steady state for problems without "detailed balance condition" or "trivial stationary solution"

My personal favorite example: the inelastic Boltzmann equation

- steady state: $\exists\, G_e \in P(\mathbb{R}^d) \cap \mathcal{S}(\mathbb{R}^d)$ solution to

$$(E) \qquad Q_e(G_e, G_e) + (1-e)\,\Delta G_e = 0, \qquad \int G_e\, v\, dv = 0$$

- Q_e Boltzmann kernel associated to $e \in [0,1)$ inelastic coefficient
- elastic collision: $e = 1$
- ΔG_e diffuse forcing
- $G_e \approx e^{-|v|^{3/2}} \notin L^2(G^{-1})$!
- See Gamba, Panferov, Villani CMP (2004) & Bobylev, Gamba, Panferov JSP (2004)

Step 1 : uniqueness of the steady state G_e ...

- $G_e \to G_1$ when $e \to 1$ with

$$G_1 \in P(\mathbb{R}^d) \cap \mathcal{S}(\mathbb{R}^d), \quad Q(G_1, G_1) = 0, \quad \int G_1 \, v \, dv = 0.$$

$\Rightarrow$ $G_1(v) = (2\pi\theta)^{-d/2} \, e^{-\frac{|v|^2}{2\theta}}$ for some $\theta > 0$.

- $(E) \times |v|^2$ implies

$$-(1-e^2)\, D_{\mathcal{E}}(G_e) + (1-e)\, 2d \int G_e \, dv = 0$$

and in the limit $e \to 1$:

$$D_{\mathcal{E}}(G_1) := \int \int |v - v_*|^3 \, G_1(v) \, G_1(v_*) \, dv dv_* = d \;\Rightarrow\; \theta = \bar\theta.$$

- We prove more: $\exists ! \, \bar G_1$ for "any" strong norm $\| \cdot \|$ $\exists \, C$

$$\forall \, G_e \text{ solution} \quad \| G_e - \bar G_1 \| \le C \, \eta(1-e) \to 0$$

Step 1 : ... by a "implicit function argument"

- $\Phi(e, G_e) = 0$ when we define

$$\Phi(e,g) := (D_{\mathcal{E}}(g) - \frac{2d}{1+e}, Q_e(g,g) + (1-e)\,\Delta g).$$

- We define $A : \mathcal{E} \to \mathbb{R} \times \mathcal{E}_0$ invertible, $\mathcal{E} = \mathcal{E}_0 \oplus \mathcal{E}_1$, by

$$Ah := D_2\Phi(1, \bar{G}_1)\, h = [2\, D_{\mathcal{E}}(g, \bar{G}_1), \mathcal{L}\, h], \quad \mathcal{L}h := 2\, Q(\bar{G}_1, h).$$

- For two given solutions G_e and H_e of (E) :

$$\begin{aligned} G_e - H_e &= A^{-1}\, [A\, G_e - \Phi(e, G_e) + \Phi(e, H_e) - A\, H_e] \\ \Rightarrow \|G_e - H_e\| &\le \|A^{-1}\|\, \eta(1-e)\, \|G_e - H_e\| \end{aligned}$$

$\|G_e - \bar{G}_1\| = 0 \quad \text{if} \quad \|A^{-1}\|\, \eta(1-e) < 1$ we note it $\bar{G}_e$

Step 2 : linear and nonlinear stability of $\bar G_e$

- Define the inelastic linearized operator

$$\mathcal{L}_e h \;:=\; 2\,Q_e(\bar G_e, h) + (1-e)\,\Delta h \approx 2\,Q_1(\bar G_1, h) = \mathcal{L}_1 h$$

- Introduce a decomposition $\mathcal{L} = \mathcal{A} + \mathcal{B}$, $\mathcal{B}(\xi) = \mathcal{B} - \xi$, $L_1(\xi) = L_1 - \xi$, and $\mathcal{U}(\xi) := \mathcal{B}(\xi)^{-1} - L_1(\xi)^{-1}\,\mathcal{A}\,\mathcal{B}(\xi)^{-1}$, we get

$$(\mathcal{L}_e - \xi)\,\mathcal{U}(\xi) \;=\; Id - (\mathcal{L}_e - \mathcal{L}_1)\,L_1(\xi)\,A\,\mathcal{B}(\xi) \approx Id$$

if $\mathcal{A}\,h := Q^{+,*}_{e,\delta}(\bar G_e, h)$, $\mathcal{B}\,h := r_{e,\delta}(h) - \nu(\bar G_e)\,h - (1-e)\,\Delta h$

- We conclude with
  - $\Sigma(\mathcal{L}_e) \cap \Delta_a = \{\lambda_{\mathcal{E}}(e), 0\}, \quad \lambda_{\mathcal{E}}(e) \approx -(1-e)\,\bar\lambda_{\mathcal{E}} < 0$
  - $e^{t\,\mathcal{L}_e}\,(Id - \Pi_{\mathcal{L}_e,a}) = \mathcal{O}(e^{a\,t})$

Plan

Statement of the theorem

$E \subset \mathcal{E}$ Banach spaces

L, $\mathcal{L}$ generators s.t. $\mathcal{L}_{|\mathcal{E}} = L$
Hypothesis. For $a < 0$

(H0) E is a Hilbert space

(H1) **L is coercive:** $\leftarrow$ known

(i) $\Sigma(L) \cap \Delta_a = \Sigma_d(L) \cap \Delta_a = \{0\}$ (localization of the spectrum);
(ii) $L - a$ is dissipative on $R(I - \Pi_{L,a})$;

(H2) **Decomposition of $\mathcal{L}$:** $\exists\, \mathcal{A}, \mathcal{B}$ s.t. $\mathcal{L} = \mathcal{A} + \mathcal{B}$ and

(i) $\mathcal{B} - a$ is dissipative ($\Rightarrow \Sigma(\mathcal{B}) \cap \Delta_a = \emptyset$) $\leftarrow$ to be proved;
(ii) $\mathcal{A} \in B(\mathcal{E}, E)$ $\leftarrow$ to be proved;

Theorem

$$(i) \qquad \Sigma(\mathcal{L}) \cap \Delta_a = \Sigma_d(\mathcal{L}) \cap \Delta_a = \{0\}, \ \Pi_{\mathcal{L},0}|_E = \Pi_{L,0}$$

$$(ii) \qquad \forall\, a' > a, \exists\, C_{a'}\, \forall\, t \geq 0 \qquad \|e^{t\mathcal{L}} - \Pi_{\mathcal{L},0}\|_{B(\mathcal{E})} \leq C_{a'}\, e^{a'\,t}$$

Proof of the theorem
Step 1: right inverse of $\mathcal{L} - \xi$

Define $\Omega := \Delta_a \backslash \{0\}$ and on Ω: $\Lambda(\xi) := \Lambda - \xi$, $\mathcal{R}_\Lambda(\xi) = \Lambda(\xi)^{-1}$ and

$$\mathcal{U}(\xi) := \underbrace{\mathcal{R}_\mathcal{B}(\xi)}_{(\mathcal{B}-\xi)^{-1}} - \underbrace{\mathcal{R}_L(\xi)}_{(L-\xi)^{-1}} \, \mathcal{A} \, \underbrace{\mathcal{R}_\mathcal{B}(\xi)}_{(\mathcal{B}-\xi)^{-1}}$$

• $\mathcal{U}(\xi)$ is the right inverse of $\mathcal{L} - \xi$. For $\xi \in \Omega$, the operators $\mathcal{A}\mathcal{B}(\xi)^{-1}$ and $R(\xi)$ being bounded, we deduce

$$\begin{aligned} (\mathcal{L} - \xi)\mathcal{U}(\xi) &= (\mathcal{A} + (\mathcal{B} - \xi))\,\mathcal{R}_\mathcal{B}(\xi) - (\mathcal{L} - \xi)\,\mathcal{R}_L(\xi)\,\mathcal{A}\,\mathcal{R}_\mathcal{B}(\xi) \\ &= \mathcal{A}\,\mathcal{R}_\mathcal{B}(\xi) + Id_\mathcal{E} - (L - \xi)\,\mathcal{R}_L(\xi)\,\mathcal{A}\,\mathcal{R}_\mathcal{B}(\xi) \\ &= \mathcal{A}\,\mathcal{R}_\mathcal{B}(\xi) + Id_\mathcal{E} - \mathcal{A}\,\mathcal{R}_\mathcal{B}(\xi) = Id_\mathcal{E}. \end{aligned}$$

Step 2: $\mathcal{L} - \xi$ is one-to-one on Ω

- $\mathcal{L}$ generates a semigroup $\Rightarrow \exists\, \xi_0 \in \Delta_a$ s.t. $\mathcal{L} - \xi_0$ is invertible

- Neumann series:

$\mathcal{R}_{\mathcal{L}}(z_0)$ exists / is bounded by M

$$\Rightarrow \mathcal{R}_{\mathcal{L}}(z) \text{ exists on } B(z_0, 1/M)$$

$$\Rightarrow \mathcal{R}_{\mathcal{L}}(z) = \mathcal{U}(z) \text{ on } B(z_0, 1/M)$$

- A priori bound on $\mathcal{U}(\xi)$

$$\begin{aligned} \|\mathcal{U}(\xi)\|_{B(\mathcal{E})} &\le \|\mathcal{R}_{\mathcal{B}}(\xi)\|_{B(\mathcal{E})} + \|\mathcal{R}_L(\xi)\|_{B(E)} \, \|\mathcal{A}\|_{B(\mathcal{E},E)} \, \|\mathcal{R}_{\mathcal{B}}(\xi)\|_{B(\mathcal{E})} \\ &\le M \quad \text{on} \quad \Delta_a \backslash B(0,r) \end{aligned}$$

- Conclusion by a continuation argument:

$$\boxed{\mathcal{R}_{\mathcal{L}}(z) = \mathcal{R}_{\mathcal{B}}(z) - \mathcal{R}_L(z)\, \mathcal{A}\, \mathcal{R}_{\mathcal{B}}(z) \quad \text{on} \quad \Delta_a}$$

Step 3: discrete spectrum

• On Δ_a spectrum of $\mathcal{L} =$ poles of $\mathcal{R}_\mathcal{L}$ = poles of $\mathcal{R}_L = \{0\}$

• eigenspace and eigenprojector: write the Laurent series

$$\mathcal{R}_L(z) = \sum_{\ell=1}^{\ell_0} z^{-\ell} L_{-\ell} + \sum_{\ell=0}^{\infty} z^\ell L_0^\ell, \qquad \mathcal{A}\,\mathcal{R}_\mathcal{B}(z) = \sum_{j=0}^{\infty} \mathcal{C}_j z^j.$$

Then

$$\begin{aligned} \Pi_{\mathcal{L},0} &:= \frac{i}{2\pi} \int_{|z|=r} \mathcal{R}_\mathcal{L}(z)\, dz \\ &= \frac{1}{2i\pi} \int_{|z|=r} \mathcal{R}_L(z)\, \mathcal{A}\,\mathcal{R}_\mathcal{B}(z)\, dz \\ &= L_{-1}\,\mathcal{C}_0 + L_{-2}\,\mathcal{C}_1 + ... + L_{-\ell_0}\,\mathcal{C}_{\ell_0-1} \\ R(\Pi_{\mathcal{L},0}) &\subset \textit{algebraic eigenspace of } L \end{aligned}$$

Step 4: representation formula

$$\forall\, f_0 \in \mathcal{D}(\mathcal{L}),\ \forall\, t \geq 0 \qquad e^{t\mathcal{L}}\, f_0 = \Pi_{\mathcal{L},0}\, f_0 + e^{t\,\mathcal{B}}\, f_0 + T_1(t)\, f_0,$$

where

$$T_1(t) := \lim_{M\to\infty} \frac{1}{2i\pi} \int_{a-iM}^{a+iM} e^{zt}\, \mathcal{R}_L(z)\, \mathcal{A}\, \mathcal{R}_B B(z)\, dz$$

Step 4: representation formula

$$\forall\, f_0 \in \mathcal{D}(\mathcal{L}),\ \forall\, t \geq 0 \qquad e^{t\mathcal{L}}\, f_0 = \Pi_{\mathcal{L},0}\, f_0 + e^{t\,\mathcal{B}}\, f_0 + \, T_1(t)\, f_0,$$

where

$$T_1(t) := \lim_{M\to\infty} \frac{1}{2i\pi} \int_{a-iM}^{a+iM} e^{zt}\, \mathcal{R}_L(z)\, \mathcal{A}\, \mathcal{R}_B B(z)\, dz$$

proof : $\qquad e^{t\mathcal{L}}\, f_0 \ \approx\ \int_{b-i\infty}^{b+i\infty} e^{zt}\, \mathcal{R}_{\mathcal{L}}(z)\, f_0\, dz$

Step 4: representation formula

$$\forall\, f_0 \in \mathcal{D}(\mathcal{L}),\ \forall\, t \geq 0 \qquad e^{t\mathcal{L}}\, f_0 = \Pi_{\mathcal{L},0}\, f_0 + e^{t\,\mathcal{B}}\, f_0 + T_1(t)\, f_0,$$

where

$$T_1(t) := \lim_{M\to\infty} \frac{1}{2i\pi} \int_{a-iM}^{a+iM} e^{zt}\, \mathcal{R}_L(z)\, \mathcal{A}\, \mathcal{R}_B B(z)\, dz$$

proof :

$$\begin{aligned} e^{t\mathcal{L}}\, f_0 &\approx \int_{b-i\infty}^{b+i\infty} e^{zt}\, \mathcal{R}_{\mathcal{L}}(z)\, f_0\, dz \\ &\approx \Pi_{\mathcal{L},0}\, f_0 + \int_{a-i\infty}^{a+i\infty} e^{zt}\, \mathcal{R}_{\mathcal{L}}(z)\, f_0\, dz \end{aligned}$$

Step 4: representation formula

$$\forall f_0 \in \mathcal{D}(\mathcal{L}),\ \forall t \geq 0 \qquad e^{t\mathcal{L}} f_0 = \Pi_{\mathcal{L},0} f_0 + e^{t\mathcal{B}} f_0 + T_1(t) f_0,$$

where

$$T_1(t) := \lim_{M\to\infty} \frac{1}{2i\pi} \int_{a-iM}^{a+iM} e^{zt} \mathcal{R}_L(z) \mathcal{A} \mathcal{R}_B B(z)\, dz$$

proof :

$$\begin{aligned} e^{t\mathcal{L}} f_0 &\approx \int_{b-i\infty}^{b+i\infty} e^{zt} \mathcal{R}_{\mathcal{L}}(z) f_0\, dz \\ &\approx \Pi_{\mathcal{L},0} f_0 + \int_{a-i\infty}^{a+i\infty} e^{zt} \mathcal{R}_{\mathcal{L}}(z) f_0\, dz \\ &\approx \Pi_{\mathcal{L},0} f_0 + \int_{a-i\infty}^{a+i\infty} e^{zt} \mathcal{R}_{\mathcal{B}}(z) f_0\, dz \\ &\quad + \int_{a-i\infty}^{a+i\infty} e^{zt} \mathcal{R}_L(z) \mathcal{A} \mathcal{R}_{\mathcal{B}}(z) f_0\, dz \end{aligned}$$

Step 5: control of the reminder term

Thanks to Cauchy-Schwarz inequality for any $\phi \in E^* = E$

$$
\begin{aligned}
|\langle \phi, T_1(t) f_0 \rangle| &= \frac{1}{2\pi} \lim_{M\to\infty} \left| \int_{a-iM}^{a+iM} e^{z\,t} \, \langle \mathcal{R}_{L^*}(z)\,\phi, \mathcal{A}\,\mathcal{R}_{\mathcal{B}}(z)\, f_0 \rangle \, dz \right| \\
&\le \frac{e^{at}}{2\pi} \int_{-\infty}^{\infty} \|\mathcal{R}_{L^*}(a+iy)\,\phi\|_{E^*} \, \|\mathcal{A}\,\mathcal{R}_{\mathcal{B}}(a+iy)\, f_0\|_E \, dy \\
&\le \frac{e^{at}}{2\pi} \left(\int_{\mathbb{R}} \|\mathcal{R}_{L^*}(a+iy)\,\phi\|^2 \, dy \right)^{1/2} \left(\int_{\mathbb{R}} \|\mathcal{A}\,\mathcal{R}_{\mathcal{B}}(a+iy)\, f_0\|^2 \, ds \right)^{1/2}
\end{aligned}
$$

▸ 1stTerm

Step 5: control of the reminder term

Thanks to Cauchy-Schwarz inequality for any $\phi \in E^* = E$

$$
\begin{aligned}
|\langle \phi, T_1(t) f_0 \rangle| &= \frac{1}{2\pi} \lim_{M\to\infty} \left| \int_{a-iM}^{a+iM} e^{z\,t} \, \langle \mathcal{R}_{L^*}(z)\,\phi, \mathcal{A}\,\mathcal{R}_{\mathcal{B}}(z)\,f_0 \rangle \, dz \right| \\
&\le \frac{e^{at}}{2\pi} \int_{-\infty}^{\infty} \|\mathcal{R}_{L^*}(a+iy)\,\phi\|_{E^*} \, \|\mathcal{A}\,\mathcal{R}_{\mathcal{B}}(a+iy)\,f_0\|_E \, dy \\
&\le \frac{e^{at}}{2\pi} \left(\int_{\mathbb{R}} \|\mathcal{R}_{L^*}(a+iy)\,\phi\|^2 \, dy \right)^{1/2} \left(\int_{\mathbb{R}} \|\mathcal{A}\,\mathcal{R}_{\mathcal{B}}(a+iy)\,f_0\|^2 \, ds \right)^{1/2}
\end{aligned}
$$

1stTerm

$$
\le \frac{e^{at}}{2\pi} K_1 \, \|\phi\|_{E^*} \, K_2 \, \|f_0\|_E
$$

from which we conclude

$$
\|T_1(t) f_0\| \frac{e^{at}}{2\pi} K_1 \, K_2 \, \|f_0\|_E.
$$

Back

First term

Using the identity

$$R_{L^*}(a' + iy) = (Id_{E^*} + (a' - b)\, R_{L^*}(a' + iy))\, R_{L^*}(b + iy),$$

$\|R_{L^*}(a' + iy)\|_{B(E^*)} = \|R_L(a' + iy)\|_{B(E)}$ is uniformly bounded for $y \in \mathbb{R}$, the Plancherel's identity in the Hilbert space $E = E^*$ and $\|e^{tL^*}\|_{B(E^*)} = \|e^{tL}\|_{B(E)} \le C_b\, e^{b\,t}$ (taking $b > 0$), we get

$$\begin{aligned}
\int_{\mathbb{R}} \|\mathcal{R}_{L^*}(a + iy)\phi\|^2\, dy &\le C_1 \int_{\mathbb{R}} \|\mathcal{R}_{L^*}(b + iy)\phi\|^2\, dy \\
&\le 2\pi\, C_1 \int_0^{+\infty} \left\|e^{-b\,t}\, e^{t\,L^*}\,\phi\right\|^2 dt \\
&\le 2\pi\, C_1 \left(\int_0^{+\infty} \left\|e^{-b\,t}\, e^{t\,L^*}\right\|^2 dt\right) \|\phi\|^2 \\
&\le C_2\, \|\phi\|^2_{E^*}.
\end{aligned}$$

Second term

Introduce the C^1 function $\varphi_1 : \mathbb{R}_+ \to E$, $\varphi_1(t) := \mathcal{A}\, e^{t\,\mathcal{B}}\, f_0$. Its Laplace transform r_1 satisfies

$$\begin{aligned} r_1(z) &= \int_0^\infty e^{-z\,t}\,\varphi_1(t)\,dt \quad \forall\, z \in \Delta_a, \\ \varphi_1(t) &= \frac{1}{2i\pi}\int_{a-i\infty}^{a+i\infty} e^{zt}\, r_1(z)\,dz \quad \forall\, t \ge 0, \\ r_1(z) &= \mathcal{A}\,\mathcal{R}_{\mathcal{B}}(z)\, f_0. \end{aligned}$$

The Plancherel's identity in E gives ▸ Back

$$\begin{aligned} \int_{\mathbb{R}} \|\mathcal{A}\,\mathcal{R}_{\mathcal{B}}(a+iy) f_0\|_E^2\,dy &= \int_{\mathbb{R}} \|r(a'+iy)\|_E^2\,dy \\ &= 2\pi \int_0^\infty \|\varphi_1(t)\,e^{-a'\,t}\|_E^2\,dt = 2\pi \int_0^\infty \|e^{-a'\,t}\,\mathcal{A}\,e^{t\,\mathcal{B}}\,f_0\|_E^2\,dt \\ &= 2\pi \int_0^\infty K_1^2\,e^{2(a-a')\,t}\,dt\,\|f\|_{\mathcal{E}}^2 \le C_3\,\|f\|_{\mathcal{E}}^2. \end{aligned}$$

Putting together these three estimates, we conclude ▸ Step 5

Plan

What about space inhomogeneous equation?

Consider the linear Boltzmann equation

$$\partial_t f = \mathcal{L} f = -v \cdot \nabla_x f + \int k(v,v')\, f(v')\, dv' - K(v)\, f$$

Assuming k very smooth

$$\mathcal{A} f := \int k(v,v')\, f(v') \quad : \quad H^k_x(L^1_v(m^{-1})) \to H^k_x(H^\infty_v(G^{-1}))$$

$$\textit{but} \quad : \quad \mathcal{E} := W^{k,1}_x(L^1_v(m^{-1})) \not\to E := H^k_x(H^\infty_v(G^{-1}))!$$

first idea $\mathcal{A} : D(\mathcal{L}^\alpha) \subset \mathcal{E} \to E$

Implies $\|e^{t\mathcal{L}} - \Pi_{\mathcal{L},0}\|_{D(\mathcal{L}^\alpha)\to\mathcal{E}} \le C_{a'}\, e^{a'\,t}$

Implies a "conditionally exponential H-Theorem"

but not the stability result for weakly space inhomogeneous initial data

$E \subset \mathcal{E}$ Banach spaces, L, $\mathcal{L}$ generators s.t. $\mathcal{L}_{|E} = L$

Hypothesis. For $a < 0$

(H0) E is a Hilbert space

(H1) **L is hypocoercive:** ← known

(i) $\Sigma(L) \cap \Delta_a = \Sigma_d(L) \cap \Delta_a = \{0\}$ (localization of the spectrum);

(ii) $L - a$ is hypodissipative on $R(I - \Pi_{L,a})$;

(H2) **Decomposition of $\mathcal{L}$:** $\exists\ \mathcal{A}, \mathcal{B}$ s.t. $\mathcal{L} = \mathcal{A} + \mathcal{B}$ ← to be proved

(i) $\mathcal{B} - a$ is hypodissipative : $\|\mathcal{S}_B(t)\|_{B(\mathcal{E})} \le C_a\, e^{a\,t}$

(ii) $\mathcal{A} \in B(\mathcal{E})$

(iii) $T_n := (\mathcal{A}\,\mathcal{S}_{\mathcal{B}})^{(*n)}$ satisfies $\|T_n(t)\|_{B(\mathcal{E},E)} \le C_a\, e^{a\,t}$ for some $n \in \mathbb{N}^*$

n=1

Theorem

$$\forall\, a' > a, \exists\, C_{a'}\, \forall\, t \ge 0 \qquad \|e^{t\mathcal{L}} - \Pi_{\mathcal{L},0}\|_{B(\mathcal{E})} \le C_{a'}\, e^{a'\,t}$$

About the statement and the proof

Definition of the convolution:

$$(S_1 * S_2)(t) := \int_0^t S_1(s) \circ S_2(t-s)\, ds$$

Remark 1: $S_1 * S_2$ is not a semigroup but it has the good decay
Remark 2: Convolution behaves well with respect to Laplace transform

Cornerstone of the proof

$$(*) \qquad \mathcal{R}_{\mathcal{L}} = \mathcal{R}_{\mathcal{B}} \sum_{\ell=0}^{n-1} (-1)^\ell \, (\mathcal{A}\, \mathcal{R}_{\mathcal{B}})^\ell + (-1)^n \, \mathcal{R}_L \, (\mathcal{A}\, \mathcal{R}_{\mathcal{B}})^n$$

because the r.h.s. is bounded thanks to (iii)

(iii) and (*) are related to Dyson-Phillips expansion for semigroups / related to Mokhtar-Kharroubi & Sbihi works on linear Boltzmann equation

Plan

Extensions / Open problems

extensions

- $\Sigma(L) \cap \Delta_a = \Sigma_d(L) \cap \Delta_a = \{\xi_1, ..., \xi_k\}$
- E can be a general Banach space

open problems

- Existence of global solution for the free cooling inelastic inhomogeneous Boltzmann equation in the torus for weakly inhomogeneous initial data?
- Existence of global solution for the non cut-off inhomogeneous Boltzmann equation in the torus for weakly inhomogeneous initial data?