Weak KAM from a PDE point of view: viscosity solutions of the Hamilton-Jacobi equation and Aubry set

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Course CANPDE 17-19 February 2011

Contents

We will mainly introduce the notion of viscosity solutions for the Hamilton-Jacobi equation which is a first-order PDE. There also is an extensive literature on viscosity solutions of second-order PDE's, we do not touch this topic at all, see for example [CIL92].

The notion of viscosity solution is due to Crandall and Lions, see [CL83]. There are two excellent books on the subject one by by Guy Barles [Bar94] and another one by Martino Bardi and Italo Capuzzo-Dolceta [BCD97]. A first introduction to viscosity solutions can be found in Craig Evans' book [Eva98]. Our treatment has been extremely influenced by the content of these three books. Although many things are standard, we will do the theory on general manifolds since this is the right setting for weak KAM theory. This is probably the first time that a general introduction on viscosity solutions on manifolds appears in print. Whatever is not in the standard references comes from joint work with Antonio Siconolfi, see [FS04] and [FS05]. Of course, our treatment follows some of the unpublished notes [Fat08]. We hardly touch the dynamical implications of the theory, and refer the reader to Patrick Bernard's companion notes [Ber11]

We would like to apologize for the small number of references. In a work of this size, to give a fair and large set of references in the subject is nowadays an impossible task. A look at the references in [BCD97] shows that already fifteen years ago that would have been very difficult. We feel however that a larger set of references can easily be found on the web.

In these notes, we denote by M a connected, paracompact C^∞ manifold without boundary. For any $x \in M$, the tangent and cotangent spaces of M at x are T_xM and T_x^*M, respectively. The tangent and cotangent bundle are TM and T^*M, respectively. A point in TM (resp. T^*M) will be denoted by (x, v) (resp. (x, p)) where $x \in M$, and $v \in T_xM$ (resp. $p \in T_x^*M$). With this notation the canonical projection $\pi : TM \to M$ (resp. $\pi^* : T^*M \to M$) is nothing but $(x, v) \mapsto x$ (resp. $(x, p) \mapsto x$).

We will assume in the sequel that M is endowed with a C^∞ Riemannian metric g. For $v \in T_xM$, we will set $\|v\|_x = (g_x(v, v))^{1/2}$. We will also denote by $\|\cdot\|_x$ the norm on T_x^*M dual to $\|\cdot\|_x$ on T_xM.

1 The different forms of the Hamilton-Jacobi Equation

We will suppose that M is a fixed manifold, and that $H : T^*M \to \mathbb{R}$ is a continuous function, which we will call the Hamiltonian.

Definition 1.1 (Stationary HJE)**.** The Hamilton-Jacobi equation associated to H is the equation

$$H(x, d_xu) = c,$$

where c is some constant.

A first good example to keep in mind is

$$H(x, p) = \frac{1}{2}\|p\|_x^2 + V(x),$$

where the norm comes from the Riemannian metric on the manifold M, and $V : M \to \mathbb{R}$ is a continuous (even C^∞ function). An even better example is to modify H in the following way: consider a continuous vector field $X : M \to TM$, and define H by

$$H(x,p) = \frac{1}{2}\|p\|_x^2 + V(x) + p(X(x)).$$

A classical solution of the Hamilton-Jacobi equation $H(x, d_x u) = c$ (HJE in short) on the open subset U of M is a C^1 map $u : U \to \mathbb{R}$ such that $H(x, d_x u) = c$, for each $x \in U$.

We will deal usually only with the case $H(x, d_x u) = 0$, since we can reduce the general case to that case if we replace the Hamiltonian H by H_c defined by $H_c(x,p) = H(x,p) - c$.

Definition 1.2 (Evolutionary HJE)**.** The evolutionary Hamilton-Jacobi equation associated to the Hamiltonian H is the equation

$$\frac{\partial u}{\partial t}(t,x) + H\big(x, \frac{\partial u}{\partial x}(t,x)\big) = 0.$$

A classical solution to this evolutionary Hamilton-Jacobi equation on the open subset W of $\mathbb{R} \times T^*M$ is a C^1 map $u : W \to \mathbb{R}$ such that $\frac{\partial u}{\partial t}(t,x) + H\big(x, \frac{\partial u}{\partial x}(t,x)\big) = 0$, for each $(t,x) \in W$.

The evolutionary form can be reduced to the stationary form by introducing the Hamiltonian $\hat{H} : T^*(\mathbb{R} \times M)$ defined by

$$\hat{H}(t,x,s,p) = s + H(x,p),$$

where $(t,x) \in \mathbb{R} \times M$, and $(s,p) \in T^*_{(t,x)}(\mathbb{R} \times M) = \mathbb{R} \times T^*_x M$.

It is also possible to consider a time dependent Hamiltonian defined on an open subset of $\mathbb{R} \times M$. Consider for example a Hamiltonian $H : \mathbb{R} \times TM^* \to \mathbb{R}$, the evolutionary form of the HJE for that Hamiltonian is

$$\frac{\partial u}{\partial t}(t,x) + H\big(t, x, \frac{\partial u}{\partial x}(t,x)\big) = 0.$$

A classical solution of that equation on the open subset W of $\mathbb{R} \times M$ is, of course, a C^1 map $u : W \to \mathbb{R}$ such that $\frac{\partial u}{\partial t}(t,x) + H\big(t, x, \frac{\partial u}{\partial x}(t,x)\big) = 0$, for each $(t,x) \in W$. This form of the Hamilton-Jacobi equation can also be reduced to the stationary form by introducing the Hamiltonian $\tilde{H} : T^*(\mathbb{R} \times M) \to \mathbb{R}$ defined by

$$\tilde{H}(t,x,s,p) = s + H(t,x,p).$$

It is usually impossible to find global C^1 solutions of the Hamilton-Jacobi equation $H(x, d_x u) = c$. For example, if the Hamiltonian is of the form

$$H(x,p) = \frac{1}{2}\|p\|_x^2 + V(x),$$

and u is a classical solution of $H(x, d_x u) = c$, we get $c = \frac{1}{2}\|d_x u\|_x^2 + V(x) \geq V(x)$, hence $c \geq \sup_M V$. If we assume that M is compact, then u has at least two distinct critical points (minimum and maximum) x_1, x_2. At these critical points we get $c = H(x, d_{x_i}u) = V(x_i)$, since $d_{x_i}u = 0$. Therefore, on the compact manifold M, a classical solution of $H(x, d_u) = c$ for such a Hamiltonian can only occur at $c = \max V$. Moreover, if this equation has a classical solution V, then this solution V must necessarily achieve its maximum at two distinct points. In particular, if we choose V such that its maximum on the compact manifold M is achieved at a single point, then the Hamilton-Jacobi equation does not have classical solutions.

2 Viscosity Solutions

We will suppose in this section that M is a manifold and $H : T^*M \to M$ is a Hamiltonian.

Since it is generally impossible to find C^1-solutions to the Hamilton-Jacobi equation, one has to admit more general functions. A first attempt is to consider Lipschitz functions.

Definition 2.1 (Very Weak Solution)**.** We will say that $u : M \to \mathbb{R}$ is a very weak solution of $H(x, d_xu) = c$, if it is Lipschitz, and $H(x, d_xu) = c$ almost everywhere (this makes sense since the derivative of u exists almost everywhere by Rademacher's theorem).

This is too general because it gives too many solutions. A notion of weak solution is useful if it gives a unique, or at least a small number of solutions. This is not satisfied by this notion of very weak solution as can be seen in the following example.

Example 2.2. We suppose $M = \mathbb{R}$, so $T^*M = \mathbb{R} \times \mathbb{R}$, and we take $H(x, p) = p^2 - 1$. Then any continuous piecewise C^1 function u with derivative taking only the values ± 1 is a very weak solution of $H(x, d_xu) = 0$. This is already too huge, but there are even more very weak solutions. In fact, if A is any measurable subset of $\mathbb{R}$, then the function

$$f_A(x) = \int_0^x 2\chi_A(t) - 1\,dt,$$

where χ_A is the characteristic function of A, is Lipschitz with derivative ± 1 almost everywhere.

Therefore we have to define a more stringent notion of solutions. Crandall and Lions have introduced the notion of viscosity solutions, see [CL83] and [CEL84].

Definition 2.3 (Viscosity solution)**.** A function $u : V \to \mathbb{R}$ is a viscosity *subsolution* of $H(x, d_xu) = c$ on the open subset $V \subset M$, if for every C^1 function $\phi : V \to \mathbb{R}$ and every point $x_0 \in V$ such that $u - \phi$ has a *maximum* at x_0, we have $H(x_0, d_{x_0}\phi) \leq c$.

A function $u : V \to \mathbb{R}$ is a viscosity *supersolution* of $H(x, d_xu) = c$ on the open subset $V \subset M$, if for every C^1 function $\psi : V \to \mathbb{R}$ and every point $y_0 \in V$ such that $u - \psi$ has a *minimum* at y_0, we have $H(y_0, d_{y_0}\psi) \geq c$.

A function $u : V \to \mathbb{R}$ is a viscosity *solution* of $H(x, d_xu) = c$ on the open subset $V \subset M$, if it is *both* a subsolution and a supersolution.

This definition is reminiscent of the definition of distributions: since we cannot restrict to differentiable functions, we use *test* functions (namely ϕ or ψ) which are smooth and on which we can test the condition. We first see that this is indeed a generalization of classical solutions.

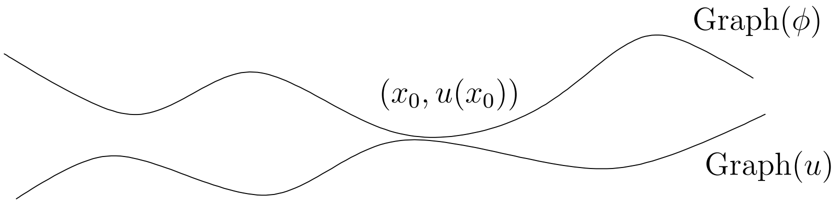

Figure 1: Subsolution: $\phi \geq u, u(x_0) = \phi(x_0) \Rightarrow H(x_0, d_{x_0}\phi) \leq c$

Theorem 2.4. *A* C^1 *function* $u : V \to \mathbb{R}$ *is a viscosity solution of* $H(x, d_x u) = c$ *on* V *if and only if it is a classical solution.*

In fact, the C^1 *function* u *is a viscosity subsolution (resp. supersolution) of* $H(x, d_x u) = c$ *on* V *if and only* $H(x, d_x u) \leq c$ *(resp.* $H(x, d_x u) \geq c$*), for each* $x \in V$*.*

Proof. We will prove the statement about the subsolution case. Suppose that the C^1 function u is a viscosity subsolution. Since u is C^1, we can use it as a test function. But $u - u = 0$, therefore every $x \in V$ is a maximum, hence $H(x, d_x u) \leq c$ for each $x \in V$.

Conversely, suppose $H(x, d_x u) \leq c$ for each $x \in V$. If $\phi : V \to \mathbb{R}$ is C^1 and $u - \phi$ has a maximum at x_0, then the differentiable function $u - \phi$ must have derivative 0 at the maximum x_0. Therefore $d_{x_0}\phi = d_{x_0}u$, and $H(x, d_{x_0}\phi) = H(x, d_{x_0}u) \leq c$. □

To get a feeling for these viscosity notions, it is better to restate slightly the definitions. We first remark that the condition imposed on the test functions (ϕ or ψ) in the definition above is on the derivative, therefore, to check the condition, we can change our test function by a constant. Suppose now that ϕ (resp. ψ) is C^1 and $u - \phi$ (resp. $u - \psi$) has a maximum (resp. minimum) at x_0 (resp. y_0), this means that $u(x_0) - \phi(x_0) \geq u(x) - \phi(x)$ (resp. $u(y_0) - \psi(y_0) \leq u(x) - \psi(x)$). As we said, since we can add to ϕ (resp. ψ) the constant $u(x_0) - \phi(x_0)$ (resp. $u(y_0) - \psi(y_0)$), these conditions can be replaced by $\phi \geq u$ (resp. $\psi \leq u$) and $u(x_0) = \phi(x_0)$ (resp. $u(y_0) = \psi(y_0)$). Therefore we obtain an equivalent definition for subsolution and supersolution.

Definition 2.5 (Viscosity Solution)**.** A function $u : V \to \mathbb{R}$ is a viscosity *subsolution* of $H(x, d_x u) = c$ on the open subset $V \subset M$, if for every C^1 function $\phi : V \to \mathbb{R}$, with $\phi \geq u$ everywhere, at every point $x_0 \in V$ where $u(x_0) = \phi(x_0)$ we have $H(x_0, d_{x_0}\phi) \leq c$, see figure 1.

A function $u : V \to \mathbb{R}$ is a viscosity *supersolution* of $H(x, d_x u) = c$ on the open subset $V \subset M$, if for every C^1 function $\psi : V \to \mathbb{R}$, with $u \geq \psi$ everywhere, at every point $y_0 \in V$ where $u(y_0) = \psi(y_0)$ we have $H(y_0, d_{y_0}\psi) \geq c$, see figure 2.

To see what the viscosity conditions mean we test them on the example 2.2 given above.

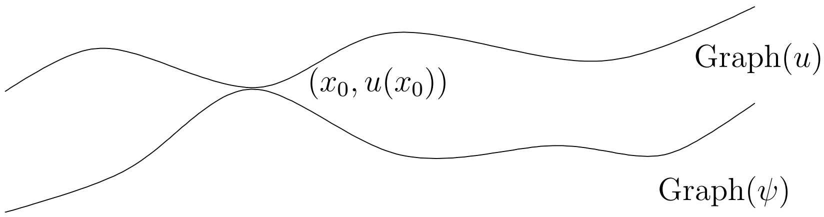

Figure 2: Supersolution: $\psi \leq u, u(x_0) = \psi(x_0) \Rightarrow H(x_0, d_{x_0}\psi) \geq c$

Example 2.6. We suppose $M = \mathbb{R}$, so $T^*M = \mathbb{R} \times \mathbb{R}$, and we take $H(x,p) = p^2 - 1$. Any Lipschitz function $u : \mathbb{R} \to \mathbb{R}$ with Lipschitz constant ≤ 1 is in fact a viscosity subsolution of $H(x, d_x u) = 0$. To check this consider ϕ a C^1 function and $x_0 \in \mathbb{R}$ such that $\phi(x_0) = u(x_0)$ and $\phi(x) \geq u(x)$, for $x \in \mathbb{R}$. We can write

$$\phi(x) - \phi(x_0) \geq u(x) - u(x_0) \geq \ -|x - x_0|.$$

For $x > x_0$, this gives

$$\frac{\phi(x) - \phi(x_0)}{x - x_0} \geq -1,$$

hence passing to the limit $\phi'(x_0) \geq -1$. On the other hand, if $x < x_0$ we obtain

$$\frac{\phi(x) - \phi(x_0)}{x - x_0} \leq 1,$$

hence $\phi'(x_0) \leq 1$.This yields $|\phi'(x_0)| \leq 1$, and therefore

$$H(x_0, \phi'(x_0)) = |\phi'(x_0)|^2 - 1 \leq 0.$$

So in fact, any very weak subsolution (i.e. a Lipschitz function u such that $H(x, d_x, u) \leq 0$ almost everywhere) is a viscosity subsolution. This is due to the fact that, in this example, the Hamiltonian is convex in p, see 10.6 below.

Of course, the two smooth functions $x \mapsto x$, and $x \mapsto -x$ are the only two classical solutions in that example. It is easy to check that the absolute value function $x \mapsto |x|$, which is a subsolution and even a solution on $\mathbb{R} \setminus \{0\}$ (where it is smooth and a classical solution), is not a viscosity solution on the whole of $\mathbb{R}$. In fact the constant function equal to 0 is less than the absolute value everywhere with equality at 0, but we have $H(0, 0) = -1 < 0$, and this violates the supersolution condition.

The function $x \mapsto -|x|$ is a viscosity solution. It is smooth and a classical solution on $\mathbb{R} \setminus \{0\}$. It is a subsolution everywhere. Moreover, any function ϕ with $\phi(0) = 0$ and $\phi(x) \leq -|x|$ everywhere cannot be differentiable at 0. This is obvious on a picture of the graphs, see figure 3. Formally, it results from the fact that both $\phi(x) - x$ and $\phi(x) + x$ have a maximum at 0.

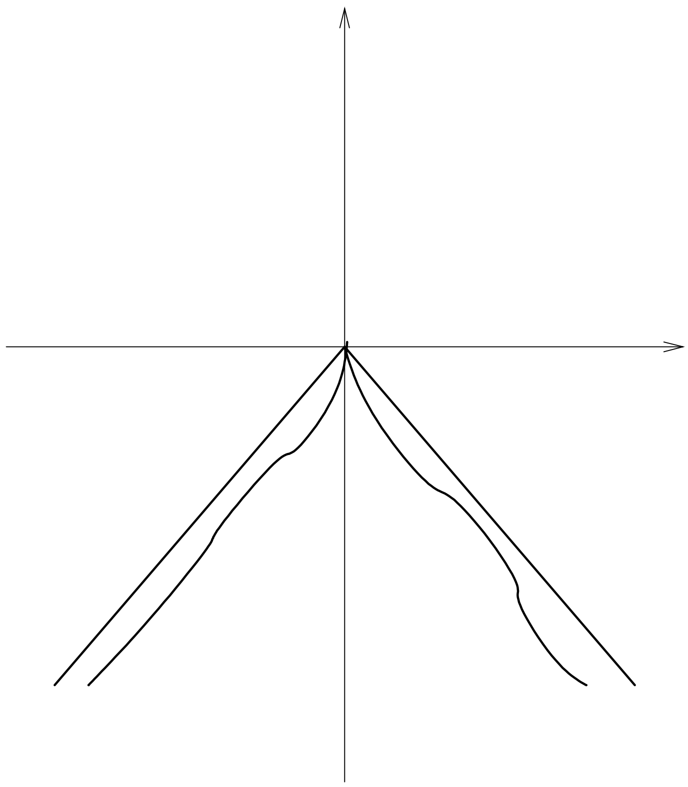

Figure 3: Graphs of $\psi(x) \leq -|x|$ with $\psi(0) = 0$.

Exercise 2.7. *Suppose $H : T^*M \to \mathbb{R}$ is a continuous Hamiltonian on M. For, $c \in \mathbb{R}$, define the Hamiltonian $H_c : M \to \mathbb{R}$ by*

$$H_c(x,p) = H(x,p) - c.$$

Show that $u : M \to \mathbb{R}$ is a viscosity subsolution (resp. supersolution, solution) of

$$H(x, d_x u) = c$$

if and only if it is a viscosity subsolution (resp. supersolution, solution) of

$$H_c(x, d_x u) = 0.$$

Exercise 2.8. *If we consider an open interval $I \subset \mathbb{R}$, then its cotangent space is canonically identified to $I \times \mathbb{R}$. We consider the Hamiltonian $H : I \times \mathbb{R} \to \mathbb{R}$ defined by $H(t,p) = p$. In this cas, for $c \in \mathbb{R}$, the Hamilton-Jacobi equation $H(t, d_t u) = c$ can be written as*

$$u'(t) = c.$$

1) Show that $u : \mathbb{R} \to \mathbb{R}$ is a viscosity subsolution (resp. supersolution) of $u'(t) = c$ if and only if $v(t) = u(t) - ct$ is a viscosity subsolution (resp. supersolution) of $v'(t) = 0$.

2) Show that any non-increasing (resp. non-decreasing) function $u : I \to \mathbb{R}$ is a viscosity subsolution (resp. supersolution) of $u'(t) = 0$.

3) More generally, for $c \in \mathbb{R}$, show that any function continuous $\rho : \mathbb{R} \to \mathbb{R}$ such that $t \mapsto \rho(t) - ct$ is non-increasing is a subsolution of $u'(t) = c$.

4) Find the classical subsolutions, supersolutions, and solutions of $u'(t) = c$.

Exercise 2.9. *Suppose $H : T^*M \to \mathbb{R}$ is a Hamiltonian, and $\phi : M \to \mathbb{R}$ is a C^1 function. Define the Hamiltonian $H_\phi : M \to \mathbb{R}$ by*

$$H_\phi(x,p) = H(x, p + d_x\phi).$$

Show that v is a subsolution (resp. supersolution, or solution) of $H_\phi(x, d_x v) = c$ if and only if $u = v + \phi$ is a a subsolution (resp. supersolution, or solution) of $H(x, d_x u) = c$.

Exercise 2.10. *Suppose $H : T^*M \to \mathbb{R}$ is a Hamiltonian. Let $u : M \to \mathbb{R}$ be a continuous function, and let $c \in \mathbb{R}$ be a constant. We define $U : \mathbb{R} \times M \to \mathbb{R}$ by*

$$U(x,t) = u(x) - ct.$$

1) Show that if u is a subsolution (resp. supersolution or solution) of the Hamilton-Jacobi equation

$$H(x, d_x u) = c, \tag{HJ}$$

then U is a viscosity subsolution (resp. supersolution or solution) of the evolutionary Hamilton-Jacobi equation

$$\frac{\partial U}{\partial t}(t,x) + H(x, \frac{\partial U}{\partial x}(t,x)) = 0, \tag{EHJ}$$

on $\mathbb{R} \times M$.

2) Conversely, if $a, b \in \mathbb{R}$, with $a < b$, and U is a viscosity subsolution (resp. supersolution, or solution) of (EHJ) on $]a, b[\times M$, then u is a a subsolution (resp. supersolution, or solution) of (HJ) on M.

3 Lower and upper differentials

We need to introduce the notion of lower and upper differentials.

Definition 3.1. If $u : M \to \mathbb{R}$ is a map defined on the manifold M, we say that the linear form $p \in T^*_{x_0}M$ is a lower (resp. upper) differential of u at $x_0 \in M$, if we can find a neighborhood V of x_0 and a function $\phi : V \to \mathbb{R}$, differentiable at x_0, with $\phi(x_0) = u(x_0)$ and $d_{x_0}\phi = p$, and such that $\phi(x) \leq u(x)$ (resp. $\phi(x) \geq u(x)$), for every $x \in V$.

We denote by $D^-u(x_0)$ (resp. $D^+u(x_0)$) the set of lower (resp. upper) differentials of u at x_0.

Exercise 3.2. *Consider the function $u : \mathbb{R} \to \mathbb{R}, x \mapsto |x|$, for each $x \in \mathbb{R}$, find $D^-u(x)$, and $D^+u(x)$. Same question with $u(x) = -|x|$.*

Definition 3.1 is not the one usually given for M an open set of an Euclidean space, see [Bar94], [BCD97] or [Cla90]. It is nevertheless equivalent to the usual definition as we now show.

Proposition 3.3. *Let $u : U \to \mathbb{R}$ be a function defined on the open subset U of $\mathbb{R}^n$, then the linear form p is in $D^-u(x_0)$ if and only if*

$$\liminf_{x \to x_0} \frac{u(x) - u(x_0) - p(x - x_0)}{\|x - x_0\|} \geq 0.$$

In the same way $p \in D^+u(x_0)$ if and only if

$$\limsup_{x \to x_0} \frac{u(x) - u(x_0) - p(x - x_0)}{\|x - x_0\|} \leq 0.$$

Proof. Suppose $p \in D^-u(x_0)$, we can find a neighborhood V of x_0 and a function $\phi : V \to \mathbb{R}$, differentiable at x_0, with $\phi(x_0) = u(x_0)$ and $d_{x_0}\phi = p$, and such that $\phi(x) \leq u(x)$, for every $x \in V$. Therefore, for $x \in V$, we can write

$$\frac{\phi(x) - \phi(x_0) - p(x - x_0)}{\|x - x_0\|} \leq \frac{u(x) - u(x_0) - p(x - x_0)}{\|x - x_0\|}.$$

Since $p = d_{x_0}\phi$ the left hand side tends to 0, when $x \to x_0$, therefore

$$\liminf_{x \to x_0} \frac{u(x) - u(x_0) - p(x - x_0)}{\|x - x_0\|} \geq 0.$$

Suppose conversely, that $p \in \mathbb{R}^{n*}$ satisfies

$$\liminf_{x \to x_0} \frac{(u(x) - u(x_0) - p(x - x_0))}{\|x - x_0\|} \geq 0.$$

We pick $r > 0$ such that the ball $\mathring{B}(x_0, r) \subset U$, and for $h \in \mathbb{R}^n$ such that $0 < \|h\| < r$, we set

$$\epsilon(h) = \min(0, \frac{u(x_0 + h) - u(x_0) - p(h)}{\|h\|}).$$

It is easy to see that $\lim_{h\to 0}\epsilon(h) = 0$. We can therefore set $\epsilon(0) = 0$. The function $\phi : \mathring{B}(x_0, r) \to \mathbb{R}$, defined by $\phi(x) = u(x_0) + p(x - x_0) + \|x - x_0\|\epsilon(x - x_0)$, is differentiable at x_0, with derivative p, it is equal to u at x_0 and satisfies $\phi(x) \le u(x)$, for every $x \in \mathring{B}(x_0, r)$. □

Proposition 3.4. *Let $u : M \to \mathbb{R}$ be a function defined on the manifold M.*

(i) *For each x in M, we have $D^+u(x) = -D^-(-u)(x) = \{-p \mid p \in D^-(-u)(x)\}$ and $D^-u(x) = -D^+(-u)(x)$.*

(ii) *For each x in M, both sets $D^+u(x), D^-u(x)$ are closed convex subsets of T^*_xM.*

(iii) *If u is differentiable at x, then $D^+u(x) = D^-u(x) = \{d_xu\}$.*

(iv) *If both sets $D^+u(x), D^-u(x)$ are non-empty then u is differentiable at x.*

(v) *if $v : M \to \mathbb{R}$ is a function with $v \le u$ and $v(x) = u(x)$, then $D^-v(x) \subset D^-u(x)$ and $D^+v(x) \supset D^+u(x)$.*

(vi) *If U is an open convex subset of an Euclidean space and $u : U \to \mathbb{R}$ is convex then $D^-u(x)$ is the set of subdifferentials of u at $x \in U$. In particular $D^+u(x) \neq \emptyset$ if and only if u is differentiable at x.*

(vii) *Suppose that d is the distance obtained from the Riemannian metric g on M. If $u : M \to \mathbb{R}$ is Lipschitz for d with Lipschitz constant $\mathrm{Lip}(u)$, then for any $p \in D^\pm u(x)$ we have $\|p\|_x \le \mathrm{Lip}(u)$.*

In particular, if M is compact then the sets $D^\pm u = \{(x, p) \mid p \in D^\pm u(x), x \in M\}$ are compact.

Proof. Part (i) and the convexity claim in part (ii) are obvious from the definition 3.1.

To prove the fact that $D^+u(x_0)$ is closed for a given for $x_0 \in M$, we can assume that M is an open subset of $\mathbb{R}^k$. We will apply proposition 3.3. If $p_n \in D^+u(x_0)$ converges to $p \in \mathbb{R}^{k*}$, we can write

$$\frac{u(x) - u(x_0) - p(x - x_0)}{\|x - x_0\|} \le \frac{u(x) - u(x_0) - p_n(x - x_0)}{\|x - x_0\|} + \|p_n - p\|.$$

Fixing n, and letting $x \to x_0$, we obtain

$$\limsup_{x\to x_0} \frac{u(x) - u(x_0) - p(x - x_0)}{\|x - x_0\|} \le \|p_n - p\|.$$

If we let $n \to \infty$, we see that $p \in D^+u(x_0)$.

We now prove (iii) and (iv) together. If u is differentiable at $x_0 \in M$ then obviously $d_{x_0}u \in D^+u(x_0) \cap D^-u(x_0)$. Suppose now that both $D^+u(x_0)$ and $D^-u(x_0)$ are both not empty, pick $p_+ \in D^+u(x_0)$ and $p_- \in D^-u(x_0)$. For h small, we have

$$p_-(h) + \|h\|\epsilon_-(h) \le u(x_0 + h) - u(x_0) \le p_+(h) + \|h\|\epsilon_+(h), \tag{*}$$

where both $\epsilon_-(h)$ and $\epsilon_+(h)$ tend to 0, a $h \to 0$. If $v \in \mathbb{R}^n$, for $t > 0$ small enough, we can replace h by tv in the inequalities (*) above. Forgetting the middle term and dividing by t, we obtain

$$p_-(v) + \|v\|\epsilon_-(tv) \leq p_+(v) + \|v\|\epsilon_+(tv),$$

letting t tend to 0, we see that $p_-(v)+ \leq p_+(v)$, for every $v \in \mathbb{R}^n$. Replacing v by $-v$ gives the reverse inequality $p_+(v) \leq p_-(v)$, therefore $p_- = p_+$. This implies that both $D^+u(x_0)$ and $D^-u(x_0)$ are reduced to the same singleton $\{p\}$. The inequality (*) above now gives

$$p(h) + \|h\|\epsilon_-(h) \leq u(x_0 + h) - u(x_0) \leq p(h) + \|h\|\epsilon_+(h),$$

this clearly implies that p is the derivative of u at x_0.

Part (v) follows routinely from the definition.

To prove (vi), we remark that by convexity $u(x_0 + th) \leq (1 - t)u(x_0) + tu(x_0 + h)$, therefore

$$u(x_0 + h) - u(x_0) \geq \frac{u(x_0 + th) - u(x_0)}{t}.$$

If p is a linear form we obtain

$$\frac{u(x_0 + h) - u(x_0) - p(h)}{\|h\|} \geq \frac{u(x_0 + th) - u(x_0) - p(h)}{\|th\|}.$$

If $p \in D^-u(x_0)$, then the $\liminf$ as $t \to 0$ of the right hand side is ≥ 0, therefore $u(x_0 + h) - u(x_0) - p(h) \geq 0$, which shows that p is a subdifferential. Conversely, a subdifferential is clearly a lower differential.

It remains to prove (vii). Suppose, for example that $\phi : V \to \mathbb{R}$ is defined on some neighborhood V of a given $x_0 \in M$, that it is differentiable at x_0, and that $\phi \geq u$ on V, with equality at x_0. If $v \in T_{x_0}M$ is given, we pick a C^1 path $\gamma : [0, \delta] \to V$, with $\delta > 0, \gamma(0) = x_0$, and $\dot\gamma(0) = v$. We have

$$\begin{aligned}\forall t \in [0, \delta], |u(\gamma(t)) - u(x_0)| &\leq \operatorname{Lip}(u) d(\gamma(t), x_0)\\ &\leq \operatorname{Lip}(u) \int_0^t \|\dot\gamma(s)\|\, ds.\end{aligned}$$

Therefore $u(\gamma(t)) - u(x_0) \geq -\operatorname{Lip}(u) \int_0^t \|\dot\gamma(s)\|\, ds$. Since $\phi \geq u$ on V, with equality at x_0, it follows that

$$\phi(\gamma(t)) - \phi(x_0) \geq -\operatorname{Lip}(u) \int_0^t \|\dot\gamma(s)\|\, ds.$$

Dividing by $t > 0$, and letting $t \to 0$, we get

$$d_{x_0}\phi(v) \geq -\operatorname{Lip}(u)\|v\|.$$

Since $v \in T_{x_0}M$ is arbitrary, we can change v into $-v$ in the inequality above to conclude that we also have

$$d_{x_0}\phi(v) \leq \operatorname{Lip}(u)\|v\|.$$

It then follows that $\|d_{x_0}\phi\| \leq \operatorname{Lip}(u)$. □

Exercise 3.5. *Suppose V is an open subset of M, and $u : V \to \mathbb{R}$ is a continuous function.*

1) Show that we can find a C^∞ function $\phi : V \to \mathbb{R}$ such that $\phi \geq u$ (resp. $\phi \leq u$) everywhere. [Indication: Pick a C^∞ partition of unity $(\varphi_i)_{i\in I}$ such that the support of each φ_i is compact, and consider c_i the maximum of u on the compact support of φ_i.]

2) Suppose that moreover $\epsilon : V \to]0, +\infty[$ is a continuous function, show that one can find a C^∞ function $\phi : V \to \mathbb{R}$ such that $u \leq \phi \leq u + \epsilon$.

Lemma 3.6. *If $u : M \to \mathbb{R}$ is continuous and $p \in D^+u(x_0)$ (resp. $p \in D^-u(x_0)$), there exists a C^1 function $\phi : M \to \mathbb{R}$, such that $\phi(x_0) = u(x_0), d_{x_0}\phi = p$, and $\phi(x) > u(x)$ (resp. $\phi(x) < u(x)$) for $x \neq x_0$.*

Moreover, if W is any neighborhood of x_0 and $C > 0$, we can choose ϕ such that $\phi(x) \geq u(x) + C$, for $x \notin W$ (resp. $\phi(x) \leq u(x) - C$).

Proof. Assume first $M = \mathbb{R}^k$. To simplify notations, we can assume $x_0 = 0$. Moreover, subtracting from u the affine function $x \mapsto u(0) + p(x)$. We can assume $u(0) = 0$ and $p = 0$. The fact that $0 \in D^+u(0)$ gives

$$\limsup_{x\to 0} \frac{u(x)}{\|x\|} \leq 0.$$

If we take the non-negative part $u^+(x) = \max(u(x), 0)$ of u, this gives

$$\lim_{x\to 0} \frac{u^+(x)}{\|x\|} = 0. \tag{$\spadesuit$}$$

If we set

$$c_n = \sup\{u^+(x) \mid 2^{-(n+1)} \leq \|x\| \leq 2^{-n}\}$$

then c_n is finite and ≥ 0, because $u^+ \geq 0$ is continuous. Moreover, using that $2^n u_+(x) \leq u^+(x)/\|x\|$, for $\|x\| \leq 2^{-n}$, and the limit in ($\spadesuit$) above, we obtain

$$\lim_{n\to\infty} [\sup_{m\geq n} 2^m c_m] = 0. \tag{$\heartsuit$}$$

We now consider $\theta : \mathbb{R}^k \to \mathbb{R}$ a C^∞ bump function with $\theta = 1$ on the set $\{x \in \mathbb{R}^k \mid 1/2 \leq \|x\| \leq 1\}$, and whose support is contained in $\{x \in \mathbb{R}^k \mid 1/4 \leq \|x\| \leq 2\}$. We define the function $\psi : \mathbb{R}^k \to \mathbb{R}$ by

$$\psi(x) = \sum_{n\in\mathbb{Z}} (c_n + 2^{-2n})\theta(2^n x).$$

This function is well defined at 0 because every term is then 0. For $x \neq 0$, we have $\theta(2^n x) \neq 0$ only if $1/4 < \|2^n x\| < 2$. Taking the logarithm in base 2, we see that this can happen only if $-2 - \log_2\|x\| < n < 1 - \log_2\|x\|$. Therefore this can happen for at most 3 consecutive integers n, hence the sum is also well defined for $x \neq 0$. Moreover, if $x \neq 0$, the set $V_x = \{y \neq 0 \mid -1 - \log_2\|x\| < -\log_2\|y\| < 1 - \log_2\|x\|\}$ is a neighborhood of x and

$$\forall y \in V_y, \psi(y) = \sum_{-3-\log_2\|x\|<n<2-\log_2\|x\|} (c_n + 2^{-2n})\theta(2^n y). \tag{*}$$

This sum is finite with at most 5 terms, therefore θ is C^∞ on $\mathbb{R}^k \setminus \{0\}$.

We now check that ψ is continuous at 0. Using equation (*), and the limit ($\heartsuit$) we see that

$$\begin{aligned} 0 \leq \psi(x) &\leq \sum_{-3-\log_2\|x\|<n<2-\log_2\|x\|} (c_n + 2^{-2n}) \\ &\leq 5 \sup_{n>-3-\log_2\|x\|} (c_n + 2^{-2n}) \to 0 \text{ as } x \to 0. \end{aligned}$$

To show that ψ is C^1 on the whole of $\mathbb{R}^k$ with derivative 0 at 0, it suffices to show that $d_x\psi$ tends to 0 as $\|x\| \to 0$. Differentiating equation (*) we see that

$$d_x\psi = \sum_{-3-\log_2\|x\|<n<2-\log_2\|x\|} (c_n + 2^{-2n})2^n d_{2^n x}\theta.$$

Since θ has compact support $K = \sup_{x\in\mathbb{R}^n} \|d_x\theta\|$ is finite. The equality above and the limit in ($\heartsuit$) give

$$\|d_x\psi\| \leq 5K \sup\{2^n c_n + 2^{-n} \mid n > -3 - \log_2\|x\|\},$$

but the right hand side goes to 0 when $\|x\| \to 0$.

We now show $\psi(x) > u(x)$, for $x \neq 0$. There is an integer n_0 such that $\|x\| \in [2^{-n_0+1}, 2^{-n_0}]$, hence $\theta(2^{n_0}x) = 1$ and $\psi(x) \geq \theta(2^{n_0}x)(c_{n_0} + 2^{-2n_0}) \geq c_{n_0} + 2^{-2n_0}$, since $c_{n_0} = \sup\{u^+(y) \mid \|y\| \in [2^{(-n_0+1)}, 2^{-n_0}]\}$, we obtain $c_{n_0} \geq u^+(x)$ and therefore $\psi(x) > u^+(x) \geq u(x)$.

It remains to show that we can get rid of the assumption $M = \mathbb{R}^k$, and to show how to obtain the desired inequality on the complement of W. We pick a small open neighborhood $U \subset W$ of x_0 which is diffeomorphic to an Euclidean space. By what we have done, we can find a C^1 function $\psi : U \to \mathbb{R}$ with $\psi(x_0) = u(x_0), d_{x_0}\psi = p$, and $\psi(x) > u(x)$, for $x \in U \setminus \{x_0\}$. We then take a C^∞ bump function $\varphi : M \to [0,1]$ which is equal to 1 on a neighborhood of x_0 and has compact support contained in $U \subset W$. By exercise 3.5, we can find a C^∞ function $\tilde{\psi} : M \to \mathbb{R}$ such that $\tilde{\psi} \geq u + C$. It is easy to check that the function $\phi : M \to \mathbb{R}$ defined by $\phi(x) = (1 - \varphi(x))\tilde{\psi}(x) + \varphi(x)\psi(x)$ has the required property. □

The following simple lemma is very useful.

Lemma 3.7. *Suppose $\psi : M \to \mathbb{R}$ is C^r, with $r \geq 0$. If $x_0 \in M, C \geq 0$, and W is a neighborhood of x_0, there exist two C^r functions $\psi_+, \psi_- : M \to \mathbb{R}$, such that $\psi_+(x_0) = \psi_-(x_0) = \psi(x_0)$, and $\psi_+(x) > \psi(x) > \psi_-(x)$, for $x \neq x_0$. Moreover $\psi_+(x) - C > \psi(x) > \psi_-(x) + C$, for $x \notin W$. If $r \geq 1$, then necessarily $d_{x_0}\psi_+ = d_{x_0}\psi_- = d_{x_0}\tilde{\psi}$*

Proof. The last fact is clear since $\psi_+ - \psi$ (resp. $\psi_- - \psi$) achieves a minimum (resp. maximum) at x_0.

Using the same arguments as in the end of the proof in the previous lemma to obtain the general case, it suffices to assume $C = 0$ and $M = \mathbb{R}^n$. In that case, we can take $\psi_\pm(x) = \psi(x) \pm \|x - x_0\|^2$. □

4 Criteria for viscosity solutions

We fix in this section a continuous function $H : T^*M \to \mathbb{R}$.

Theorem 4.1. *Let $u : M \to \mathbb{R}$ be a continuous function.*

(i) *u is a viscosity subsolution of $H(x, d_x u) = 0$ if and only if for each $x \in M$, and each $p \in D^+u(x)$, we have $H(x, p) \leq 0$.*

(ii) *u is a viscosity supersolution of $H(x, d_x u) = 0$ if and only if for each $x \in M$, and each $p \in D^-u(x)$, we have $H(x, p) \geq 0$.*

Proof. Suppose that u is a viscosity subsolution. If $p \in D^+u(x)$, since u is continuous, it follows from 3.6 that there exists a C^1 function $\phi : M \to \mathbb{R}$, with $\phi \geq u$ on M, $u(x) = \phi(x)$ and $d_x\phi = p$. By the viscosity subsolution condition $H(x, p) = H(x, d_x\phi) \leq 0$, .

Suppose conversely that for each $x \in M$ and each $p \in D^+u(x_0)$ we have $H(x, p) \leq 0$. If $\phi : M \to \mathbb{R}$ is C^1 with $u \leq \phi$, then at each point x where $u(x) = \phi(x)$, we have $d_x\phi \in D^+u(x)$ and therefore $H(x, d_x\phi) \leq 0$. □

Since $D^{\pm}u(x)$ depends only on the values of u in a neighborhood of x, the following corollary is now obvious. It shows the local nature of the viscosity conditions.

Corollary 4.2. *Let $u : M \to \mathbb{R}$ be a continuous function.*

If u is a viscosity subsolution (resp. supersolution, solution) of $H(x, d_x u) = 0$ on M, then any restriction $u_{|U}$ to an open subset $U \subset M$ is itself a viscosity subsolution (resp. supersolution, solution) of $H(x, d_x u) = 0$ on U.

Conversely, if there exists an open cover $(U_i)_{i \in I}$ of M such that every restriction $u_{|U_i}$ is a viscosity subsolution (resp. supersolution, solution) of $H(x, d_x u) = 0$ on U_i, then u itself is a viscosity subsolution (resp. supersolution, solution) of $H(x, d_x u) = 0$ on M.

Here is another straightforward consequence of theorem 4.1.

Corollary 4.3. *Let $u : M \to \mathbb{R}$ be a locally Lipschitz function. If u is a viscosity subsolution (resp. supersolution, solution) of $H(x, d_x u) = 0$, then $H(x, d_x u) \leq 0$ (resp. $H(x, d_x u) \geq 0, H(x, d_x u) = 0$) for almost every $x \in M$.*

In particular, a locally Lipschitz viscosity solution is always a very weak solution.

Exercise 4.4. *Let $I \subset \mathbb{R}$, and consider $u : I \to \mathbb{R}$ a viscosity subsolution of*

$$u'(t) = 0.$$

We want to show that u is non-increasing.

Fix $a < b$ with $a, b \in I$. For every $\epsilon > 0$ consider the function $\theta_\epsilon : [a, b[\to \mathbb{R}$ defined by

$$\theta_\epsilon(t) = \frac{\epsilon}{b - t}.$$

1) Show that $u - \theta_\epsilon$ cannot have a local maximum in the open interval $]a, b[$.

2) Show that $u(t) \leq u(a) + \theta_\epsilon(t) - \theta_\epsilon(a)$, for every $t \in [a, b[$. Conclude that u is non-increasing.

3) What are the supersolutions (resp. solutions) of $u'(t) = 0$.

4) For $c \in \mathbb{R}$, characterize the viscosity subsolutions, supersolutions, and solutions of $u'(t) = c$.

We end this section with one more characterization of viscosity solutions.

Proposition 4.5 (Criterion for viscosity solution)**.** *Suppose that $u : M \to \mathbb{R}$ is continuous. To check that u is a viscosity subsolution (resp. supersolution) of $H(x, d_x u) = 0$, it suffices to show that for each C^∞ function $\phi : M \to \mathbb{R}$ such that $u - \phi$ has a unique strict global maximum (resp. minimum), attained at x_0, we have $H(x_0, d_{x_0}\phi) \leq 0$ (resp. $H(x_0, d_{x_0}\phi) \geq 0$).*

Proof. We treat the subsolution case. We first show that if $\phi : M \to \mathbb{R}$ is a C^∞ function such that $u - \phi$ achieves a (not necessarily strict) maximum at x_0, then we have $H(x_0, d_{x_0}\phi) \leq 0$. In fact applying 3.7, we can find a C^∞ function $\phi_+ : M \to \mathbb{R}$ such that $\phi_+(x_0) = \phi(x_0), d_{x_0}\phi_+ = d_{x_0}\phi, \phi_+(x) > \phi(x)$, for $x \neq x_0$. The function $u - \phi_+$ has a unique strict global maximum achieved at x_0, therefore $H(x_0, d_{x_0}\phi_+) \leq 0$. Since $d_{x_0}\phi_+ = d_{x_0}\phi$, this finishes our claim.

Suppose now that $\psi : M \to \mathbb{R}$ is C^1 and that $u - \psi$ has a global maximum at x_0, we must show that $H(x_0, d_{x_0}\psi) \leq 0$. We fix a *relatively compact* open neighborhood W of x_0. By Lemma 3.7, applied to the continuous function ψ, there exists a C^1 function $\psi_+ : M \to \mathbb{R}$ such that $\psi_+(x_0) = \psi(x_0), d_{x_0}\psi_+ = d_{x_0}\psi, \psi_+(x) > \psi(x)$, for $x \neq x_0$, and even $\psi_+(x) > \psi(x) + 3$, for $x \notin W$. It is easy to see that $u - \psi_+$ has a strict global maximum at x_0, and that $u(x) - \psi_+(x) < u(x_0) - \psi_+(x_0) - 3$, for $x \notin W$. By smooth approximations, we can find a sequence of C^∞ functions $\phi_n : M \to \mathbb{R}$ such that ϕ_n converges to ψ_+ in the C^1 topology uniformly on compact subsets, and $\sup_{x \in M} |\phi_n(x) - \psi_+(x)| < 1$. This last condition together with $u(x) - \psi_+(x) < u(x_0) - \psi_+(x_0) - 3$, for $x \notin W$, gives $u(x) - \phi_n(x) < u(x_0) - \phi_n(x_0) - 1$, for $x \notin W$. This implies that the maximum of $u - \phi_n$ on the compact set $\bar{W}$ is a global maximum of $u - \phi_n$. Choose $y_n \in \bar{W}$ where $u - \phi_n$ attains its global maximum. Since ϕ_n is C^∞, from the beginning of the proof we must have $H(y_n, d_{y_n}\phi_n) \leq 0$. Extracting a subsequence, if necessary, we can assume that y_n converges to $y_\infty \in \bar{W}$. Since ϕ_n converges to ψ_+ uniformly on the compact set $\bar{W}$, necessarily $u - \psi_+$ achieves its maximum on $\bar{W}$ at y_∞. This implies that $y_\infty = x_0$, because the strict global maximum of $u - \tilde{\psi}$ is precisely attained at $x_0 \in W$. The convergence of ϕ_n to ψ_+ is in the C^1 topology, therefore $(y_n, d_{y_n}\phi_n) \to (x_0, d_{x_0}\psi_+)$, and hence $H(y_n, d_{y_n}\phi_n) \to H(x_0, d_{x_0}\psi_+)$, by continuity of H. But, using $H(y_n, d_{y_n}\phi_n) \leq 0$ and $d_{x_0}\psi = d_{x_0}\psi_+$, we get $H(x_0, d_{x_0}\psi) \leq 0$. □

5 Coercive Hamiltonians

Definition 5.1 (Coercive)**.** A continuous function $H : T^*M \to \mathbb{R}$ is said to be coercive above every compact subset, if for each compact subset $K \subset M$ and each $c \in \mathbb{R}$ the set $\{(x, p) \in T^*M \mid x \in K, H(x, p) \leq c\}$ is compact.

Choosing any Riemannian metric on M, it is not difficult to see that H is coercive, if and only if for each compact subset $K \subset M$, we have $\lim_{\|p\|_x \to \infty} H(x, p) = +\infty$ the limit being uniform in $x \in K$.

Theorem 5.2. *Suppose that $H : T^*M \to \mathbb{R}$ is coercive above every compact subset, and $c \in \mathbb{R}$ then a viscosity subsolution of $H(x, d_x u) = c$ is necessarily locally Lipschitz, and therefore satisfies $H(x, d_x u) \leq c$ almost everywhere.*

Proof. Since this is a local result we can assume $M = \mathbb{R}^k$, and prove only that u is Lipschitz on a neighborhood of the origin 0. We will consider the usual distance d given by $d(x,y) = \|y-x\|$, where we have chosen the usual Euclidean norm on $\mathbb{R}^k$. We set

$$\ell_0 = \sup\{\|p\| \mid p \in \mathbb{R}^{k*}, \exists x \in \mathbb{R}^k, \|x\| \leq 3, H(x,p) \leq c\}.$$

We have $\ell_0 < +\infty$ by the coercivity condition. Suppose $u : \mathbb{R}^k \to \mathbb{R}$ is a subsolution of $H(x, d_x u) = c$. Choose $\ell \geq \ell_0 + 1$ such that

$$2\ell > \sup\{|u(y) - u(x)| \mid x, y \in \mathbb{R}^k, \|x\| \leq 3, \|y\| \leq 3\}.$$

Fix x, with $\|x\| \leq 1$, and define $\phi : \mathbb{R}^k \to \mathbb{R}$ by $\phi(y) = \ell\|y-x\|$. Pick $y_0 \in \bar{B}(x,2)$ where the function $y \mapsto u(y) - \phi(y)$ attains its maximum for $y \in \bar{B}(x,2)$. We first observe that y_0 is not on the boundary of $\bar{B}(x,2)$. In fact, if $\|y-x\| = 2$, we have $u(y) - \phi(y) = u(y) - 2\ell < u(x) = u(x) - \phi(x)$. In particular, the point y_0 is a local maximum of $u - \phi$. If y_0 is not equal to x, then $d_{y_0}\phi$ exists, with $d_{y_0}\phi(v) = \ell\langle y_0 - x, v\rangle / \|y_0 - x\|$, and we obtain $\|d_{y_0}\phi\| = \ell$. On the other hand, since $u(y) \leq u(y_0) - \phi(y_0) + \phi(y)$, for y in a neighborhood of y_0, we get $d_{y_0}\phi \in D^+u(y_0)$, and therefore have $H(y_0, d_{y_0}\phi) \leq c$. By the choice of ℓ_0, this gives $\|d_{y_0}\phi\| \leq \ell_0 < \ell_0 + 1 \leq \ell$. This contradiction shows that $y_0 = x$, hence $u(y) - \ell\|y-x\| \leq u(x)$, for every x of norm ≤ 1, and every $y \in \bar{B}(x,2)$. This implies that u has Lipschitz constant $\leq \ell$ on the unit ball of $\mathbb{R}^k$. □

It is important to notice that for the evolutionary Hamilton-Jacobi Equation there are subsolutions which are not locally Lipschitz even if the coercive Hamiltonian is very simple.

Exercise 5.3. *We consider the coercive Hamiltonian $H : T^*M \to \mathbb{R}$ defined by*

$$H(x,p) = \frac{1}{2}\|p\|_x^2.$$

If $\rho : \mathbb{R} \to \mathbb{R}$ is a non-increasing function, show that $u(x,t) = \rho(t)$ is a viscosity subsolution of

$$\frac{\partial u}{\partial t}(x,t) + H\left(x, \frac{\partial u}{\partial x}(x,t)\right) = 0.$$

Give an example of such a ρ which is not locally Lipschitz.

6 Stability

Theorem 6.1 (Stability)**.** *Suppose that the sequence of continuous functions $H_n : T^*M \to \mathbb{R}$ converges uniformly on compact subsets to $H : T^*M \to \mathbb{R}$. Suppose also that $u_n : M \to \mathbb{R}$ is a sequence of continuous functions converging uniformly on compact subsets to $u : M \to \mathbb{R}$. If, for each n, the function u_n is a viscosity subsolution (resp. supersolution, solution) of $H_n(x, d_x u_n) = 0$, then u is a viscosity subsolution (resp. supersolution, solution) of $H(x, d_x u) = 0$.*

Proof. We show the subsolution case. We use the criterion 4.5. Suppose that $\phi : M \to \mathbb{R}$ is a C^∞ function such that $u - \phi$ has a unique strict global maximum, achieved at x_0, we have to show $H(x_0, d_{x_0}\phi) \leq 0$. We pick a relatively compact open neighborhood W of x_0. For each n, choose $y_n \in \bar{W}$ where $u_n - \phi$ attains its maximum on the compact subset $\bar{W}$. Extracting a subsequence, if necessary, we can assume that y_n converges to $y_\infty \in \bar{W}$. Since u_n converges to u uniformly on the compact set $\bar{W}$, necessarily $u - \phi$ achieves its maximum on $\bar{W}$ at y_∞. But $u - \phi$ has a strict global maximum at $x_0 \in W$ therefore $y_\infty = x_0$. By continuity of the derivative of ϕ, we obtain $(y_n, d_{y_n}\phi) \to (x_0, d_{x_0}\phi)$. Since W is an open neighborhood of x_0, dropping the first terms if necessary, we can assume $y_n \in W$, this implies that y_n is a local maximum of $u_n - \phi$, therefore $d_{y_n}\phi \in D^+u_n(y)$. Since u_n is a viscosity subsolution of $H_n(x, d_x u_n) = 0$, we get $H_n(y_n, d_{y_n}\phi) \leq 0$. The uniform convergence of H_n on compact subsets now implies $H(x_0, d_{x_0}\phi) = \lim_{n\to\infty} H_n(y_n, d_{y_n}\phi) \leq 0$. □

Exercise 6.2. *We consider the Hamiltonian $H : T^*M \to \mathbb{R}$ on the manifold M. Suppose $U : [0, +\infty[\times M$ is a viscosity subsolution of the evolutionary Hamilton-Jacobi equation*

$$\frac{\partial U}{\partial t}(t, x) + H(x, \frac{\partial U}{\partial x}(t, x)) = 0. \tag{EHJ}$$

on $]0, +\infty[\times M$.

1) If $\rho : [0, +\infty[\to \mathbb{R}$ is a non-increasing C^1function show that $U_\rho : [0, +\infty[\times M \to \mathbb{R}$ defined by

$$U_\rho(t, x) = U(x, t) + \rho(t)$$

is also a viscosity subsolution of (EHJ) on $]0, +\infty[\times M$.

2) If $\rho : [0, +\infty[\to \mathbb{R}$ is an arbitrary non-increasing continuous function show that it can be uniformly approximated on compact subsets by C^∞ non-increasing functions. [Indication: Use a convolution argument.]

3) Show that 1) remains true for arbitrary non-increasing continuous function $\rho : [0, +\infty[\to \mathbb{R}$.

4) Show that U can be uniformly approximated on compact subsets by viscosity subsolutions of (HEJ) which are not locally Lipschitz.

7 Uniqueness

Our goal here is to obtain some uniqueness results especially for the evolutionary Hamilton-Jacobi Equation.

Theorem 7.1. *Let $H : T^*M \to \mathbb{R}$ be a Hamiltonian on the manifold M. Suppose that $u : M \to \mathbb{R}$ is a viscosity subsolution of $H(x, d_x u) = c_1$, and $v : M \to \mathbb{R}$ is a viscosity supersolution of $H(x, d_x v) = c_2$. Assume further that either u or v is locally Lipschitz on M. If $u - v$ has a local maximum, then necessarily $c_2 \leq c_1$.*

Proof. Call $x_0 \in M$ a point where $u - v$ achieves a local maximum. Changing u (or v) by adding an appropriate constant, we can assume that this local maximum of $u - v$ is 0. This means that $u \leq v$ in a neighborhood of x_0, with equality at x_0. If both u and

v where differentiable at x_0, we would have $d_{x_0}(u-v)=0$, therefore $d_{x_0}u=d_{x_0}v$, and $c_2\le H(x_0,d_{x_0}v)=H(x_0,d_{x_0}u)\le c_1$. Since we do not know that these derivatives exist, we have to go around this difficulty. The following argument is known in viscosity theory as the doubling argument. The problem is essentially local around x_0. Hence, choosing a chart we can assume $x_0=0$, and $M=\mathbb{R}^n$.

Call $\|\cdot\|$ the usual Euclidean norm in $\mathbb{R}^n$, denote by $\mathbb{B}^n$ the usual unit ball in $\mathbb{R}^n$. We will also use the canonical identification $T^*\mathbb{R}^n=\mathbb{R}^n\times\mathbb{R}^n$. In this identification, the differential of a function is nothing but its gradient.

Since either u or v are locally Lipschitz on $M=\mathbb{R}^n$, and $\mathbb{B}^n$ is a compact subset we can assume that there exits a a constant $K<+\infty$ such that either u or v is Lipschitz on $\mathbb{B}^n$ with Lipschitz constant K.

We know that $u\le v$ with equality at 0. Also $u:\mathbb{R}^n\to\mathbb{R}$ is a viscosity subsolution of $H(x,d_xu)=c_1$, and $v:\mathbb{R}^n\to\mathbb{R}$ is a viscosity supersolution of $H(x,d_xu)=c_2$. We want to show that $c_2\le c_1$. For $\ell\ge 1$, we set

$$m_\ell=\sup_{x,y\in\mathbb{B}^n} u(x)-v(y)-\|x\|^2-\ell\|x-y\|^2. \tag{a}$$

Note that $m_\ell\ge 0$, since $u(0)=v(0)$. By compactness of $\mathbb{B}^n$, we can find $x_\ell,y_\ell\in\mathbb{B}^n$ such that

$$0\le m_\ell=u(x_\ell)-v(y_\ell)-\|x_\ell\|^2-\ell\|x_\ell-y_\ell\|^2. \tag{b}$$

By compactness of $\mathbb{B}^n$, we have $A=\sup_{x,y\in\mathbb{B}^n}u(x)-v(y)<+\infty$. It follows that

$$0\le m_\ell\le A-\ell\|x_\ell-y_\ell\|^2.$$

This implies that $\|x_\ell-y_\ell\|^2\le A/\ell$, hence $x_\ell-y_\ell\to 0$. Again by compactness of $\mathbb{B}^n$, we can find an extracted subsequence such that x_{ℓ_i} converges to x_∞. Necessarily we also have $y_{\ell_i}\to x_\infty$. By inequality (b) above $u(x_\ell)-v(y_\ell)-\|x_\ell\|^2\ge 0$. Passing to the limit we get $u(x_\infty)-v(y_\infty)-\|x_\infty\|^2\ge 0$. Since $u\le v$, we find that $x_\infty=0$. Therefore both x_{ℓ_i} and y_{ℓ_i} converge to 0. In particular, for i large enough x_{ℓ_i} and y_{ℓ_i} are in $\mathring{\mathbb{B}}^n$. We can therefore drop some of the first ℓ_i's and assume $x_{\ell_i},y_{\ell_i}\in\mathring{\mathbb{B}}^n$, for all i.

It follows from (a) and (b) above, that $u(x)-[v(y_{\ell_i})+\|x\|^2+\ell_i\|x-y_{\ell_i}\|^2]$ has a local maximum at x_{ℓ_i}. But the function $\varphi(x)=v(y_{\ell_i})+\|x\|^2+\ell_i\|x-y_{\ell_i}\|^2$ is C^∞ with gradient $2x+2\ell_i(x-y_{\ell_i})$. Therefore $2x_{\ell_i}+2\ell_i(x_{\ell_i}-y_{\ell_i})\in D^+u(x_{\ell_i})$, and using the fact that $u:\mathbb{R}^n\to\mathbb{R}$ is a viscosity subsolution of $H(x,d_xu)=c_1$, we obtain

$$H\big(x_{\ell_i},2x_{\ell_i}+\ell_i(x_{\ell_i}-y_{\ell_i})\big)\le c_1. \tag{c}$$

In the same way, we get that $v(y)-[u(x_{\ell_i})-\|x_{\ell_i}\|^2-\ell_i\|x_{\ell_i}-y\|^2]$ has a local minimum at y_{ℓ_i}. Therefore $2\ell_i(x_{\ell_i}-y_{\ell_i})\in D^-v(y_{\ell_i})$, and using that $v:\mathbb{R}^n\to\mathbb{R}$ is a viscosity supersolution of $H(x,d_xu)=c_2$, we obtain

$$H\big(y_{\ell_i},2\ell_i(x_{\ell_i}-y_{\ell_i})\big)\ge c_2. \tag{d}$$

Since x_{ℓ_i},y_{ℓ_i} are in $\mathbb{B}^n$, and either u or v has Lipschitz constant $\le K$ on $\mathbb{B}^n$, using $2x_{\ell_i}+2\ell_i(x_{\ell_i}-y_{\ell_i})\in D^+u(x_{\ell_i}), 2\ell_i(x_{\ell_i}-y_{\ell_i})\in D^-v(y_{\ell_i})$, from Proposition 3.4.(vi) we obtain that either $\|2x_{\ell_i}+2\ell_i(x_{\ell_i}-y_{\ell_i})\|\le K$ or $\|2\ell_i(x_{\ell_i}-y_{\ell_i})\|\le K$. Since $x_{\ell_i}\in\mathbb{B}^n$, we conclude that $\|2\ell_i(x_{\ell_i} - y_{\ell_i})\| \leq K + 2$, for all i. Therefore, up to extraction, we assume that $2\ell_i(x_{\ell_i} - y_{\ell_i})$ converges to $p \in \mathbb{R}^n$. Since both x_{ℓ_i} and y_{ℓ_i} converge to 0, passing to the limit in (c) and (d), we get $c_2 \leq H(0, p) \leq c_1$. □

Corollary 7.2. *Let $H : T^*M \to \mathbb{R}$ be a Hamiltonian coercive above every compact subset of the manifold M. Suppose that $u : M \to \mathbb{R}$ is viscosity subsolution of $H(x, d_x u) = c_1$, and $v : M \to \mathbb{R}$ be a viscosity supersolution of $H(x, d_x v) = c_2$. If $u - v$ has a local maximum, then necessarily $c_2 \leq c_1$.*

Proof. In that case Theorem 5.2 implies that u is locally Lipschitz. Therefore we can apply Theorem 7.1 □

Corollary 7.3. *Suppose $H : T^*M \to \mathbb{R}$ is a coercive Hamiltonian on the compact manifold M. If there exists a viscosity subsolution of $H(x, d_x u) = c_1$ and a viscosity supersolution of $H(x, d_x u) = c_2$, then necessarily $c_2 \leq c_1$.*

In particular, there exists at most one c for which the Hamilton-Jacobi equation $H(x, d_x u) = c$ has a global viscosity solution $u : M \to \mathbb{R}$. This only possible value is the smallest c for which $H(x, d_x u) = c$ admits a global viscosity subsolution $u : M \to \mathbb{R}$.

Proof. Call $u : M \to \mathbb{R}$ a viscosity subsolution of $H(x, d_x u) = c_1$, and call $v : M \to \mathbb{R}$ a viscosity supersolution of $H(x, d_x v) = c_2$. By compactness of M, we can find a point $x_0 \in M$ where $u - v$ achieves its maximum. Therefore by Corollary 7.2, we have $c_2 \leq c_1$. □

Theorem 7.4. *Let $H : M \to \mathbb{R}$ be a continuous Hamiltonian on the compact manifold M. Suppose $U, V : [0, +\infty[\times M \to \mathbb{R}$ are two continuous functions with $U(x, 0) = V(x, 0)$, for all $x \in M$. Assume that U (resp. V) is a viscosity subsolution (resp. supersolution) of the evolutionary Hamilton Jacobi Equation*

$$\frac{\partial u}{\partial t}(t, x) + H\big(x, \frac{\partial u}{\partial x}(t, x)\big) = 0,$$

on $]0, +\infty[\times M$. If either U or V is locally Lipschitz on $]0, +\infty[\times M$, then $U \leq V$ on the whole of $[0, +\infty[\times M$.

Proof. We introduce the Hamiltonian $\hat{H}$ on $\mathbb{R} \times M$ defined by

$$\hat{H}(t, x, s, p) = s + H(x, p),$$

where $(t, x) \in \mathbb{R} \times M$ and $(s, p) \in T^*_{(t,x)}(\mathbb{R} \times M) = \mathbb{R} \times T^*_x M$. With this notation U (resp. V) becomes a viscosity subsolution (resp. supersolution) of the Hamilton-Jacobi equation

$$\hat{H}\big((t, x), d_{(t,x)} u\big) = 0.$$

Fix $a, \epsilon > 0$, we will show that

$$\forall t \in [0, a[, \forall x \in M, U(t, x) + \frac{\epsilon}{t - a} \leq V. \tag{$\diamond$}$$

The theorem follows because we can let $\epsilon \to 0$, and $a > 0$ is arbitrary. To simplify notation define $\rho : [0, a] \to \mathbb{R}$ by

$$\rho(t) = \frac{\epsilon}{t - a}.$$

Since $\rho'(t) = -\epsilon/(t-a)^2 \leq -\epsilon/a^2$, it is not difficult to see that the continuous function $\hat{U} : [0, a[\times M \to$ defined by

$$\hat{U}(t,x) = U(t,x) + \rho(t)$$

is a viscosity subsolution of the Hamilton-Jacobi equation

$$\hat{H}((t,x), d_{(t,x)}u) = -\frac{\epsilon}{a^2}.$$

Since ρ is C^∞, it follows from the hypothesis that either $\hat{U}$ or V is locally Lipschitz on $]0, a[\times M$. Since $-\epsilon a^2 < 0$, we can apply Theorem 7.1 to conclude that $\hat{U} - V$ has no local maximum on $]0, a[\times M$. But $\rho(t) \to -\infty$, as $t \to a$, hence, by the compactness of M, the continuous function $\hat{U} - V = U + \rho - V$ must attain its maximum in $[0, a[\times M$. Since this maximum can only be in $\{0\} \times M$, and $\hat{U} - V$ is equal to $\rho(0) = -\epsilon/a$ on $\{0\} \times M$, we obtain that $\hat{U} - V \leq -\epsilon/a \leq 0$ on $[0, a[\times M$. This is precisely the inequality ($\diamond$) that we are seeking. □

Corollary 7.5. *Let $H : M \to \mathbb{R}$ be a continuous Hamiltonian on the compact manifold M. Suppose that the continuous function $U : [0, +\infty[\times M \to \mathbb{R}$ is locally Lipschitz on $]0, +\infty[\times M$, and is a viscosity solution of the evolutionary Hamilton Jacobi Equation*

$$\frac{\partial u}{\partial t}(t,x) + H\big(x, \frac{\partial u}{\partial x}(t,x)\big) = 0, \tag{EHJ}$$

on $]0, +\infty[\times M$.

Any other continuous function $V : [0, +\infty[\times M \to \mathbb{R}$, which is a viscosity solution of (EHJ) on $]0, +\infty[\times M$, and coincides with U on $\{0\} \times M$ coincides with U on the whole of $[0, +\infty[\times M$.

8 Construction of viscosity solutions

In this section, we will introduce the Perron Method for constructing viscosity solutions.

Proposition 8.1. *Let $H : T^*M \to \mathbb{R}$ be a continuous function. Suppose $(u_i)_{i\in I}$ is a family of continuous functions $u_i : M \to \mathbb{R}$ such that each u_i is a subsolution (resp. supersolution) of $H(x, d_x u) = 0$. If $\sup_{i\in I} u_i$ (resp. $\inf_{i\in u} u_i$) is finite and continuous everywhere, then it is also a subsolution (resp. supersolution) of $H(x, d_x u) = 0$.*

Proof. Set $u = \sup_{i\in I} u_i$. Suppose $\phi : M \to \mathbb{R}$ is C^1, with $\phi(x_0) = u(x_0)$ and $\phi(x) > u(x)$, for every $x \in M \setminus \{x_0\}$. We have to show $H(x_0, d_{x_0}\phi) \leq 0$. Fix some distance d on M. By continuity of the derivative of ϕ, it suffices to show that for each $\epsilon > 0$ small enough there exists $x \in \mathring{B}(x_0, \epsilon)$, with $H(x, d_x\phi) \leq 0$.

For $\epsilon > 0$ small enough, the closed ball $\bar{B}(x_0, \epsilon)$ is compact. Fix such an $\epsilon > 0$. There is a $\delta > 0$ such that $\phi(y) - \delta \geq u(y) = \sup_{i\in I} u_i(y)$, for each $y \in \partial B(x_0, \epsilon)$.

Since $\phi(x_0) = u(x_0)$, we can find $i_\epsilon \in I$ such that $\phi(x_0) - \delta < u_{i_\epsilon}(x_0)$. It follows that the maximum of the continuous function $u_{i_\epsilon} - \phi$ on the compact set $\bar{B}(x_0, \epsilon)$ is not attained on the boundary, therefore $u_{i_\epsilon} - \phi$ has a local maximum at some $x_\epsilon \in \mathring{B}(x_0, \epsilon)$. Since the function u_{i_ϵ} is a viscosity subsolution of $H(x, d_x u) = 0$, we have $H(x_\epsilon, d_{x_\epsilon}\phi) \leq 0$. □

Theorem 8.2 (Perron Method)**.** *Suppose the Hamiltonian $H : TM \to \mathbb{R}$ is coercive above every compact subset. Assume that M is connected and there exists a viscosity subsolution $u : M \to \mathbb{R}$ of $H(x, d_x u) = 0$. Then for every $x_0 \in M$, the function $S_{x_0} : M \to \mathbb{R}$ defined by $S_{x_0}(x) = \sup_v v(x)$, where the supremum is taken over all viscosity subsolutions v satisfying $v(x_0) = 0$, has indeed finite values and is a viscosity subsolution of $H(x, d_x u) = 0$ on M.*

Moreover, it is a viscosity solution of $H(x, d_x u) = 0$ on $M \setminus \{x_0\}$.

Proof. Call $\mathcal{SS}_{x_0}$ the family of viscosity subsolutions $v : M \to \mathbb{R}$ of $H(x, d_x v) = 0$ satisfying $v(x_0) = 0$.

Since H is coercive above every compact subset of M, by theorem 5.2, we know that each element of this family is locally Lipschitz. Moreover, since for each compact set K, the set $\{(x, p) \mid x \in K, H(x, p) \leq 0\}$ is compact, it follows that the family of restrictions $v_{|K}, v \in \mathcal{SS}_{x_0}$ is equi-Lipschitzian. We now show, that S_{x_0} is finite everywhere. Since M is connected, given $x \in M$, there exists a compact connected set K_{x,x_0} containing both x and x_0. By the equicontinuity of the family of restrictions $\{v_{|K_{x,x_0}} \mid v \in \mathcal{SS}_{x_0}\}$, we can find $\delta > 0$, such that for each $y, z \in K_{x,x_0}$ with $d(y, z) \leq \delta$, we have $|v(y) - v(z)| \leq 1$, for each $v \in \mathcal{SS}_{x_0}$.

By its choice, the set K_{x,x_0} is connected , we can find a sequence $x_0, x_1, \cdots, x_n = x$ in K_{x,x_0} with $d(x_i, x_{i+1}) \leq \delta$. It follows that $|v(x)| = |v(x) - v(x_0)| \leq \sum_{i=0}^{n-1} |v(x_{i+1}) - v(x_i)| \leq n$, for each $v \in \mathcal{SS}_{x_0}$. Therefore $\sup_{v \in \mathcal{SS}_{x_0}} v(x)$ is finite everywhere. Moreover, as a finite-valued supremum of a family of locally equicontinuous functions, it is continuous.

By the previous proposition 8.1, the function S_{x_0} is a viscosity subsolution on M itself. It remains to show that it is a viscosity solution of $H(x, d_x u)$ on $M \setminus \{x_0\}$.

Suppose $\psi : M \to \mathbb{R}$ is C^1 with $\psi(x_1) = S_{x_0}(x_1)$, where $x_1 \neq x_0$, and $\psi(x) < S_{x_0}(x)$ for every $x \neq x_1$. We want to show that necessarily $H(x_1, d_{x_1}\psi) \geq 0$. If this were false, by continuity of the derivative of ψ, endowing M with a distance defining its topology, we could find $\epsilon > 0$ such that $H(y, d_y\psi) < 0$, for each $y \in \bar{B}(x_1, \epsilon)$. Taking $\epsilon > 0$ small enough, we assume that $\bar{B}(x_1, \epsilon)$ is compact and $x_0 \notin \bar{B}(x_1, \epsilon)$. Since $\psi < S_{x_0}$ on the boundary $\partial B(x_1, \epsilon)$ of $\bar{B}(x_1, \epsilon)$, we can pick $\delta > 0$, such that $\psi(y) + \delta \leq S_{x_0}(y)$, for every $y \in \partial B(x_1, \epsilon)$. We define $\tilde{S}_{x_0}$ on $\bar{B}(x_1, \epsilon)$ by $\tilde{S}_{x_0}(x) = \max(\psi(x) + \delta/2, S_{x_0}(x))$. The function $\tilde{S}_{x_0}$ is a viscosity subsolution of $H(x, d, u)$ on $\mathring{B}(x_1, \epsilon)$ as the maximum of the two viscosity subsolutions $\psi + \delta/2$ and S_{x_0}. Moreover, this function $\tilde{S}_{x_0}$ coincides with S_{x_0} outside $K = \{x \in \mathring{B}(x_1, \epsilon) \mid \psi(x) + \delta/2 \geq S_{x_0}(x))\}$ which is a compact subset of $\mathring{B}(x_1, \epsilon)$, therefore we can extend it to M itself by $\tilde{S}_{x_0} = S_{x_0}$ on $M \setminus K$. It is a viscosity subsolution of $H(x, d_x u)$ on M itself, since its restrictions to both open subsets $M \setminus K$ and $\mathring{B}(x_1, \epsilon)$ are viscosity subsolutions and $M = \mathring{B}(x_1, \epsilon) \cup (M \setminus K)$.

But $\tilde{S}_{x_0}(x_0) = S_{x_0}(x_0) = 0$ because $x_0 \notin \bar{B}(x_1, \epsilon)$. Moreover $\tilde{S}_{x_0}(x_1) = \max(\psi(x_1) + \delta/2, S_{x_0}(x_1)) = \max(S_{x_0}(x_1) + \delta/2, S_{x_0}(x_1)) = S_{x_0}(x_1) + \delta/2 > S_{x_0}(x_1)$. This contradicts the definition of S_{x_0}. □

The next argument is inspired by the construction of Busemann functions in Riemannian Geometry, see [BGS85].

Corollary 8.3. *Suppose that $H : T^*M \to \mathbb{R}$ is a continuous Hamiltonian coercive above every compact subset of the connected* non-compact *manifold M. If there exists a viscosity subsolution of $H(x, d_x u) = 0$ on M, then there exists a viscosity* solution *on M.*

Proof. Fix $\hat{x} \in M$, and pick a sequence $x_n \to \infty$ (this means that each compact subset of M contains only a finite number of points in the sequence).

By arguments analogous to the ones used in the previous proof, the sequence S_{x_n} is locally equicontinuous and moreover, for each $x \in M$, the sequence $S_{x_n}(x) - S_{x_n}(\hat{x})$ is bounded. Therefore, by Ascoli's theorem, extracting a subsequence if necessary, we can assume that $S_{x_n} - S_{x_n}(\hat{x})$ converges uniformly to a continuous function $u : M \to \mathbb{R}$. It now suffices to show that the restriction of u to an arbitrary open relatively compact subset V of M is a viscosity solution of $H(x, d_x u) = 0$ on V. Since $\{n \mid x_n \in \bar{V}\}$ is finite, for n large enough, the restriction of $S_{x_n} - S_{x_n}(\hat{x})$ to V is a viscosity solution; therefore by the stability theorem 6.1, the restriction of the limit u to V is also a viscosity solution. □

The situation is different for compact manifolds as can be seen from Corollary 7.3.

9 Strict subsolutions

Definition 9.1 (strict subsolution)**.** Let $H : T^*M \to \mathbb{R}$ be a continuous function. We say that a viscosity subsolution $u : M \to \mathbb{R}$ of $H(x, d_x u) = c$ is strict at $x_0 \in M$ if there exists an open neighborhood V_{x_0} of x_0, and $c_{x_0} < c$ such that $u|V_{x_0}$ is a viscosity subsolution of $H(x, d_x u) = c_{x_0}$ on V_{x_0}.

Here is a way to construct viscosity subsolutions which are strict at some point.

Proposition 9.2. *Let $H : T^*M \to \mathbb{R}$ be a continuous function. Suppose that $u : M \to R$ is a viscosity subsolution of $H(y, d_y u) = c$ on M, that is also a viscosity solution on $M \setminus \{x\}$. If u is not a viscosity solution of $H(y, d_y u) = c$ on M itself then there exists a viscosity subsolution of $H(y, d_y u) = c$ on M which is strict at x.*

Proof. If u is not a viscosity solution, since it is a subsolution on M, it is the supersolution condition that is violated. Moreover, since u is a supersolution on $M \setminus \{x\}$, the only possibility is that there exists $\psi : M \to R$ of class C^1 such that $\psi(x) = u(x), \psi(y) < u(y)$, for $y \neq x$, and $H(x, d_x\psi) < c$. By continuity of the derivative of ψ, we can find a compact ball $\bar{B}(x, r)$, with $r > 0$, and a $c_x < c$ such that $H(y, d_y\psi) < c_x$, for every $y \in \bar{B}(0, r)$. In particular, the C^1 function ψ is a subsolution of $H(z, d_z v) = c_x$ on $\mathring{B}(x, r)$, and therefore also of $H(z, d_z v) = c$ on the same set since $c_x < c$.

We choose $\delta > 0$ such that for every $y \in \partial B(x, r)$ we have $u(y) > \psi(y) + \delta$. This is possible since $\partial B(x, r)$ is a compact subset of $M \setminus \{x\}$ where we have the strict inequality $\psi < u$.

If we define $\tilde{u} : M \to R$ by $\tilde{u}(y) = u(y)$ if $y \notin \bar{B}(x, r)$ and $\tilde{u}(y) = \max(u(y), \psi(y) + \delta)$, we obtain the desired viscosity subsolution of $H(y, d_y u) \leq c$ which is strict at x. In fact, by the choice of $\delta > 0$, the subset $K = \{y \in \bar{B}(x, r) \mid \psi(y) + \delta \leq u(y)\}$ is compact and contained in the open ball $\mathring{B}(x, r)$. Therefore M is covered by the two open subsets $M \setminus K$ and $\mathring{B}(x, r)$. On the first open subset $\tilde{u}$ is equal to u, it is therefore a subsolution of $H(y, d_y u) = c$ on that subset. On the second open subset $\mathring{B}(x, r)$, the function $\tilde{u}$ is the maximum of u and $\psi + \delta$ which are both subsolutions of $H(y, d_y u) = c$ on $\mathring{B}(x, r)$, by proposition 8.1, it is therefore a subsolution of $H(y, d_y u) = c$ on that second open subset. Since $u(x) = \psi(x)$; we have $\tilde{u}(x) = \psi(x) + \delta > u(x)$, therefore by continuity $\tilde{u} = \psi + \delta$

on a neighborhood $N \subset \mathring{B}(x,r)$ of x. On that neighborhood $H(y, d_y\psi) < c_x$, hence $\tilde{u}$ is strict at x. ☐

Here is another useful result on strict subsolutions.

Proposition 9.3. *Let $H : T^*M \to \mathbb{R}$ be a continuous function. Suppose that $u : M \to R$ (resp. $v : M \to R$) is a viscosity subsolution (resp. supersolution) of $H(y, d_y u) = c$ on M. Assume further that either u or v is locally Lipschitz. Then u cannot be strict at any local maximum of $u - v$.*

Proof. We argue by contradiction. Assume x_0 is a local maximum of $u-v$. If u was strict at x_0, we could find an open set V containing x_0, and a $c' < c$ such that $u|V$ is a viscosity subsolution of $H(x, d_x u) = c' < c$. But if we apply Theorem 7.1 to the restrictions $u|V$ and $v|V$, we see that we must have $c \leq c'$, which contradicts the choice of c'. ☐

10 Quasi-convexity and viscosity subsolutions

We first recall the definition of a quasi-convex function.

Definition 10.1. The function $f : C \to \mathbb{R}$, defined on the convex C subset of the real vector space E, is said to be quasi-convex, if for every $t \in \mathbb{R}$ the sublevel $\{x \in C \mid f(x) \leq t\}$ is convex.

Exercise 10.2. *Suppose $f : C \to \mathbb{R}$ is defined on the convex subset C of the real vector space E.*

1) Show that f is quasi-convex if and only if fore every sequence $\alpha_1 \ldots, \alpha_\ell \in [0,1]$ with $\sum_{i=1}^{\ell} \alpha_i = 1$, and every sequence $x_1, \ldots, x_\ell \in C$, we have $f(\sum_{i=1}^{\ell} \alpha_i x_i) \leq \max_{i=1}^{\ell} f(x_i)$.

2) Suppose moreover that E is a topological vector space, and that f is continuous and quasi-convex, show that for any sequence $(\alpha_i)_{i\in\mathbb{N}}$ with $\alpha_i \in [0,1]$ such that $\sum_{i=0}^{\infty} \alpha_i = 1$, and every sequence $(x_i)_{i\in\mathbb{N}}$ such that $\sum_{i=0}^{\infty} \alpha_i x_i$ exists and is in C, we have $f(\sum_{i\in\mathbb{N}} \alpha_i x_i) \leq \sup_{i\in\mathbb{N}} f(x_i)$.

3) (Difficult) Suppose further that E is a finite dimensional vector, and that the convex set C is Borel measurable. If μ is a Borel probability measure on E with $\mu(C) = 1$, show that $\int_E x\, d\mu(x) \in C$. [Indication: One can assume that this is true for a vector space whose dimension is strictly lower than that of E, then argue by contradiction: if $x_0 = \int_E x\, d\mu(x) \notin C$, by Hahn-Banach theorem and the finite dimensionality of E, find a linear map $\theta : E \to \mathbb{R}$ such that $\theta(x) \leq \theta(x_0)$.]

4) If E is finite dimensional show that 2) remains true even when f is only assumed Borel measurable on the Borel measurable convex set C.

In this section we will be mainly interested in Hamiltonians $H : T^*M \to \mathbb{R}$ quasi-convex in the fibers, i.e. for each $x \in M$, the function $p \mapsto H(x,p)$ is quasi-convex on the vector space T_x^*M.

Our first goal in this section is to prove the following theorem:

Theorem 10.3. *Suppose that the continuous Hamiltonian $H : T^*M \to \mathbb{R}$ is quasi-convex in the fibers. If $u : M \to \mathbb{R}$ is locally Lipschitz and $H(x, d_x u) \leq c$ almost everywhere, for some fixed $c \in \mathbb{R}$, then u is a viscosity subsolution of $H(x, d_x u) = c$.*

Before giving the proof of the theorem we need some preliminary material.

If $u : U \to \mathbb{R}$ is a locally Lipschitz function defined on the open subset of M, it is convenient to introduce the Hamiltonian constant $\mathbb{H}_U(u)$ as the essential supremum on U of $H(x, d_x u)$, i.e. the constant $\mathbb{H}_U(u)$ is the smallest $c \in \mathbb{R}$ such that $H(x, d_x u) \leq c$ for almost every $x \in U$

We will use some classical facts about convolution. Let $(\rho_\delta)_{\delta>0}$ be a family of functions $\rho_\delta : \mathbb{R}^k \to [0, \infty[$ of class C^∞, with $\rho_\delta(x) = 0$, if $\|x\| \geq \delta$, and $\int_{\mathbb{R}^k} \rho_\delta(x)\, dx = 1$. Suppose that V, U are open subsets of $\mathbb{R}^k$, with $\bar{V}$ compact and contained in U. Call $2\delta_0$ the Euclidean distance of the compact set $\bar{V}$ to the boundary of U, we have $\delta_0 > 0$, therefore the closed δ_0-neighborhood

$$\bar{N}_{\delta_0}(\bar{V}) = \{y \in \mathbb{R}^k \mid \exists x \in \bar{V}, \|y - x\| \leq \delta_0\}$$

of $\bar{V}$ is compact and contained in U.

If $u : U \to \mathbb{R}$ is a continuous function, then for $\delta < \delta_0$, the convolution

$$u_\delta(x) = \rho_\delta * u(x) = \int_{\mathbb{R}^k} \rho_\delta(y) u(x - y)\, dy.$$

makes sense and is of class C^∞ on a neighborhood of $\bar{V}$. Moreover, the family u_δ converges uniformly on $\bar{V}$ to u, as $t \to 0$.

Lemma 10.4. *Under the hypothesis above, suppose that $u : U \to \mathbb{R}$ is a locally Lipschitz function. Given any Hamiltonian $H : T^*U \to \mathbb{R}$ quasi-convex in the fibers and any $\epsilon > 0$, for every $\delta > 0$ small enough, we have* $\sup_{x \in V} |u_\delta(x) - u(x)| \leq \epsilon$ *and* $\mathbb{H}_V(u_\delta) \leq \mathbb{H}_U(u) + \epsilon$.

Proof. Because u is locally Lipschitz the derivative $d_z u$ exists for almost every $z \in U$. We first show that, for $\delta < \delta_0$, we must have

$$\forall x \in V, d_x u_\delta = \int_{\mathbb{R}^k} \rho_\delta(y) d_{x-y} u\, dy. \tag{*}$$

In fact, since u_δ is C^∞, it suffices to check that

$$\lim_{t \to 0} \frac{u_\delta(x + th) - u_\delta(x)}{t} = \int_{\mathbb{R}^k} \rho_\delta(y) d_{x-y} u(h)\, dy, \tag{**}$$

for $x \in V, \delta < \delta_0$, and $h \in \mathbb{R}^k$. Writing

$$\frac{u_\delta(x + th) - u_\delta(x)}{t} = \int_{\mathbb{R}^k} \rho_\delta(y) \frac{u(x + th - y) - u(x - y)}{t}\, dy,$$

We see that we can obtain (**) from Lebesgue's dominated convergence theorem, since ρ_δ has a compact support contained in $\{y \in \mathbb{R}^k \mid \|y\| < \delta\}$, and for $y \in \mathbb{R}^k, t \in \mathbb{R}$ such that $\|y\| < \delta, \|th\| < \delta_0 - \delta$, the two points $x + th - y, x - y$ are contained in the compact set $\bar{N}_{\delta_0}(\bar{V})$ on which u is Lipschitz. Equation (*) yields

$$H(x, d_x u_\delta) = H(x, \int_{\mathbb{R}^k} \rho_\delta(y) d_{x-y} u\, dy). \tag{***}$$

Since $\bar{N}_{\delta_0}(\bar{V})$ is compact and contained in U, and u is locally Lipschitz, we can find $K < \infty$ such that $\|d_z u\| \leq K$, for each $z \in \bar{N}_{\delta_0}(\bar{V})$ for which $d_z u$ exists. Since H is continuous, by a compactness argument, we can find $\delta_\epsilon \in]0, \delta_0[$, such that for $z, z' \in \bar{N}_{\delta_0}(\bar{V})$, with $\|z - z'\| \leq \delta_\epsilon$, and $\|p\| \leq K$, we have $|H(z', p) - H(z, p)| \leq \epsilon$. If $\delta \leq \delta_\epsilon$, since $\rho_\delta(y) = 0$, if $\|y\| \geq \delta$, we deduce that for all x in V and almost every y with $\|y\| \leq \delta$, we have

$$H(x, d_{x-y}u) \leq H(x - y, d_{x-y}u) + \epsilon \leq \mathbb{H}_U(u) + \epsilon.$$

The quasi-convexity of H in the fibers implies that the set $C = \{p \in T_x^*M \mid H(x,p) \leq \mathbb{H}_U(u) + \epsilon\}$ is convex and closed. Since $\rho_\delta \, dy$ is a probability measure whose support is contained in $\bar{B}(0, \delta) = \{y \in \mathbb{R}^k \mid \|y\| \leq \delta\}$, and $d_{x-y}u \in C$, for every $y \in \bar{B}(0, \delta)$, we obtain that the average $\int_{\mathbb{R}^k} \rho_\delta(y) d_{x-y}u \, dy$ is also in C. Hence we obtain

$$\forall \delta \leq \delta_\epsilon, H(x, \int_{\mathbb{R}^k} \rho_\delta(y) d_{x-y}u \, dy) \leq \mathbb{H}_U(u) + \epsilon.$$

It follows from inequality (***) above that $H(x, d_x u_\delta) \leq \mathbb{H}_U(u) + \epsilon$, for $\delta \leq \delta_\epsilon$ and $x \in V$. This gives $\mathbb{H}_V(u_\delta) \leq \mathbb{H}_U(u) + \epsilon$, for $\delta \leq \delta_\epsilon$. The inequality $\sup_{x \in \bar{V}} |u_\delta(x) - u(x)| < \epsilon$ also holds for every δ small enough, since u_δ converges uniformly on $\bar{V}$ to u, as $t \to 0$. □

Proof of theorem 10.3. We have to prove that for each $x_0 \in M$, there exists an open neighborhood V of x_0 such that $u_{|V}$ is a viscosity subsolution of $H(x, d_x u) = c$ on V. In fact, if we take V any open neighborhood such that $\bar{V}$ is contained in a domain of a coordinate chart, we can apply Lemma 10.4 to obtain a sequence $u_n : V \to \mathbb{R}, n \geq 1$, of C^∞ functions such that u_n converges uniformly to $u_{|V}$ on V and $H(x, d_x u_n) \leq c + 1/n$. If we define $H_n(x, p) = H(x, p) - c - 1/n$, we see that u_n is a smooth classical, and hence viscosity, subsolution of $H_n(x, d_x w) = 0$ on V. Since H_n converges uniformly to $H - c$, the stability theorem 6.1 implies that $u_{|V}$ is a viscosity subsolution of $H(x, d_x u) - c = 0$ on V. □

Corollary 10.5. *Suppose that the Hamiltonian $H : T^*M \to \mathbb{R}$ is continuous and quasi-convex in the fibers. For every $c \in \mathbb{R}$, the set of Lipschitz functions $u : M \to \mathbb{R}$ which are viscosity subsolutions of $H(x, d_x u) = c$ is convex.*

Proof. If $u_1, \ldots, u_n$ are such viscosity subsolutions. By 4.3, we know that at every x where $d_x u_j$ exists we must have $H(x, d_x u_j) \leq c$. If we call A the set of points x where $d_x u_j$ exists for each $j = 1, \ldots, n$, then A has full Lebesgue measure in M. If $a_1, \ldots, a_n \geq 0$, and $a_1 + \cdots + a_n = 1$, then $u = a_1 u_1 + \cdots + a_n u_n$ is differentiable at each point of $x \in A$ with $d_x u = a_1 d_x u_1 + \cdots + a_n d_x u_n$. Therefore by the quasi-convexity of $H(x,p)$ in the variable p, for every $x \in A$, we obtain $H(x, d_x u) = H(x, a_1 d_x u_1 + \cdots + a_n d_x u_n) \leq \max_{i=1}^n H(x, d_x u_i) \leq c$. Since A is of full measure, by theorem 10.3, we conclude that u is also a viscosity subsolution of $H(x, d_x u) = c$. □

The next corollary shows that the viscosity subsolutions are the same as the very weak subsolutions, at least in the geometric cases we have in mind. This corollary is clearly a consequence of Theorems 5.2 and 10.3.

Corollary 10.6. *Suppose that the Hamiltonian $H : T^*M \to \mathbb{R}$ is continuous, coercive, and quasi-convex in the fibers. A continuous function $u : M \to \mathbb{R}$ is a viscosity subsolution of $H(x, d_xu) = c$, for some $c \in \mathbb{R}$ if and only if u is locally Lipschitz and $H(x, d_xu) \leq c$, for almost every $x \in M$.*

We now give a global version of Lemma 10.4.

Theorem 10.7. *Suppose that $H : T^*M \to \mathbb{R}$ is a Hamiltonian, which is quasi-convex in the fibers. Let $u : M \to \mathbb{R}$ be a locally Lipschitz viscosity subsolution of $H(x, d_xu) = c$ on M. For every couple of continuous functions $\delta, \epsilon : M \to]0, +\infty[$, we can find a C^∞ function $v : M \to \mathbb{R}$ such that $|u(x) - v(x)| \leq \delta(x)$ and $H(x, d_xv) \leq c + \epsilon(x)$, for each $x \in M$.*

Proof. We pick up a locally finite countable open cover $(V_i)_{i\in\mathbb{N}}$ of M such that each closure $\bar{V}_i$ is compact and contained in the domain U_i of a chart which has a compact closure $\bar{U}_i$ in M. The local finiteness of the cover $(V_i)_{i\in\mathbb{N}}$ and the compactness of $\bar{V}_i$ imply that, for each $i \in \mathbb{N}$, the set $J(i) = \{j \in \mathbb{N} \mid V_i \cap V_j \neq \emptyset\}$ is finite. Therefore, denoting by $\#A$ for the number of elements in a set A, we obtain

$$
\begin{aligned}
j(i) =& \#J(i) = \#\{j \in \mathbb{N} \mid V_i \cap V_j \neq \emptyset\} < +\infty, \\
\tilde{j}(i) =& \max_{\ell \in J(i)} j(\ell) < +\infty.
\end{aligned}
$$

We define $R_i = \sup_{x\in\bar{U}_i} \|d_xu\|_x < +\infty$, where the sup is in fact taken over the subset of full measure of $x \in U_i$ where the locally Lipschitz function u has a derivative. It is finite because $\bar{U}_i$ is compact. Since $J(i)$ is finite, the following quantity $\tilde{R}_i$ is also finite

$$\tilde{R}_i = \max_{\ell \in J(i)} R_\ell < +\infty.$$

We now choose $(\theta_i)_{i\in\mathbb{N}}$ a C^∞ partition of unity subordinated to the open cover $(V_i)_{i\in\mathbb{N}}$. We also define

$$K_i = \sup_{x\in M} \|d_x\theta_i\|_x < +\infty,$$

which is finite since θ_i is C^∞ with support in V_i which is relatively compact.

Again by compactness, continuity, and finiteness routine arguments the following numbers are > 0

$$
\begin{aligned}
\delta_i &= \inf_{x\in\bar{V}_i} \delta(x) > 0, \tilde{\delta}_i = \min_{\ell\in J(i)} \delta_\ell > 0 \\
\epsilon_i &= \inf_{x\in\bar{V}_i} \epsilon(x) > 0, \tilde{\epsilon}_i = \min_{\ell\in J(i)} \epsilon_\ell > 0.
\end{aligned}
$$

Since $\bar{V}_i$ is compact, the subset $\{(x, p) \in T^*M \mid x \in \bar{V}_i, \|p\|_x \leq \tilde{R}_i + 1\}$ is also compact, therefore by continuity of H, we can find $\eta_i > 0$ such that

$$
\begin{aligned}
\forall x \in \bar{V}_i, \forall p, p' \in T_x^*M, \|p\|_x \leq \tilde{R}_i + 1, \|p'\|_x \leq \eta_i, H(x, p) \leq c + \frac{\epsilon_i}{2} \\
\Rightarrow H(x, p + p') \leq c + \epsilon_i.
\end{aligned}
$$

We can now choose $\tilde{\eta}_i > 0$ such that $\tilde{j}(i)K_i\tilde{\eta}_i < \min_{\ell\in J(i)} \eta_\ell$. Noting that $H(x,p)$ and $\|p\|_x$ are both quasi-convex in p, and that $\bar{V}_i$ is compact and contained in the domain U_i of a chart, by Lemma 10.4, for each $i \in \mathbb{N}$, we can find a C^∞ function $u_i : V_i \to \mathbb{R}$ such that

$$\begin{aligned} \forall x \in V_i, |u(x) - u_i(x)| \leq \min(\tilde{\delta}_i, \tilde{\eta}_i),& \\ H(x, d_x u_i) \leq \sup_{z\in V_i} H(z, d_z u) + \frac{\tilde{\epsilon}_i}{2} \leq c + \frac{\tilde{\epsilon}_i}{2}& \\ \|d_x u_i\|_x \leq \sup_{z\in V_i} \|d_z u\|_z + 1 = R_i + 1,& \end{aligned}$$

where the sup in the last two lines is taken over the set of points $z \in V_i$ where $d_z u$ exists.

We now define $v = \sum_{i\in\mathbb{N}} \theta_i u_i$, it is obvious that v is C^∞. We fix $x \in M$, and choose $i_0 \in \mathbb{N}$ such that $x \in V_{i_0}$. If $\theta_i(x) \neq 0$ then necessarily $V_i \cap V_{i_0} \neq \emptyset$ and therefore $i \in J(i_0)$. Hence $\sum_{i\in J(i_0)} \theta_i(x) = 1$, and $v(x) = \sum_{i\in J(i_0)} \theta_i(x)u_i(x)$.We can now write

$$\begin{aligned} |u(x) - v(x)| &\leq \sum_{i\in J(i_0)} \theta_i(x)|u(x) - u_i(x)| \leq \sum_{i\in J(i_0)} \theta_i(x)\tilde{\delta}_i \\ &\leq \sum_{i\in J(i_0)} \theta_i(x)\delta_{i_0} = \delta_{i_0} \leq \delta(x). \end{aligned}$$

We now estimate $H(x, d_x u)$. First we observe that $\sum_{i\in J(i_0)} \theta_i(y) = 1$, and $v(x) = \sum_{i\in J(i_0)} \theta_i(y)u_i(y)$, for every $y \in V_{i_0}$. Since V_{i_0} is a neighborhood of x, we can differentiate to obtain $\sum_{i\in J(i_0)} d_x\theta_i = 0$, and

$$d_x v = \underbrace{\sum_{i\in J(i_0)} \theta_i(x) d_x u_i}_{p(x)} + \underbrace{\sum_{i\in J(i_0)} u_i(x) d_x \theta_i}_{p'(x)}.$$

Using the quasi-convexity of H in p, we get

$$H(x, p(x)) \leq \max_{i\in J(i_0)} H(x, d_x u_i) \leq \max_{i\in J(i_0)} c + \frac{\tilde{\epsilon}_i}{2} \leq c + \frac{\epsilon_{i_0}}{2}, \tag{*}$$

where for the last inequality we have used that $i \in J(i_0)$ means $V_i \cap V_{i_0} \neq \emptyset$, and therefore $i_0 \in J(i)$, which implies $\tilde{\epsilon}_i \leq \epsilon_{i_0}$, by the definition of $\tilde{\epsilon}_i$.

In the same way, we have

$$\|p(x)\|_x \leq \max_{i\in J(i_0)})\|d_x u_i\|_x \leq \max_{i\in J(i_0)} R_i + 1 \leq \tilde{R}_{i_0} + 1. \tag{**}$$

We now estimate $\|p'(x)\|_x$. Using $\sum_{i\in J(i_0)} d_x\theta_i = 0$, we get

$$p'(x) = \sum_{i\in J(i_0)} u_i(x) d_x \theta_i = \sum_{i\in J(i_0)} (u_i(x) - u(x)) d_x \theta_i.$$

Therefore

$$\|p'(x)\|_x = \|\sum_{i\in J(i_0)} (u_i(x) - u(x))d_x\theta_i\|_x \leq \sum_{i\in J(i_0)} |u_i(x) - u(x)|\|d_x\theta_i\|_x$$
$$\leq \sum_{i\in J(i_0)} \tilde{\eta}_i K_i. \qquad (***)$$

From the definition of $\tilde{\eta}_i$, we get $K_i\tilde{\eta}_i \leq \frac{\eta_{i_0}}{j(i_0)}$, for all $i \in J(i_0)$. Hence $\|p'(x)\|_x \leq \sum_{i\in J(i_0)} \frac{\eta_{i_0}}{j(i_0)} = \eta_{i_0}$. The definition of η_{i_0}, together with the inequalities (*), (**) and (***), above implies $H(x, d_x v) = H(x, p(x) + p'(x)) \leq c + \epsilon_{i_0} \leq c + \epsilon(x)$. □

Theorem 10.8. *Suppose $H : T^*M \to \mathbb{R}$ is a Hamiltonian quasi-convex in the fibers. Let $u : M \to \mathbb{R}$ be a locally Lipschitz viscosity subsolution of $H(x, d_x u) = c$ which is strict at every point of an open subset $U \subset M$. For every continuous function $\epsilon : U \to]0, +\infty[$, we can find a viscosity subsolution $u_\epsilon : M \to \mathbb{R}$ of $H(x, d_x u) = c$ such that $u = u_\epsilon$ on $M \setminus U, |u(x) - u_\epsilon(x)| \leq \epsilon(x)$, for every $x \in M$, and the restriction $u_{\epsilon|U}$ is a C^∞ with $H(x, d_x u) < c$ for each $x \in U$.*

Proof. We define $\tilde{\epsilon} : M \to \mathbb{R}$ by $\tilde{\epsilon}(x) = \min(\epsilon(x), d(x, M \setminus U)^2)$, for $x \in U$, and $\tilde{\epsilon}(x) = 0$, for $x \notin U$. It is clear that $\tilde{\epsilon}$ is continuous on M and $\tilde{\epsilon} > 0$ on U.

For each $x \in U$, we can find $c_x < c$, and $V_x \subset V$ an open neighborhood of x such that $H(y, d_y u) \leq c_x$, for almost every $y \in V_x$. The family $(V_x)_{x\in U}$ is an open cover of U, therefore we can find a locally finite partition of unity $(\varphi_x)_{x\in U}$ on U submitted to the open cover $(V_x)_{x\in U}$. We define $\delta : U \to]0, +\infty[$ by $\delta(g) = \sum_{x\in U} \varphi_x(y)(c - c_x)$, for $y \in U$. It is not difficult to check that $H(y, d_y u) \leq c - \delta(y)$ for almost every $y \in U$.

We can apply Theorem 10.7 to the Hamiltonian $\tilde{H} : T^*U \to \mathbb{R}$ defined by $\tilde{H}(y, p) = H(y, p) + \delta(y)$ and $u|U$ which satisfies $\tilde{H}(y, d_y u) \leq c$ for almost every $y \in U$, we can therefore find a C^∞ function $u_\epsilon : U \to \mathbb{R}$, with $|u_\epsilon(y) - u(y)| \leq \tilde{\epsilon}(y)$, and $\tilde{H}(y, d_y u_\epsilon) \leq c + \delta(y)/2$, for each $y \in U$. Therefore, we obtain $|u_\epsilon(y) - u(y)| \leq \epsilon(y)$, and $H(y, d_y u_\epsilon) \leq c - \delta(y)/2 < c$, for each $y \in U$. Moreover, since $\tilde{\epsilon}(y) \leq d(y, M \setminus U)^2$, it is clear that we can extend continuously u_ϵ by u on $M\setminus U$. This extension satisfies $|u_\epsilon(x) - u(x)| \leq d(x, M\setminus U)^2$, for every $x \in M$. We must verify that u_ϵ is a viscosity subsolution of $H(x, d_x u_\epsilon) = c$. This is clear on U, since u_ϵ is C^∞ on U, and $H(y, d_y u_\epsilon) < c$, for $y \in U$. It remains to check that if $\phi : M \to \mathbb{R}$ is such that $\phi \geq u_\epsilon$ with equality at $x_0 \notin U$ then $H(x_0, d_{x_0}\phi) \leq c$. For this, we note that $u_\epsilon(x_0) = u(x_0)$, and $u(x) - u_\epsilon(x) \leq d(x, M \setminus U)^2 \leq d(x, x_0)^2$. Hence $u(x) \leq \phi(x) + d(x, x_0)^2$, with equality at x_0. The function $x \to \phi(x) + d(x, x_0)^2$ has a derivative at x_0 equal to $d_{x_0}\phi$, therefore $H(x_0, d_{x_0}\phi) \leq c$, since u is a viscosity solution of $H(x, d_x u) \leq c$. □

11 The viscosity semi-distance

We will suppose that $H : T^*M \to \mathbb{R}$ is a continuous Hamiltonian coercive above every compact subset of the connected manifold M.

Definition 11.1 (Mañé Critical Value)**.** Let $H : T^*M \to \mathbb{R}$ be a continuous Hamiltonian. We define c[0] as the infimum of all $c \in \mathbb{R}$, such that $H(x, d_x u) = c$ admits a global subsolution $u : M \to \mathbb{R}$.

Note that $c[0]$ is always well defined and finite if, for some $c \in \mathbb{R}$, there is a viscosity subsolution of $H(x, d_x u) = c$. If M is compact this is always the case. If M is non-compact and there is no c for which $H(x, d_x u) = c$ admits a viscosity subsolution, we will set $c[0] = +\infty$.

Exercise 11.2. *1) Suppose that $H : T^*M \to \mathbb{R}$ is a continuous Hamiltonian coercive above every compact subset of the connected manifold M. Show that*

$$c[0] \geq \sup_{x \in M} \inf_{p \in T_x^*M} H(x, p) > -\infty.$$

[Indication: Use the fact that a viscosity subsolution of $H(x, d_x u) = c$ is necessarily differentiable a.e. on M.]

2) Supose that $V : M \to \mathbb{R}$ is continuous. If the Hamiltonian H on M is defined by $H(x, p) = \frac{1}{2}\|p\|_x^2 + V(x)$, where $\|\cdot\|_x$ is the norm associated to a Riemannian on M, show that $c[0] = \sup_M V$.

We denote by $\mathcal{SS}^c$ the set of viscosity subsolutions of $H(x, d_x u) = c$, and by $\mathcal{SS}^c_{\hat{x}} \subset \mathcal{SS}^c$ the subset of subsolutions vanishing at a given $\hat{x} \in M$. Of course, since we can always add a constant to a viscosity subsolution and still obtain a subsolution, we have $\mathcal{SS}^c_{\hat{x}} \neq \emptyset$ if and only if $\mathcal{SS}^c \neq \emptyset$, and in that case $\mathcal{SS}^c = \mathbb{R} + \mathcal{SS}^c_{\hat{x}}$.

Proposition 11.3. *Suppose that $H : T^*M \to \mathbb{R}$ is a continuous Hamiltonian coercive above every compact subset of the connected manifold M. Assume that there is a $c \in \mathbb{R}$, such that $H(x, d_x u) = c$ has a viscosity subsolution on the whole of M (in particular, the Mañé critical value $c[0]$ is finite). Then there exists a global $u : M \to \mathbb{R}$ viscosity subsolution of $H(x, d_x u) = c[0]$.*

Proof. Fix a point $\hat{x} \in M$. Subtracting $u(\hat{x})$ if necessary, we will assume that all the viscosity subsolutions of $H(x, d_u) = c$ we consider vanish at $\hat{x}$. Since H is coercive above every compact subset of M, for each c the family of functions in $\mathcal{SS}^c_{\hat{x}}$ is locally equi-Lipschitzian. Therefore, using that M is connected and the fact that every $v \in \mathcal{SS}^c_{\hat{x}}$ vanish at $\hat{x}$, we obtain

$$\forall x \in M, \sup_{v \in \mathcal{SS}^c_{\hat{x}}} |v(x)| < +\infty.$$

We pick a sequence $c_n \searrow c[0]$, with $c_n \leq c$, and a sequence $u_n \in \mathcal{SS}^{c_n}_{\hat{x}}$. Since, by Ascoli's theorem, the family $\mathcal{SS}^c_{\hat{x}}$ is relatively compact in the topology of uniform convergence on each compact subset, extracting a sequence if necessary, we can assume that u_n converges uniformly to u on each compact subset of M. By the Stability Theorem 6.1, since u_n is a viscosity subsolution of $H(x, d_x u_)) = c_n$, the limit u is a viscosity subsolution of $H(x, d_x u) = c[0]$. □

For $c \geq c[0]$, we define

$$S^c(x, y) = \sup_{u \in \mathcal{SS}^c} u(y) - u(x) = \sup_{u \in \mathcal{SS}^c_x} u(y).$$

It follows from Theorem 8.2, that for each $x \in M$ the function $S^c(x, .)$ is a viscosity subsolution of $H(y, d_y u) = c$ on M itself, and a viscosity solution on $M \setminus \{x\}$.

Theorem 11.4. *For each $c \geq c[0]$, the function S^c is a semi-distance, i.e. it satisfies*

(i) *for each $x \in M, S^c(x,x) = 0$,*

(ii) *for each $x, y, z \in M$, $S^c(x,z) \leq S^c(x,y) + S^c(y,z)$*

Moreover, for $c > c[0]$, the symmetric semi-distance, $\hat{S}^c(x,y) = S^c(x,y) + S^c(y,x)$ is a distance which is locally Lipschitz-equivalent to any distance coming from a Riemannian metric.

Proof. The fact that S^c is a semi-distance follows easily from the definition

$$S^c(x,y) = \sup_{u \in \mathcal{SS}^c} u(y) - u(x).$$

Fix a Riemannian metric on the connected manifold M whose associated norm is denoted by $\|\cdot\|$, and associated distance is d. Given a compact subset $K \subset M$, the constant $\sup\{\|p\| \mid x \in K, p \in T_xM, H(x,p) \leq c\}$, is finite since H is coercive above compact subsets of M. It follows from this that for each compact subset $K \subset M$, there exists a constant $L_K < \infty$ such that.

$$\forall x, y \in K, S^c(x,y) \leq L_K d(x,y).$$

It remains to show a reverse inequality for $c > c[0]$. Fix such a c, and a compact set $K \subset M$. Choose $\delta > 0$, such that $\bar{N}_\delta(K) = \{x \in M \mid d(x,K) \leq \delta\}$ is also compact. By the compactness of the set

$$\{(x,p) \mid x \in \bar{N}_\delta(K), H(x,p) \leq c[0]\},$$

and the continuity of H, we can find $\epsilon > 0$ such that

$$\begin{aligned} \forall x \in \bar{N}_\delta(K), \forall p, p' \in T_xM, &H(x,p) \leq c[0] \text{ and } \|p'\| \leq \epsilon \\ &\Rightarrow H(x, p+p') \leq c. \end{aligned} \tag{*}$$

We can find $\delta_1 > 0$, such that the radius of injectivity of the exponential map, associated to the Riemannian metric, is at least δ_1 at every point x in the compact subset $\bar{N}_\delta(K)$. In particular, the distance function $x \mapsto d(x,x_0)$ is C^∞ on $\mathring{B}(x_0,\delta_1) \setminus \{x_0\}$, for every $x_0 \in \bar{N}_\delta(K)$. The derivative of $x \mapsto d(x,x_0)$ at each point where it exists has norm 1, since this map has (local) Lipschitz constant equal to 1. We can assume $\delta_1 < \delta$. We now pick $\phi : \mathbb{R} \to \mathbb{R}$ a C^∞ function, with support in $]1/2, 2[$, and such that $\phi(1) = 1$. If $x_0 \in K$ and $0 < d(y,x_0) \leq \delta_1/2$, the function

$$\phi_y(x) = \phi(\frac{d(x,x_0)}{d(y,x_0)})$$

is C^∞. In fact, if $d(x,x_0) \geq \delta_1$, then ϕ_y is zero in a neighborhood of x, since $d(x,x_0)/d(y,x_0) \geq \delta_1/(\delta_1/2) = 2$; if $0 < d(x,x_0) < \delta_1 < \delta$, then it is C^∞ on a neighborhood of x; finally $\phi_y(x) = 0$ for x such that $d(x,x_0) \leq d(y,x_0)/2$. In particular, we obtained that $d_x\phi_y = 0$, unless $0 < d(x,x_0) < \delta$, but at each such x, the derivative of $z \mapsto d(z,x_0)$ exists and has norm 1. It is then not difficult to see that $\sup_{x\in M}\|d_x\phi_y\| \leq A/d(y,x_0)$, where $A = \sup_{t\in\mathbb{R}}|\phi'(t)|$.

Therefore if we set $\lambda = \epsilon d(y,x_0)/A$, we see that $\|\lambda d_x\phi_y\| \leq \epsilon$, for $x \in M$. Since ϕ is 0 outside the ball $B(x_0,\delta_1) \subset N_{\delta_1}(K)$, it follows from the property (*) characterizing ϵ that we have

$$\forall (x,p) \in T^*M, H(x,p) \leq c[0] \Rightarrow H(x,p+\lambda d_x\phi_y) \leq c.$$

Since $S^{c[0]}(x_0,\cdot)$ is a viscosity subsolution of $H(x,d_xu) = c[0]$, and ϕ_y is C^∞, we conclude that the function $u(.) = S^{c[0]}(x_0,.)+\lambda\phi_y(.)$ is a viscosity subsolution of $H(x,d_xu) = c$. But the value of u at x_0 is 0, and its value at y is $S^{c[0]}(x_0,y)+\lambda\phi_y(y) = S^{c[0]}(x_0,y)+\epsilon d(y,x_0)/A$, since $\phi_y(y) = \phi(1) = 1$. Therefore $S^c(x_0,y) \geq S^{c[0]}(x_0,y) + \epsilon d(y,x_0)/A$. Hence we obtained

$$\forall x,y \in K, d(x,y) \leq \delta_1/2 \Rightarrow S^c(x,y) \geq S^{c[0]}(x,y) + \epsilon A^{-1}d(x,y).$$

Adding up and using $S^{c[0]}(x,y) + S^{c[0]}(y,x) \geq S^{c[0]}(x,x) = 0$, we get

$$\forall x,y \in K, d(x,y) \leq \delta_1/2 \Rightarrow S^c(x,y) + S^c(y,x) \geq \frac{2\epsilon}{A}d(x,y).$$ □

12 The projected Aubry set

Theorem 12.1. *Assume that $H : T^*M \to \mathbb{R}$ is a Hamiltonian coercive above every compact subset of the connected manifold M, with $c[0] < +\infty$. For each $c \geq c[0]$, and each $x \in M$, the following two conditions are equivalent:*

(i) *The function $S^c(x,\cdot)$ is a viscosity solution of $H(z,d_zu) = c$ on the whole of M.*

(ii) *There is no viscosity subsolution of $H(z,d_zu) = c$ on the whole of M which is strict at x.*

In particular, for every $c > c[0]$, the function $S^c(x,\cdot)$ is not a viscosity solution of $H(z,d_zu) = c$.

Proof. The implication (ii)⇒(i) follows from proposition 9.2.

To prove (i)⇒(ii), fix $x \in M$ such that $S^c_x(\cdot) = S^c(x,\cdot)$ is a viscosity solution on the whole of M, and suppose that $u : M \to \mathbb{R}$ is a viscosity subsolution of $H(y,d_yu) = c$ which is strict at x. Therefore we can find an open neighborhood V_x of x, and a $c_x < c$ such that $u_{|V_x}$ is a viscosity subsolution of $H(y,d_yu) = c_x$ on V_x. By definition of S, we have $u(y) - u(x) \leq S^c_x(y)$ with equality at $y = x$. This implies $u - S^c_x$ has a global maximum at x. Applying Theorem 7.2 to the restrictions of u and S^c_x to V_x, we see that we must have $c \leq c_x < c$, a contradiction.

Since a viscosity subsolution of $H(x,d_xu) = c[0]$ is a strict viscosity subsolution of $H(x,d_xu) = c$ for any $c > c[0]$, we obtain the last part of the theorem. □

The above theorem yields the following definition.

Definition 12.2 (Projected Aubry set)**.** If $H : T^*M \to \mathbb{R}$ is a continuous Hamiltonian, coercive above every compact subset of the connected manifold M. We define the projected Aubry set as the set of $x \in M$ such that that $S^{c[0]}(x, \cdot)$ is a viscosity solution of $H(z, d_z u) = c[0]$.

To be able to go further in our discussion we will restrict to Hamiltonian convex in the fibers.

Proposition 12.3. *Assume that $H : T^*M \to \mathbb{R}$ is a continuous Hamiltonian, convex in the fibers, and coercive above every compact subset of the connected manifold M. There exists a viscosity subsolution $v : M \to \mathbb{R}$ of $H(x, d_x v) = c[0]$, which is strict at every $x \in M \setminus \mathcal{A}$.*

Proof. We fix some base point $\hat{x} \in M$. For each $x \notin \mathcal{A}$, we can find $u_x : M \to \mathbb{R}$, an open subset V_x containing x, and $c_x < c[0]$, such that u_x is a viscosity subsolution of $H(y, d_y u_x) = c[0]$ on M, and $u_x|V_x$ is a viscosity subsolution of $H(y, d_y u_x) \leq c_x$, on V_x. Subtracting $u_x(\hat{x})$ if necessary, we will assume that $u_x(\hat{x}) = 0$. Since $U = M \setminus \mathcal{A}$ is covered by the family of open sets $V_x, x \notin \mathcal{A}$, we can extract a countable subfamily $(V_{x_i})_{i \in \mathbb{N}}$ covering U. Since H is coercive above every compact set the sequence $(u_{x_i})_{i \in \mathbb{N}}$ is locally equi-Lipschitzian. Therefore, since M is connected, and all the u_{x_i} vanish at $\hat{x}$, the sequence $(u_{x_i})_{i \in \mathbb{N}}$ is uniformly bounded on every compact subset of M. It follows that the sum $V = \sum_{i \in \mathbb{N}} \frac{1}{2^{i+1}} u_{x_i}$ is uniformly convergent on each compact subset. If we set $u_n = (1 - 2^{-(n+1)})^{-1} \sum_{0 \leq i \leq n} \frac{1}{2^{i+1}} u_{x_i}$, then u_n is a viscosity subsolution of $H(x, d_x u_n) = c[0]$ as a convex combination of viscosity subsolutions, see proposition 10.5. Since u_n converges uniformly on compact subsets to u, the stability theorem 6.1 implies that v is also a viscosity subsolution of $H(x, d_x v) = c[0]$.

On the set $V_{x_{n_0}}$, we have $H(x, d_x u_{x_{n_0}}) \leq c_{x_{n_0}}$, for almost every $x \in V_{x_{n_0}}$. Therefore, if we fix $n \geq n_0$, we see that for almost every $x \in V_{x_{n_0}}$ we have

$$\begin{aligned} H(x, d_x u_n) &\leq (1 - 2^{-(n+1)})^{-1} \sum_{i=0}^{n} \frac{1}{2^{i+1}} H(x, d_x u_{x_i}) \\ &\leq (1 - 2^{-(n+1)})^{-1} \left[\sum_{i=0}^{n} \frac{1}{2^{i+1}} c[0] + \frac{(c_{x_{n_0}} - c[0])}{2^{n_0+1}} \right]. \end{aligned}$$

Therefore $u_n|V_{x_{n_0}}$ is a viscosity subsolution of $H(x, d_x u_n) \leq c[0] + (c_{x_{n_0}} - c[0])/2^{n_0+1}$.

By the stability theorem 6.1, this is also true for $v|V_{x_{n_0}}$. Since $c_{x_{n_0}} - c[0] < 0$, we conclude that $u|V_{x_{n_0}}$ is a strict subsolution of $H(x, d_x v) = c[0]$, for each $x \in V_{x_{n_0}}$, and therefore at each $x \in U \subset \cup_{n \in \mathbb{N}} V_{x_n}$. □

Corollary 12.4. *Assume that $H : T^*M \to \mathbb{R}$ is a Hamiltonian convex in the fibers and coercive, where M is a compact connected manifold. Its projected Aubry set $\mathcal{A}$ is not empty.*

Proof. We argue by contradiction. If $\mathcal{A} = \emptyset$ then by Proposition 12.3 above, we can find a viscosity subsolution u of $H(x, d_x u) = c[0]$ which is strict everywhere. In particular for every $x \in M$, we can find an open neighborhood V_x of x and $c_x < c[0]$ such that $u|V_x$ is a viscosity solution of $H(y, d_y v) = c_x$. By compactness of M, we can find a finite number of points $x_1, \dots, x_\ell$) of M such that $M = V_{x_1} \cup \cdots \cup V_{x_\ell}$. It follows from Corollary 4.2 that u is a viscosity subsolution of $H(x, d_x u) = \max(c_{x_1}, \dots, c_{x_\ell})$ on the whole of M. This is in contradiction of the definition of $c[0]$ since $\max(c_{x_1}, \dots, c_{x_\ell}) < c[0]$. □

Theorem 12.5. *Assume that $H : T^*M \to \mathbb{R}$ is a Hamiltonian convex in the fibers and coercive, where M is a compact connected manifold. Suppose $u_1, u_2 : M \to \mathbb{R}$ are respectively a viscosity subsolution and a viscosity supersolution of $H(x, d_x u) = c[0]$. If $u_1 \le u_2$ on the projected Aubry set $\mathcal{A}$, then $u_1 \le u_2$ everywhere on M.*

In particular, if two viscosity solutions of $H(x, d_x u) = c[0]$ coincide on $\mathcal{A}$, they coincide on M.

Proof. By Proposition 12.3 above, we can find a viscosity subsolution u_0 of $H(x, d_x u) = c[0]$ which is strict at every point of $M \setminus \mathcal{A}$. We interpolate between u_0 and u_1 by defining $u_t = (1-t)u_0 + tu_1$. Like in the proof of Proposition 12.3, we can show that u_t is a viscosity subsolution of $H(x, d_x u) = c[0]$, for any $t \in [0, 1]$. Moreover, for $t < 1$, the viscosity subsolution u_t is strict at each point of $M \setminus \mathcal{A}$. By the coercivity condition all subsolutions are locally Lipschitz. Since M is compact $u_t - u_2$ achieves a maximum on M. By Proposition 9.3, for $t < 1$, this maximum is achieved at a point of the compact subset $\mathcal{A}$. Since u_t converges uniformly to u_1, it follows that $u_1 - u_2$ achieves also its maximum on M in the same compact subset $\mathcal{A}$. But $u_1 - u_2 \le 0$ on $\mathcal{A}$. Therefore $u_1 - u_2 \le 0$ everywhere on M. □

We now give an example to show that Theorem 12.3 and Corollary 12.4 are not necessarily valid for a Hamiltonian quasi-convex in the fibers

Example 12.6. Define the quasi-convex function $h : \mathbb{R} \to \mathbb{R}$ by

$$h(t) = \begin{cases} -t-1, \text{ for } t \le -1, \\ t+1, \text{ for } -1 \le t \le 0, \\ 1, \text{ for } 0 \le t \le 1, \\ t, \text{ for } t \ge 1. \end{cases}$$

We define a Hamiltonian H on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We use the usual identification of the cotangent space $T^*\mathbb{T}$ with $\mathbb{T} \times \mathbb{R}$. In this usual identification the derivative du of a function $u : \mathbb{T} \to \mathbb{R}$, as a section, is exactly $t \mapsto (t, u'(t))$. The Hamiltonian $H : \mathbb{T} \times \mathbb{R}$ is defined by $H(t, s) = h(s)$. Obviously the constant function $u_0 \equiv 0$ obviously satisfies $H(t, u_0'(t)) = H(t, 0) = h(0) = 1$, therefore $c[0] = 1$, by Corollary 7.3. For any $t_0 \in \mathbb{T}$, the function $v_{t_0}(t) = (2\pi)^{-1} \sin(2\pi t + \pi - 2\pi t_0)$ has a derivative $v_{t_0}'(t) = \cos(2\pi t + \pi - 2\pi t_0)$ which is between -1 and 1 everywhere. Therefore v_{t_0} is a subsolution of $H(t, v'(t)) = 1$. Moreover, its derivative at t_0 is $\cos(\pi) = -1$, Hence $H(t_0, v_{t_0}'(t_0)) = h(-1) = 0 < 1$. By continuity of the derivative of v_{t_0}, it follows that v_{t_0} is strict at t_0. Since t_0 is arbitrary in $\mathbb{T}$, it follows from Theorem 12.1 that the Aubry set of H is empty. This shows that Corollary 12.4 cannot be true for general quasi-convex Hamiltonian. We now show that Proposition 12.4 cannot be true for H. In fact, if it were true we would obtain a viscosity subsolution which is strict at every point of $\mathbb{T}$. Using the compactness of $\mathbb{T}$ like in the proof of Corollary 12.4, we see that this yields a viscosity subsolution of $H(t, v'(t)) = c$, for some $c < 1$. This is impossible since $c[0] = 1$.

13 The representation formula

We still assume that M is compact, and that $H : T^*M \to \mathbb{R}$ is a coercive Hamiltonian convex in the fibers.

Theorem 13.1. *Any viscosity solution $u : M \to \mathbb{R}$ for $H(x, d_x u) = c[0]$ satisfies*

$$\forall x \in M, u(x) = \inf_{x_0 \in \mathcal{A}} u(x_0) + S^{c[0]}(x_0, x)$$

.

This theorem follows easily from the uniqueness theorem 12.5 and the following one:

Theorem 13.2. *For any function $v : \mathcal{A} \to \mathbb{R}$ bounded below, the function*

$$\tilde{v}(x) = \inf_{x_0 \in \mathcal{A}} v(x_0) + S^{c[0]}(x_0, x)$$

is a viscosity solution of $H(x, d_x v) = c[0]$. Moreover, we have $\tilde{v}_{|\mathcal{A}} = v$, if and only if

$$\forall x, y \in \mathcal{A}, v(y) - v(x) \leq S^{c[0]}(x, y).$$

We start with a lemma.

Lemma 13.3. *Suppose $H : T^*M \to \mathbb{R}$ is a continuous Hamiltonian convex in the fibers, and coercive above each compact subset of the connected manifold M. Let $u_i : M \to \mathbb{R}, i \in I$ be a family of viscosity subsolutions of $H(x, d_x u) = c$. If $\inf_{i \in I} u_i(x_0)$, is finite for some $x_0 \in M$, then $\inf_{i \in I} u_i$ is finite everywhere. In that case, the function $u = \inf_{i \in I} u_i$ is a viscosity subsolution of $H(x, d_x u) = c$.*

In particular, if each $u_i, i \in I$ is a viscosity solution so is $u = \inf_{i \in I} u_i$.

Proof. We fix an auxiliary Riemannian metric on M, and we use as a distance on M its associated distance.

By the coercivity condition, the family $(u_i)_{i \in I}$ is locally equi-Lipschitzian, therefore if K is a compact connected subset of M, there exists a constant $C(K)$ such that

$$\forall x, y \in K, \forall i \in I, |u_i(x) - u_i(y)| \leq C(K).$$

If $x \in M$ is given, we can find a compact connected subset K_x containing x_0 and x, it follows that

$$\inf_{i \in I} u_i(x_0) \leq \inf_{i \in I} u_i(x) + C(K_x)$$

therefore $\inf_{i \in I} u_i$ is finite everywhere. It now suffices to show that for a given $\tilde{x} \in M$, we can find an open neighborhood V of $\tilde{x}$ such that $\inf_{i \in I} u_i|V$ is a viscosity subsolution of $H(x, d_x u) = c$ on V. We choose an open neighborhood V of $\tilde{x}$ such that its closure $\bar{V}$ is compact. Since $\mathcal{C}^0(\bar{V}, \mathbb{R})$ is metric and separable in the topology of uniform convergence, we can find a countable subset $I_0 \subset I$ such that $u_{i|\bar{V}}, i \in I_0$ is dense in $\{u_{i|\bar{V}} \mid i \in I\}$, for the topology of uniform convergence. Therefore $\inf_{i \in I} u_i = \inf_{i \in I_0} u_i$ on $\bar{V}$. Since I_0 is countable, we have reduced the proof to the cases where $I_0 = \{0, \cdots, N\}$, or $I_0 = \mathbb{N}$.

Let us start with the first case. Since $u_0, \cdots, u_N$, and $u = \inf_{i=0}^N u_i$ are all Lipschitzian on V, we can find $E \subset V$ of full Lebesgue measure such that $d_x u, d_x u_0, \cdots, d_x u_N$ exists, for each $x \in E$. At each such $x \in E$, we necessarily have $d_x u \in \{d_x u_0, \ldots, d_x u_N\}$. In fact, if n is such that $u(x) = u_n(x)$, since $u \leq u_n$ with equality at x and both derivative at x exists, they must be equal. Since each u_i is a viscosity subsolution of $H(x, d_x v) = c$, we obtain $H(x, d_x u) \leq c$, for every x in the subset E of full measure in V. The convexity of H in the fibers imply that u is a viscosity subsolution of $H(x, d_x u) = c$ in V. It remains to consider the case $I_0 = \mathbb{N}$. Define $u^N(x) = \inf_{0 \leq i \leq N} u_i(x)$, by the previous case, u^N is a viscosity subsolution of $H(x, d_x u^N) = c$ on V.

Now $u^N(x) \to \inf_{i \in I_0} u_i(x)$, for each $x \in \bar{V}$, the convergence is in fact, uniform on $\bar{V}$ since $(u_i)_{i \in I_0}$ is equi-Lipschitzian on the compact set $\bar{V}$. It remains to apply the stability theorem 6.1.

To prove the last part of the lemma, it suffices to recall that from Proposition 8.1 an infimum of a family of supersolutions is itself supersolution. □

Proof of Theorem 13.2. By definition of the projected Aubry set, for every, $x_0 \in \mathcal{A}$, the function $v(x_0) + SS^{c[0]}(x_0, \cdot)$ is a viscosity solution. It follows from Lemma 13.3 above that $\tilde{v}$ is a viscosity solution.

Since $\tilde{v}$ is in particular a subsolution, it satisfies everywhere $\tilde{v}(y) - \tilde{v}(x) \leq S^{c[0]}(x, y)$. Therefore if $v = \tilde{v}$ on $\mathcal{A}$, we must have that

$$\forall x, y \in \mathcal{A}, v(y) - v(x) \leq S^{c[0]}(x, y).$$

Conversely, if v satifies the property above, from the definition of $\tilde{v}$, it is obvious that $v = \tilde{v}$ on $\mathcal{A}$. □

14 Tonelli Hamiltonians and Lagrangians

We now establish part of the relationship between viscosity solutions, weak KAM solutions, and the Lax-Oleinik semi-group for a Tonelli Hamiltonian. A reference for this part is of course Patrick Bernard's companion lectures [Ber11]. Another reference is [Fat08].

In this section we will always suppose that the manifold is compact. We first recall the definition of a Tonelli Hamiltonian.

Definition 14.1. Let M be a compact manifold. A Hamiltonian $H : T^*M \to \mathbb{R}$ is said to be Tonelli if it is at least C^2, and satisfies the following two conditions :

(1) (Superlinearity) for every $K \geq 0$, there exists $C^*(K) < \infty$ such that

$$\forall (x, p) \in T^*M, H(x, p) \geq K\|p\|_x - C^*(K) \; ;$$

(2) (C^2 strict convexity in the fibers) for every $(x, p) \in T^*M$, the second derivative along the fibers $\partial^2 H/\partial p^2(x, p)$ is (strictly) positive definite.

Note that condition (1) is independent of the choice of a Riemannian metric on M. In fact, all Riemannian metrics on the compact manifold M are equivalent. Moreover, condition (1) implies that H is coercive.

To such a Hamiltonian is associated a Lagrangian $L : TM \to \mathbb{R}$ defined by

$$\forall (x,v) \in TM, L(x,v) = \max_{p \in T_x^*M} \langle p, v\rangle - H(x,p).$$

Since H is of class C^2 finite everywhere, superlinear and strictly convex in each fiber T_x^*M, it is well known that L is finite everywhere of class C^2, strictly convex and superlinear in each fiber T_xM, and satisfies

$$\forall (x,p) \in T^*M, H(x,p) = \max_{v \in T_xM} \langle p, v\rangle - L(x,v).$$

Definition 14.2 (Evolution Dominated Function)**.** A function $U : [0,+\infty[\times M \to \mathbb{R}$ is said to be evolution dominated by the Tonelli Lagrangian L associated to the Tonelli Hamiltonian H if for every continuous piecewise C^1 curve $\gamma : [a,b] \to M$,with $0 \leq a \leq b$, we have

$$U(b,\gamma(b)) - U(a,\gamma(a)) \leq \int_a^b L(\gamma(s),\dot\gamma(s))\, ds.$$

Note that an evolution dominated function is not necessarily continuous. In fact, since L is superlinear, we have $c = \inf L > -\infty$. It $\rho : [0,+\infty[\to \mathbb{R}$ is any non-increasing (not necessarily continuous) function, then $U(t,x) = ct + \rho(t)$ is evolution dominated by L.

Exercise 14.3. *1) Show that a function $U : [0,+\infty[\times M \to \mathbb{R}$ is evolution dominated by L if and only if for every continuous piecewise C^1 curve $\gamma : [\alpha,\beta] \to M$,with $\alpha,\beta \in \mathbb{R}, \alpha \leq \beta$, and every $a \geq 0$, we have*

$$U(a+\beta-\alpha,\gamma(\beta)) - U(a,\gamma(\alpha)) \leq \int_\alpha^\beta L(\gamma(s),\dot\gamma(s))\, ds.$$

[Indication: Reparametrize the curve γ by a shift in time.]

2) Suppose that $U : [0,+\infty[\times M \to \mathbb{R}$ is evolution dominated by L. If $a \geq 0$ show that $V(t,x) = U(t+a,x)$ is also evolution dominated by L.

Proposition 14.4. *If a continuous function $U : [0,+\infty[\times M \to \mathbb{R}$ is evolution dominated by the Tonelli Lagrangian L associated to the Tonelli Hamiltonian H, then U is a viscosity subsolution of*

$$\frac{\partial U}{\partial t}(t,x) + H(x, \frac{\partial U}{\partial x}(t,x)) = 0,$$

on the open set $]0,+\infty[\times M$.

Proof. Suppose $\phi \geq U$, with ϕ of class C^1 and $(t_0,x_0) = U(t_0,x_0)$, where $t_0 > 0$. Fix $v \in T_{x_0}M$, and pick a C^1 curve $\gamma : [0,t_0] \to M$ such that $(\gamma(t_0),\dot\gamma(t_0)) = (x,v)$.

If $0 \leq t \leq t_0$, we have

$$U(t_0,\gamma(t_0)) - U(t,\gamma(t)) \leq \int_t^{t_0} L(\gamma(s),\dot\gamma(s))\, ds. \qquad (*)$$

Since $\phi \geq U$, with equality at (t_0,x_0), noticing that $\gamma(t_0) = x_0$, we obtain from $(*)$

$$\forall t \in]0,t_0[, \phi(t_0,\gamma(t_0)) - \phi(t,\gamma(t)) \leq \int_t^{t_0} L(\gamma(s),\dot\gamma(s))\, ds.$$

Dividing by $t_0 - t > 0$, and letting $t \to t_0$, we get

$$\forall v \in T_{x_0}M, \frac{\partial \phi}{\partial t}(t_0, x_0) + \frac{\partial \phi}{\partial x}(t_0, x_0)(v) \leq L(x_0, v).$$

Since

$$H(x_0, \frac{\partial \phi}{\partial x}(t_0, x_0)) = \sup_{v \in T_{x_0}M} \frac{\partial \phi}{\partial x}(t_0, x_0)(v) - L(x_0, v),$$

we obtain

$$\frac{\partial \phi}{\partial t}(t_0, x_0) + H(x_0, \frac{\partial \phi}{\partial x}(t_0, x_0)) \leq 0.$$

This finishes the proof. □

An important object of the theory is the Lax-Oleinik semi-group. We recall its definition and some of its properties, and send the reader to the last section of Patrick Bernard's companion lectures [Ber11], or to [Fat08].

If $u : M \to \mathbb{R}$ is a continuous function, and $t > 0$, we define $T_t^- u : M \to \mathbb{R}$ by

$$T_t^- u(x) = \inf_{\gamma} \{ u(\gamma(0)) + \int_0^t L(\gamma(s), \dot\gamma(s))\, ds \},$$

where the infimum is taken over all the continuous piecewise C^1 curves $\gamma : [0, t] \to M$ such that $\gamma(t) = x$.

In fact, for each $t > 0$ the function $T_t^- u$ is continuous (and even Lipschitz). Moreover, setting $T_0^- u = u$, the function $(t, x) \mapsto T_t^- u(x)$ is continuous on $[0, +\infty[\times M$, and is locally Lipschitz on $]0, +\infty[\times M$.

Moreover, the family $T_t^-, t \geq 0$ is a semi-group, i.e.

$$\forall t, t' \geq 0, \forall u \in C^0(M, \mathbb{R}), T_{t+t'}^- u = T_t^- T_{t'}^- u.$$

Exercise 14.5. *1) Suppose that $U : [0, +\infty[\times M \to \mathbb{R}$ is a continuous function. For $a \geq 0$, set $U_a(x) = U(a, x)$. Show that U is evolution dominated by L, if and only if for every $t, a \geq 0$, we have $U_{t+a} \leq T_t^- U_a$.*

2) If $u \in C^0(M, \mathbb{R})$, and $U(t, x) = T_t^- u(x)$, show that U is evolution dominated by L.

Theorem 14.6. *If $u \in C^0(M, \mathbb{R})$, and $U(t, x) = T_t^- u(x)$, then U is a viscosity solution of*

$$\frac{\partial U}{\partial t}(t, x) + H(x, \frac{\partial U}{\partial x}(t, x)) = 0, \tag{EHJ}$$

on the open subset $]0, +\infty[\times M$.

Proof. By Proposition 14.4, and part 2) of Exercise 14.5, the function U is a viscosity subsolution of (EHJ) on $]0, +\infty[\times M$.

To prove that U is a supersolution, we consider $\psi \leq U$, with ψ of class C^1. Suppose $U(t_0, x_0) = \psi(t_0, x_0)$, with $t_0 > 0$.

As is well-known, by Tonelli's theorem, the infimum in the definition of $T_t^- u(x)$ is attained by a curve which is a minimizer, hence at least C^2. Therefore, we can pick a C^2 curve $\gamma : [0, t_0] \to M$ such that $\gamma(t_0) = x_0$ and

$$U(t_0, x_0) = T_{t_0}^- u(x_0) = u(\gamma(0)) + \int_0^{t_0} L(\gamma(s), \dot\gamma(s))\, ds.$$

Since $U(0,\gamma(0)) = u(\gamma(0))$, this can be rewritten as

$$U(t_0, x_0) - U(0, \gamma(0)) = \int_0^{t_0} L(\gamma(s), \dot\gamma(s))\, ds. \qquad (**)$$

Applying twice the fact that U is evolution dominated by L, for every $t \in [0, t_0]$, we obtain

$$U(t_0, x_0) - U(t, \gamma(t)) \le \int_t^{t_0} L(\gamma(s), \dot\gamma(s))\, ds$$
$$U(t, \gamma(t)) - U(0, \gamma(0)) \le \int_0^{t} L(\gamma(s), \dot\gamma(s))\, ds.$$

Adding these two inequalities, by (**), we get in fact an equality. Hence we must have

$$\forall t \in [0, t_0], U(t_0, \gamma(t_0)) - U(t, \gamma(t)) = \int_t^{t_0} L(\gamma(s), \dot\gamma(s))\, ds.$$

Since $\psi \le U$, with equality at (t_0, x_0), for every $t \in [0, t_0]$, we obtain

$$\psi(t_0, \gamma(t_0)) - \psi(t, \gamma(t)) \ge \int_t^{t_0} L(\gamma(s), \dot\gamma(s))\, ds.$$

Dividing by $t_0 - t > 0$, and letting $t \to t_0$, we get

$$\frac{\partial \psi}{\partial t}(t_0, x_0) + \frac{\partial \psi}{\partial x}(t_0, x_0)(\dot\gamma(t_0)) \ge L(x_0, \dot\gamma(t_0)).$$

By definition of L, we have

$$L(x_0, \dot\gamma(t_0)) \ge \frac{\partial \psi}{\partial x}(t_0, x_0)(\dot\gamma(t_0)) - H(x_0, \frac{\partial \psi}{\partial x}(t_0, x_0)).$$

It follows that

$$\frac{\partial \psi}{\partial t}(t_0, x_0) + \frac{\partial \psi}{\partial x}(t_0, x_0)(\dot\gamma(t_0)) \ge \frac{\partial \psi}{\partial x}(t_0, x_0)(\dot\gamma(t_0)) - H(x_0, \frac{\partial \psi}{\partial x}(t_0, x_0)).$$

Therefore

$$\frac{\partial \psi}{\partial t}(t_0, x_0) + H(x_0, \frac{\partial \psi}{\partial x}(t_0, x_0)) \ge 0. \qquad \square$$

Since the continuous function $(t,x) \mapsto T_t^- u(x), (t,x) \in [0, +\infty[\times M$, is locally Lipschitz on $]0, +\infty[\times M$, and is a viscosity solution of

$$\frac{\partial U}{\partial t}(t, x) + H(x, \frac{\partial U}{\partial x}(t, x)) = 0,$$

on $]0, +\infty[\times M$, we can apply the uniqueness statement of Corollary 7.5 to obtain the following theorem.

Theorem 14.7. *Let $H : T^*M \to \mathbb{R}$ is a Tonelli Hamiltonian on the compact manifold M. Suppose that the continuous function $U : [0, +\infty[\times M \to \mathbb{R}$ is a viscosity solution of*

$$\frac{\partial U}{\partial t}(t,x) + H(x, \frac{\partial U}{\partial x}(t,x)) = 0,$$

on the open set $]0, +\infty[\times M$. Then $U(t,x) = T_t^- u(x)$, for every $(t,x) \in [0, +\infty[\times M$, where $u : M \to \mathbb{R}$ is defined by $u(x) = U(x,0)$.

We now conclude with the characterization of the solutions of the Hamilton-Jacobi equation by the Lax-Oleinik semi-group.

Theorem 14.8. *Let $H : T^*M \to \mathbb{R}$ be a Tonelli Hamiltonian on the compact manifold M. A continuous function $u : M \to \mathbb{R}$ is a viscosity solution of $H(x, d_x u) = c$ if and only if $u = T_t^- u + ct$, for all $t \geq 0$.*

Proof. We set $U(t,x) = u(x) - ct$. By Exercise 2.10, the function u is a viscosity solution of

$$H(x, d_x u) = c$$

on M if and only if U is a viscosity solution of

$$\frac{\partial U}{\partial t}(t,x) + H(x, \frac{\partial U}{\partial x}(t,x)) = 0$$

on $]0, +\infty[\times M$. It now follows from Theorem 14.7 that u is a viscosity solution of $H(x, d_x u) = c$ if and only if $U(t,x) = T_t^- u(x)$, for all $x \in M$, and $t \geq 0$. □

Exercise 14.9. *Suppose that $H : T^*M \to \mathbb{R}$ is a Tonelli Hamiltonian on the compact manifold M. Assume that the continuous function $U, V : [0, +\infty[\times M \to \mathbb{R}$ are respectively a viscosity subsolution and supersolution of the evolutionary Hamilton-Jacobi equation*

$$\frac{\partial U}{\partial t}(t,x) + H(x, \frac{\partial U}{\partial x}(t,x)) = 0, \tag{EHJ}$$

on the open set $]0, +\infty[\times M$. For $a \geq 0$ define $U_a, V_a : M \to \mathbb{R}$ by $U_a(x) = U(a,x)$ and $V_a(x) = V(a,x)$.

1) Show that for all $t \geq 0$, we have

$$U_{t+a} \leq T_t^- U_a \text{ and } T_t^- V_a \leq V_{t+a}.$$

2) Show that U is evolution dominated by L.

3) Conclude that a continuous function on $[0, +\infty[\times M$ is a viscosity subsolution of (EHJ) on $]0, +\infty[\times M$ if and only if it is evolution dominated by L.

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