The Lax-Oleinik semi-group: a Hamiltonian point of view.

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The Weak KAM theory was developed by Fathi in order to study the dynamics of convex Hamiltonian systems. It somehow makes a bridge between viscosity solutions of the Hamilton-Jacobi equation and Mather invariant sets of Hamiltonian systems, although this was fully understood only a posteriori. These theories converge under the hypothesis of convexity, and the richness of applications mostly comes from this remarkable convergence. In the present course, we provide an elementary exposition of some of the basic concepts of weak KAM theory. In a companion lecture, Albert Fathi exposes the aspects of his theory which are more directly related to viscosity solutions. Here on the contrary, we focus on dynamical applications, even if we also discuss some viscosity aspects to underline the connections with Fathi's lecture. The fundamental reference on Weak KAM theory is the still unpublished book of Albert Fathi *Weak KAM theorem in Lagrangian dynamics*. Although we do not offer new results, our exposition is original in several aspects. We only work with the Hamiltonian and do not rely on the Lagrangian, even if some proofs are directly inspired from the classical Lagrangian proofs. This approach is made easier by the choice of a somewhat specific setting. We work on $\mathbb{R}^d$ and make uniform hypotheses on the Hamiltonian. This allows us to replace some compactness arguments by explicit estimates. For the most interesting dynamical applications however, the compactness of the configuration space remains a useful hypothesis and we retrieve it by considering periodic (in space) Hamiltonians. Our exposition is centered on the Cauchy problem for the Hamilton-Jacobi equation and the Lax-Oleinik evolution operators associated to it. Dynamical applications are reached by considering fixed points of these evolution operators, the Weak KAM solutions. The evolution operators can also be used for their regularizing properties, this opens a second way to dynamical applications.

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1 The method of characteristics, existence and uniqueness of regular solutions.

We consider a C^2 Hamiltonian

$$H(t,q,p):\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^{d*}\longrightarrow\mathbb{R}$$

and study the associated Hamiltonian system

$$\dot{q}(t)=\partial_p H(t,q(t),p(t)) \quad , \quad \dot{p}(t)=-\partial_q H(t,q(t),p(t)) \tag{HS}$$

and Hamilton-Jacobi equation

$$\partial_t u + H(t,q,\partial_q u(t,q))=0. \tag{HJ}$$

We denote by $X_H(x)=X_H(q,p)$ the Hamiltonian vector field $X_H=JdH$, where J is the matrix

$$J=\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

The Hamiltonian system can be written in condensed terms $\dot{x}(t)=X_H(t,x(t))$. We will always assume that that the solutions extend to $\mathbb{R}$. We denote by

$$\varphi_\tau^t=(Q_\tau^t,P_\tau^t):\mathbb{R}^d\times\mathbb{R}^{d*}\longrightarrow\mathbb{R}^d\times\mathbb{R}^{d*}$$

the flow map which, to a point $x\in\mathbf{T}^*\mathbb{R}^d$, associate the value at time t of the solution $x(s)$ of (HS) which satisfies $x(\tau)=x$.

If $u(t,q)$ solves (HJ), and if $q(s)$ is a curve in $\mathbb{R}^d$, then the formula

$$u(t_1,q(t_1))-u(t_0,q(t_0))=\int_{t_0}^{t_1}\partial_q u(s,q(s))\cdot\dot{q}(s)-H(s,\partial_q u(s,q(s)))ds \tag{1}$$

follows from an obvious computation. The integral on the right hand side is the **Hamiltonian action** of the curve $s\longmapsto(q(s),\partial_q u(s,q(s)))$. The **Hamiltonian action** of the curve $(q(s),p(s))$ on the interval $[t_0,t_1]$ is the quantity

$$\int_{t_0}^{t_1} p(s)\cdot\dot{q}(s)-H(s,q(s),p(s))ds.$$

A classical and important property of the Hamiltonian actions is that orbits are critical points of this functional. More precisely, we have:

Proposition 1. *The C^2 curve $x(t)=(q(t),p(t)):[t_0,t_1]\longrightarrow\mathbb{R}^d\times\mathbb{R}^{d*}$ solves (HS) if and only if the equality*

$$\frac{d}{ds}\Big|_{s=0}\left(\int_{t_0}^{t_1} p(t,s)\cdot\dot{q}(t,s)-H(t,q(t,s),p(t,s))dt\right)=0,$$

where the dot is the derivative with respect to t, holds for each C^2 variation $x(t,s)=(q(t,s),p(t,s)):[t_0,t_1]\times\mathbb{R}\longrightarrow\mathbb{R}^d\times\mathbb{R}^{d*}$ fixing the endpoints, which means that $x(t,0)=x(t)$ for each t and that $q(t_0,s)=q(t_0)$ and $q(t_1,s)=q(t_1)$ for each s.

Proof. We set $\theta(t) = \partial_s q(t,0)$, $\zeta(t) = \partial_s p(t,0)$ and compute:

$$\begin{aligned}
&\frac{d}{ds}\Big|_{s=0} \left(\int_{t_0}^{t_1} p(t,s)\dot{q}(t,s) - H(t,q(t,s),p(t,s))dt \right) \\
&= \int_{t_0}^{t_1} p(t)\dot{\theta}(t) + \zeta(t)\dot{q}(t) - \partial_q H(t,q(t),p(t))\theta(t) - \partial_p H(t,q(t),p(t))\zeta(t)dt \\
&= p(t_1)\theta(t_1) - p(t_0)\theta(t_0) + \int_{t_0}^{t_1} \big(\dot{q}(t) - \partial_p H(t,q(t),p(t))\big)\zeta(t)dt \\
&- \int_{t_0}^{t_1} \big(\dot{p}(t) + \partial_q H(t,q(t),p(t))\big)\theta(t)dt.
\end{aligned}$$

As a consequence, the derivative of the action vanishes if $(q(t),p(t))$ is a Hamiltonian trajectory and if the variation $q(t,s)$ is fixing the boundaries. Conversely, this computation can be applied to the variation $q(t,s) = q(t) + s\theta(t), p(t,s) = p(t) + s\zeta(t)$, and implies that

$$\int_{t_0}^{t_1} \big(\dot{q}(t) - \partial_p H(t,q(t),p(t))\big)\zeta(t)dt - \int_{t_0}^{t_1} \big(\dot{p}(t) + \partial_q H(t,q(t),p(t))\big)\theta(t)dt = 0$$

for each C^2 curve $\theta(t)$ vanishing on the boundary and each C^2 curve $\zeta(t)$. This implies that $\dot{q}(t) - \partial_p H(t,q(t),p(t)) \equiv 0$ and $\dot{p}(t) + \partial_q H(t,q(t),p(t)) \equiv 0$. □

We now return to the connections between (HS) and (HJ). A function is said of class $C^{1,1}$ if it is differentiable and if its differential is Lipschitz. It is said of class $C^{1,1}_{loc}$ if it is differentiable with a locally Lipschitz differential. The Theorem of Rademacher states that a locally Lipschitz function is differentiable almost everywhere.

Theorem 1. *Let $\Omega \subset \mathbb{R} \times \mathbb{R}^d$ be an open set, and let $u(t,q) : \Omega \longrightarrow \mathbb{R}$ be a $C^{1,1}_{loc}$ solution of (HJ). Let $q(t) : [t_0,t_1] \longrightarrow \mathbb{R}^d$ be a C^1 curve such that $(t,q(t)) \in \Omega$ and*

$$\dot{q}(t) = \partial_p H\big(t, q(t), \partial_q u(t,q(t))\big)$$

for each $t \in [t_0,t_1]$. Then, setting $p(t) = \partial_q u(t,q(t))$, the curve $(q(t),p(t))$ is C^1 and it solves (HS).

The curves $q(t)$ satisfying the hypothesis of the theorem, as well as the associated trajectories $(q(t),p(t))$ are called the **characteristics** of u.

Proof. Let $\theta(t) : [t_0,t_1] \longrightarrow \mathbb{R}^d$ be a smooth curve vanishing on the boundaries. We define $q(t,s) := q(t) + s\theta(t)$ and $p(t,s) := \partial_q u(t,q(t,s)))$. Our hypothesis is that $\dot{q}(t) = \partial_p H(t,q(t),p(t))$, which is the first part of (HS). For each s, we have

$$u(t_1,q(t_1)) - u(t_0,q(t_0)) = \int_{t_0}^{t_1} p(t,s)\cdot\dot{q}(t,s) - H(t,q(t,s),p(t,s))ds$$

hence $\frac{d}{ds}\big|_{s=0} \left(\int_{t_0}^{t_1} p(t,s)\dot{q}(t,s) - H(t,q(t,s),p(t,s))dt \right) = 0$. We now claim that

$$\int_{t_0}^{t_1} \partial_q H(t,q(t),p(t))\cdot\theta(t) - p(t)\dot{\theta}(t)\,dt = \frac{d}{ds}\Big|_{s=0} \left(\int_{t_0}^{t_1} p(t,s)\dot{q}(t,s) - H(t,q(t,s),p(t,s))dt \right).$$

Assuming the claim, we obtain the equality $\int_{t_0}^{t_1} \partial_q H(t,q(t),p(t))\cdot\theta(t) - p(t)\cdot\dot{\theta}(t)\,dt = 0$ for each smooth function θ vanishing at the boundary. In other words, we have

$$\dot{p}(t) = -\partial_q H(t,q(t),p(t))$$

in the sense of distributions. Since the right hand side is continuous, this implies that p is C^1 and that the equality holds for each t. We have proved the theorem, assuming the claim.

The claim can be proved by an easy computation in the case where u is C^2. Under the assumption that u is only $C^{1,1}_{loc}$, the map p is only locally Lipschitz, and some care is necessary. For each fixed θ, we have

$$\partial_q H(t,q(t,s),p(t,s))\cdot\theta(t)-p(t,s)\cdot\dot\theta(t)=\partial_q H(t,q(t),p(t))\cdot\theta(t)-p(t)\cdot\dot\theta(t)+O(s)$$
$$\partial_t q(t,s)-\partial_p H(t,q(t,s),p(t,s))=\dot q-\partial_p H(t,q(t),p(t))+O(s)=O(s)$$

where $O(s)$ is uniform in t. We then have, for small $S>0$,

$$\begin{aligned}&\int_{t_0}^{t_1}\partial_q H(t,q(t),p(t))\cdot\theta(t)-p(t)\cdot\dot\theta(t)\,dt\\
=&O(S)+\frac{1}{S}\int_{t_0}^{t_1}\int_0^S\partial_q H(t,q(t,s),p(t,s))\cdot\theta(t)-p(t,s)\cdot\dot\theta(t)\,dsdt\\
=&O(S)+\frac{1}{S}\int_{t_0}^{t_1}\int_0^S\partial_q H\cdot\partial_s q-p\cdot\partial_{st}q+\big(\partial_t q-\partial_p H\big)\cdot\partial_s p\,dsdt\\
=&O(S)+\frac{1}{S}\int_{t_0}^{t_1}\big[p\cdot\partial_t q-H\big]_0^S\,dt=O(S)+\frac{1}{S}\left[\int_{t_0}^{t_1}p\cdot\partial_t q-H\,dt\right]_0^S.\end{aligned}$$

We obtain the claimed equality at the limit $S\longrightarrow 0$. □

The following restatement of Theorem 1 has a more geometric flavor:

Corollary 2. *Let $\Omega\subset\mathbb{R}\times\mathbb{R}^d$ be an open set, and let $u(t,q):\Omega\longrightarrow\mathbb{R}$ be a $C^{1,1}_{loc}$ solution of the Hamilton Jacobi equation (HJ). Then the extended Hamiltonian vector-field $Y_H=(1,X_H)$ is tangent to the graph*

$$G:=\{(t,q,\partial_q u):(t,q)\in\Omega\}.$$

Proof. Let us fix a point (t_0,q_0) in Ω. By the Cauchy-Lipschitz theorem, there exists a solution $q(t)$ of the ordinary differential equation $\dot q=\partial_p H\big(t,q(t),\partial_q u(t,q(t))\big)$, defined on an open time interval containing t_0 and such that $q(t_0)=q_0$. Let us define as above $p(t):=\partial_q u(t,q(t))$. The curve $(t,q(t),p(t))$ is contained in the graph G, and we deduce from Theorem 1 that it solves (HS). As a consequence, the derivative Y_H of the curve $(t,q(t),p(t))$ is tangent to G. □

Corollary 3. *Let $u(t,q)$ be a $C^{1,1}_{loc}$ solution of (HJ) defined on the open set $\Omega=]t_0,t_1[\times\mathbb{R}^d$. Then, for each s and t in $]t_0,t_1[$ we have*

$$\Gamma_t=\varphi_s^t(\Gamma_s),$$

where Γ_t is defined by

$$\Gamma_t:=\{(q,du_t(q)):q\in\mathbb{R}^d\}.$$

Proof. Let (q_s,p_s) be a point in Γ_s. Let us consider the Lipschitz map

$$F(t,q):=\partial_p H(t,q,\partial_q u(t,q)),$$

and consider the differential equation $\dot q(t)=F(t,q(t))$. By the Cauchy-Peano Theorem, there exists a solution $q(t)$ of this equation, defined on the interval $]t^-,t^+[\ni s$, and such that $q(s)=q_s$. Setting $p(t)=\partial_q u(t,q(t))$, Theorem 1 implies that the curve $(q(t),p(t))$ solves (HS). We can chose t^+ such that either $t^+=t^1$ or the curve $q(t)$ is unbounded on $[s,t^+[$. The second case is not possible because $(q(t), p(t))$ is a solution of (HS), which is complete, hence we can take $t^+ = t_1$. Similarly, we can take $t^- = t_0$. We have proved that $(q(t), p(t))$ is the Hamiltonian orbit of the point (q_s, p_s). Then, for each $t \in]t_0, t_1[$, we have

$$\varphi_s^t(q_s, p_s) = (q(t), p(t)) = (q(t), \partial_q u(t, q(t))) \in \Gamma_t.$$

Since this holds for each $(q_s, p_s) \in \Gamma_s$, we conclude that $\varphi_s^t(\Gamma_s) \subset \Gamma_t$ for each $s, t \in]t_0, t_1[$. By symmetry, this inclusion is an equality. □

Let us now consider an initial condition $u_0(q)$ and study the Cauchy problem consisting of finding a solution $u(t, q)$ of (HJ) such that $u(0, q) = u_0(q)$.

Proposition 4. *Given a time interval $]t_0, t_1[$ containing the initial time $t = 0$ and a $C^{1,1}_{loc}$ initial condition u_0, there is at most one $C^{1,1}_{loc}$ solution $u(t, q) :]t_0, t_1[\times \mathbb{R}^d$ of (HJ) such that $u(0, q) = u_0(q)$ for all $q \in \mathbb{R}^d$.*

Proof. Let u and $\tilde{u}$ be two solutions of this Cauchy problem. Let us associate to them the graphs Γ_t and $\tilde{\Gamma}_t$, $t \in]t_0, t_1[$. Since $\tilde{u}(\tau, q) = u(\tau, q)$, we have $\Gamma_\tau = \tilde{\Gamma}_\tau$ hence, by Corollary 3,

$$\Gamma_t = \varphi_\tau^t(\Gamma_\tau) = \varphi_\tau^t(\tilde{\Gamma}_\tau) = \tilde{\Gamma}_t.$$

We conclude that $\partial_q u = \partial_q \tilde{u}$, and then, from (HJ), that $\partial_t u = \partial_t \tilde{u}$. The functions u and $\tilde{u}$ thus have the same differential on $]t_0, t_1[\times \mathbb{R}^d$, hence they differ by a constant. Finally, since these functions have the same value on $\{\tau\} \times \mathbb{R}^d$, they are equal. □

To study the existence problem, we lift the function u_0 to the surface Γ_0 by defining $w_0 = u_0 \circ \pi$, where π is the projection $(q, p) \longmapsto q$ (later we will also use the symbol π to denote the projection $(t, q, p) \longmapsto (t, q)$). It is then useful to work in a more general setting:

A **geometric initial condition** is the data of a subset $\Gamma_0 \subset \mathbb{R}^d \times \mathbb{R}^{d*}$ and of a function $w_0 : \Gamma_0 \longrightarrow \mathbb{R}$ such that $dw_0 = pdq$ on Γ_0. More precisely, we require that the equality $\partial_s(w_0(q(s), p(s))) = p(s)\partial_s q(s)$ holds almost everywhere for each Lipschitz curve $(q(s), p(s))$ on Γ_0. We will consider mainly two types of geometric initial conditions:

- The geometric initial condition $(\Gamma_0, w_0 = u_0 \circ \pi)$ associated to the C^1 initial condition u_0.
- The geometric initial condition $(\Gamma_0 = \{q_0\} \times \mathbb{R}^{d*}, w_0 = 0)$, for $q_0 \in \mathbb{R}^d$.

Given the geometric initial condition (Γ_0, w_0), we define:

$$G := \bigcup_{t \in]t_0, t_1[} \{t\} \times \varphi_0^t(\Gamma_0) \tag{G}$$

and, denoting by $\dot{Q}_t^s(x)$ the derivative with respect to s, the function

$$w : \quad G \longrightarrow \mathbb{R}$$

$$(t, x) \longmapsto w_0(Q_t^0(x)) + \int_0^t P_t^s(x)\dot{Q}_t^s(x) - H(s, \varphi_t^s(x))ds. \tag{w}$$

The pair (G, w) is called the **geometric solution** emanating from the geometric initial condition (Γ_0, w_0).

This definition is motivated by the following observation: Assume that a C^2 solution $u(t, q)$ of (HJ) emanating from the genuine initial condition u_0 exists. Let (Γ_0, w_0) be the geometric initial condition associated to u_0. Let G be the graph of $\partial_q u$, as defined in Corollary 3, and let w be the function defined on G by $w := u \circ \pi$. Then, (G, w) is the geometric solution emanating from the geometric initial condition Γ_0. This follows immediately from Corollary 3 and equation (1). In general, we have:

Proposition 5. *Let* (Γ_0, w_0) *be a geometric initial condition, and let* (G, w) *be the geometric solution emanating from* (Γ_0, w_0)*. Then, the function* w *satisfies* $dw = pdq - Hdt$ *on* G*. More precisely, for each Lipschitz curve* $Y(s) = (T(s), \theta(s), \zeta(s))$ *contained in* G*, then for a. e.* s*,*

$$\frac{d}{ds}\big(w(T(s), \theta(s), \zeta(s))\big) = \zeta(s)\frac{d\theta}{ds} - H(Y(s))\frac{dT}{ds}.$$

Proof. Let us first consider a C^2 curve $Y(s) = (T(s), \theta(s), \zeta(s))$ on G. We set $q(t, s) = Q^t_{T(s)}(\theta(s), \zeta(s))$ and $p(t, s) = P^t_{T(s)}(\theta(s), \zeta(s))$, and finally $x(t, s) = (q(t, s), p(t, s))$. We have

$$w(T(s), \theta(s), \zeta(s)) = w_0(q(0, s), p(0, s)) + \int_0^{T(s)} p(t, s)\dot{q}(t, s) - H(t, x(t, s))dt.$$

Since $dw_0 = pdq$ on Γ_0, the calculations in the proof of Proposition 1 imply that

$$\begin{aligned}\frac{d}{ds}(w \circ Y) &= p(0, s) \cdot \partial_s q(0, s) + p(T(s), s) \cdot \partial_s q(T(s), s) - p(0, s) \cdot \partial_s q(0, s) \\ &\quad + \Big(p(T(s), s) \cdot \partial_t q(T(s), s) - H(T(s), x(T(s), s))\Big)\frac{dT}{ds} \\ &= \zeta(s)\left(\partial_s q(T(s), s) + \partial_t q(T(s), s)\frac{dT}{ds}\right) + H(Y(s))\frac{dT}{ds}.\end{aligned}$$

The desired equality follows from the observation that $d\theta/ds = \partial_t q(T(s), s)(dT/ds) + \partial_s q(T(s), s)$, which can be seen by differentiating the equality $\theta(s) = q(T(s), s)$.

These computations, however, can't be applied directly in the case where $Y(s)$ is only C^1, or, even worse, Lipschitz. In this case, we will prove the desired equality in integral form

$$[w \circ Y]_{S_0}^{S_1} = \int_{S_0}^{S_1} \zeta(s) \cdot \partial_s\theta(s) - H \circ Y(s) \cdot \partial_s T(s)ds$$

for each $S_0 < S_1$. Fixing S_0 and S_1, we can approximate uniformly the curve $Y(s)$ by a sequence $Y_n(s) : [S_0, S_1] \longrightarrow \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d*}$ of equi-Lipschitz smooth curves such that $Y_n(S_0) = Y(S_0)$ and $Y_n(S_1) = Y(S_1)$. To the curves Y_n, we associate $x_n(t, s) = (p_n(t, s), q_n(t, s))$ as above. The functions x_n are equi-Lipschitz and converge uniformly to x. In general, we don't have $Y_n(s) \in G$ on $]S_0, S_1[$, hence we don't have $x_n(0, s) \in \Gamma_0$, and we cannot express $\partial_s w(x_n(0, s))$ as we did above. Since this is the only part of the above computation which used the inclusion $Y(s) \in G$, we can still get:

$$\frac{d}{ds}(w \circ Y_n) = \frac{d}{ds}(w_0(x_n(0, s)) - p_n(0, s) \cdot \partial_s q_n(0, s) + \zeta_n(s)\partial_s\theta_n(s) + H(Y_n(s))\partial_s T_n(s).$$

Noticing that $[w \circ Y]_{S_0}^{S_1} = [w \circ Y_n]_{S_0}^{S_1}$ and that $[w_0(x(0, .))]_{S_0}^{S_1} = [w_0(x_n(0, .))]_{S_0}^{S_1}$, we obtain

$$\begin{aligned}[w \circ Y]_{S_0}^{S_1} &= [w_0(x(0, .))]_{S_0}^{S_1} + \int_{S_0}^{S_1} -p_n(0, s) \cdot \partial_s q_n(0, s) + \zeta_n(s)\partial_s\theta_n(s) + H(Y_n(s))\partial_s T_n(s)ds \\ &= \int_{S_0}^{S_1} p(0, s) \cdot \partial_s q(0, s) - p_n(0, s) \cdot \partial_s q_n(0, s)ds \\ &\quad + \int_{S_0}^{S_1} \zeta_n(s)\partial_s\theta_n(s) + H(Y_n(s))\partial_s T_n(s)ds.\end{aligned}$$

We derive the desired formula at the limit $n \longrightarrow \infty$, along a subsequence such that

$$\partial_s q_n(0, .) \rightharpoonup \partial_s q(0, .), \quad \partial_s\theta_n \rightharpoonup \partial_s\theta, \quad \partial_s T_n \rightharpoonup \partial_s T$$

weakly-$\star$ in L^∞, taken into account that

$$p_n(0,.) \longrightarrow p(0,.), \quad \zeta_n(s) \longrightarrow \zeta(s), \quad H(Y_n(s)) \longrightarrow H(Y(s))$$

uniformly, hence strongly in L^1. Recall that a sequence of curves $f_n : [t_0,t_1] \longrightarrow \mathbb{R}^d$ is said to converge to f weakly-$\star$ in L^∞ if $\int_{t_0}^{t_1} f_n g dt \longrightarrow \int_{t_0}^{t_1} f g dt$ for each L^1 curve $g : [t_0,t_1] \longrightarrow \mathbb{R}^d$. We have used two classical properties of the weak-$\star$ convergence:

- A uniformly bounded sequence of functions has a subsequence which has a weak-$\star$ limit.
- The convergence $\int_{t_0}^{t_1} f_n g_n dt \longrightarrow \int f g dt$ holds if $f_n \rightharpoonup f$ weakly-$\star$ in L^∞ and if $g_n \to g$ strongly in L^1. □

Corollary 6. *If there exists a locally Lipschitz map $\chi : \Omega \longrightarrow \mathbb{R}^{d*}$ on some open subset Ω of $]t_0,t_1[\times\mathbb{R}^d$ such that $(t,q,\chi(t,q)) \subset G$ for all $(t,q) \in \Omega$, then the function*

$$u(t,q) := w(t,q,\chi(t,q))$$

is C^1 and it solves (HJ) on Ω. Moreover, we have $\partial_q u = \chi$.

Proof. For each C^1 curve $(T(s),Q(s))$ in Ω, the curve

$$Y(s) = (T(s),Q(s),\chi(T(s),Q(s))$$

is Lipschitz, hence, by Proposition 5, we have

$$\begin{aligned}\partial_s u(T(s),Q(s)) &= \partial_s w\big(T(s),Q(s),\chi(T(s),Q(s))\big)\\ &= \chi\big(T(s),Q(s)\big)\cdot\partial_s Q(s) - H\big(T(s),Q(s),\chi(T(s),Q(s)\big)\partial_s T(s)\end{aligned}$$

almost everywhere. Since the right hand side in this expression is continuous, we conclude that the Lipschitz functions $u(T(s),Q(s))$ is actually differentiable at each point, the equality above being satisfied everywhere. Since this holds for each C^1 curve $(T(s),Q(s))$, the function u has to be differentiable, with $\partial_q u(t,q) = \chi(t,q)$ and $\partial_t u(t,q) + H(t,q,\chi(t,q)) = 0$. □

We have reduced the existence problem to the study of the geometric solution G. We need an additional hypothesis to obtain a local existence result. We will rest on the following one, which it is stronger than would really be necessary, but will allow us to rest on simple estimates in this course.

Hypothesis 1. *There exists a constant M such that*

$$\|d^2H(t,q,p)\| \leqslant M$$

for each (t,q,p).

This hypothesis implies that the Hamiltonian vector-field is Lipschitz, hence that the Hamiltonian flow is complete. The hypothesis can be exploited further to estimate the differential

$$d\varphi_0^t = \begin{bmatrix} \partial_q Q_0^t(x) & \partial_p Q_0^t(x) \\ \partial_q P_0^t(x) & \partial_p P_0^t(x) \end{bmatrix}$$

using the variational equation

$$\begin{bmatrix} \partial_q \dot{Q}_0^t(x) & \partial_p \dot{Q}_0^t(x) \\ \partial_q \dot{P}_0^t(x) & \partial_p \dot{P}_0^t(x) \end{bmatrix} = \begin{bmatrix} \partial_{qp} H(t,x) & \partial_{pp} H(t,x) \\ -\partial_{qp} H(t,x) & -\partial_{pp} H(t,x) \end{bmatrix} \begin{bmatrix} \partial_q Q_0^t(x) & \partial_p Q_0^t(x) \\ \partial_q P_0^t(x) & \partial_p P_0^t(x) \end{bmatrix}.$$

We obtain the following estimates:

$$\|d\varphi_\tau^t - I\| \leqslant e^{M|t-\tau|} - 1$$

which implies, for $|t-\tau| \leqslant 1/M$, that

$$\|d\varphi_\tau^t - I\| \leqslant 2M|t-\tau| \tag{M}$$

or componentwise (taking $\tau = 0$, and assuming that $|t| \leqslant M$):

$$\|\partial_q Q_0^t - I\| \leqslant 2M|t| \quad , \quad \|\partial_p P_0^t - I\| \leqslant 2M|t| \quad , \quad \|\partial_q P_0^t\| \leqslant 2M|t| \quad , \quad \|\partial_p Q_0^t\| \leqslant 2M|t|.$$

We can now prove:

Theorem 2. *Let $H : \mathbb{R} \times \mathbb{R}^d \times (\mathbb{R}^d)^*$ be a C^2 Hamiltonian satisfying Hypothesis 1. Let u_0 be a $C^{1,1}$ initial condition. There exists a time $T > 0$ and a $C^{1,1}_{loc}$ solution $u(t,q) :]-T,T[\times\mathbb{R}^d \longrightarrow \mathbb{R}$ of (HJ) such that $u(0,q) = u_0(q)$. Moreover, we can take*

$$T = \Big(4M\big(1 + Lip(du_0)\big)\Big)^{-1},$$

and we have

$$Lip(du_t) \leqslant Lip(du_0) + 4|t|M\big(1 + Lip(du_0)\big)^2$$

when $|t| \leqslant T$. If the initial condition u_0 is C^2, then so is the solution $u(t,q)$.

Proof. Let (Γ_0, w_0) be the geometric initial condition associated to u_0, and let (G, w) be the geometric solution emanating from (Γ_0, w_0). We first prove that the restriction of G to $]-T,T[\times\mathbb{R}^d$ is a graph. It is enough to prove that the map

$$F(t,q) := \big(t, Q_0^t(q, du_0(q))\big)$$

is a bi-Lipschitz homeomorphism of $]-T,T[\times\mathbb{R}^d$. By (M), we have

$$Lip(F - Id) \leqslant 2|t|M\big(1 + Lip(du_0)\big) < 1,$$

provided $|t| < \Big(2M\big(1 + Lip(du_0)\big)\Big)^{-1}$. We conclude using the classical Proposition 50 of the Appendix that F is a bi-Lipschitz homeomorphism of $]-T,T[\times\mathbb{R}^d$. Moreover, if u_0 is C^2, then F is a C^1 diffeomorphism. Since F is a homeomorphism preserving t, we can denote by by $(t, Z(t,q))$ its inverse. By Proposition 50, we have

$$Lip(Z) \leqslant \frac{1}{1 - 2|t|M(1 + Lip(du_0))},$$

and, under the assumption that $|t| \leqslant T$ (as defined in the statement), we obtain

$$Lip(Z) \leqslant 1 + 4M|t|(1 + Lip(du_0)) \leqslant 2.$$

We have just used here that $(1-a)^{-1} \leqslant 1 + 2a$ for $a \in [0, 1/2]$. We set

$$\chi(t,q) = P_0^t\big(Z(t,q), du_0(Z(t,q))\big),$$

in such a way that G is the graph of χ on $]-T,T[\times\mathbb{R}^d$. Observing that χ is Lipschitz, we conclude from Corollary 6 that the function $u(t,q) := w(t,q,\chi(t,q))$ solves (HJ). Moreover, we have $u(0,q) = u_0(q)$. Corollary 6 also implies that $du_t = \chi_t$ hence, in view of (M), we have

$$\begin{aligned}
Lip(du_t) = Lip(\chi_t) &\leqslant 2M|t|Lip(Z_t) + (1 + 2M|t|)Lip(du_0)Lip(Z_t) \\
&\leqslant 4M|t| + Lip(du_0) + Lip(du_0)(4M|t|(1 + Lip(du_0))) + 4M|t|Lip(du_0) \\
&\leqslant Lip(du_0) + 4M|t|\big(1 + Lip(du_0)\big)\big(1 + Lip(du_0)\big).
\end{aligned}$$

□

1.1 Exercise :

Take $d = 1$, $H(t,q,p) = (1/2)p^2$, and $u_0(q) = -q^2$, and prove that the C^2 solution can't be extended beyond $t = 1/2$.

2 Convexity, the twist property, and the generating function.

We make an additional assumption on H. Once again, we make the assumption in a stronger form than would be necessary, this allows to obtain simpler statements:

Hypothesis 2. *There exists $m > 0$ such that*

$$\partial^2_{pp} H \geqslant mId$$

for each (t,q,p), in the sense of quadratic forms.

Let us first study the consequences of this hypothesis on the structure of the flow.

Proposition 7. *There exists $\sigma > 0$ such that the map $p \longmapsto Q^t_0(q,p)$ is $(mt/2)$-monotone when $t \in]0,\sigma]$, in the sense that the inequality*

$$(Q^t_0(q,p') - Q^t_0(q,p)) \cdot (p' - p) \geqslant mt|p' - p|^2/2$$

holds for each $q \in \mathbb{R}^d$, each $t \in [0,\sigma]$. As a consequence, it is a C^1 diffeomorphism onto $\mathbb{R}^d$.

We say that the flow has the **Twist property**.

PROOF. Fix a point q and denote by F^t the map $p \longmapsto Q^t_0(q,p)$. We have $dF^t(p) = \partial_p Q^t_0(q,p)$. In order to estimate this linear map, we recall the variational equation

$$\partial_p \dot{Q}^t_0(x) = \partial_{qp} H(t,\varphi^t_0(x))\partial_p Q^t_0(x) + \partial_{pp} H(t,\varphi^t_0(x))\partial_p P^t_0(x).$$

We deduce that

$$\partial_p \dot{Q}^t_0(x) - \partial^2_{pp} H(t,\varphi^t_0(x)) = \partial^2_{qp} H(t,\varphi^t_0(x))\partial_p Q^t_0(x) + \partial^2_{pp} H(t,\varphi^t_0(x))(\partial_p P^t_0(x) - Id)$$

and then that

$$\|\partial_p \dot{Q}^t_0(x) - \partial^2_{pp} H(t,\varphi^t_0(x))\| \leqslant 2M^2 t$$

As a consequence, for $t \leqslant \sigma = m/(4M^2)$, we have

$$\partial_p \dot{Q}^t_0 \geqslant (m - 2M^2 t)I \geqslant (m/2)I$$

in the sense of quadratic forms (note that the matrix $\partial_p \dot{Q}^t_0$ is not necessarily symmetric). Since

$$\partial_p Q^t_0(x) = \int_0^t \partial_p \dot{Q}^s_0(x) ds,$$

we conclude that

$$dF^t(p) = \partial_p Q^t_0(q,p) \geqslant (m/2)Id,$$

which means that $(dF^t(p)z,z) \geqslant (m/2)|z|^2$ for each $z \in \mathbb{R}^{d*}$. This estimate can be integrated, and implies the monotony of the map F^t:

$$\begin{aligned}(Q^t(q,p') - Q^t(q,p)) \cdot (p' - p) &= \left(\int_0^1 \partial_p Q^t(q, p + s(p' - p)) \cdot (p' - p) ds\right) \cdot (p' - p) \\ &= \int_0^1 \big(\partial_p Q^t(q, p + s(p' - p)) \cdot (p' - p)\big) ds \\ &\geqslant \int_0^1 (m/2)t(p' - p) \cdot (p' - p) ds \geqslant (m/2)t(p' - p) \cdot (p' - p).\end{aligned}$$

It is then a classical result that the map F^t is a C^1 diffeomorphism, see Proposition 51 in the appendix.

□

Corollary 8. *The map* $(t,q,p) \longmapsto (t,q,Q_0^t(q,p))$ *is a* C^1 *diffeomorphism from* $]0,\sigma[\times\mathbb{R}^d\times\mathbb{R}^{d*}$ *onto its image* $]0,\sigma[\times\mathbb{R}^d\times\mathbb{R}^d$.

We denote by $\rho_0(t,q_0,q_1)$ the unique momentum p such that $Q_0^t(t,q_0,\rho_0(t,q_0,q_1)) = q_1$. In other words, $\rho_0(t,q_0,q_1)$ is the initial momentum $p(0)$ of the unique orbit $(q(s),p(s)) : [0,t] \longrightarrow \mathbb{R}^d\times\mathbb{R}^{d*}$ of (HS) which satisfies $q(0)=q_0$ and $q(t)=q_1$. By the Corollary we just proved, the map ρ_0 is C^1. Similarly, we denote by $\rho_1(t,q_0,q_1)$ the unique momentum p such that $Q_t^0(t,q_1,\rho_1(t,q_0,q_1)) = q_0$. We can equivalently define ρ_1 as

$$\rho_1(t,q_0,q_1) = P_0^t(t,q_0,\rho_0(t,q_0,q_1)).$$

Considering the geometric initial condition $(\Gamma_0 = \{q_0\}\times\mathbb{R}^{d*}, w_0 = 0)$, and the associated geometric solution (G,w), we see that

$$G = \{(t,q,\rho_1(t,q_0,q)), \quad (t,q)\in]0,\sigma[\times\mathbb{R}^d\}.$$

We conclude from Corollary 6 that there exists a genuine solution of (HJ) emanating from the geometric initial condition $(\{q_0\}\times\mathbb{R}^{d*},0)$. We denote by $S^t(q_0,q)$ this solution. We have

$$S^t(q_0,q) = w(t,p,\rho_1(t,q_0,q))$$

and

$$\partial_q S^t(q_0,q) = \rho_1(t,q_0,q).$$

In view of the definition of geometric solutions, the function S can be written more explicitly

$$S^t(q_0,q_1) = \int_0^t P_0^s(q_0,\rho_0(t,q_0,q_1))\dot{Q}_0^s(q_0,\rho_0(t,q_0,q_1)) - H(s,\varphi_0^s(q_0,\rho_0(t,q_0,q_1))ds.$$

In words, $S^t(q_0,q_1)$ is the action of the unique trajectory $(q(s),p(s)) : [0,t] \longrightarrow \mathbb{R}^d\times\mathbb{R}^{d*}$ of (HS) which satisfies $q(0)=q_0$ and $q(t)=q_1$.

We have defined the function $S^t(q_0,q_1)$ as the action of the unique orbit joining q_0 and q_1 between time 0 and t. We can define similarly the function $S_\tau^t(q_0,q_1)$ as the action of the unique orbit joining q_0 to q_1 between time τ and time t, all this being well-defined provided $0 < t-\tau < \sigma$. It is possible to prove as above that the function $(s,q) \longmapsto S_s^t(q,q_1)$ solves the Hamilton-Jacobi equation

$$\partial_s u + H(t,q,-\partial_q u) = 0,$$

on $s<t$, and that

$$\partial_q S^t(q,q_1) = \partial_q S_0^t(q,q_1) = -\rho_0(t,q,q_1).$$

Convention: We shall from now on denote by $\partial_0 S^t$ the partial differential with respect to the first variable (which in our notations is often q_0), and by $\partial_1 S^t$ the partial differential with respect to the second variable (which in our notations is often q_1).

The relations $\partial_0 S = -\rho_0$, $\partial_1 S = \rho_1$, $\partial_t S = -H(t,q_1,\rho_1) = -H(0,q_0,\rho_0)$ that we have proved imply that the function S is C^2. Moreover, since $\varphi_0^t(q_0,\rho_0(t,q_0,q_1)) = (q_1,\rho_1(t,q_0,q_1)$, we have

$$\varphi_0^t(q_0,-\partial_0 S(q_0,q_1)) = (q_1,\partial_1 S^t(q_0,q_1)).$$

We say that S^t is a **generating function** of the flow map φ_0^t. See [17], chapter 9, for more material on generating functions. It is useful to estimate the second differentials of S:

Lemma 9. *The function S is C^2 on $]0,\sigma[\times\mathbb{R}^d\times\mathbb{R}^d$, and the estimates*

$$\partial^2_{00}S^t \geqslant \frac{c}{t}I \quad , \quad \partial^2_{11}S^t \geqslant \frac{c}{t}I$$
$$\|\partial^2_{00}S^t\| + \|\partial^2_{01}S^t\| + \|\partial^2_{01}S^t\| \leqslant \frac{C}{t}$$

hold, with constants c and C which depend only on m and M.

Proof. Let us first observe that

$$\partial^2_{11}S^t(q_0,q_1) = \big(\partial_p P^t_0(q_0,\rho_0(t,q_0,q_1))\big)\big(\partial_p Q^t_0(q_0,\rho_0(t,q_0,q_1))\big)^{-1},$$

and recall the estimates:

$$\|\partial_p P^t_0 - Id\| \leqslant 2Mt, \quad \|\partial_p Q^t_0\| \leqslant 2Mt, \quad \partial_p Q^t_0 \geqslant (mt/2)Id.$$

We conclude that (see Lemma 52)

$$(\partial_p Q^t_0)^{-1} \geqslant \frac{m}{8M^2t}Id \quad , \quad \|(\partial_p Q^t_0)^{-1}\| \leqslant 2/(mt).$$

Finally, we obtain that

$$\partial^2_{11}S(q_0,q_1) \geqslant \left(\frac{m}{8M^2t} - \frac{4M}{m}\right)Id \geqslant \frac{m}{16M^2t}Id$$

provided $t \leqslant m^2/(64M^3)$. The other estimates can be proved similarly, using the expressions

$$\begin{aligned}
\partial^2_{00}S^t(q_0,q_1) &= -\big(\partial_p P^0_t(q_1,\rho_1(t,q_0,q_1))\big)\big(\partial_p Q^0_t(q_1,\rho_1(t,q_0,q_1))\big)^{-1},\\
\partial^2_{10}S^t(q_0,q_1) &= \big(\partial_p Q^t_0(q_0,\rho_0(t,q_0,p_0))\big)^{-1}.
\end{aligned}$$

□

Proposition 10. *Given times t_1 and t_2 such that $0 < t_1 < t_2 < \sigma$, we have the triangle inequality*

$$S^{t_2}_0(q_0,q_2) \leqslant S^{t_1}_0(q_0,q_1) + S^{t_2}_{t_1}(q_1,q_2)$$

for each q_0,q_1,q_2. Moreover, $S^{t_2}_0(q_0,q_2) = \min_q\big(S^{t_1}_0(q_0,q) + S^{t_2}_{t_1}(q,q_2)\big)$.

Proof. Let us consider the map

$$q \longmapsto f(q) = S^{t_1}_0(q_0,q) + S^{t_2}_{t_1}(q,q_2).$$

We have $d^2f \geqslant 2c$ hence the map f is convex. Now let us denote by $(q(s),p(s)) : [0,t_2] \longrightarrow \mathbb{R}^d\times\mathbb{R}^{d*}$ the unique orbit which satisfies $q(0)=q_0$ and $q(t_2)=q_2$. We can compute

$$df(q(t_1)) = \partial_1 S^{t_1}_0(q_0,q(t_1)) + \partial_0 S^{t_2}_{t_1}(q(t_1),q_2) = p(t_1) - p(t_1) = 0.$$

The point $q(t_1)$ is thus a critical point of the convex function f, hence it is a minimum of this function. We conclude that

$$S^{t_1}_0(q_0,q) + S^{t_2}_{t_1}(q,q_2) \geqslant S^{t_1}_0(q_0,q(t_1)) + S^{t_2}_{t_1}(q(t_1),q_2) = S^{t_2}_0(q_0,q_2)$$

for all q. □

Under the convexity hypothesis 2, Theorem 1 can be extended to C^1 solutions:

Theorem 3. *Let $\Omega \subset \mathbb{R} \times \mathbb{R}^d$ be an open set, and let $u(t,q) : \Omega \longrightarrow \mathbb{R}$ be a C^1 solution of the Hamilton Jacobi equation (HJ). Let $q(t) : [t_0, t_1] \longrightarrow \mathbb{R}^d$ be a C^1 curve such that $(t, q(t)) \in \Omega$ and*

$$\dot{q}(t) = \partial_p H\big(q(t), \partial_q u(t, q(t))\big)$$

for each $t \in [t_0, t_1]$. Then, setting $p(t) = \partial_q u(t, q(t))$, the curve $(q(t), p(t))$ solves (HS).

Proof. As in the proof of Theorem 1, we consider a variation $q(t,s) = q(t) + s\theta(t)$ of $q(t)$, where θ is smooth and vanishes on the endpoints. We choose the vertical variation $p(t,s)$ in such a way that the equation

$$\dot{q}(t,s) = \partial_p H(t, q(t,s), p(t,s))$$

holds. The map $p(t,s)$ defined by this relation is differentiable in s, because q and $\dot{q}$ are and because the matrix $\partial^2_{pp} H$ is invertible. It is also useful to consider the other vertical variation

$$P(t,s) := \partial_q u(t, q(t,s)).$$

Our hypothesis is that $\dot{q}(t) = \partial_p H(t, q(t), p(t))$, which is the first part of (HS). We start as in the proof of Theorem 1 with the following equality:

$$\frac{d}{ds}\Big|_{s=0} \left(\int_{t_0}^{t_1} p(t,s) \cdot \dot{q}(t,s) - H(t, q(t,s), p(t,s)) dt \right) = 0. \tag{2}$$

We deduce this equality from the observation that $s = 0$ is a local minimum of the function

$$s \longmapsto F(s) := \int_{t_0}^{t_1} p(t,s) \cdot \dot{q}(t,s) - H(t, q(t,s), p(t,s)) dt.$$

This claim follows from the equality

$$F(0) = u(t_1, q(t_1)) - u(t_0, q(t_0)) = \int_{t_0}^{t_1} P(t,s) \cdot \dot{q}(t,s) - H(t, q(t,s), P(t,s)) ds,$$

which holds for all s, and from the inequality

$$F(s) \geqslant \int_{t_0}^{t_1} P(t,s) \cdot \dot{q}(t,s) - H(t, q(t,s), P(t,s)) ds$$

which results, in view of the convexity of H, from the computation

$$\begin{aligned} H(t, q(t,s), P(t,s)) &\geqslant (P(t,s) - p(t,s)) \cdot \partial_p H(t, q(t,s), p(t,s)) + H(t, q(t,s), p(t,s)) \\ &\geqslant (P(t,s) - p(t,s)) \cdot \dot{q}(t,s) + H(t, q(t,s), p(t,s)). \end{aligned}$$

We have proved (2). As in the proof of Theorem 1, we develop the left hand side and, after a simplification, we get

$$\int_{t_0}^{t_1} p(t) \cdot \dot{\theta}(t) - \partial_q H(t, q(t), p(t)) \cdot \theta(t)\, dt = 0.$$

In other words, we have proved that $\dot{p}(t) = \partial_q H(t, q(t), p(t))$ in the sense of distributions. Since the right hand side is continuous, p is C^1 and the equality holds in the genuine sense. □

As in the C^2 case, we have the following corollary, see [12]:

Corollary 11. *Let $u(t,q) :]t_0,t_1[\times\mathbb{R}^d \longrightarrow \mathbb{R}$ be a C^1 solution of (HJ). Then, for each s and t in $]t_0,t_1[$ we have*

$$\Gamma_t = \varphi_s^t(\Gamma_s),$$

where Γ_t is defined by

$$\Gamma_t := \{(q, du_t(q)) : q \in \mathbb{R}^d\}.$$

PROOF. This corollary follows from Theorem 3 in the same way as Corollary 3 follows from Theorem 1. The only difference here is that the map

$$F(t,q) := \partial_p H(t,q,\partial_q u(t,q))$$

is only continuous. By the Cauchy-Peano Theorem, this is sufficient to imply the existence of solutions to the associated differential equation, which is what we need to develop the argument. □

A last property of the functions S will be useful. Assume that we are considering a family $H_\mu, \mu \in I$ of Hamiltonians, where $I \subset \mathbb{R}$ is an interval, such that the whole function $H(\mu,t,q,p)$ is C^2 and such that each of the Hamiltonians H_μ satisfy our hypotheses 1 and 2, with uniform constants m and M. Then, for each value of μ, we have the function $S^t(\mu;q_0,q_1)$, which is defined for $t \in]0,\sigma]$, the bound $\sigma > 0$ being independent of μ. Since everything we have done so far was based on the local inversion theorem, the function $S^t(\mu;q_0,q_1)$ is C^1 in μ, or more precisely the function $(\mu,t,q_0,q_1) \longmapsto S^t(\mu;q_0,q_1)$ is C^1. Moreover, a computation similar to the proof of Proposition 1 yields

$$\partial_\mu S^t(\mu;q_0,q_1) = -\int_0^t \partial_\mu H_\mu(s,q(\mu,s),p(\mu,s))ds,$$

where $s \longmapsto (q(\mu,s),p(\mu,s))$ is the only H_μ-trajectory satisfying $q(\mu,0)=q_0$ and $q(\mu,t)=q_1$. We can exploit this remark when H_μ is the linear interpolation $H_\mu = H_0 + \mu(H_1 - H_0)$ between two Hamiltonians H_0 and H_1, and conclude the important monotony property:

$$H_0 \leqslant H_1 \quad \Rightarrow \quad S^t(0;q,q') \geqslant S^t(1;q,q'). \tag{Monotone}$$

2.1 Exercise :

If $H(t,q,p) = h(p)$ is a function of p, then

$$S^t(q_0,q_1) = th^*\left(\frac{q_1-q_0}{t}\right),$$

where h^* is the Legendre transform of h. As an example, when $H(t,q,p) = a|p|^2/2$, then

$$S^t(q_0,q_1) = \frac{1}{2ta}|q_1-q_0|^2.$$

3 Extension of the generating function: The minimal action.

A classical problem consists in finding an orbit $(q(t),p(t))$ of the Hamiltonian system such that $q(t_0)=q_0$ and $q(t_1)=q_1$, for given $[t_0,t_1] \subset \mathbb{R}$, $q_0,q_1 \in \mathbb{R}^d$. We have seen, under Hypotheses 1 and 2, that this problem has a unique solution provided $t_0 < t_1 < t_0 + \sigma$, where σ is a constant depending only on m an M. The situation is more subtle for larger values of $t_1 - t_0$. In order to study it, it is useful to consider the function

$$\mathfrak{S} : (\theta_1,\ldots,\theta_{n-1}) \longmapsto S_0^{t/n}(q_0,\theta_1) + S_{t/n}^{2t/n}(\theta_1,\theta_2) + \cdots + S_{(n-1)t/n}^t(\theta_{n-1},q_1),$$

where we have taken $t_0 = 0$ and $t_1 = t$ to simplify notations, and where n is an integer such that $t/n \leqslant \sigma$. The critical points of $\mathfrak{S}$ are in one to one correspondence with the solutions of our problem:

Lemma 12. *The point* $(\theta_1, \ldots, \theta_{n-1})$ *is a critical point of* $\mathfrak{S}$ *if and only if there exists an orbit* $(q(s), p(s)) : [0, t] \longrightarrow \mathbb{R}^d \times \mathbb{R}^{d*}$ *such that* $q(0) = q_0$, $q(t) = q_1$, *and* $q(it/n) = \theta_i$ *for* $i = 1, \ldots, n-1$. *This orbit is then unique, and its action is* $\mathfrak{S}(\theta_1, \ldots, \theta_{n-1})$.

Proof. Let $(q(s), p(s))$ be the piecewise orbit defined on $[it/n, (i+1)t/n]$ by the constraints $q(it/n) = \theta_i$ and $q((i+1)t/n) = \theta_{i+1}$. The action of this piecewise orbit is $\mathfrak{S}(\theta_1, \cdots, \theta_{n-1})$. The statement follows from the simple computation

$$\partial_{\theta_i} \mathfrak{S} = \partial_1 S^{t/n}(\theta_{i-1}, \theta_i) + \partial_0 S^{t/n}(\theta_i, \theta_{i+1}) = p^-(it/n) - p^+(it/n).$$

□

Using this finite dimensional variational functional is usually called the method of broken geodesics, see [9]. The function $\mathfrak{S}$ can be minimized under additional assumptions, for example:

Hypothesis 3.

$$\frac{m}{2}|p|^2 - M \leqslant H(t, q, p) \leqslant \frac{M}{2}|p|^2 + M.$$

By exploiting the monotony property (Monotone), this hypothesis implies that

$$\frac{1}{2tM}|q_1 - q_0|^2 - Mt \leqslant S^t(q_0, q_1) \leqslant \frac{1}{2tm}|q_1 - q_0|^2 + Mt,$$

and then that

$$\mathfrak{S}(\theta_1, \ldots, \theta_{n-1}) \geqslant \frac{n}{2tM}(|\theta_1 - q_0|^2 + |\theta_2 - \theta_1|^2 + \cdots + |q_1 - \theta_{n-1}|^2) - Mt.$$

As a consequence, the function $\mathfrak{S}$ is coercive and C^2, hence it has a minimum. Notice that, although $\mathfrak{S}$ is convex separately in each of its variables, it is not jointly convex. It can have critical points which are not minima, and it can have several different minima. We denote by A^t the value function

$$A^t(q_0, q_1) = \min \mathfrak{S} = \min_{\theta_1, \theta_2, \ldots \theta_{n-1}} \left(S_0^{t/n}(q_0, \theta_1) + S_{t/n}^{2t/n}(\theta_1, \theta_2) + S_{(n-1)t/n}^{t}(\theta_{n-1}, q_1) \right) \tag{A}$$

where n is any integer such that $t/n < \sigma$. The functions $A^t_\tau(q_0, q_1)$ are defined similarly for each $t \geqslant \tau$. This notation is legitimate in view of the following:

Lemma 13. *The value of* A^t *does not depend on* n *provided* $t/n < \sigma$. *Moreover, we have*

$$\frac{1}{2Mt}|q_1 - q_0|^2 - Mt \leqslant A^t(q_0, q_1) \leqslant \frac{1}{2tm}|q_1 - q_0|^2 + Mt.$$

This statement implies that $A^t = S^t$ when $t < \sigma$: A^t can be seen as an extension of S^t beyond $t = \sigma$.

Proof. Since we have not yet proved the independence of n, we temporarily denote by $A^t(q_0, q_1; n)$ the value of the minimum. We have

$$\begin{aligned} A^t(q_0, q_1; n) &\geqslant \min_{\theta_1, \theta_2, \ldots \theta_{n-1}} \left(\frac{n}{2Mt}(|\theta_1 - q_0|^2 + \cdots + |q_1 - \theta_{n-1}|^2) - Mt \right) \\ &= \frac{1}{2Mt}|q_1 - q_0|^2 - Mt. \end{aligned}$$

If $t < \sigma$, then the equality $S^t(q_0, q_1) = A^t(q_0, q_1; n)$ can be proved by recurrence for each n using Proposition 10. For general t, let us prove that $A^t(n)$ is independent of n. We take two integers

n and m such that $t/n < \sigma, t/m < \sigma$ and want to prove that $A^t(n) = A^t(m)$. We will prove that $A^t(n) = A^t(nm) = A^t(m)$. Since $t/m < \sigma$, we have

$$A_\tau^{\tau+t/m}(q_0, q_1; n) = S_\tau^{\tau+t/m}(q_0, q_1)$$

for each τ and n, hence

$$\begin{aligned}
A^t(q_0, q_1; nm) = & \min_{\theta_1,\theta_2,\dots\theta_{nm-1}} \Big[S_0^{t/nm}(q_0, \theta_1) + S_{t/nm}^{2t/mn}(\theta_1, \theta_2) + \cdots + S_{(n-1)t/nm}^{t/m}(\theta_{n-1}, \theta_n) \\
& + S_{t/m}^{(n+1)t/nm}(\theta_n, \theta_{n+1}) + \cdots + S_{(2n-1)t/nm}^{2t/m}(\theta_{2n-1}, \theta_{2n}) \\
& + \cdots \\
& + S_{(m-1)t/m}^{(m-1)t/m+t/nm}(\theta_{(m-1)n}, \theta_{(m-1)n+1}) + \cdots + S_{(1-1/nm)t}^{t}(\theta_{mn-1}, q_1) \Big] \\
= & \min_{\theta_{2n},\theta_{3n},\dots,\theta_{(m-1)n}} \Big[S_0^{t/m}(q_0, \theta_n) + S_{t/m}^{2t/m}(\theta_n, \theta_{2n}) + \cdots + S_{(m-1)t/m}^{t}(\theta_{(m-1)n}, q_1) \Big] \\
= & A^t(q_0, q_1; m).
\end{aligned}$$

We have proved that $A^t(nm) = A^t(m)$, by symmetry we also have $A^t(nm) = A^t(n)$ hence $A^t(n) = A^t(m)$. Finally, we have

$$\mathfrak{S}(\theta_1, \dots, \theta_{n-1}) \leqslant \frac{n}{2mt}\big(|\theta_1 - q_0|^2 + |\theta_2 - \theta_1|^2 + |q_1 - \theta_{n-1}|^2\big) + Mt$$

hence

$$\begin{aligned}
A^t(q_0, q_1) &\leqslant \min_{\theta_1,\theta_2,\dots\theta_{n-1}} \frac{n}{2mt}\big(|\theta_1 - q_0|^2 + |\theta_2 - \theta_1|^2 + |q_1 - \theta_{n-1}|^2\big) + Mt \\
&= \frac{1}{2mt}|q_1 - q_0|^2 + Mt.
\end{aligned}$$

□

The following property concerning A follows easily from the definition :

$$A_{t_0}^{t_2}(q_0, q_2) = \min_{q_1} \big(A_{t_0}^{t_1}(q_0, q_1) + A_{t_1}^{t_2}(q_1, q_2)\big), \tag{T}$$

when $0 \leqslant t_0 \leqslant t_1 \leqslant t_2$. The following consequence of Hypothesis 3 will also be useful:

Lemma 14.

$$p \cdot \partial_p H(t, q, p) - H(t, q, p) \geqslant \frac{m}{M} H(t, q, p) - (m + M).$$

Proof. We deduce from Hypothesis 2 that

$$H(t, q, 0) \geqslant H(t, q, p) - p \cdot \partial_p H(t, q, p) + \frac{m}{2}|p|^2.$$

We deduce that

$$p \cdot \partial_p H(t, q, p) - H(t, q, p) \geqslant \frac{m}{2}|p|^2 - H(t, q, 0) \geqslant \frac{m}{M}(H(t, q, p) - M) - M$$

□

The minimal action $A^t(q_0, q_1)$ is not necessarily C^1, we need some definitions before we can study its regularity. The linear form l is called a K-super-differential of the function u at point q if the inequality

$$u(\theta) \leqslant u(q) + l(\theta - q) + K|\theta - q|^2$$

holds in a neighborhood of q. The linear form l is a proximal super-differential of u at point q if it is a K-super-differential for some K. The form l is a proximal super-differential of u at q if and only if there exists a C^2 function v such that $dv(q) = l$ and such that the difference $v - u$ has a minimum at q. More generally, we will say that l is a super-differential of u at q if there exists a C^1 function v such that $dv(q) = l$ and such that the difference $v - u$ has a minimum at q. A super-differential is not necessarily a proximal super-differential.

A function $u : \mathbb{R}^d \longrightarrow \mathbb{R}$ is called K-semi-concave if it admits a K-super-differential at each point. It is equivalent to require that the function $\theta \longmapsto u(\theta) - K|\theta|^2$ is concave. A function is called semi-concave if it is K-semi-concave for some K. If u is a K-semi-concave function, and if l is a super-differential at u, then the inequality

$$u(\theta) \leqslant u(q) + l(\theta - q) + K|\theta - q|^2$$

holds for each θ. In particular, l is a K-super-differential.

Lemma 15. *The function A^t is $C(1 + 1/t)$-semi-concave, with some constant C which depends only on m and M.*

PROOF. Let us first assume that $t \in]0, \sigma[$. In this case, $A_0^t = S_0^t$, this function is C^2 and its second derivative was estimated in Lemma 9. Let us now assume that $t \geqslant \sigma$. Then, there exists $n \in \mathbb{N}$ such that $t/n \in [\sigma/3, \sigma/2[$. We have

$$A_0^t(q, q') = \min_{\theta, \theta'} \big(S_0^{t/n}(q, \theta) + A_{t/n}^{t-t/n}(\theta, \theta') + S_{t-t/n}^t(\theta', q') \big).$$

Considering a minimizing pair (θ_0, θ_1) in the expression above at (q_0, q_1), we see that the C^2 function

$$(q, q') \longmapsto S_0^{t/n}(q, \theta_0) + A_{t/n}^{t-t/n}(\theta_0, \theta_1) + S_{t-t/n}^t(\theta_1, q')$$

is touching from above the function A_0^t at point (q_0, q_1). In view of Lemma 9, this provides a uniform (for $t \geqslant \sigma$) semi-concavity constant for A_0^t. □

4 The Lax-Oleinik operators.

Given $t_0 < t_1$, we define the Lax-Oleinik operators $\mathbf{T}_{t_0}^{t_1}$ and $\check{\mathbf{T}}_{t_1}^{t_0}$ which, to each function $u : \mathbb{R}^d \longrightarrow \mathbb{R}$ associate the functions

$$\mathbf{T}_{t_0}^{t_1} u(q) := \inf_{\theta \in \mathbb{R}^d} \big(u(\theta) + A_{t_0}^{t_1}(\theta, q) \big) \quad , \quad \check{\mathbf{T}}_{t_1}^{t_0} u(q) := \sup_{\theta \in \mathbb{R}^d} \big(u(\theta) - A_{t_0}^{t_1}(q, \theta) \big).$$

We have the Markov (or semi-group) property:

$$\mathbf{T}_{t_1}^{t_2} \circ \mathbf{T}_{t_0}^{t_1} = \mathbf{T}_{t_0}^{t_2} \quad , \quad \check{\mathbf{T}}_{t_1}^{t_0} \circ \check{\mathbf{T}}_{t_2}^{t_1} = \check{\mathbf{T}}_{t_2}^{t_0}$$

for $t_0 < t_1 < t_2$. Note however that $\mathbf{T}_{t_0}^{t_1} \circ \check{\mathbf{T}}_{t_1}^{t_0}$ and $\check{\mathbf{T}}_{t_1}^{t_0} \circ \mathbf{T}_{t_0}^{t_1}$ are not the identity. Concerning these operators, we only have the inequalities

$$\check{\mathbf{T}}_{t_1}^{t_0} \circ \mathbf{T}_{t_0}^{t_1}(u) \leqslant u \quad , \quad \mathbf{T}_{t_0}^{t_1} \circ \check{\mathbf{T}}_{t_1}^{t_0}(u) \geqslant u,$$

the easy proof of which is left to the reader. Each property concerning the Lax-Oleinik operator $\mathbf{T}$ has a counterpart for the dual operator $\check{\mathbf{T}}$, that we will not always bother to state, but never hesitate to use. The family of operators $\mathbf{T}_{t_0}^{t_1}$ is characterized by the fact that $\mathbf{T}_{t_0}^{t_1} u(q) = \inf_\theta \big(u(\theta) + S_{t_0}^{t_1}(q_0, q_1) \big)$ when $t_0 \leqslant t_1 \leqslant t_0 + \sigma$ and by the Markov property. The Lax-Oleinik operators solve (HJ) in various important ways, that will be detailed in the present section. It is useful first to settle some regularity issues.

Lemma 16. *There exists a constant C, depending only on m and M such that, for each $t \in]0, \sigma]$, the function $\mathbf{T}^t u$ is (C/t)-semi-concave provided it is finite at each point.*

PROOF. The function $\mathbf{T}^t u$ is the infimum of the functions $f = u(\theta) + S^t(\theta, .)$, which are C^2 with the uniform bound $\|d^2 f\| \leqslant C/t$. It is then an easy exercise to conclude that the function $\mathbf{T}^t u$ is C/t-semi-concave, see Lemma 54. □

Given an arbitrary function u_0, the infimum in the definition of $\mathbf{T}_0^t u_0$ is not necessarily finite, and, even if it is finite, it is not necessarily a minimum. It is clear from Proposition 13 that the infimum is a finite minimum under the assumption that u_0 is continuous and **Lipschitz in the Large**, which means that there exists a constant k such that

$$u_0(q') - u_0(q) \leqslant k(1 + |q' - q|)$$

for each q and q'.

Lemma 17. *If u_0 is Lipschitz in the large, then so are the functions $\mathbf{T}_0^t u_0$ for all $t \geqslant 0$. The function $(t, q) \longmapsto u(t, q) = \mathbf{T}_0^t u_0(q)$ is locally semi-concave, hence locally Lipschitz on $]0, \infty) \times \mathbb{R}^d$. The function u solves (HJ) at all its points of differentiability (hence almost everywhere).*

PROOF. Since u_0 is Lipschitz in the large, the function $\mathbf{T}_0^t u_0 - u_0$ is bounded for each $t > 0$, as follows from the inequalities

$$\inf_{\theta} \big(u_0(q) - k - k|\theta - q| + S^t(\theta, q)\big) \leqslant \mathbf{T}_0^t u_0 \leqslant u_0(q) + S^t(q, q)$$

which imply (setting $\Delta = \theta - q$) that

$$\inf_{\Delta \in \mathbb{R}^d} \big(-k - k|\Delta| + \frac{1}{2tM}|\Delta|^2 - tM\big) \leqslant \mathbf{T}_0^t u_0(q) - u_0(q) \leqslant Mt.$$

We conclude that the function $\mathbf{T}_0^t u_0 = (\mathbf{T}_0^t u_0 - u_0) + u_0$ is Lipschitz in the large. In the computations above, we also see that the infimum can be taken on $|\Delta| \leqslant K$, where K is a constant independent from q.

Let us now prove that the function $u(t, q) := \mathbf{T}_0^t u_0(q)$ is locally Lipschitz on $t > 0$. In view of the Markov property, it is enough to prove that the function u is Lipschitz on $]\tau, \sigma/2[\times B$ for each closed ball $B \subset \mathbb{R}^d$ and each time $\tau \in]0, \sigma/2[$. Since $u(q)$ is Lipschitz in the large, there exists a radius $R > 0$ such that

$$u(t, q) = \inf_{|\theta| \leqslant R} u(\theta) + S^t(\theta, q)$$

for $(t, q) \in]\tau, \sigma/2[\times B$. Since S is C^2, the functions $(t, q) \longmapsto u(\theta) + S^t(\theta, q), |\theta| \leqslant R$ have uniform C^2 bounds on $]\tau, \sigma/2[\times B$. Their infimum $u(t, q)$ is then semi-concave, hence Lipschitz on that set, see Lemma 54.

Finally, let (t, q) be a point of differentiability of u, and let $\tau \in]\max(0, t - \sigma), t[$ be given. Since u_τ is Lipschitz in the large and locally Lipschitz, there exists θ such that $\mathbf{T}_\tau^t u_\tau(q) = u_\tau(\theta) + S_\tau^t(\theta, q)$. For a different point (s, y), we have $\mathbf{T}_\tau^s u_\tau(y) \leqslant u_\tau(\theta) + S_\tau^t(\theta, y)$, hence the function $(s, y) \longmapsto u(s, y) - S_\tau^s(\theta, y)$ has a maximum at (t, q), which implies that the functions $u(s, y)$ and $S_\tau^s(\theta, y)$, each of which is differentiable at (t, q), have the same differential at (t, q). Since the functions $(s, y) \longmapsto S_\tau^s(\theta, y)$ solves (HJ), the function u also solves (HJ) at (t, q). □

Let us now establish the relation of our operators with regular solutions.

Proposition 18. *Let $u(t,q) :]t_0,t_1[\times\mathbb{R}^d \longrightarrow \mathbb{R}$ be a C^1 solution of HJ, then $\mathbf{T}_\tau^t u_\tau = u_t$ and $\check{\mathbf{T}}_t^\tau u_t = u_\tau$ for each $\tau \leqslant t$ in $]t_0,t_1[$. The function u is locally $C^{1,1}$.*

This property is one of the main motivations to introduce the Lax-Oleinik operators. The observation that C^1 solutions are actually locally $C^{1,1}$ comes Fathi's paper [12], itself inspired by anterior works of Herman. Another consequence of this Theorem is that uniqueness extends to C^1 solutions under the convexity assumption.

PROOF. In view of the Markov property, it is enough to prove the result for $0 < t - \tau < \sigma$. Given q and θ in $\mathbb{T}^d$, we consider the unique orbit $(q(s),p(s))$ such that $q(\tau) = \theta$ and $q(t) = q$. By the convexity of H, we have

$$H(q(s),\partial_q u(s,q(s))) \geqslant H(q(s),p(s)) + (\partial_q u(s,q(s)) - p(s)) \cdot \partial_p H(s,q(s),p(s)).$$

Noticing that $\dot{q}(s) = \partial_p H(s,q(s),p(s))$ and integrating gives:

$$\begin{aligned} S_\tau^t(\theta,q) &= \int_\tau^t p(s)\cdot\dot{q}(s) - H(s,q(s),p(s))ds \\ &\geqslant \int_\tau^t \partial_q u(s,q(s))\cdot\dot{q}(s) - H(s,q(s),\partial_q u(s,q(s)))ds \\ &= u(t,q) - u(\tau,\theta), \end{aligned}$$

with equality if $p(s) = \partial_q u(s,q(s))$ for each s. We conclude that

$$\mathbf{T}_\tau^t u_\tau(q) \geqslant u_t(q),$$

with equality if there exists an orbit $(q(s),p(s)) : [\tau,t] \longrightarrow \mathbb{R}^d\times\mathbb{R}^{d*}$ such that $p(s) = \partial_q u(s,q(s))$ and $q(t) = q$. By Corollary 11, the orbit of the point $(q,\partial_q u(t,q))$ satisfies this property, hence the equality holds.

To prove the regularity of u we consider a subinterval $[\tilde{t}_0,\tilde{t}_1] \subset]t_0,t_1[$, and prove that u is locally $C^{1,1}$ on $]\tilde{t}_0,\tilde{t}_1[$. We have

$$u(t,q) = \mathbf{T}_{\tilde{t}_0}^t u_{\tilde{t}_0}(q) = \check{\mathbf{T}}_t^{\tilde{t}_1} u_{\tilde{t}_1}(q)$$

for each $t \in]\tilde{t}_0,\tilde{t}_1[$. If the functions u_t were Lipschitz in the Large, we could apply Lemma 17 and deduce that u is both locally semi-concave and locally semi-convex, hence locally $C^{1,1}$, on $]\tilde{t}_0,\tilde{t}_1[\times\mathbb{R}^d$. Here we do not make any growth assumption, so we need a slightly different argument to prove the semi-concavity of u (and, similarly, its semi-convexity). We have seen that the infimum in the definition $\mathbf{T}_{\tilde{t}_0}^t u_{\tilde{t}_0}(q)$ is a minimum, which is attained at the point $\theta = Q_t^{\tilde{t}_0}(q,\partial_q u(t,q))$. This gives us an *a priori* bound on θ, and we can continue the proof as in Lemma 17. □

Let us sum up some properties of the Lax-Oleinik operators $\mathbf{T}_\tau^t$ associated to a Hamiltonian satisfying hypotheses 1,2,3:

Property 19.

1. *Markov property: $\mathbf{T}_s^t \circ \mathbf{T}_\tau^s = \mathbf{T}_\tau^t$ when $\tau \leqslant s \leqslant t$.*
2. *Monotony: $u \geqslant v \Rightarrow \mathbf{T}_\tau^t u \geqslant \mathbf{T}_\tau^t v$ for each $t \geqslant \tau$.*
3. *Compatibility with (HJ): If $u(t,q) :]t_0,t_1[\times\mathbb{R}^d \longrightarrow \mathbb{R}$ is a C^2 solution of (HJ), then $\mathbf{T}_\tau^t u_\tau = u_t$ when $t_0 < \tau < t < t_1$.*

4. *Boundedness: If u_τ is Lipschitz in the large, then the functions $\mathbf{T}_\tau^t u_\tau$, $t \in [\tau, T]$ are uniformly Lipschitz in the large for each $T \geqslant \tau$.*

5. *Regularity: If u_τ is Lipschitz in the large, the function $(t,q) \longmapsto \mathbf{T}_\tau^t u_\tau(q)$ is locally Lipschitz on $]\tau, \infty) \times \mathbb{R}^d$.*

6. *Translation invariance: $\mathbf{T}_\tau^t(c+u) = c + \mathbf{T}_\tau^t u$ for each constant $c \in \mathbb{R}$.*

The Lax-Oleinik operators solve the Cauchy problem for (HJ) in the viscosity sense. Actually, this follows from Property 19:

Proposition 20. *Let H be a Hamiltonian satisfying Hypothesis 1. Assume that there exists a family $\mathbf{T}_\tau^t$, $0 \leqslant \tau \leqslant t$ of operators satisfying the Markov property, the monotony, the compatibility with (HJ), and the boundedness as expressed in Property 19. Then if u_0 is an initial condition which is Lipschitz in the large, the function*

$$(t,q) \longmapsto u(t,q) = \mathbf{T}_0^t u_0(q)$$

is a viscosity solution of (HJ) on $]0, \infty) \times \mathbb{R}^d$.

Notice that we did not make any convexity assumption. This kind of axiomatic characterization of viscosity solutions is reminiscent from [1], see also [8]. It may also help to understand the links between viscosity solutions and variational solutions in the non-convex setting. Such links were suggested by Claude Viterbo and established in her thesis by Qiaolin Wei, [20].

Proof of Proposition 20: Let us prove that u is a viscosity sub-solution, a similar proof yields that it is also a super-solution. We consider a point $(T,Q) \in]0,\infty) \times \mathbb{R}^d$ and a super-differential (h,p) of the function u at (T,Q). To prove that $h + H(T,Q,p) \leqslant 0$, we assume, by contradiction, that

$$h + H(T,Q,p) > 0.$$

As is usual for viscosity solutions we will use a test function ϕ. We will assume that $\phi : \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ is smooth and satisfies the following properties:

- $\phi(T,Q) = u(T,Q), \quad \partial_t \phi(T,Q) = h, \quad \partial_q \phi(T,Q) = p,$
- $\phi \geqslant u$ on $[-T/2, 2T] \times \mathbb{R}^d$,
- There exists a constant $C > 0$ such that $\phi(t,q) = C\sqrt{1+|q|^2}$ when $|q| + |t| \geqslant C$.

Note that $d^2\phi$ is bounded. Such a test function exists because the functions u_t, $t \in [T/2, 2T]$, are uniformly Lipschitz in the large, as follows from the boundedness property assumed on the operators.

Claim: There exists $S > 0$ and a C^2 function $w(\tau, t, q)$ defined on the open set

$$\left\{(\tau,t,q) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d : \tau - S < t < \tau + S\right\} \subset \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$$

such that, for each fixed τ, the function $w_\tau : (t,q) \longmapsto w(\tau,t,q)$ is the solution of the Cauchy problem

$$\begin{cases} \partial_t w_\tau + H(t,q,\partial_q w_\tau) = 0 \\ \quad w_\tau(\tau,q) = \phi(\tau,q). \end{cases}$$

The existence of a solution w_τ to this problem follows from Theorem 2. However, to see that w is C^2 in all its variables, we find it more convenient to consider the Cauchy problem

$$\begin{cases} \partial_s u + \Big(\partial_z u + H(z,q,\partial_q u(s,z,q))\Big) = 0 \\ \qquad u(0,z,q) = \phi(z,q). \end{cases}$$

By Theorem 2, applied to the Hamiltonian

$$\hat{H}(s,z,q,\xi,p) : \mathbb{R}\times(\mathbb{R}\times\mathbb{R}^d)\times(\mathbb{R}\times\mathbb{R}^d)^* \longrightarrow \mathbb{R}$$
$$(s,z,q,\xi,p) \longmapsto \xi + H(z,q,p)$$

there exists $S>0$ and a C^2 solution $u(s,z,q) :]-S,S[\times\mathbb{R}\times\mathbb{R}^d \longrightarrow \mathbb{R}$ of this Cauchy problem. Setting

$$w(\tau,t,q) := u(t-\tau,t,q),$$

we verify that

$$\partial_t w(t,q) + H(t,q,\partial_q w(t,q)) = \partial_s u(t-\tau,t,q) + \partial_z u(t-\tau,t,q) + H(t,q,\partial_q u(t-\tau,t,q)) = 0$$

and that $w(\tau,\tau,q) = u(0,\tau,q) = \phi(\tau,q)$.
Claim : There exists $\tau \in]T-S,T[$ such that $w(\tau,T,Q) < \phi(T,Q)$.
Since $w(T,T,q) = \phi(T,q)$, we have

$$\partial_t w(T,T,Q) = -H(T,Q,\partial_q w(T,Q)) = -H(T,Q,\partial_q \phi(T,Q)) < \partial_t \phi(T,Q).$$

As a consequence, there exists $\delta > 0$ such that

$$\partial_t w(\tau,t,Q) - \partial_t \phi(t,Q) < 0$$

for $\tau, t \in]T-\delta,T[$. Since $w(\tau,\tau,Q) = \phi(\tau,Q)$, we deduce by integration that

$$w(\tau,T,Q) - \phi(T,Q) = \int_\tau^T \partial_t w(\tau,t,Q) - \partial_t \phi(t,Q) dt < 0$$

provided $\tau \in]T-\delta,T[$, which proves our claim.
Conclusion : Since we are considering monotone operators compatible with (HJ) we have

$$w(\tau,T,Q) = \mathbf{T}_\tau^T w_\tau(Q) = \mathbf{T}_\tau^T \phi_\tau(Q) \geqslant \mathbf{T}_\tau^T u_\tau(Q) = u(T,Q)$$

hence $\phi(T,Q) > u(T,Q)$, which is a contradiction. □

This parenthesis through viscosity solutions being closed, let us turn our attention to more geometric aspects of the Lax-Oleinik operators. We denote by Γ_u the graph of the differential of u on its domain of definition,

$$\Gamma_u := \{(q, du(q)) : \quad q \in \mathbb{R}^d, \quad du(q)\,\text{exists}\}.$$

Proposition 21. *Let u be a semi-concave and Lipschitz function. The set*

$$\varphi_t^0\left(\bar{\Gamma}_{\mathbf{T}_0^t u}\right)$$

is contained in Γ_u for each $t>0$, and it is a Lipschitz graph.

Proof. In view of the Markov property, it is enough to prove the result for $t \in]0,\sigma]$. Let (q,p) be a point of $\Gamma_{\mathbf{T}_0^t u}$, which means that the function $\mathbf{T}_0^t u$ is differentiable at q and that $d(\mathbf{T}_0^t u)(q) = p$. Let Θ be a minimizing point in the expression $\mathbf{T}_0^t u(q) = \min_\theta u(\theta) + S_0^t(\theta,q)$. Since each of the functions u and $S_0^t(.,q)$ are semi-concave, this implies that they are both differentiable at Θ, and that $du(\Theta) + \partial_0 S_0^t(\Theta,q) = 0$. Moreover, this implies that the function $u(\Theta) + S_0^t(\Theta,.)$ touches the function $\mathbf{T}_0^t u$ from above at point q, hence that $S_0^t(\Theta,.)$ is differentiable at q, with a differential equal to p. We then have

$$\varphi_t^0(q,p) = \varphi_t^0(q, \partial_1 S_0^t(\Theta,q)) = (\Theta, -\partial_0 S_0^t(\Theta,q) = (\Theta, du(\Theta)) \subset \Gamma_u.$$

We have proved that $\varphi_t^0(\Gamma_{\mathbf{T}_0^t u}) \subset \Gamma_u$. Moreover, we have $Q_t^0(\Gamma_{\mathbf{T}_0^t u}) \subset \mathcal{I}$, where $\mathcal{I} \subset \mathbb{R}^d$ is the set of points θ which are minimizing in the definition of $\mathbf{T}_0^t u(q)$ for some point q.
Claim: The function u is $C^{1,1}$ on $\mathcal{I}$. This means that u is differentiable at each point of $\mathcal{I}$, and that the map $\theta \longmapsto du(\theta)$ is Lipschitz on $\mathcal{I}$. In other words, the projection of Γ_u to $\mathbb{R}^d$ contains $\mathcal{I}$, and the set

$$\Gamma_{u|\mathcal{I}} := \{(\theta, du(\theta), \theta \in \mathcal{I}\}$$

is a Lipschitz graph.

To prove the claim, we first prove that u has C-super-differentials and C-sub-differentials at each point of $\mathcal{I}$, where C is a common semi-concavity constant of all the functions $-S_0^t(.,q)$ and of the function u. The existence of a C-super-differential follows from the C-semi-concavity of u. To prove the existence of a C-sub-differential at a point $\Theta \in \mathcal{I}$, we consider a point q such that $u(\Theta) + S_0^t(\Theta, q) = T_0^t u(q)$. Such a point exists by definition of $\mathcal{I}$. This implies that the function $\theta \longmapsto u(\theta) + S_0^t(\theta, q)$ has a minimum at $\theta = \Theta$, hence each C-sub-differential of $-S_0^t(.,q)$ is a C-sub-differential of u. The claim then follows from a result of Fathi, see Proposition 53 in the Appendix.

Let now (q,p) be a point in the closure $\bar{\Gamma}_{\mathbf{T}_0^t u}$ of $\Gamma_{\mathbf{T}_0^t u}$. There exists a sequence (q_n, p_n) of points of $\Gamma_{\mathbf{T}_0^t u}$ which converges to (q,p). By definition, the function $\mathbf{T}_0^t u$ is differentiable at q_n, and $p_n = d(\mathbf{T}_0^t u)(q_n)$. Let $\Theta_n = Q_t^0(q_n, p_n)$ be the sequence of points such that

$$\mathbf{T}_0^t u(q_n) = u(\Theta_n) + S_0^t(\Theta_n, q_n).$$

The sequence Θ_n is converging to $\Theta = Q_t^0(q,p)$, and, at the limit, we see that

$$\mathbf{T}_0^t u(q) = u(\Theta) + S_0^t(\Theta, q).$$

We conclude that $\Theta \in \mathcal{I}$. Since we have already proved the Lipschitz regularity of du on $\mathcal{I}$, we deduce that $\varphi_t^0(q,p) = \lim(\varphi_t^0(q_n,p_n)) = \lim(\Theta_n, du(\Theta_n)) = (\Theta, du(\Theta)) \in \Gamma_{u|\mathcal{I}} \subset \Gamma_u$. □

The action of the Lax-Oleinik operators on semi-convex functions also has a remarkable property, see [4]. It is useful to denote by L_u the set of point (Q,P) such that P is a sub-differential of u at Q. Note that $\Gamma_u \subset L_u$.

Proposition 22. *If u is K-semi-convex, then for each $\delta \in]0,1[$ there exists $T > 0$ such that $\mathbf{T}_0^t u$ is $(K+\delta)$-semi-convex, hence $C^{1,1}$, for each $t \in]0,T]$. One can take*

$$T = \frac{\delta}{2M(3+2K)^2}.$$

PROOF. Since u is K-semi-convex, for each $(Q,P) \in L_u$, we have

$$u(q) \geqslant u(Q) + P(q-Q) - K|q-Q|^2.$$

We denote by $l_{Q,P}(q)$ the function on the right in this inequality, so that

$$u = \max_{(Q,P)\in L_u} l_{Q,P}.$$

Taking T as in the statement, it follows from Theorem 2 that the functions $\mathbf{T}_0^t(l_{Q,P}), t \in [-T,T]$ are C^2 with a second derivative bounded by $2K + 4tM(1+2K)^2 \leqslant 2K + 2\delta$. We claim that

$$\mathbf{T}_0^t u = \max_{(Q,P)\in L} \mathbf{T}_0^t(l_{Q,P}),$$

for $t \in [0,T] \cap [0,\sigma]$, which implies that $\mathbf{T}_0^t u$ is $(K + 4tM(1+2K)^2)$-semi-convex. We prove the claim in two steps. First, the inequality

$$\mathbf{T}_0^t u \geqslant \max_{(Q,P)\in L} \mathbf{T}_0^t(l_{Q,P})$$

follows immediately from the fact that $u \geqslant l_{Q,P}$ for each $(Q,P) \in L$ in view of the monotony of $\mathbf{T}_0^t$, see Property 19. Let us fix a point (t,q) and prove the converse inequality at this point. Since

$$u(\theta) + S_0^t(\theta,q) \geqslant u(q) + P(\theta - q) - K(\theta-q)^2 + \frac{1}{2tM}|\theta - q|^2 - tM$$

and since $K \leqslant 1/2tM$, there exists a point θ such that $\mathbf{T}_0^t u(q) = u(\theta) + S_0^t(\theta,q)$. Assuming that $t \leqslant \sigma$, this implies that the point $(\theta,\zeta) = (\theta, -\partial_0 S_0^t(\theta,q))$ belongs to L_u, and that $q = Q_0^t(\theta,\zeta)$. Then, we have

$$\mathbf{T}_0^t(l_{\theta,\zeta})(q) = l_{\theta,\zeta}(\theta) + S_0^t(\theta,q) = u(\theta) + S_0^t(\theta,q) = \mathbf{T}_0^t u(q)$$

hence

$$\mathbf{T}_0^t u(q) \leqslant \max_{(Q,P)\in L} \mathbf{T}_0^t(l_{Q,P})(q)$$

provided $t \leqslant \sigma$. We conclude that $\mathbf{T}_0^t u$ is semi-concave with constant $K + 2tM(1+2K)^2$ for $t \in [0,\sigma] \cap [0,T]$. We can then apply this result to $\mathbf{T}_0^\sigma u$, and, since $K + tM(1+2K)^2 \leqslant K+1$, we conclude that the function $\mathbf{T}_\sigma^t \mathbf{T}_0^\sigma u$ is semi-concave with constant

$$K + 2\sigma M(1+2K)^2 + 2tM(3+2K)^2 \leqslant K + 2(\sigma+t)M(3+2K)^2 \leqslant K+1$$

for $t \in [0,\sigma] \cap [0, T-\sigma]$. In other words, the functions $\mathbf{T}_0^t u$ are semi-concave with constant $K + 2tM(3+2K)^2$ for $t \in [0,2\sigma] \cap [0,T]$. We can apply this argument as many times as necessary and obtain that, the functions $\mathbf{T}_0^t u$ are semi-concave with constant $K + 2tM(3+2K)^2$ for each $t \in [0,T]$. □

The following was first stated explicitly by Marie-Claude Arnaud in [2].

Addendum 1. *Under the hypotheses of Proposition 22, we have $L_u = \varphi_t^0(\Gamma_{\mathbf{T}_0^t u})$ for each $t \in]0,T[$. Moreover, for each q, we have $\mathbf{T}_0^t u(q) = u(\theta) + S_0^t(\theta,q)$, with $\theta = Q_t^0(q, d(\mathbf{T}_0^t u)(q))$.*

Proof. For each $q \in \mathbb{R}^d$, we have seen that there exists $(\theta,\zeta) \in L_u$ such that $\mathbf{T}_0^t u(q) = u(\theta) + S_0^t(\theta,q)$ and $\zeta = -\partial_0 S_0^t(\theta,q)$. Since we know that $\mathbf{T}_0^t u$ is C^1, the first of these equalities implies that $d(\mathbf{T}_0^t u)(q) = \partial_1 S_0^t(\theta,q)$, while the second implies that $\varphi_0^t(\theta,\zeta) = (q, \partial_1 S_0^t(\theta,q))$. We conclude that $\varphi_t^0(\Gamma_{\mathbf{T}_0^t u}) \subset L_u$. Moreover, $\theta = Q_t^0(q, d(\mathbf{T}_0^t u)(q))$.

Conversely, let us consider a point $(\theta,\zeta) \in L$, and denote by l the associated function $l_{\theta,\zeta}$. By Proposition 18, the function $(t,q) \longmapsto \mathbf{T}_0^t l(q)$ is the restriction to $]0,T[\times\mathbb{R}^d$ of the C^2 solution of (HJ) emanating from l. As a consequence, we have

$$\mathbf{T}_0^t l(Q_0^t(\theta,\zeta)) = l(\theta) + S_0^t(\theta, Q_0^t(\theta,\zeta)) = u(\theta) + S_0^t(\theta, Q_0^t(\theta,\zeta)) \geqslant \mathbf{T}_0^t u(Q_0^t(\theta,\zeta)).$$

Since we know from the monotony property that $\mathbf{T}_0^t l \leqslant \mathbf{T}_0^t u$, we conclude that this last inequality is actually an equality. Setting $q_1 = Q_0^t(\theta,\zeta)$, this implies that

$$(\theta,\zeta) = (\theta, -\partial_0 S_0^t(\theta,q_1)) = \varphi_t^0(q_1, \partial_1 S_0^t(\theta,q_1)) = \varphi_t^0(q_1, d\mathbf{T}_0^t u(q_1)) \subset \varphi_t^0(\Gamma_{\mathbf{T}_0^t u}).$$

We conclude that $L_u \subset \varphi_t^0(\Gamma_{\mathbf{T}_0^t u})$. □

Addendum 2. *Under the hypotheses of Proposition 22, we have $\check{\mathbf{T}}_t^0 \circ \mathbf{T}_0^t u = u$ for each $t \in]0,T[$.*

Proof. Let us define the map $F : q \longmapsto Q_t^0(q, d(\mathbf{T}_0^t u(q))$. By the first addendum, the image of F is equal to the projection of L_u on $\mathbb{R}^d$, hence the map F is onto. Given a point $\theta \in \mathbb{R}^d$, we consider a preimage q of θ by F, and write

$$\check{\mathbf{T}}_t^0 \circ \mathbf{T}_0^t u(\theta) \geqslant \mathbf{T}_0^t u(q) - S_0^t(\theta, q) = u(\theta)$$

where the last equality comes from the first addendum. We conclude that $\check{\mathbf{T}}_t^0 \circ \mathbf{T}_0^t u \geqslant u$, hence $\check{\mathbf{T}}_t^0 \circ \mathbf{T}_0^t u = u$. □

The following extrapolates on [7]. For $t_0 \in \mathbb{R}$ and $\delta, t > 0$, let us define the operators

$$\mathbf{R}^t := \check{\mathbf{T}}_{t_0+\delta t}^{t_0} \circ \mathbf{T}_{t_0-t}^{t_0+\delta t} \circ \check{\mathbf{T}}_{t_0}^{t_0-t} \quad , \quad \check{\mathbf{R}}^t := \mathbf{T}_{t_0-\delta t}^{t_0} \circ \check{\mathbf{T}}_{t_0+t}^{t_0-\delta t} \circ \mathbf{T}_{t_0}^{t_0+t}.$$

Theorem 4. *There exists $\delta \in]0,1[$, which depends only on m and M such that the operators $\mathbf{R}^t, \check{\mathbf{R}}^t$ have the following properties:*

- *For each $t_0 \in \mathbb{R}$ and $t \in]0,1[$, the finite valued functions in the images of $\mathbf{R}^t$ and $\check{\mathbf{R}}^t$ are uniformly $C^{1,1}$.*
- *For each semi-concave function u, there exists $T > 0$ such that $\mathbf{R}^t u \leqslant u$ and $\check{\mathbf{R}}^t u \leqslant u$ for each $t_0 \in \mathbb{R}$ and $t \in]0, T[$.*
- *For each semi-convex function u, there exists $T > 0$ such that $\mathbf{R}^t u \geqslant u$ and $\check{\mathbf{R}}^t u \geqslant u$ for each $t_0 \in \mathbb{R}$ and $t \in]0, T[$.*
- *For each $C^{1,1}$ function u, there exists $T > 0$ such that $\mathbf{R}^t u = u$ and $\check{\mathbf{R}}^t u = u$ for each $t_0 \in \mathbb{R}$ and $t \in]0, T[$.*

Proof. The finite valued functions in the image of $\mathbf{T}_{t_0-t}^{t_0+\delta t}$ are C/t-semi-concave, by Lemma 16 (we assume that $t \in]0,1[$). Then, by Proposition 22, the finite valued functions in the image of $\check{\mathbf{T}}_{t_0+\delta t}^{t_0} \circ \mathbf{T}_{t_0-t}^{t_0+\delta t}$ are $(2C/t)$-semi-concave provided

$$\delta t \leqslant \frac{C}{tM(3 + 2C/t)^2} = \frac{Ct}{M(3t + 2C)^2},$$

which holds it $\delta \leqslant C/(M(3 + 2C))$. For such a δ, the finite valued functions in the image of $\mathbf{R}^t$ are uniformly semi-concave. They are also uniformly semi-convex, hence uniformly $C^{1,1}$. The proof is similar for $\check{\mathbf{R}}$. Let us now write

$$\mathbf{R}^t := (\check{\mathbf{T}}_{t_0+\delta t}^{t_0} \circ \mathbf{T}_{t_0}^{t_0+\delta t}) \circ (\mathbf{T}_{t_0-t}^{t_0} \circ \check{\mathbf{T}}_{t_0}^{t_0-t}),$$

which implies, using the monotony, that $\mathbf{R}^t u \geqslant \check{\mathbf{T}}_{t_0+\delta t}^{t_0} \circ \mathbf{T}_{t_0}^{t_0+\delta t} u$ and $\mathbf{R}^t u \leqslant \mathbf{T}_{t_0-t}^{t_0} \circ \check{\mathbf{T}}_{t_0}^{t_0-t} u$. By Addendum 2 we conclude that $\mathbf{R}^t u \geqslant u$ for small t when u is semi-convex. All the statements of the second and third point follow by similar considerations. The last point follows from the second and third one.

□

5 Sub-solutions of the stationary Hamilton-Jacobi equation.

We assume from now on that the Hamiltonian does not explicitly depend on time. Then, in addition to (HJ), we can consider the stationary Hamilton-Jacobi equation

$$H(q, du(q)) = a, \tag{HJa}$$

for each real parameter a. This stationary equation is the main character of Fathi's joined lecture. Formally, a function $u(q)$ solves (HJa) if and only if the function $(t,q)\longmapsto u(q)-at$ solves (HJ). It is not hard to check that this also holds in the sense of viscosity solutions: The function $u(q)$ is a viscosity solution of (HJa) if and only if the function $(t,q)\longmapsto u(q)-at$ is a viscosity solution of (HJ). Let us explicit for later references:

Hypothesis 4. *We say that H is autonomous if it does not depend on the time variable.*

In this autonomous context, we have $\mathbf{T}_{\tau}^{\tau+t}=\mathbf{T}_0^t$. We will denote by $\mathbf{T}^t$ this operator. The Markov property turns to the equality $\mathbf{T}^t\circ\mathbf{T}^s=\mathbf{T}^{t+s}$. In other words, the Lax Oleinik operators form a semi-group, the famous Lax-Oleinik semi-group. Another important specificity of the autonomous context is that the Hamiltonian H is constant along Hamiltonian orbits, as can be checked by an easy computation.

Proposition 23. *Given a Hamiltonian H satisfying Hypotheses 1,2,3,4, the following properties are equivalent for a function u:*

1. *The function u is Lipschitz and it solves the inequation $H(q,du(q))\leqslant a$ almost everywhere.*
2. *The inequality $u(q_1)-u(q_0)\leqslant A^t(q_0,q_1)+at$ holds for each $q_0\in\mathbb{R}^d, q_1\in\mathbb{R}^d, t>0$.*
3. *The inequality $u\leqslant \mathbf{T}^t u+ta$ holds for each $t\geqslant 0$.*
4. *The function u is a viscosity sub-solution of the Hamilton-Jacobi equation $H(q,du(q))=a$.*
5. *The function u is Lipschitz and the inequation $H(q,du(q))\leqslant a$ holds at each point of differentiability q of u (by Rademacher Theorem, the set of points of differentiability has full measure).*

The function u is called a sub-solution at level a, or a sub-solution of (HJa), if it satisfies these properties.

Proof. It is tautological that $5\Rightarrow 1$ and easy that $2\Leftrightarrow 3$. Let us prove that $1\Rightarrow 2$, following Fathi. If 1 holds, then there exists a set $M\subset\mathbb{R}^d$ of full measure composed of points of differentiability q of u such that $H(q,du(q))\leqslant a$. We first assume that $t<\sigma$ and prove 2 (recall that $A^t=S^t$). Let us consider the map

$$(q_0,q_1,\tau)\longmapsto (q(\tau),q_1,\tau),$$

where $q(\tau)$ is the value at time τ of the unique orbit $(q(s),p(s))$ which satisfies $q(0)=q_0$ and $q(t)=q_1$. This map is a diffeomorphism of $\mathbb{R}^d\times\mathbb{R}^d\times]0,t[$, the inverse diffeomorphism being

$$(\theta,q_1,\tau)\longmapsto (q(0),q_1,\tau),$$

where $(q(s),p(s))$ is the unique orbit such that $q(\tau)=\theta$ and $q(t)=q_1$. As a consequence, for almost each pair (q_0,q_1), the function u is differentiable at the point $q(s)$ for almost every $s\in]0,t[$. If (q_0,q_1) is such a pair, we have, using the convexity of H in p,

$$\begin{aligned}u(q_1)-u(q_0)=u(q(t))-u(q(0))&=\int_0^t du_{q(s)}\cdot\dot q(s)ds=\int_0^t du_{q(s)}\cdot\partial_p H(q(s),p(s))ds\\&\leqslant\int_0^t H(q(s),du_{q(s)})+\partial_pH(q(s),p(s))\cdot p(s)-H(q(s),p(s))ds\\&\leqslant at+S^t(q(0),q(t))=at+A^t(q_0,q_1).\end{aligned}$$

We have proved the desired inequality for almost every pair (q_0, q_1), hence on a dense subset of pairs. Since both sides of the inequality are continuous, we deduce that the inequality holds for all pairs (q_0, q_1), provided $t < \sigma$. In order to deduce the inequality when $t \geqslant \sigma$, we write, for n large enough,

$$\begin{aligned} A^t(q_0, q_1) + at &= \min_{\theta_1, \ldots, \theta_{n-1}} \big(S^{t/n}(q_0, \theta_1) + at/n + \cdots + S^{t/n}(q_{n-1}, q_1) + at/n\big) \\ &\geqslant \min_{\theta_1, \ldots, \theta_{n-1}} \big(u(\theta_1) - u(q_0) + \cdots + u(q_1) - u(\theta_{n-1})\big) = u(q_1) - u(q_0). \end{aligned}$$

Let us now prove that $3 \Rightarrow 4$. Let u be a function satisfying 3. This function then satisfies 2, hence it is Lipschitz. We consider a C^2 function $v(q)$ which touches u from above at some point θ, which means that $v - u$ has a global minimum at θ. Since the function u is Lipschitz, we can modify v at infinity and assume that it has bounded second differential. Then, there exists a C^2 solution $V(t, q)$ of (HJ) defined on $]-T, T[\times \mathbb{R}^d$ with $T > 0$, and such that $V(0, q) = v(q)$. For $t \geqslant 0$, we have $V_t = \mathbf{T}^t v$, by Proposition 18. Since $v \geqslant u$, we obtain that

$$V(t, q) = \mathbf{T}^t v(q) \geqslant \mathbf{T}^t u(q) \geqslant u(q) - at$$

for $t \in]0, T[$, hence $\partial_t V(0, \theta) \geqslant -a$ (recall that θ is the point of contact between u and v). Since we know that V solves (HJ), we conclude that

$$H(\theta, \partial_q V(0, \theta)) = H(\theta, dv(\theta)) \leqslant a.$$

The proof that $4 \Rightarrow 5$ is very classical and can be found in Fathi's lecture, but we recall it here for completeness. If q is a point of differentiability of u, then $du(q)$ is a super-differential (but not necessarily a proximal super-differential) of u at q, hence $H(q, du(q)) \leqslant a$. We will now prove that the function u is locally Lipschitz. The estimate $H(q, du(q)) \leqslant a$, which holds at each point of differentiability of u, then implies that it is globally Lipschitz in view of Hypothesis 3.

Let $B(Q, 1)$ be a closed ball, of radius one. Let us set $r = \max_{\theta \in B(Q,2), q \in B(Q,1)} (u(\theta) - u(q))$. Let k be a positive number greater that r and such that $|p| \geqslant k \Rightarrow H(q, p) > a$ for each q. Such a k exists by Hypothesis 3. Given q in $B(Q, 1)$, the function

$$\theta \longmapsto k|\theta - q| - u(\theta)$$

has then a local minimum in the interior of the ball $B(Q, 2)$. If this minimum is reached at a point q_1 different from q, then the function $v(\theta) := k|\theta - q|$ is smooth at q_1, and, since u is a viscosity sub-solution, we have $H(q_1, dv(q_1)) \leqslant a$, which is in contradiction with the fact that $|dv(q_1)| = k$. Hence the minimum must be reached at q, which implies that $k|\theta - q| - u(\theta) \geqslant -u(q)$ or equivalently that

$$u(\theta) - u(q) \leqslant k|\theta - q|$$

for each $\theta \in B(Q, 2)$ and all $q \in B(Q, 1)$. We conclude that u is k-Lipschitz on $B(Q, 1)$. □

Corollary 24. *If u is a sub-solution of (HJa), then, for each $t \geqslant 0$, $\mathbf{T}^t u$ is a sub-solution of (HJa), and so is $\check{\mathbf{T}}^t u$.*

Proof. The function u is a sub-solution if and only if $\mathbf{T}^s u + as \geqslant u$ for each $t \geqslant 0$. Applying $\mathbf{T}^t$, we obtain $\mathbf{T}^t \mathbf{T}^s u + as = \mathbf{T}^s \mathbf{T}^t u + as \geqslant \mathbf{T}^t u$. Since this inequality holds for each $s \geqslant 0$, we conclude that $\mathbf{T}^t u$ is a sub-solution. □

Corollary 25. *If the function u is Lipschitz, and if the Hamiltonian is autonomous, then the functions $\mathbf{T}^t u, t \geqslant 0$ are equi-Lipschitz.*

PROOF. If the function u is k-Lipschitz, then $du(q) \leqslant k$ almost everywhere, hence u is a sub-solution to (HJa) for some a (one can take $a = \sup_{|p|\leqslant k} H(q,p)$). As a consequence, the functions $\mathbf{T}^t u, t \geqslant 0$ are all sub-solutions to (HJa), hence they are K-Lipschitz, with $K = \sup\{|p|, H(q,p) \leqslant a\}$. □

6 Weak KAM solutions and invariant sets.

We derive here the first dynamical consequences from the theory.

Definition 26. *The function u is called a Weak KAM solution at level a if $\mathbf{T}^t u + ta = u$ for each $t \geqslant 0$. Weak KAM solutions at level a are viscosity solutions of (HJa). We say that the function u is a Weak KAM Solution if it is a Weak KAM solution at some level a.*

If u is a weak KAM solution, then it is semi-concave (with a semi-concavity constant which depends only on M and m). By Theorem 21, for $t > 0$, we have the inclusion

$$\varphi^{-t}(\bar{\Gamma}_u) \subset \Gamma_u$$

and this set is a Lipschitz graph. The set

$$\mathcal{I}^*(u) := \cap_{n\in\mathbb{N}} \varphi^{-n}(\bar{\Gamma}_u)$$

is a closed invariant set contained in a Lipschitz graph. It would be a very nice result to have obtained a distinguished closed invariant subsets of our Hamiltonian system contained in a Lipschitz graph. Unfortunately, at this point, we can't prove (because it is not necessarily true) that the set $\mathcal{I}^*(u)$ is not empty. In order to obtain interesting dynamical consequences from this theory, we need an additional assumption.

Hypothesis 5. *We say that the Hamiltonian H is periodic if $H(q+w,p) = H(q,p)$ for each $w \in \mathbb{Z}^d$, $q \in \mathbb{R}^d$ and $p \in \mathbb{R}^{d*}$.*

Under this hypothesis, we should see the Hamiltonian system as defined on the phase space $\mathbb{T}^d \times \mathbb{R}^{d*}$, with $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. Indeed, the flow φ^t commutes with the translations $(q,p) \longmapsto (q+w,p)$, $w \in \mathbb{Z}^d$. The compactness of this new configuration space has remarkable consequences, summed up in the following Theorem. We assume in the rest of this section that the Hamiltonian H satisfies Hypotheses 1, 2, 3, 4, 5.

Theorem 5. *If the Hamiltonian is autonomous and periodic, then there exists a periodic Weak KAM solution. The corresponding set $\mathcal{I}^*(u)$ is a non-empty closed invariant set which is contained in a Lipschitz graph and which is invariant under the translations $(q,p) \longmapsto (q+w,p)$, $w \in \mathbb{Z}^d$.*

This last property on the invariance under translations means that $\mathcal{I}^*(u)$ naturally gives rise to an invariant space on the quotient phase space $\mathbb{T}^d \times \mathbb{R}^{d*}$.

PROOF. Let us first prove the second part of the Theorem. If u is a periodic Weak KAM solution, then the set $\bar{\Gamma}_u$ is contained in $\{|p| \leqslant C\}$ for some constant C, and it is invariant under the integral translations, hence it descends to a compact subset of $\mathbb{T}^d \times \mathbb{R}^{d*}$, that we still denote by $\bar{\Gamma}_u$. Then the sets $\varphi^{-n}(\bar{\Gamma}_u)$ form a decreasing sequence of non-empty compact sets, hence their intersection is a non-empty compact set.

Let us now prove that there exists a periodic Weak KAM solution. We follow the proof of [6], which is slightly different from the original proof of Fathi. Observe first that the functions $A^t(q_0,q_1)$ are periodic in the sense that $A^t(q_0+w,q_1+w) = A^t(q_0,q_1)$ for each $w \in \mathbb{Z}^d$. This implies that $\mathbf{T}^t u$ is periodic when u is periodic. Considering the Cauchy problem for (HJ) with initial condition equal to zero, we define $v(t,q) := \mathbf{T}^t 0(q)$. The quantities $a^+(t) = \max_q v_t(q)$ and $a^-(t) = \min_q v_t(q)$ will be useful. Since the functions $v_t, t \geqslant 0$ are equi-Lipschitz, there exists a constant K such that $a^+(t) - a^-(t) \leqslant K$ for all $t \geqslant 0$. We have

$$a^+(t+s) = \max \mathbf{T}^{t+s} 0 = \max \mathbf{T}^t(\mathbf{T}^s 0) \leqslant \mathbf{T}^t(a^+(s)) = a^+(s) + \mathbf{T}^t(0) \leqslant a^+(s) + a^+(t),$$

and similarly

$$a^-(t+s) \geqslant a^-(t) + a^-(s).$$

By standard results on sub-additive functions, we conclude that $a^+(t)/t$ and $a^-(t)/t$ converge respectively to $\inf_{t \geqslant 0} a^+(t)/t$ and $\sup_{t \geqslant 0} a^-(t)/t$. Since $a^+ - a^-$ is bounded, these two limits have the same value, let us call it $-a$. We have

$$K - ta \geqslant a^-(t) + K \geqslant a^+(t) \geqslant -ta \geqslant a^-(t) \geqslant a^+(t) - K \geqslant K - at$$

for all $t \geqslant 0$, hence

$$K \geqslant v(t,q) + ta \geqslant -K.$$

We can now define

$$u(q) := \liminf_{t \longrightarrow \infty} (v(t,q) + ta).$$

We claim that u is a Weak KAM solution at level a. Since the functions $v_t + ta$ are equi-Lipschitz and equi-bounded, the function u is well-defined and Lipschitz. We have to prove that $\mathbf{T}^t u + ta = u$ for all $t \geqslant 0$.

We have

$$v(t+s, q_1) + (t+s)a \leqslant v(s, q_0) + sa + A^t(q_0, q_1) + ta$$

for each q_0, q_1 and $t \geqslant 0, s \geqslant 0$. Taking the lim inf in s yields

$$u(q_1) \leqslant u(q_0) + A^t(q_0, q_1) + ta.$$

We have proved that u is a sub-solution to (HJa).

Conversely, we have to prove that $\mathbf{T}^t u + ta \geqslant u$. Let us pick a point q and consider a sequence t_n such that $v(t_n, q) + t_n a \longrightarrow u(q)$. Fixing $t > 0$, we consider a sequence q_n in $\mathbb{R}^d$ such that

$$v(t_n, q) + t_n a = v(t_n - t, q_n) + (t_n - t)a + A^t(q_n, q) + ta.$$

This equality implies that the sequence q_n is bounded, and we assume by taking a subsequence that it has a limit q'. We can also assume that the sequence $v(t_n - t, q') + (t_n - t)a$ has a limit, that we denote by l. Note that $l \geqslant u(q')$. Since the functions v_t are equi-Lipschitz, we have $v(t_n - t, q_n) + (t_n - t)a \longrightarrow l$ hence, taking the limit in the equality above,

$$u(q) = l + A^t(q', q) + at \geqslant u(q') + A^t(q', q) + at \geqslant \mathbf{T}^t u(q) + at.$$

We have proved that u is a periodic Weak KAM solution at level a. □

The periodic Weak KAM solutions at level a are the periodic viscosity solutions of (HJa), as is proved in Fathi's joined lecture. The existence of periodic viscosity solutions was first obtained by Lions, Papanicolaou and Varadhan in a famous unpublished preprint, [15]. The most important aspect of Fathi's weak KAM theorem that we just exposed is that these viscosity solutions have a dynamical relevance and give rise to invariant sets.

Let us comment a bit further in that direction, and explain the name "Weak KAM". Consider a periodic Lipschitz function u, and the associated set Γ_u, that we consider here as a subspace of $\mathbb{T}^d \times \mathbb{R}^{d*}$.

Assume first that u is C^2, so that Γ_u is a C^1 graph. This graph is invariant if and only if there exists a such that u solves (HJa). This follows from Section 1: If u solves (HJa), then the function $U(t,q) = u(q) - at$ solves HJ, hence

$$\varphi^t(\Gamma_u) = \Gamma_{U_t} = \Gamma_u.$$

Conversely, if Γ_u is invariant, then $\Gamma_{\mathbf{T}^t u} = \varphi^t(\Gamma_u) = \Gamma_u$, by Corollay 3, hence $\mathbf{T}^t u$ is equal to u up to an additive constant $a(t)$. Since $\mathbf{T}^t$ is a semi-group, it is easy to deduce that $a(t) = at$ for some $a \in \mathbb{R}$. As a consequence, u is a C^2 Weak KAM solution, hence a classical solution of (HJa).

The classical KAM theorem gives the existence, in certain very specific settings, of some invariant C^1 graphs of the form Γ_u. From what we just explained, it can be interpreted as giving the existence of C^2 solutions of (HJa), although this point of view is not the right one to obtain its proof. It is natural to expect that the Hamilton-Jacobi equation could be used to produce invariant sets in more general situations. Since we do not know any direct method to prove the existence of C^2 solutions of (HJa), we should deal with some kind of weak solutions. However, if u is just a Lipschitz solution almost everywhere, we can't say much about the dynamical properties of Γ_u. It is remarkable that the inclusion $\varphi^t(\Gamma_u) \supset \bar{\Gamma}_u$ holds for viscosity solutions (or, equivalently Weak KAM solutions) in the convex case. This is the starting point of Fathi's construction of the invariant set $\mathcal{I}^*(u)$ that we exposed in the present section.

7 Regular sub-solutions and the Aubry set.

We abandon for a moment the hypothesis 5, and consider a Hamiltonian satisfying Hypotheses 1, 2, 3, 4. We describe a new construction of invariant sets based on the study of regular sub-solutions, and define the Aubry set. We mostly follow [4] in this section. The following result is at the base of our constructions, see [4, 2, 13].

Theorem 6. *If (HJa) admits a sub-solution, then it admits a $C^{1,1}$ sub-solution. Moreover, the set of $C^{1,1}$ sub-solutions is dense in the set of all sub-solutions for the uniform topology.*

PROOF. Let u be a sub-solution at level a. We use the operator $\mathbf{R}^t = \breve{\mathbf{T}}^{\delta t} \circ \mathbf{T}^{(\delta+1)t} \circ \breve{\mathbf{T}}^t$ of Theorem 4 to regularize u. Since the operators $\mathbf{T}^t$ and $\breve{\mathbf{T}}^t$ preserve sub-solutions, so does $\mathbf{R}^t$. We claim that

$$u - (C+a)(1+\delta)t \leqslant \mathbf{R}^t u \leqslant u + (C+a)(1+\delta)t$$

with a constant C which depends only on m and M. This implies that the function $\mathbf{R}^t u$ is finite valued. If the parameter δ has been chosen small enough, then, by Theorem 4, the functions $\mathbf{R}^t$ are $C^{1,1}$ sub-solutions, which converge uniformly to u as $t \longrightarrow 0$. The bound on $\mathbf{R}^t u$ claimed above follows from the following ones in view of Property 19,

$$v - sa \leqslant \mathbf{T}^s v \leqslant v + Cs, \quad v - Cs \leqslant \breve{\mathbf{T}}^s v \leqslant v + sa$$

which hold for each $s \geqslant 0$ and each sub-solution v at level a. The first one can be seen by writing

$$u(q) - as \leqslant \mathbf{T}^s u(q) \leqslant u(q) + A^s(q,q) \leqslant u(q) + Cs.$$

This ends the proof of Theorem 6. Observe that we could have used the simpler operator $\breve{\mathbf{T}}^{\delta t} \circ \mathbf{T}^t$, as was done in [4], but the operator $\mathbf{R}^t$ deserves attention for some nicer properties. □

Definition 27. *The critical value of H is the real number α (or $\alpha(H)$) defined as the infimum of all real numbers a such that (HJa) has a sub-solution. The sub-solutions of (HJα) are called critical sub-solutions.*

Lemma 28. *We have the estimate $-M \leqslant \alpha \leqslant M$.*

PROOF. The function $u = 0$ is a sub-solution at level M, hence $\alpha \leqslant M$. Conversely, since $H \geqslant -M$ there exists no sub-solution at level a when $a < -M$. □

Proposition 29. *There exists a $C^{1,1}$ sub-solution of (HJα).*

PROOF. Let a_n be a sequence decreasing to α. Since $a_n > \alpha$, the Hamilton-Jacobi equation at level a_n has a sub-solution u_n. The sequence u_n is equi-Lipschitz, and we can assume by adding constants that it is also equi-bounded. Taking a subsequence, we can also assume that it converges locally uniformly to a limit u. Taking the limit $n \longrightarrow \infty$ in the inequalities $u_n(q_1) - u_n(q_0) \leqslant A^t(q_0, q_1) + ta_n$ gives $u(q_1) - u(q_0) \leqslant A^t(q_0, q_1) + ta_n$. This holds for all q_0, q_1 and $t > 0$, hence u is a sub-solution at level α, or in other words a critical sub-solution. Since there exists a critical sub-solution, Theorem 6 implies that there exists a $C^{1,1}$ critical sub-solution. □

Definition 30. *The projected Aubry set is the set $\mathcal{A} \subset \mathbb{R}^d$ of points q such that the equality $H(q, du(q)) = \alpha$ holds for all C^1 critical sub-solutions u.*

We point out that $\mathcal{A}$ might be empty without additional hypotheses.

Lemma 31. *If $q \in \mathcal{A}$, then all C^1 critical sub-solutions u have the same differential at q. In other words, the restriction $\Gamma_{u|\mathcal{A}}$ does not depend on the C^1 critical sub-solution u.*

PROOF. Let u and v be two critical sub-solutions, and q a point in $\mathcal{A}$. We have to prove that $du(q) = dv(q)$. Assume, by contradiction, that this equality does not hold and consider the sub-solution $w = (u + v)/2$. Since $H(q, du(q)) = H(q, dv(q)) = \alpha$, the strict convexity of $H(q, .)$ implies that $H(q, dw(q)) < \alpha$, which contradicts the definition of $\mathcal{A}$. □

Lemma 32. *There exists a $C^{1,1}$ sub-solution u_0 which satisfies the strict inequality $H(q, du_0(q)) < \alpha$ for all q in the complement of $\mathcal{A}$.*

PROOF. The set of C^1 functions is separable for the topology of uniform C^1 convergence on compact sets. This topology can be defined for example by the distance

$$d(u, v) = \sum_n \frac{\sup_{|q| \leqslant n} \arctan(|u(q)| + |du(q)|)}{2^n}.$$

Since a subset of a separable space is separable, there exists a sequence u_n of C^1 critical sub-solutions which is dense for this topology in the set of all C^1 critical sub-solutions. Let us set

$$a_n = \frac{a_0}{2^n \sup_{k \leqslant n, |q| \leqslant n}(1 + |u_k(q)| + |du_k(q)|)}$$

and choose a_0 such that $\sum_{n \geqslant 1} a_n = 1$. The sum $\sum_{n \geqslant 1} a_n u_n$ converges uniformly with its differentials on each compact sets to a C^1 limit v_0. The function v_0 is a critical sub-solution, and we claim that $H(q, dv_0(q)) = \alpha$ if and only if q belongs to $\mathcal{A}$. Indeed, this equality holds only if all the inequalities $H(q, du_n(q)) \leqslant \alpha$ are equalities, which, in view of the density of the sequence u_n, implies that $H(q, du(q)) = \alpha$ for all C^1 sub-solutions u. By definition, this implies that q belongs to $\mathcal{A}$. We have constructed a C^1 sub-solution v_0 such that

$$H(q, dv_0(q)) < \alpha$$

outside of $\mathcal{A}$. We have to prove the existence of a $C^{1,1}$ critical sub-solution with the same property. We consider a smooth function $V(q)$ which is bounded in C^2, which is positive outside of $\mathcal{A}$, and such that

$$0 \leqslant V(q) \leqslant \alpha - H(q, dv_0(q))$$

for all $q \in \mathbb{R}^n$. The modified Hamiltonian $\tilde{H}(q,p) = H(q,p) + V(q)$ satisfies all our hypotheses. Since $\tilde{H} \geqslant H$, the corresponding critical value $\tilde{\alpha}$ satisfies $\tilde{\alpha} \geqslant \alpha$. Since v_0 is a sub-solution of the inequation

$$\tilde{H}(q, dv_0(q)) \leqslant \alpha,$$

we can apply Theorem 6 to $\tilde{H}$ at level α, and obtain the existence of a $C^{1,1}$ sub-solution u_0 to the same inequation. The inequality

$$H(q, du_0(q)) \leqslant \alpha - V(q)$$

implies that u_0 is a critical sub-solution for H which is strict on the set $\{V > 0\}$ which, from our construction of V, is the complement of $\mathcal{A}$. □

Definition 33. *The Aubry set $\mathcal{A}^*$ is defined as:*

$$\mathcal{A}^* = \cap_u \Gamma_{u|\mathcal{A}} = \cap_u \Gamma_u,$$

where the intersections are taken on the set of C^1 critical sub-solutions.

In view of Lemma 31 we have $\mathcal{A}^* = \Gamma_{u|\mathcal{A}}$ for each C^1 sub-solution u, hence $\pi(\mathcal{A}^*) = \mathcal{A}$, where $\pi : \mathbb{R}^d \times \mathbb{R}^{d*} \longrightarrow \mathbb{R}^d$ is the projection on the first factor. To check the second inequality, it is sufficient to prove that $\cap_u \Gamma_u \subset \mathcal{A}^*$. Let u_0 be a C^1 critical sub-solution such that $H(q, du_0(q)) < \alpha$ outside of $\mathcal{A}$. Given a point (q_0, p_0) in $\Gamma_{u_0} - \mathcal{A}^*$, we can slightly perturb the critical sub-solution u_0 around q_0 to a critical sub-solution u_1 such that $du_1(q_0) \neq du_0(q_0)$ (we use the strict inequality $H(q, du_0(q)) < \alpha$). The point (q_0, p_0) does not belong to Γ_{u_1}, hence it does not belong to $\cap_u \Gamma_u$, which ends our proof.

The set $\mathcal{A}^*$ is contained in the Lipschitz graph Γ_{u_0} for each $C^{1,1}$ sub-solution u_0. As in Section 6, we have obtained an invariant set contained in a Lipschitz graph, but which may be empty in general:

Proposition 34. *The Aubry set is a closed invariant set.*

Proof. Let u_0 be a $C^{1,1}$ critical solution such that $H(q, du_0(q)) < \alpha$ outside of $\mathcal{A}$. By Proposition 22, there exists $T > 0$ such that $\mathbf{T}^t u_0$ is still $C^{1,1}$ for $t \in [-T, T]$. Given $(q,p) \in \mathcal{A}^*$, we conclude that, for $t \in [0, T]$, we have $p = d(\mathbf{T}^t u_0)(q)$. Setting $\theta = Q^{-t}(q,p)$, the addendum to Proposition 22 implies that $\mathbf{T}^t u_0(q) = u_0(\theta) + S^t(\theta, q)$, and that

$$\varphi^t(\theta, du_0(\theta)) = (q,p).$$

Since the flow preserves the Hamiltonian, we get that $H(\theta, du_0(\theta)) = \alpha$, hence the point θ belongs to $\mathcal{A}$, and then

$$\varphi^{-t}(q,p) = \big(\theta, du_0(\theta)\big) \in \mathcal{A}^*.$$

We have proved that $\varphi^{-t}(\mathcal{A}^*) \subset \mathcal{A}^*$ for $t \in [0,T]$. We can prove in a similar way, using the $C^{1,1}$ sub-solution $\check{\mathbf{T}}^t u_0$ instead of $\mathbf{T}^t u_0$, that $\varphi^t(\mathcal{A}^*) \subset \mathcal{A}^*$ for $t \in [0,T]$, and hence that

$$\varphi^t(\mathcal{A}^*) = \mathcal{A}^*$$

for each $t \in [-T, T]$, which clearly implies that this equality holds for all t. We have proved the invariance of $\mathcal{A}^*$. □

Proposition 35. *The equality*

$$\check{\mathbf{T}}^t u(q) - t\alpha = u(q) = \mathbf{T}^t u(q) + t\alpha$$

holds for each critical sub-solution u, each $t \geqslant 0$ and each $q \in \mathcal{A}$. The inclusion $\mathcal{A}^* \subset \Gamma_u$ holds for each critical sub-solution, hence the inclusion $\mathcal{A}^* \subset \mathcal{I}^*(u)$ holds for each weak KAM solution at level α.

Proof. Let $(q(s), p(s))$ be a trajectory contained in $\mathcal{A}^*$, and $t \geqslant 0$ be given. For each C^1 critical sub-solution u, we have $p(s) = du_{q(s)}$, and

$$\begin{aligned} u(q(t)) - u(q(0)) = &\int_0^t du_{q(s)}\dot{q}(s)ds = t\alpha + \int_0^t du_{q(s)}\dot{q}(s) - H(q, du_{q(s)})ds \\ &\geqslant A^t(q(0), q(t)) + t\alpha. \end{aligned}$$

Since u is a critical sub-solution, the second point in Proposition 23 implies that the last inequality must be an equality, hence

$$u(q(t)) - u(q(s)) = A^{t-s}(q(s), q(t)) + (t-s)\alpha$$

for each $t \geqslant s$. In the terminology of Fathi, we have proved that the curve $q(s)$ is calibrated by the sub-solution u. We can now write

$$u(q(t)) \leqslant \mathbf{T}^t u(q(t)) + t\alpha \leqslant u(q(0)) + A^t(q(0), q(t)) + t\alpha = u(q(t)).$$

This implies that $\mathbf{T}^t u + t\alpha = u$ on $\mathcal{A}$, and, similarly, $\check{\mathbf{T}}^t u - t\alpha = u$ on $\mathcal{A}$. Let us now fix $t \in]0, \sigma[$. Given an orbit $(q(s), p(s))$ in $\mathcal{A}^*$, we have

$$u(q(0)) \leqslant u(\theta) + S^t(\theta, q(0)) + t\alpha$$

for each sub-solution u and each θ, with equality at $\theta = q(-t)$. This implies that $\partial_1 S(q(-t), q(0))$ is a super-differential of u at $q(0)$. This holds in particular for C^1 sub-solutions, which satisfy $du(q(0)) = p(0)$, hence $\partial_1 S(q(-t), q(0)) = p(0)$. We have proved that $p(0)$ is a super-differential of u at $q(0)$. Similarly, using the inequality

$$u(q(0)) \geqslant u(\theta) - S^t(q(0), \theta) - t\alpha,$$

with equality at $\theta = q(t)$, we conclude that $p(0)$ is a sub-differential of u at $q(0)$. This implies that u is differentiable at $q(0)$, and that $du(q(0)) = p(0)$. As a consequence, $\mathcal{A}^* \subset \Gamma_u$ for each sub-solution u. □

In the course of the above proof, we have established the following lemma, which will be needed later:

Lemma 36. *Let u be a sub-solution at level a, and let $(q(s), p(s))$ be a Hamiltonian trajectory contained in $\Gamma_u \cap \{H = a\}$ (note that this set is not necessarily invariant in general), then, the equality $\check{\mathbf{T}}^t u(q(s)) - ta = u(q(s)) = \mathbf{T}^t u(q(s)) + ta$ holds, for each $t \geqslant 0$ and each $s \in \mathbb{R}$.*

8 The Mañé Potential.

In this section, we work with a Hamiltonian satisfying Hypotheses 1, 2, 3, 4. The Mañé Potential at level a is the function

$$\Phi^a(q_0, q_1) := \inf_{t>0} \left(A^t(q_0, q_1) + at\right).$$

This function was first introduced by Ricardo Mañé, see [16]. We leave as an easy exercise for the reader to prove the triangle inequality

$$\Phi^a(q_0, q_1) \leqslant \Phi^a(q_0, \theta) + \Phi^a(\theta, q_1).$$

In view of Proposition 23, each sub-solution u at level a satisfies

$$u(q_1) - u(q_0) \leqslant \Phi^a(q_0, q_1)$$

for each q_0 and q_1. We conclude that Φ^a is finite if there exists a sub-solution at level a, which holds if and only if $a \geqslant \alpha$. Conversely, If the function Φ^a is finite, then we see from the triangle inequality that the function $q \longmapsto \Phi^a(q_0, q)$ is a sub-solution at level a, which implies that $a \geqslant \alpha$. The estimates of Lemma 13 imply that

$$\Phi^a(q_0, q_1) \leqslant 2\sqrt{2m(M+a)}|q_1 - q_0|$$

provided $a \geqslant \alpha$ (note that $\alpha \geqslant -M$). We have proved that the Mañé Potential is the function called the viscosity semi-distance in Fathi's lecture:

Proposition 37. *If $a \geqslant \alpha$, then the function $q \longmapsto \Phi^a(q_0, q)$ is the maximum of all sub-solutions u at level a which vanish at q_0. If $a < \alpha$, then there is no such sub-solution and Φ^a is identically equal to $-\infty$.*

This statement also implies that the Mañé Potential at level a only depends on the energy level $\{H = a\}$. More precisely, let G be another Hamiltonian satisfying our hypotheses and such that $H = a \Leftrightarrow G = a$. Then, the sets $\{H \leqslant a\}$ and $\{G \leqslant a\}$ are equal, which implies in view of the first characterization of sub-solutions in Proposition 23 that G and H have the same sub-solutions at level a. As a consequence, they have the same Mañé potential at level a. This is also reflected in the following Proposition by the fact that the involved orbits are contained in the set $\{H = a\}$.

Proposition 38. *Given $q_0 \neq q_1$, there exists $\tau \in]0, \infty]$ and an orbit $(q(s), p(s)) : (-\tau, 0] \longrightarrow \mathbb{R}^d \times \mathbb{R}^{d*}$ such that $q(0) = q_1$, $A_s^0(q_0, q(s)) - as = \Phi^a(q(s), q_1)$,*

$$\Phi^a(q_0, q(s)) + \Phi^a(q(s), q_1) = \Phi^a(q_0, q_1)$$

and $H(q(s), p(s)) = a$ for each $s \in (-\tau, 0]$. If moreover τ is finite, then $q(-\tau) = q_0$.

Proof. If $q_0 \neq q_1$, then either the functions $t \longmapsto A^t(q_0, q_1) + at$ reaches its minimum at some finite time $\tau > 0$, or it has a minimizing sequence $\tau_n \longrightarrow \infty$. This follows from Lemma 13.

In the first case, there exists an orbit $(q(t), p(t)) : [-\tau, 0] \longrightarrow \mathbb{R}^d \times \mathbb{R}^{d*}$ such that $q(-\tau) = q_0$, $q(0) = q_1$, and

$$\int_{-\tau}^{0} p \cdot \dot{q} - H(q, p) dt = A^\tau(q_0, q_1) = \Phi^a(q_0, q_1) - \tau a.$$

We obtain, for each $s \in [-\tau, 0]$, that

$$\begin{aligned} \Phi^a(q_0, q_1) - a\tau = \int_{-\tau}^{0} p \cdot \dot{q} - H(q, p) dt &= \int_{-\tau}^{s} p \cdot \dot{q} - H(q, p) dt + \int_{s}^{0} p \cdot \dot{q} - H(q, p) dt \\ &\geqslant A^{s+\tau}(q_0, q(s)) + A^{-s}(q(s), q_1) \\ &\geqslant \Phi^a(q_0, q(s)) - a(s+\tau) + \Phi^a(q(s), q_1) + as \\ &\geqslant \Phi^a(q_0, q_1) - a\tau. \end{aligned}$$

We conclude that all these inequalities are equalities, hence

$$\Phi^a(q_0, q(s)) + \Phi^a(q(s), q_1) = \Phi^a(q_0, q_1).$$

We also deduce from the above chain of inequalities that $A^{-s}(q(s), q_1) - as = \Phi^a(q(s), q_1)$, which implies that the function $t \longmapsto A^t(q(s), q_1) + at$ is minimal for $t = -s$. Taking $s \in]-\sigma, 0[$, we can differentiate with respect to t at $t = -s$ and get

$$\partial_{t|t=-s} S^t(q(s), q_1) + a = 0.$$

Recalling the equality

$$\partial_t S^{-s}(q(s), q_1) + H(q_1, p(0)) = 0$$

(because $p(0) = \rho_1(-s, q(s), q_1)$ in the notations of Section 2), we conclude that $H(q_1, p(0)) = a$, and, since the Hamiltonian is constant on Hamiltonian orbits, $H(q(t), p(t)) = a$ for each t.

In the second case, there exists a sequence of orbits $(q_n(t), p_n(t))$ on $[-\tau_n, 0]$ such that

$$\int_{-\tau_n}^0 p_n \cdot \dot{q}_n - H(q_n, p_n)dt + a\tau_n = A^{\tau_n}(q_0, q_1) + a\tau_n \leqslant \Phi^a(q_0, q_1) + \delta_n,$$

where $\delta_n \longrightarrow 0$. Let us denote $h_n := H(q_n(s), p_n(s))$, it does not depend on s. By Lemma 14 and the above inequality, we have

$$\frac{m}{M}\tau_n h_n - (m + M)\tau_n \leqslant \int_{-\tau_n}^0 p_n \cdot \partial_p H(q_n, p_n) - H(q_n, p_n)dt \leqslant \Phi^a(q_0, q_1) + \delta_n$$

hence the sequence h_n is bounded. As a consequence, the curves $p_n(s)$ are uniformly bounded, hence so is $\dot{q}_n(s) = \partial_p H(q_n(s), p_n(s))$. On each compact interval of time $[s, 0]$, the curves $x_n(t) = (q_n(t), p_n(t))$ are thus uniformly bounded, hence uniformly Lipschitz. Up to taking a subsequence, we can thus assume that the curves $x_n(t)$ converge, uniformly on compact time intervals, to a Hamiltonian orbit $x(t) = (q(t), p(t)) : (-\infty, 0] \longrightarrow \mathbb{R}^d \times \mathbb{R}^{d*}$. Passing at the limit in the inequality

$$\Phi^a(q_0, q_n(s)) + \Phi^a(q_n(s), q_1) \leqslant \Phi^a(q_0, q_1) + \delta_n,$$

which holds for each $s \in [-\tau_n, 0]$, yields

$$\Phi^a(q_0, q(s)) + \Phi^a(q(s), q_1) \leqslant \Phi^a(q_0, q_1),$$

which must actually be an equality. We prove as in the first case that $H(q_1, p(0)) = a$, thus $H(q(s), p(s)) \equiv a$. □

The projected Aubry set $\mathcal{A}$ can be characterized in terms of the Mañé potential (see also Fathi's lecture):

Proposition 39. *The following statements are equivalent for a point q_0 and a real number a, where we denote by u the function $\Phi^a(q_0, .)$:*

1. *$q_0 \in \mathcal{A}$ and $a = \alpha$.*
2. *$\mathbf{T}^t u(q_0) + ta = u(q_0) = 0$ for each $t \geqslant 0$.*
3. *The function u is a Weak KAM solution at level a.*
4. *u is differentiable at q_0.*

Proof. $1 \Rightarrow 2$. This follows from Proposition 35 since u is a sub-solution at level $a = \alpha$.

$2 \Rightarrow 3$. Let us fix $t > 0$ and q_1. We have to prove that there exists θ such that $u(q_1) \geqslant u(\theta) + A^t(\theta, q_1) + ta$ (this inequality is then an equality). If $q_1 = q_0$, the existence of this point follows from the equality $\mathbf{T}^t u(q_0) + ta = u(q_0)$.

If $q_1 \neq q_0$, we can apply Proposition 38 to this pair of points. With the notations of Proposition 38, if $\tau \geqslant t$, then the point $\theta = q(-t)$ fulfills our demand. If $\tau < t$, then we set $s = t - \tau$. We have $q(-\tau) = q_0$ and $A^\tau(q_0, q_1) + a\tau = u(q_1)$. Since $\mathbf{T}^s u(q_0) + sa = u(q_0)$, there exists θ such that $u(\theta) + A^s(\theta, q_0) + sa = u(q_0) = 0$. The infimum in the definition of $\mathbf{T}^s u(q_0)$ exists because u is Lipschitz. We conclude that

$$u(\theta) + A^t(\theta, q_1) + at \leqslant u(\theta) + A^s(\theta, q_0) + sa + A^\tau(q_0, q_1) + a\tau = u(q_1).$$

$3 \Rightarrow 4$. If u is a Weak KAM solution, then it has a proximal super-differential at each point. Conversely, if v is a C^1 sub-solution, then $u - v$ has a minimum at q_0 hence $dv(q_0)$ is a sub-differential of u at q_0. The function u both has a super-differential and a sub-differential at q_0, hence it is differentiable at q_0.

$4 \Rightarrow 1$. If $a > \alpha$ or if q_0 does not belong to $\mathcal{A}$, then there exists a C^1 sub-solution v at level a which is strict near q_0. We can then slightly perturb the function v near q_0 and build a sub-solution w such that $dw(q_0) \neq dv(q_0)$. In view of the characterization of u as the largest sub-solution vanishing at q_0, we conclude that $dv(q_0)$ as well as $dw(q_0)$ are sub-differentials of u at q_0, hence u is not differentiable at this point. □

The Mañé potential also allows to build Weak KAM solutions in the non periodic case by the Busemann method, see [11] and Fathi's Lecture. Let q_n be a sequence of points of $\mathbb{R}^d$ such that $|q_n| \geqslant n$. We consider the sequence of functions

$$u_n(q) = \Phi^a(q_n, q) - \Phi^a(q_n, q_0).$$

By construction, $u_n(q_0) = 0$, and it follows from the triangle inequality that the functions u_n are equi-Lipschitz. We can then assume, without loss of generality, that the functions u_n converge, uniformly on compact sets, to a Lipschitz limit $u(q)$.

Proposition 40. *The limit function $u(q)$ is a Weak KAM solution at level a.*

Proof. The functions u_n are all sub-solutions at level a, which means that $u_n(q_1) - u_n(q_0) \leqslant A^t(q_0, q_1) + ta$ for each $t \geqslant 0$, q_0, q_1. At the limit $n \longrightarrow \infty$, we obtain that that $\mathbf{T}^t u + ta \geqslant u$ for each $t \geqslant 0$.

We have to prove that $\mathbf{T}^t u + ta \leqslant u$ for all $t \geqslant 0$. Let us fix a point q and a time $t \geqslant 0$, and consider a sequence t_n such that

$$A^{t_n}(q_n, q) + at_n \leqslant \Phi^a(q_n, q) + 1/n.$$

This inequality implies that

$$\frac{1}{2Mt_n}|q_n - q|^2 \leqslant 1 + (M - a)t_n + 2\sqrt{2m(M + a)}|q_n - q|$$

and, since $|q_n - q| \longrightarrow \infty$, that $t_n \longrightarrow \infty$. When n is large enough, we have $t_n \geqslant t$ and there exists $\theta_n \in \mathbb{R}^d$ such that $A^{t_n}(q_n, q) = A^{t_n - t}(q_n, \theta_n) + A^t(\theta_n, q)$. This implies that

$$\begin{aligned}\Phi^a(q_n, q) &\geqslant A^{t_n}(q_n, q) + at_n - 1/n \\ &\geqslant A^{t_n - t}(q_n, \theta_n) + a(t_n - t) + A^t(\theta_n, q) + at - 1/n \\ &\geqslant \Phi^a(q_n, \theta_n) + A^t(\theta_n, q) + at - 1/n.\end{aligned}$$

This inequality implies that

$$u_n(q) \geqslant u_n(\theta_n) + A^t(\theta_n, q) + at - 1/n.$$

Since the functions u_n are equi-Lipschitz, this implies that the sequence θ_n is bounded, by Lemma 13. By taking a subsequence, we assume that θ_n has a limit θ, and, at the limit, we obtain

$$u(q) \geqslant u(\theta) + A^t(\theta, q) + at,$$

which implies that $u(q) \geqslant \mathbf{T}^t u(q) + ta$. □

9 Return to the periodic case.

A more precise link can be established between the contents of Sections 6 and 7 under the assumption that H is periodic (see Hypothesis 5). It is useful first to expose a variant of Section 7 adapted to the periodic case. We leave as exercises the proofs which are direct adaptations of the ones given above. From now on, we assume Hypotheses 1, 2, 3, 4, 5.

Theorem 7. *If (HJa) admits a periodic sub-solution, then it admits a periodic $C^{1,1}$ sub-solution. Moreover, the set of periodic $C^{1,1}$ sub-solutions is dense in the set of all periodic sub-solutions for the uniform topology.*

Definition 41. *The periodic critical value of H is the real number $\alpha(0)$ defined as the infimum of all real numbers a such that (HJa) has a periodic sub-solution. The periodic sub-solutions at level $\alpha(0)$ are called critical periodic sub-solutions.*

Definition 42. *The projected periodic Aubry set is the set $\mathcal{A}(0) \subset \mathbb{T}^d$ of points q such that the equality $H(q, du(q)) = \alpha(0)$ holds for all C^1 periodic critical sub-solutions u.*

Lemma 43. *If $q \in \mathcal{A}(0)$, then all C^1 critical periodic sub-solutions u have the same differential at q. In other words, the restriction $\Gamma_{u|\mathcal{A}}$ does not depend on the C^1 critical periodic sub-solution u.*

Proposition 44. *There exists a $C^{1,1}$ periodic critical sub-solution u_0 such that $H(q, du_0(q)) < \alpha(0)$ outside of $\mathcal{A}(0)$.*

Without surprise, we define the periodic Aubry set $\mathcal{A}^*(0)$ as

$$\mathcal{A}^*(0) := \Gamma_{u_0|\mathcal{A}},$$

with u_0 given by the proposition (there is not a single u_0, but the Aubry set is well defined).

Proposition 45. *The set $\mathcal{A}^*(0) \subset \mathbb{T}^d \times \mathbb{R}^{d*}$ is compact, non empty, and invariant.*

Proof. Let us prove that $\mathcal{A}(0)$, hence $\mathcal{A}^*(0)$ is not empty. Assuming by contradiction that it was empty, then the equality $H(q, du_0(q)) < \alpha(0)$ would hold for all $q \in \mathbb{R}^d$. Since the function $q \longmapsto H(q, du_0(q))$ is periodic, we could conclude that $\sup_q H(q, du_0(q)) < \alpha(0)$, which is in contradiction with the definition of $\alpha(0)$. □

We are now in a position to specify the connection with the invariant sets introduced in Section 6:

Proposition 46. *In the periodic case, we have the equality*

$$\mathcal{A}^*(0) = \cap_u \mathcal{I}^*(u),$$

where the intersection is taken on all periodic weak KAM solutions.

Proof. The inclusion $\mathcal{A}^*(0) \subset \cap_u \mathcal{I}^*(u)$ is proved as in Section 7. Our goal is to prove the other inclusion. Let u_0 be a $C^{1,1}$ periodic sub-solution which is strict outside of $\mathcal{A}(0)$. The map $t \longmapsto \mathbf{T}^t u_0 + t\alpha(0)$ is non-decreasing. In addition, the functions $\mathbf{T}^t u_0 + t\alpha(0)$ are equi-Lipschitz, and they coincide with u_0 on $\mathcal{A}$, hence they are equi-bounded. As a consequence, $\mathbf{T}^t u_0 + t\alpha \longrightarrow u_\infty$ uniformly as $t \longrightarrow \infty$.

Claim: The limit u_∞ is a periodic weak KAM solution such that $u_0 < u_\infty$ outside of $\mathcal{A}(0)$.

In order to prove that u_∞ is a weak KAM solution, it is enough to notice that the function $\mathbf{T}^{t+s} u_0 + (t+s)\alpha(0)$ converges both to u_∞ and to $\mathbf{T}^s u_\infty + s\alpha(0)$ when $t \longrightarrow \infty$. This implies, as desired, that $\mathbf{T}^s u_\infty + s\alpha(0) = u_\infty$ for each $s \geqslant 0$.

We know that $u_\infty \geqslant u_0$, with equality on $\mathcal{A}(0)$. Conversely, let us consider a point q such that $u_\infty(q) = u_0(q)$. The point q is minimizing the difference $u_\infty - u_0$. Since u_∞ is semi-concave and u_0 is C^1, the function u_∞ must be differentiable at q with $du_\infty(q) = du_0(q)$. Since u_∞ solves the Hamilton-Jacobi equation at its points of differentiability, we conclude that $H(q, du_0(q)) = H(q, du_\infty(q)) = \alpha(0)$, hence $q \in \mathcal{A}(0)$. We have proved the claim.

Let us now establish that $\mathcal{I}(u_\infty) = \mathcal{A}(0)$, which implies the proposition. By Lemma 36, we have $\check{\mathbf{T}}^t u_\infty - t\alpha = u_\infty$ on $\mathcal{I}(u_\infty)$ for each $t \geqslant 0$. Setting $\epsilon(t) = \sup(u_\infty - \mathbf{T}^t u_0 - t\alpha(0))$, we have

$$u_\infty \geqslant u_0 \geqslant \check{\mathbf{T}}^t \circ \mathbf{T}^t u_0 \geqslant \check{\mathbf{T}}^t (u_\infty - \epsilon(t) - t\alpha(0)) \geqslant \check{\mathbf{T}}^t u_\infty - \epsilon(t) - t\alpha(0) = u_\infty - \epsilon(t)$$

on $\mathcal{I}(u_\infty)$. Since this holds for all $t \geqslant 0$, and since $\lim_{t \longrightarrow \infty} \epsilon(t) = 0$, we conclude that $u_0 = u_\infty$ on $\mathcal{I}(u_\infty)$. On the other hand, we have seen that $u_0 < u_\infty$ outside of $\mathcal{A}(0)$, hence $\mathcal{I}(u_\infty) \subset \mathcal{A}(0)$. □

We finish with an easy remark which is specific to the periodic case:

Proposition 47. *All periodic weak KAM solutions have level $\alpha(0)$.*

Proof. Let u_0 be a critical periodic sub-solution, and let u be a periodic weak KAM solution at level a. Since u is a periodic sub-solution at level a, the definition of $\alpha(0)$ implies that $a \geqslant \alpha(0)$. On the other hand, there exists a constant C such that $u - C \leqslant u_0 \leqslant u + C$, which implies

$$u = \mathbf{T}^t u + ta \geqslant \mathbf{T}^t u_0 - C + ta \geqslant u_0 + t(a - \alpha(0)) - C \geqslant u + t(a - \alpha(0)) - 2C.$$

We obtain that $t(a - \alpha(0)) \leqslant 2C$ for each $t \geqslant 0$, hence $a - \alpha(0) \leqslant 0$. □

10 The Lagrangian.

In most expositions of weak KAM theory, the Lagrangian plays an important role. In the present section, we relate it to our main objects in order to facilitate the connection with the core of the literature, where what we state here as properties is usually taken as definitions. We define the Lagrangian as

$$\begin{aligned} L : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d &\longrightarrow \mathbb{R} \\ (t, q, v) &\longmapsto \sup_{p \in (\mathbb{R}^d)^*} \big(p \cdot v - H(t, q, p)\big). \end{aligned}$$

By standard results on convex Analysis, see[19] for example, we then have

$$H(t,q,p)=\sup_{v\in\mathbb{R}^d}\big(p\cdot v-L(t,q,v)\big).$$

We obviously have the Legendre inequality

$$H(t,q,p)+L(t,q,v)\geqslant p\cdot v$$

for all t,q,p,v. This inequality is an equality if and only if

$$p=\partial_v L(t,q,v)\text{ or equivalently }v=\partial_p H(t,q,p).$$

Let $q(t):]t_0,t_1[\longrightarrow M$ be a curve, The **action** of q is the number

$$\int_{t_0}^{t_1}L(t,q(t),\dot q(t))dt.$$

We can also call it Lagrangian action if we want to distinguish from the previously defined Hamiltonian action. The Lagrangian and Hamiltonian actions are related as follows:

The Hamiltonian action of a curve $(q(t),p(t))$ is smaller than the Lagrangian action of its projection $q(t)$, with equality if and only if $p(t)\equiv\partial_v L(t,q(t),\dot q(t))$. In particular, the Hamiltonian action of an orbit is equal to the Lagrangian action of its projection.

Lemma 48. *let q_0 and q_1 be two points of $\mathbb{R}^d$, and t_0,t_1 be two times, with $0<t_1-t_0<\sigma$. If $(q(s),p(s))$ is the orbit satisfying $q(t_0)=q_0,q(t_1)=q_1$, we have*

$$S_{t_0}^{t_1}(q_0,q_1)=\int_{t_0}^{t_1}L(s,q(s),\dot q(s))ds=\min_{\theta(s)}\int_{t_0}^{t_1}L(s,\theta(s),\dot\theta(s))ds,$$

where the minimum is taken on the set of Lipschitz curves $\theta:[t_0,t_1]\longrightarrow\mathbb{R}^d$ which satisfy $\theta(t_0)=q_0$ and $\theta(t_1)=q_1$.

Proof. Since $S_{t_0}^{t_1}(q_0,q_1)$ is the Hamiltonian action of the unique orbit $(q(t),p(t))$, it is also the Lagrangian action of the curve $q(t)$:

$$S_{t_0}^{t_1}(q_0,q_1)=\int_{t_0}^{t_1}L(s,q(s),\dot q(s))ds.$$

The function $u(t,q):=S_{t_0}^{t}(q_0,q)$ solves (HJ) on $]t_0,t_1[\times\mathbb{R}^d$. Let us now consider any Lipschitz curve $\theta(s):[t_0,t_1]\longrightarrow\mathbb{R}^d$ satisfying $\theta(t_0)=q_0$ and $\theta(t_1)=q_1$, and write

$$\begin{aligned}\int_{t_0}^{t_1}L(s,\theta(s),\dot\theta(s))ds&\geqslant\int_{t_0}^{t_1}\partial_q u(s,\theta(s))\cdot\dot\theta(s)-H(s,\theta(s),\partial_q u(s,\theta(s))ds\\&=\int_{t_0}^{t_1}\partial_q u(s,\theta(s))\cdot\dot\theta(s)-\partial_t u(s,\theta(s))ds\\&=u(t_1,q_1)-u(t_0,q_0)=S_{t_0}^{t_1}(q_0,q_1).\end{aligned}$$

□

The following proposition is usually taken as the definition of A:

Proposition 49. *Given two points q_0 and q_1 and two times $t_0<t_1$, we have*

$$A_{t_0}^{t_1}(q_0,q_1)=\min_{\theta(s)}\int_{t_0}^{t_1}L(s,\theta(s),\dot\theta(s))ds,$$

where the minimum is taken on the set of Lipschitz curves $\theta:[t_0,t_1]\longrightarrow\mathbb{R}^d$ which satisfy $\theta(t_0)=q_0$ and $\theta(t_1)=q_1$.

It is part of the statement that the minimum is achieved. This is usually called the Theorem of Tonelli. The statement can be extended to absolutely continuous curves instead of Lipschitz curves, but this setting is not useful for our discussion.
PROOF. For n large enough, we have $(t_1 - t_0)/n < \sigma$, hence, setting $\tau_i = t_0 + i(t_1 - t_0)/n$,

$$\begin{aligned} A_{t_0}^{t_1}(q_0, q_1) &= \min_{(\theta_1,\dots,\theta_{n-1})} \big(S_{t_0}^{\tau_1}(q_0, \theta_1) + S_{\tau_1}^{\tau_2}(\theta_1, \theta_2) + \dots + S_{\tau_{n-1}}^{t_1}(\theta_{n-1}, q_1)\big) \\ &= \min_{(\theta_1,\dots,\theta_{n-1})} \big(\min_{\theta(s)} \int_{t_0}^{\tau_1} L(s, \theta(s), \dot\theta(s))ds + \ldots + \min_{\theta(s)} \int_{\tau_{n-1}}^{t_1} L(s, \theta(s), \dot\theta(s))ds\big) \\ &= \min_{\theta(s)} \int_{t_0}^{t_1} L(s, \theta(s), \dot\theta(s))ds. \end{aligned}$$

□

A Some technical results.

Proposition 50. *A Lipschitz map $F : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ which satisfies $Lip(F - Id) < 1$ is a bi-Lipschitz homeomorphism of $\mathbb{R}^d$. Its inverse is Lipschitz and $Lip(F^{-1}) \leqslant (1-k)^{-1}$. If F is C^1, then so is F^{-1}.*

PROOF. The equation $F(q) = \theta$ can be rewritten

$$\theta - (F(q) - q) = q$$

The map on the left being contracting, we conclude that F is invertible. We now write

$$|x_1 - x_0| - |F(x_1) - F(x_0)| \leqslant |(F(x_1) - x_1) - (F(x_0) - x_0)| \leqslant k|x_1 - x_0|$$

and deduce that $|F(x_1) - F(x_2)| \geqslant (1-k)|x_1 - x_0|$. □

Proposition 51. *Let $F : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be a C^1, c-monotone map on $\mathbb{R}^d$, with $c > 0$. Then F is a diffeomorphism from $\mathbb{R}^d$ onto itself.*

PROOF. Let us consider a point $\theta \in \mathbb{R}^d$, and the line $\theta(s) = F(0) + s(\theta - F(0))$. Since F is a local diffeomorphism around 0, the points $\theta(s)$ for small s have a unique preimage $p(s)$. Let S be the infimum of the positive real numbers s such that the equation $F(p) = \theta(s)$ does not have a solution in $\mathbb{R}^d$. The curve $p(s)$ is well-defined, C^1, and Lipschitz on $[0, S[$, hence, if S is finite, it extends at S with with $F(p(S)) = \theta(S)$. Since F is a local diffeomorphism at $p(S)$, the points near $\theta(S)$ have preimages, which contradicts the definition of S. Hence S can't be finite. □

Lemma 52. *Let A be a $d \times d$ matrix, such that $A \geqslant aId$ in the sense of quadratic forms, and $\|A\| \leqslant b$. Then $A^{-1} \geqslant (a/b^2)I$ in the sense of quadratic forms.*

PROOF. We have

$$(A^{-1}v, v) = (AA^{-1}v, A^{-1}v) \geqslant a|A^{-1}v|^2 \geqslant a(|v|/b)^2.$$

□

The following important result appears in Fathi's book on Weak KAM theory (the proof is also his):

Proposition 53. *Let $u : \mathbb{R}^d \longrightarrow \mathbb{R}$ be a function and K be a positive number. Let $\mathcal{I} \in \mathbb{R}^d$ be the set of points where u has both a K-super-differential and a K-sub-differential. Then, the function u is differentiable at each point of $\mathcal{I}$ and the function $q \longmapsto du(q)$ is $6K$-Lipschitz on $\mathcal{I}$.*

PROOF. For each $q \in \mathcal{I}$, there exists a unique $l(q) \in \mathbb{R}^{d*}$ such that

$$|u(q+\theta) - u(q) - l(q) \cdot \theta| \leqslant K\|\theta\|^2.$$

We conclude that $l(q)$ is the differential of u at q, and we have to prove that the map $q \longmapsto l(q)$ is Lipschitz on $\mathcal{I}$. We have, for each q, θ and y in H:

$$l(q) \cdot (y+\theta) - K\|y+\theta\|^2 \leqslant u(q+y+\theta) - u(q) \leqslant l(q) \cdot (y+\theta) + K\|y+\theta\|^2$$

$$l(q+y) \cdot (-y) - K\|y\|^2 \leqslant u(q) - u(q+y) \leqslant l(q+y) \cdot (-y) + K\|y\|^2$$

$$l(q+y) \cdot (-\theta) - K\|\theta\|^2 \leqslant u(q+y) - u(q+y+\theta) \leqslant l(q+y) \cdot (-\theta) + K\|\theta\|^2.$$

Taking the sum, we obtain

$$\big|(l(q+y) - l(q)) \cdot (y+\theta)\big| \leqslant K\|y+\theta\|^2 + K\|y\|^2 + K\|z\|^2.$$

By a change of variables, we get

$$\big|(l(q+y) - l(q)) \cdot \theta\big| \leqslant K\|\theta\|^2 + K\|y\|^2 + K\|\theta - y\|^2.$$

Taking $\|\theta\| = \|y\|$, we obtain

$$\big|(l(q+y) - l_(q)) \cdot (\theta)\big| \leqslant 6K\|\theta\|\|y\|$$

for each θ such that $\|\theta\| = \|y\|$, we conclude that

$$\|l(q+y) - l(q)\| \leqslant 6K\|y\|.$$

□

Lemma 54. *Let u be a finite valued function which is the infimum of a family $\mathcal{F}$ of equi-semi-concave functions: $u = \inf_{f \in \mathcal{F}} f$. Then the function u is semi-concave.*

It is important in the statement to assume that u is really finite valued at each point.
PROOF. Let us assume that the functions in $\mathcal{F}$ are k-semi-concave. Given a point $q_0 \in \mathbb{R}^d$ let $f_n(q) = a_n + p_n \cdot q + k/2\|q\|^2$ be a sequence of functions of $\mathcal{F}$ such that $f_n(q_0) \longrightarrow u(q_0)$. We have $f_n(q) \leqslant f_n(q_0) + p_n \cdot (q - q_0) + k/2\|q - q_0\|^2$ for some sequence $p_n \in \mathbb{R}^*$. If the sequence p_n is bounded, then we can take the limit along a subsequence and get the inequality

$$u(q) \leqslant u(q_0) + p \cdot (q - q_0) + k/2\|q - q_0\|^2.$$

If this holds for each q_0, we conclude that u is k-semi-concave. Let us now prove that p_n is bounded. If this was not true, there would exist a point q such that $p_n \cdot (q - q_0)$ is not bounded from below. This would imply that

$$u(q) = \inf_{f \in \mathcal{F}} f(q) \leqslant \inf_n f_n(q) \leqslant \inf_n \big(f_n(q_0) + p_n \cdot (q - q_0) + k/2\|q - q_0\|^2\big) = -\infty,$$

which would contradict the finiteness of u at q. □

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