Connecting orbits of time dependent Lagrangian systems

Patrick Bernard

Spring 2002

Résumé: On donne une généralisation à la dimension supérieure des résultats obtenus par Birkhoff et Mather sur l'existence d'orbites errant dans les zones d'instabilité des applications de l'anneau déviant la verticale. Notre généralisation s'inspire fortement de celle proposée par Mather dans [9]. Elle présente cependant l'avantage de contenir effectivement l'essentiel des resultats de Birkhoff et Mather sur les difféomorphismes de l'anneau.

Abstract: We generalize to higher dimension results of Birkhoff and Mather on the existence of orbits wandering in regions of instability of twist maps. This generalization is strongly inspired by the one proposed by Mather in [9]. However, its advantage is that it contains most of the results of Birkhoff and Mather on twist maps.

A very natural class of problems in dynamical systems is the existence of orbits connecting prescribed regions of phase space. There are several important open questions in this line, like the one posed by Arnold : Is a generic Hamiltonian system transitive on its energy shells?

Birkhoff's theory of regions of instability of twists maps, recently extended by Mather using variational methods and by Le Calvez, provide very relevant results in that direction. In short, these works establish the existence, for a certain class of mappings of the annulus, of orbits visiting in turn prescribed regions of the annulus under the hypothesis that these regions are not separated by a rotational invariant circle.

John Mather has opened the way to a generalization in higher dimension of this celebrated theory by proposing what seems to be the appropriate setting *i.e.* time dependent positive definite Lagrangian systems. In this setting, he has obtained the existence of families of invariant sets generalizing the well known Aubry-Mather invariant sets of twist maps. Then he stated in 1993 a result on the existence of orbits visiting in turn neighborhoods of an arbitrary sequence of these invariant sets. However, the work of Mather is not a complete achievement since there is no relevant example in high dimension to which it can be applied, and since it is not completely optimal even in the case of Twist maps. There are examples where two Aubry-Mather sets of a twist map are not separated by a rotational invariant circle, hence can be connected by an orbit, but where this can't be seen by the result of Mather.

In the present paper, we state a new result on the existence of connecting orbits in higher dimension, with a full self-contained proof. This result is very close to the one of Mather, and the main ideas of the proof are the ones he introduced. Our result has the advantage that it is optimal when applied to the twist map case, but it does not contain the result of Mather, which we were not able to prove. [1]

It is still an open question whether these results may be applied to interesting example in higher dimension [2]. On one hand, it is encouraging that this result is optimal when restricted

[1] As it is written in [11], the proof contains a gap which I am not able to fill.

[2] Just before I finished this text, John Mather has announced that he had been able to prove an important result on Arnold diffusion, so the full achievement of the method may soon be reached.

to the case of twist maps, but on the other hand we will prove that the result is useless in the autonomous case. Additional work will be required both to weaken the abstract hypotheses needed to prove the existence of connections, and to understand when these hypotheses are satisfied.

I am grateful to Lucien Guillou who patiently encouraged me while I was writing this paper, and to Albert Fathi for his careful reading of the manuscript and for valuable comments which helped me to shorten some of the proofs. It is also a pleasure to thank Daniel Massart for all the discussions we had, which were very helpful to me.

0.1 Let M be a smooth, compact, connected manifold, $TM \xrightarrow{\pi} M$ its tangent bundle. In the following, we note $S = \mathbb{R}/\mathbb{Z}$. We choose once and for all a Riemann metric g on M. It is classical that there is a canonical way to associate to it a metric on TM. Let us fix a C^2 Lagrangian function $L : TM \times \mathbb{R} \longrightarrow \mathbb{R}$. Given any compact interval I, we have an action functional defined on $C^1(I, M)$ by

$$A(\gamma) = \int_I L(d\gamma(t), t)dt.$$

Here and in the following, we note $d\gamma(t)$ for the curve $d\gamma_t(1) : I \longrightarrow TM$. The extremals of L on I are the critical points of A with fixed endpoints. We want to study the Lagrangian system associated with L, that is the extremal curves of L. We suppose that L satisfies the following conditions introduced by Mather [10]:

Periodicity The Lagrangian L is 1-periodic in time *i.e.* $L(z, t) = L(z, t+1)$ for all $z \in TM$ and all $t \in \mathbb{R}$.

Positive Definiteness For each $x \in M$ and each $t \in \mathbb{R}$, the restriction of L to $T_xM \times t$ is strictly convex with non degenerate Hessian.

Superlinear Growth For each $t \in \mathbb{R}$,

$$L(z, t)/\|z\| \longrightarrow \infty \text{ as } \|z\| \longrightarrow \infty.$$

Under these hypotheses, there exists a continuous vector field E_L on $TM \times S$, the Euler-Lagrange vector-field, which has the property that a C^1 curve γ is an extremal of L if and only if the curve $(d\gamma(t), t \bmod 1)$ is an integral curve of E_L. Although this vector field is only continuous, it has a local flow ϕ_t on $TM \times S$ called the Euler-Lagrange flow. We assume :

Completeness The local flow ϕ_t is complete *i.e.* any trajectory $X : I \longrightarrow TM \times S$ of the flow can be extended to a trajectory $\bar{X} : \mathbb{R} \longrightarrow TM \times S$.

0.2 Let $I = [a, b]$ be a compact interval of time. A curve $\gamma \in C^1(I, M)$ is called a minimizer or a minimal curve if it minimizes the action among all curves $\xi \in C^1(I, M)$ which satisfy $\gamma(a) = \xi(a)$ and $\gamma(b) = \xi(b)$. If J is a non compact interval, the curve $\gamma \in C^1(J, M)$ is called a minimizer if $\gamma|_I$ is minimal for any compact interval $I \subset J$. An orbit $X(t)$ of ϕ_t is called minimizing if the curve $\pi \circ X$ is minimizing, a point $(z, s) \in TM \times S$ is minimizing if its orbit $\phi_t((z, s))$ is minimizing. Let us call $\tilde{\mathcal{G}}$ the set of minimizing points of $TM \times S$. We shall see that $\tilde{\mathcal{G}}$ is a nonempty compact subset of $TM \times S$, invariant for the Euler-Lagrange flow.

0.3 Let η be a 1-form of $M \times S$. We associate to this form a function on $TM \times \mathbb{R}$, still denoted η, and defined by

$$\eta(z, t) = \langle \eta, (z, t \bmod 1, 1) \rangle_{(\pi(z), t \bmod 1)},$$

where $\langle .,. \rangle_{(x,s)}$ is the usual coupling between forms and vectors of $T_{(x,s)}(M \times S)$. If the form η is closed, then the Euler-Lagrange vector field of $L-\eta$ is the Euler-Lagrange vector field of L, and $L-\eta$ satisfies all the hypotheses of 0.1 if L does. Let us define the mapping

$$\begin{aligned} i_s : M &\longrightarrow M \times S \\ x &\longmapsto (x,s). \end{aligned}$$

For any 1-form η on $M \times S$, let us define the form η_s on M by

$$\eta_s = i_s^* \eta.$$

If η is a closed 1-form, we define its class $[\eta] = [\eta_s] \in H^1(M,\mathbb{R})$, which does not depend on s. Let η and μ be two closed forms such that $[\eta] = [\mu]$. It is clear that the minimizing curves of $L-\eta$ and $L-\mu$ are the same. Let us call $\tilde{\mathcal{G}}(c)$ the set of minimizing points associated to the Lagrangian $L-\eta$, where η is any closed one-form such that $[\eta] = c$. Let us also define, for each $s \in S$, the set $\tilde{\mathcal{G}}_s(c) \subset TM$ of points $z \in TM$ such that $(z,s) \in \tilde{\mathcal{G}}(c)$. We will also call $\mathcal{G}(c)$ and $\mathcal{G}_s(c)$ the projections of $\tilde{\mathcal{G}}(c)$ and $\tilde{\mathcal{G}}_s(c)$ on $M \times S$ and M.

0.4 Let $\tilde{\omega}(c)$ be the union of ω-limit points of minimizing trajectories $X : [0,\infty) \longrightarrow TM \times S$. Let $\tilde{\alpha}(c)$ be the union of α-limit points of minimizing trajectories $X : (-\infty, 0] \longrightarrow TM \times S$. In both definitions above, minimization is considered with Lagrangians $L-\eta$, where η is any closed one-form on $M \times \mathbb{R}$ satisfying $[\eta] = c$. We will consider the invariant set

$$\tilde{\mathcal{L}}(c) = \tilde{\omega}(c) \cup \tilde{\alpha}(c).$$

We will see that $\tilde{\mathcal{L}}(c) \subset \tilde{\mathcal{G}}(c)$. In addition, $\tilde{\mathcal{L}}$ is contained in the classical Aubry set $\tilde{\mathcal{A}}(c)$, hence satisfies the Lipschitz graph property, see section 3 for more details.

0.5 We associate to any subset A of M the subspace

$$V(A) = \bigcap \big\{ i_{U*} H_1(U,\mathbb{R}) : U \text{ is an open neighborhood of } A \big\} \subset H_1(M,\mathbb{R}),$$

where $i_{U*} : H_1(U,\mathbb{R}) \longrightarrow H_1(M,\mathbb{R})$ is the mapping induced by the inclusion. There exists an open neighborhood U of A such that $V(A) = i_{U*} H_1(U)$. We can now define, for each $c \in H^1(M,\mathbb{R})$ the following subspace of $H^1(M,\mathbb{R})$:

$$R(c) = \sum_{t \in S} \big(V\big(\mathcal{G}_t(c)\big)\big)^{\perp}.$$

Our improvement compared with [11] is that $R(c)$ may be bigger than $V\big(\mathcal{G}_0(c)\big)^{\perp}$, which was considered there. To be more precise, the minimizing curves used in Mather's work satisfy stronger conditions than belonging to $\tilde{\mathcal{G}}$, and their union is a smaller set called the Mañe set $\tilde{\mathcal{N}}$, see section 3 for the definition. As a consequence, our result does not contain the result stated in [11]. However, the proof is only sketched in Mather's paper, and it is not clear to me how it should be completed.

0.6 We say that a continuous curve $c : \mathbb{R} \longrightarrow H^1(M, \mathbb{R})$ is admissible if for each $s_0 \in \mathbb{R}$, there exists $\delta > 0$ such that $c(s) - c(s_0) \in R(c(s_0))$ for all $s \in [s_0 - \delta, s_0 + \delta]$. We say $c_0, c_1 \in H^1(M, \mathbb{R})$ are C-equivalent if there exists an admissible continuous curve $c : \mathbb{R} \longrightarrow H^1(M, \mathbb{R})$ such that $c(s) = c_0$ when $s \leqslant 0$ and $c(1) = c_1$ when $s \geqslant 1$. This is precisely the definition of Mather [11] except that our $R(c)$ is different from Mather's one. We are now in a position to state our main result :

THEOREM : Let us fix a C-equivalence class C in $H^1(M, \mathbb{R})$. Let $(c_i)_{i \in \mathbb{Z}}$ be a bi-infinite sequence of elements of C and $(\epsilon_i)_{i \in \mathbb{Z}}$ be a bi-infinite sequence of positive numbers. There exist a trajectory $X(t)$ of the Euler-Lagrange flow and a bi-infinite increasing sequence t_i of times such that

$$d\big(X(t_i), \tilde{\mathcal{L}}(c_i)\big) \leqslant \epsilon_i.$$

If in addition there exists a class c_∞ such that $c_i = c_\infty$ for large i, or a class $c_{-\infty}$ such that $c_i = c_{-\infty}$ for small i, then the trajectory X is ω-asymptotic to $\tilde{\mathcal{L}}(c_\infty)$ or α-asymptotic to $\tilde{\mathcal{L}}(c_{-\infty})$.

We shall state and prove in section 2 slightly refined theorems, which imply the following corollaries :

COROLLARY 1 : Let c_0 and c_1 be two C-equivalent classes. There exists a trajectory of the Euler Lagrange flow the α-limit of which lies in $\tilde{\mathcal{L}}(c_0)$ and the ω-limit of which lies in $\tilde{\mathcal{L}}(c_1)$.

COROLLARY 2 : If there exist two C-equivalent classes c_0 and c_1 such that $\tilde{\mathcal{L}}(c_0)$ and $\tilde{\mathcal{L}}(c_1)$ are disjoint, then the time one map of the Euler-Lagrange flow has positive topological entropy. There is another statemement using the function α of Mather, see section 4. If there exist two C-equivalent classes c_0 and c_1 such that $\alpha_{[c_0,c_1]}$ is not affine, then the time one map of the Euler-Lagrange flow has positive topological entropy.

0.7 Let us insist on the relations between our theorem and the theorem of Mather in [11]. The only difference between these two results lies in the definition of C-equivalence, and more precisely in the definition of $R(c)$. We replaced

$$V\big(\mathcal{N}_0(c)\big)^\perp$$

as the subspace of allowed directions in [11], §12, by

$$R(c) = \sum_{t \in S} \big(V\big(\mathcal{G}_t(c)\big)\big)^\perp,$$

where $\mathcal{N}$ is the set of semi-static curves, see section 3. The bigger the subspace of allowed directions is, the stronger the result. Our result do not contain the result of Mather because we had to replace the set $\mathcal{N}$ of semi-static orbits (see section 3) by the larger set $\mathcal{G}$ of minimizing orbits in order to fill the proof. On the other hand our subspace is bigger in certain cases for example in the twist map case. An important consequence is that our result is optimal in the case $M = S$ while the result of Mather was not. In this case, two cohomology classes c and c' are C-equivalent in our sense if and only if the associated sets $\tilde{\mathcal{G}}(c)$ belong to the same region of instability, that is if they are not separated by an invariant graph. See section 6 for the details. Our result is equivalent to the result of Mather in the autonomous case, however, as we shall explain in 4.11 it is of no interest in this case.

0.8 In order to apply the theorem, it is necessary to be able to describe the C-equivalence classes. This is not an easy task even in the case $M = S$. It requires a good understanding of the set $\mathcal{G}(c)$ of minimizing curves. A lot of literature is devoted to the study of globally minimizing orbits. We give a review in Section 3. We focus in Section 4 on the dependence on the cohomology, and introduced the function α of Mather. All these results provide a good description of a smaller set, the Mañe set. In section 5, we see that the difference between the Mañe set and the set $\mathcal{G}$ is linked with the asymptotic behavior of the so called Lax-Oleinik semi-group. We exploit this remark to obtain some results on the shape of the set $\mathcal{G}$. In section 6, we apply these results to the case of twist maps, and obtain that our theorem is optimal in this case. Unfortunately, there is no hope to apply our result in the autonomous case, as is explained in 4.11.

0.9 For the convenience of the reader, we list the sets of minimizing orbits that will be considered in the sequel:
$\tilde{\mathcal{G}}$ is the set of minimizing orbits, defined in 0.2.
$\tilde{\mathcal{L}}$ is defined in 0.4.
$\tilde{\mathcal{M}}$ is the Mather set, defined in 2.4.
$\tilde{\mathcal{N}}$ is the Mañe set, defined in 3.4.
$\tilde{\mathcal{A}}$ is the Aubry set, defined in 3.4.
$\tilde{\mathcal{S}}$ are the static classes, defined in 3.11. They partition $\tilde{\mathcal{A}}$.
We will prove in 3.9 the inclusion

$$\tilde{\mathcal{M}} \subset \tilde{\mathcal{L}} \subset \tilde{\mathcal{A}} \subset \tilde{\mathcal{N}} \subset \tilde{\mathcal{G}}.$$

The sets $\mathcal{M}, \mathcal{L}, \ldots$ are the projections onto M of the corresponding invariant sets in TM.

1 Minimization

It is useful to work in a slightly more general setting. In this section, we will consider a Lagrangian $L : TM \times \mathbb{R} \longrightarrow \mathbb{R}$, not necessarily time-periodic, satisfying positive definiteness and superlinearity, but not necessarily completeness.

1.1 If the positive definiteness and superlinear growth are satisfied, there is a continuous local flow ψ_t on TM such that the curve γ is a C^1 extremal of L if and only if the curve $X(t) = d\gamma(t)$ is a trajectory of ψ_t. This local flow, called the Euler-Lagrange flow, is not assumed to be complete in the present section.

1.2 The variational study of L relies on some standard results proved in [10].
Lemma : Given a real number K and a compact interval $[a, b]$, the set of all absolutely continuous curves $\gamma : [a, b] \longrightarrow M$ for which $A(\gamma) \leqslant K$ is compact for the topology of uniform convergence.
Tonelli's theorem : Let $[a, b]$ be a compact interval, and let us fix two points x_a and x_b in M. The action takes a finite minimum over the set of absolutely continuous curves $\gamma : [a, b] \longrightarrow M$ satisfying $\gamma(a) = x_a$ and $\gamma(b) = x_b$. If in addition the Euler-Lagrange local flow is complete, then any curve γ realizing this minimum is C^1 and $d\gamma(t)$ is a trajectory of the Euler-Lagrange flow.

Let $I = [a, b]$ be a compact interval of time. A curve $\gamma \in C^{ac}(I, M)$ is called a minimizer or a minimal curve if it is minimizing the action among all curves $\xi \in C^{ac}(I, M)$ which satisfy $\gamma(a) = \xi(a)$ and $\gamma(b) = \xi(b)$. Let J be any interval of $\mathbb{R}$. A curve $\gamma \in C^{ac}(J, M)$ is called a minimizer if $\gamma|_I$ is minimal for any compact interval $I \subset J$. Let us notice that if the completeness is not assumed, the absolutely continuous minimizers need not be C^1, an example of this is given in [1].

1.3 Proposition There exist absolutely continuous minimizers $\gamma \in C^{ac}(\mathbb{R}, M)$. If the flow is complete, these minimizers are C^1 extremals and the curves $d\gamma(t)$ are trajectories of the Euler-Lagrange flow.

This proposition follows from the following lemmas, which are stated in higher generality for later use.

1.4 Lemma Let us fix a positive definite superlinear Lagrangian L, a compact interval of time $[a, b]$ and a positive constant C. There exists a constant K with the following property: If $\tilde{L}$ is a positive definite superlinear Lagrangian such that

$$|\tilde{L}(z, t) - L(z, t)| \leqslant C(1 + \|z\|)$$

for all $z \in TM$ and all $t \in [a, b]$, and if $\gamma : [a, b] \longrightarrow M$ is a minimizer of $\tilde{L}$, then

$$\int_a^b \|d\gamma(t)\|\, dt \leqslant K \quad \text{and} \quad \int_a^b L(d\gamma(t), t)\, dt \leqslant K.$$

Proof There exists a constant B depending on L, C and $[a, b]$ such that all minimizer γ of $\tilde{L}$ satisfies $\tilde{A}(\gamma) \leqslant B$, where $\tilde{A}$ is the action associated to $\tilde{L}$. Since L is superlinear, there exists a constant D such that

$$L(z, t) \geqslant (C + 1)\|z\| - D$$

for all $z \in TM$ and $t \in [a, b]$. It follows that $\tilde{L} \geqslant \|z\| - C - D$, and we get the first estimate

$$\int \|d\gamma\| \leqslant B + (b - a)(C + D).$$

We get the second estimate thanks to the inequality

$$A(\gamma) \leqslant \tilde{A}(\gamma) + C \int \|d\gamma\| + C(b - a).$$

This ends the proof of the lemma. □

1.5 Lemma Let L be a positive definite superlinear Lagrangian, and let $[a, b]$ be a compact interval of time. Let L_n be a sequence of positive definite superlinear Lagrangians, such that $|L_n(z, t) - L(z, t)| \leqslant \epsilon_n(1 + \|z\|)$ for all $z \in TM$ and all $t \in [a, b]$, where ϵ_n is a sequence converging to 0. If $\gamma_n : [a, b] \longrightarrow M$ is a sequence of minimizers of L_n converging uniformly to $\gamma : [a, b] \longrightarrow M$, then

$$A(\gamma) = \lim \int_a^b L_n(d\gamma_n(t), t)\, dt$$

and γ is a minimizer of L on $]a, b[$.

PROOF In view of Lemma 1.4, the sequence $A(\gamma_n)$ is bounded and $A(\gamma_n) - A_n(\gamma_n) \longrightarrow 0$. By Lemma 1.2, the curve γ is absolutely continuous, and satisfies

$$A(\gamma) \leqslant \liminf A(\gamma_n) = \liminf A_n(\gamma_n).$$

In order to prove the lemma, it is thus sufficient to prove that if $x : [a, b] \longrightarrow M$ is an absolutely continuous curve such that $\gamma(t) = x(t)$ in a neighborhood of a and b, then $A(x) \geqslant \limsup A_n(\gamma_n)$. Let $x(t)$ be such a curve. Recall that x is differentiable almost everywhere. Let us consider an interval $[a', b'] \subset [a, b]$ such that x is differentiable at a' and b' and such that $\gamma(a') = x(a')$ and $\gamma(b') = x(b')$. There exist positive constants δ_0 and K such that, for all $\delta \in]0, \delta_0[$,

$$d\big(x(a'), x(a' + \delta)\big) \leqslant K\delta \quad \text{and} \quad d\big(x(b' - \delta), x(b')\big) \leqslant K\delta.$$

As a consequence, there exists an integer $N(\delta)$ such that

$$d\big(\gamma_n(a'), x(a' + \delta)\big) \leqslant 2K\delta \quad \text{and} \quad d\big(x(b' - \delta), \gamma_n(b')\big) \leqslant 2K\delta$$

for all $n \geqslant N(\delta)$. Now let us consider the geodesic $\xi : [a', a' + \delta] \longrightarrow M$ connecting $\gamma_n(a')$ and $x(a' + \delta)$, and the geodesic $\zeta : [b' - \delta, b'] \longrightarrow M$ connecting $x(b' - \delta)$ and $\gamma_n(b')$. If $\delta \leqslant \delta_0$ and $n \geqslant N(\delta)$, they satisfy $\|d\xi\| \leqslant 2K$ and $\|d\zeta\| \leqslant 2K$, hence there exists a constant B such that $A_n(\xi) \leqslant B\delta$ and $A_n(\zeta) \leqslant B\delta$. Since γ_n is minimizing on $[a', b']$, it follows that

$$A_n\big(x|_{[a'+\delta, b'-\delta]})\big) + 2B\delta \geqslant A_n\big(\gamma_n|_{[a',b']}\big).$$

Taking the limit, we obtain

$$A\big(x|_{[a'+\delta, b'-\delta]})\big) + 2B\delta \geqslant \limsup A\big(\gamma_n|_{[a',b']}\big).$$

since this holds for all $\delta \leqslant \delta_0$, we get that $A\big(\gamma|_{[a',b']}\big) \geqslant \limsup A\big(x_n|_{[a',b']}\big)$. At the limit $a' \longrightarrow a$, $b' \longrightarrow b$, we obtain that $A(x) \geqslant \limsup A(\gamma_n)$. □

1.6 LEMMA Let $I_n = [a_n, b_n]$ be a nondecreasing sequence of compact intervals and let $J = \cup_n I_n$. Let L_n be a sequence of positive definite superlinear Lagrangians, such that

$$\big|L_n(z, t) - L(z, t)\big| \leqslant \epsilon_n(1 + \|z\|)$$

for all $z \in TM$ and all $t \in I_n$, where $\epsilon_n \longrightarrow 0$. If $\gamma_n : I_n \longrightarrow M$ is a sequence of minimizers of L_n, then there is an absolutely continuous curve $\gamma : J \longrightarrow M$ which is minimizing for L on the interior of J, and a subsequence of γ_n which converges uniformly on compact sets of J to γ.
PROOF In view of Lemma 1.4, the sequence

$$k \longmapsto A\big(\gamma_k|_{I_n}\big)$$

is bounded for each n. It follows from Lemma 1.2 that there is a subsequence of $k \longmapsto \gamma_k|_{I_n}$ converging uniformly. By diagonal extraction, we can build a subsequence of γ_n which converges uniformly on compact sets to an absolutely continuous limit $\gamma : J \longrightarrow M$. By Lemma 1.5, this limit is a minimizer of L on the interior of J. □

1.7 We will have in the following to consider one-forms on $M \times \mathbb{R}$ which are neither periodic nor closed. Let μ be a 1-form of $M \times \mathbb{R}$. We associate to this form a function on $TM \times \mathbb{R}$, still denoted μ, and defined by

$$\mu(z,t) = \langle \mu, (z,t,1)\rangle_{(\pi(z),t)}.$$

The new Lagrangian $L - \mu$ is positive definite and superlinear if L is. If μ is closed, then the Euler-Lagrange flows of L and $L - \mu$ are the same. Let us define the mapping

$$\begin{aligned} i_t : M &\longrightarrow M \times \mathbb{R} \\ x &\longmapsto (x,t), \end{aligned}$$

and the form $\mu_t = i_t^* \mu$. If μ is closed, we define its homology $[\mu] = [\mu_t] \in H^1(M, \mathbb{R})$. We will often identify a form η on $M \times S$ with its periodic pull-back on $M \times \mathbb{R}$.

2 Connecting orbits

In this section, we prove Theorem 0.6. In fact, we will prove more precise results, Theorems 2.10, which clearly imply Theorem 0.6 and the corollaries. We suppose from now on that L satisfies all the hypotheses of 0.1.

2.1 Proposition The set $\tilde{\mathcal{G}}(c)$ as defined in 0.3 is a non empty compact subset of $TM \times S$. It is invariant under the Euler-Lagrange flow. The mapping $c \longmapsto \tilde{\mathcal{G}}(c)$ is upper semi-continuous.
Proof That $\tilde{\mathcal{G}}(c)$ is not empty follows from Proposition 1.3. The other statements are consequences of the following lemma.

2.2 Lemma : Let us consider a sequence $c_n \longrightarrow c$ of cohomology classes, a sequence $T_n \longrightarrow \infty$ of times, and a sequence $\gamma_n : [-T_n, T_n] \longrightarrow M$ of curves minimizing $L - c_n$. Then there exists a curve $\gamma \in C^1(\mathbb{R}, M)$ minimizing $L - c$ and a subsequence γ_k of γ_n such that the sequence $d\gamma_k$ is converging uniformly on compact sets to $d\gamma$.

Proof This lemma is mainly a special case of Lemma 1.6. However, we have to prove that the convergence of γ_n to γ holds in C^1 topology. Since all the curves γ_n satisfy the Euler-Lagrange equation associated to L, the sequence γ_n is relatively compact in the C^1 topology if it is bounded in this topology. This in turns follows from :

2.3 Lemma : Let us consider a compact set $Q \subset H^1(M, \mathbb{R})$. There exists a constant $K > 0$ such that, if $b \geqslant a + 1$, all curve $\gamma : [a,b] \longrightarrow M$ minimizing $L + c$, with any $c \in Q$ satisfy $\|d\gamma(t)\| \leqslant K$ for each t.

Proof Let $\gamma : [a,b] \longrightarrow M$ be a curve minimizing $L + c$, with $c \in Q$. Let I be an interval of the form $[a+i, a+i+1]$ in $[a,b]$, with $i \in \mathbb{Z}$. By Lemma 1.4, there exists a constant K' independent of I such that $\int_I \|d\gamma\|\, dt \leqslant K'$. It follows that $(d\gamma, t \bmod 1)$ intersects the compact set $\{\|z\| \leqslant K'\} \subset TM \times S$ on each of the intervals I. Let us set

$$\tilde{\mathcal{K}} = \cup_{t \in [-2,2]} \phi_t\big(\{\|z\| \leqslant K'\}\big) \ \subset TM \times S,$$

we have $(d\gamma, t \bmod 1) \in \tilde{\mathcal{K}}$ for all $t \in [a,b]$. On the other hand, the set $\tilde{\mathcal{K}}$ is compact in view of the continuity of the Euler-Lagrange flow, hence contained in $\{\|z\| \leqslant K\}$ for some K. $\square$

2.4 The restriction of the Euler-Lagrange flow defines a continuous flow on the compact set $\tilde{\mathcal{G}}(c)$. By the Krylov Bogolioubov theorem, this flow has invariant probability measures. The Mather set $\tilde{\mathcal{M}}(c)$ is the closure of the union of all the supports of these invariant probability measures. We have the following lemma, which is a straightforward result of topological dynamics in compact spaces:

LEMMA : For all positive number ϵ, there exists a positive number T such that, if $X : [0,T] \longrightarrow \tilde{\mathcal{G}}(c)$ is a trajectory of the Euler-Lagrange flow, there exists a time $t \in [0,T]$ such that $d(X(t), \tilde{\mathcal{M}}(c)) \leqslant \epsilon$.

2.5 Let U be an open subset of $M \times S$. We also note U the open subset in $M \times \mathbb{R}$ of points (x,t) such that $(x, t \bmod 1) \in U$. The one form μ of $M \times \mathbb{R}$ is called a U-step form if there exist a closed form $\bar{\mu}$ on $M \times S$, also considered as a periodic one-form on $M \times \mathbb{R}$, such that the restriction of μ to $t \leqslant 0$ is 0, the restriction of μ to $t \geqslant 1$ is $\bar{\mu}$, and such that the restriction of μ to the set $U \cup \{t \leqslant 0\} \cup \{t \geqslant 1\}$ is closed.

2.6 Let $R(c) \subset H^1(M,\mathbb{R})$ be the vector subspace defined in 0.5. If (and only if) the class d belongs to $R(c)$, then there exist an open neighborhood U of $\mathcal{G}(c)$ and a U-step form μ such that $[\bar{\mu}] = d$. Since $H^1(M,\mathbb{R})$ is finite dimensional, there exists an open neighborhood U of $\mathcal{G}(c)$ such that, for each $d \in R(c)$, there exists an U-step form satisfying $[\bar{\mu}] = d$. Such a neighborhood U will be called an adapted neighborhood.

PROOF We shall only prove that if a class d belong to $R(c)$ then an appropriate step form exists. The other implication will not be used, and its proof is left to the reader. Let us fix a time $t \in [0,1]$ and a chomology class $d \in V\big(\mathcal{G}_t(c)\big)^{\perp}$. There exist an open neighborhood Ω of $\mathcal{G}_t(c)$ and a $\delta > 0$ such that $V(\Omega) = V(\mathcal{G}_t(c))$ and such that $\mathcal{G}_s(c) \subset \Omega$ for all $s \in [t-\delta, t+\delta]$. Let us take a closed 1-form $\bar{\mu}$ on M the support of which is disjoint from Ω and such that $[\bar{\mu}] = d$. We can consider this one-form on M as a form on $M \times S$. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a smooth function such that $f = 0$ on $(-\infty, t-\delta]$ and $f = 1$ on $[t+\delta, \infty)$. It is not hard to see that the form

$$\mu = f(t)\bar{\mu}$$

is an U-step form satisfying $[\bar{\mu}] = d$, where U is the open set $M \times [0, t-\delta[\cup \Omega \times S \cup M \times]t+\delta, 1]$.

2.7 PROPOSITION : Let us fix a cohomology class c in $H^1(M,\mathbb{R})$, and let U be an adapted neighborhood of $\mathcal{G}(c)$. There exists a positive numbers δ and an integer T_0 with the following property : If $d \in R(c)$ satisfies $|d| \leqslant \delta$, then there exists an U-step form μ satisfying $[\bar{\mu}] = d$ and such that all the minimizers $\gamma : [-T_0, T_0+1] \longrightarrow M$ of $L - \mu - \eta_0$ are C^1 extremals of L, for each closed one-form η_0 of $M \times \mathbb{R}$ satisfying $[\eta_0] = c$.

PROOF The minimizers of $L - \eta_0 - \mu$ do not depend on the choice of the form η_0 satisfying $[\eta_0] = c$. As a consequence, it is enough to prove the proposition for a fixed form η_0. Since $H^1(M,\mathbb{R})$ is finite dimensional, it is possible to take a finite dimensional subspace E of the space of all U-steps forms on $M \times S$ such that the restriction to E of the linear map $\mu \longmapsto [\bar{\mu}]$ is onto. We shall prove by contradiction that, if $\mu \in E$ is sufficiently small, there exists a minimizer $\gamma : [-T_0, T_0+1] \longrightarrow M$ of $L - \eta_0 - \mu$ such that $(\gamma(t), t) \in U$ for all $t \in [0,1]$. Else, there would exist a sequence μ_n of elements of E such that $\mu_n \longrightarrow 0$ (this is meaningful in the finite dimensional vector space E) and a sequence $\gamma_n : [-T_n, T_n+1] \longrightarrow M$, with $T_n \longrightarrow \infty$, of absolutely continuous curves minimizing $L - \eta_0 - \mu_n$, such that $(\gamma_n(t_n), t_n) \notin U$ for some

$t_n \in [0,1]$. There exists a sequence ϵ_n of positive numbers such that $\epsilon_n \longrightarrow 0$ and

$$|\mu_n(z,t)| \leqslant \epsilon_n \|z\|$$

for all $(z,t) \in TM \times \mathbb{R}$. By Lemma 1.6, there exist a curve $\gamma \in C^1(\mathbb{R}, M)$ minimizing for $L - \eta_0$ and a subsequence of γ_n converging uniformly on compact sets to γ. This implies that $(\gamma_n(t), t \bmod 1) \in U$ for all $t \in [0,1]$ when n is large enough, which is a contradiction. This ends the proof of the existence of a minimizer $\gamma : [-T_0, T_0+1] \longrightarrow M$ of $L - \eta_0 - \mu$ such that $(\gamma(t), t) \in U$ for all $t \in [0,1]$. The form $\eta_0 + \mu$ is closed in a neighborhood of the set $\big\{(\gamma(t), t)_{t\in\mathbb{R}}\big\} \subset M \times \mathbb{R}$, hence γ is an extremal of L, hence is C^1. □

2.8 LEMMA Let c_0 and c_1 be two C-equivalent classes as defined in 0.6. There exist an integer $T(c_0, c_1)$ and a form μ on $M \times \mathbb{R}$ such that :

ι. The restriction of μ to $\{t \leqslant 0\}$ is 0 and the restriction of μ to $\{t \geqslant T(c_0,c_1)\}$ is a closed periodic one form $\bar{\mu}$ satisfying $[\bar{\mu}] = c_1 - c_0$.

ιι. If η_0 is a closed periodic one form such that $[\eta_0] = c_0$, then any absolutely continuous curve $\gamma : I \longrightarrow M$ minimizing for $L - \eta_0 - \mu$ is an extremal of L if I contains $[0, T(c_0,c_1)]$.

PROOF Let $c(t) : \mathbb{R} \longrightarrow H^1(M,\mathbb{R})$ be an admissible curve such that $c(t) = c_0$ for all $t \leqslant 0$ and $c(t) = c_1$ for all $t \geqslant 1$. Let us fix, for each $t \in [0,1]$, an adapted neighborhood $U(t)$ of $\mathcal{G}(c(t))$, and let $\delta(t)$ and $T_0(t)$ be the numbers given by applying Proposition 2.7 to $c(t)$ and $U(t)$. For each t, there is a positive number $\delta'(t)$ such that $c(s) - c(t) \in R(c(t))$ and $|c(s) - c(t)| \leqslant \delta(t)$ for all $t \in]t - 10\delta'(t), t + 10\delta'(t)[$. There is a finite increasing sequence $(t_i)_{0 \leqslant i \leqslant N}$ of times such that the intervals $]t_i - \delta'(t_i), t_i + \delta'(t_i)[$ cover $[0,1]$. We require in addition that $t_0 = 0$ and $t_N = 1$. To sum up, we have constructed a finite increasing sequence $(t_i)_{0 \leqslant i \leqslant N}$ such that

$$c(t_{i+1}) - c(t_i) \in R(c(t_i)) \text{ and } |c(t_{i+1}) - c(t_i)| \leqslant \delta(t_i).$$

Let us call μ_i the step form given by Proposition 2.7 applied with $d = c(t_{i+1}) - c(t_i)$ for $0 \leqslant i < N$. Let us set $T_i = 1 + \max\big(T_0(t_i), T_0(t_{i+1})\big)$, $0 \leqslant i \leqslant N-1$ and $T_{-1} = T_0(t_0) + 1$ and define the integers $(\tau_i)_{-1 \leqslant i \leqslant N}$ by $\tau_{-1} = 0$ and $\tau_{i+1} = \tau_i + T_i$. We also consider τ_i as the time translation $(q,t) \longmapsto (q, t + \tau_i)$ on $M \times \mathbb{R}$. Let us define the one form

$$\mu = \sum_{i=0}^{N-1} (-\tau_i)^* \mu_i.$$

Setting $T(c_0, c_1) = \tau_N$, we consider an interval I containing $[0, T(c_0,c_1)]$. We have to prove that if $\gamma : I \longrightarrow M$ is a minimizer of $L - \eta_0 - \mu$, then γ is an extremal of L. To check this we consider, for each $1 \leqslant i \leqslant N-1$, the curve

$$\gamma(t + \tau_i) : [\tau_{i-1} - \tau_i + 1, \tau_{i+1} - \tau_i] \longrightarrow M,$$

which is a minimizer of

$$L - \eta_0 - \sum_{j=0}^{i-1} (\tau_i - \tau_j)^* \bar{\mu}_j \; - \; \mu_i,$$

where $\eta_0 + \sum_{j=0}^{i-1} (\tau_i - \tau_j)^* \bar{\mu}_j$ is a closed form satisfying

$$\left[\eta_0 + \sum_{j=0}^{i-1} (\tau_i - \tau_j)^* \bar{\mu}_j \right] = c(t_i).$$

Since $\tau_{i-1} - \tau_i + 1 = 1 - T_{i-1} \leqslant -T_0(t_i)$ and since $\tau_{i+1} - \tau_i = T_i \geqslant T_0(t_i) + 1$, we are in a position to apply Proposition 2.7 and obtain that γ is an extremal of L on $[\tau_{i-1}+1, \tau_{i+1}]$ for each i satisfying $1 \leqslant i \leqslant N-1$. It follows that L is an extremal of L on $[0, \tau_N]$. Since η is a closed periodic one-form on $I - [T_1, \tau_n - T_{N-1}]$, the curve γ is an extremal. $\square$

2.9 LEMMA : For each cohomology class c and each positive number ϵ, there exists a positive number $\mathcal{T}_\epsilon(c)$ with the following properties :
If $X : [0, \mathcal{T}_\epsilon(c)] \longrightarrow TM \times S$ is a trajectory of the Euler-Lagrange flow minimizing $L - c$, then there exists a time t in $[0, \mathcal{T}_\epsilon(c)]$ such that

$$d\big(X(t), \tilde{\mathcal{M}}(c)\big) \leqslant \epsilon.$$

If $X : [-\mathcal{T}_\epsilon(c), \mathcal{T}_\epsilon(c)] \longrightarrow TM \times S$ is a trajectory of the Euler-Lagrange flow minimizing $L - c$, then

$$d\big(X(0), \tilde{\mathcal{G}}(c)\big) \leqslant \epsilon.$$

PROOF : Let us fix $\epsilon > 0$, and consider a sequence $X_i : [0, 2i] \longrightarrow TM \times S$ of trajectories minimizing $L + c$. By lemma 2.2, there exists a minimizing trajectory $X \in C(\mathbb{R}, TM \times S)$ such that the curves $Y_k(t) = X_k(t+k)$ are converging uniformly on compact sets to $X(t)$. On the other hand, by Lemma 2.4, there exists a time t such that

$$d\big(X(t), \tilde{\mathcal{M}}(c)\big) \leqslant \epsilon/2.$$

It follows that

$$d\big(X_k(t+k), \tilde{\mathcal{G}}(c)\big) \leqslant \epsilon$$

when k is large enough, which proves the first part. The second part follows from Lemma 2.2. $\square$

2.10 THEOREM (A): Let us fix a C-equivalence class C in $H^1(M, \mathbb{R})$. Let $(c_i)_{i \in \mathbb{Z}}$ be a bi-infinite sequence of elements of C and $(\epsilon_i)_{i \in \mathbb{Z}}$ be a bi-infinite sequence of positive numbers. If t'_i and t''_i are bi-infinite sequences of real numbers such that $t''_i - t'_i \geqslant \mathcal{T}_{\epsilon_i}(c_i)$ and $t'_{i+1} - t''_i \geqslant T(c_i, c_{i+1})$, then there exist a trajectory $X(t)$ of the Euler-Lagrange flow and a bi-infinite sequence $t_i \in]t'_i, t''_i[$ such that

$$d\big(X(t_i), \tilde{\mathcal{M}}(c_i)\big) \leqslant \epsilon_i.$$

If in addition there exists a class c_∞ such that $c_i = c_\infty$ for large i, or a class $c_{-\infty}$ such that $c_i = c_{-\infty}$ for small i, then the trajectory X is ω-asymptotic to $\tilde{\mathcal{L}}(c_\infty)$ or α-asymptotic to $\tilde{\mathcal{L}}(c_{-\infty})$. Recall that the sets $\tilde{\mathcal{L}}$ have been defined in 0.4.

THEOREM (B) : Let us fix a C-equivalence class C in $H^1(M, \mathbb{R})$. Let $(c_i)_{i \in \mathbb{Z}}$ be a bi-infinite sequence of elements of C and $(\epsilon_i)_{i \in \mathbb{Z}}$ be a bi-infinite sequence of positive numbers. If t_i is a bi-infinite sequence of real numbers such that and $t_{i+1} - t_i \geqslant T(c_i, c_{i+1}) + \mathcal{T}_{\epsilon_i}(c_i) + \mathcal{T}_{\epsilon_{i+1}}(c_{i+1})$, then there exists a trajectory $X(t)$ of the Euler-Lagrange flow such that

$$d\big(X(t_i), \tilde{\mathcal{G}}(c_i)\big) \leqslant \epsilon_i.$$

If in addition there exists a class c_∞ such that $c_i = c_\infty$ for large i, or a class $c_{-\infty}$ such that $c_i = c_{-\infty}$ for small i, then the trajectory X is ω-asymptotic to $\tilde{\mathcal{L}}(c_\infty)$ or α-asymptotic to $\tilde{\mathcal{L}}(c_{-\infty})$.

Proof : Let us prove Theorem (A). Using Lemma 2.8, one can build a 1-form η on $M \times \mathbb{R}$ such that the minimizers of $L - \eta$ are extremals of L, and such that, for each i, the form $\eta|_{[t'_i, t''_i]}$ is closed and periodic and satisfies

$$\big[\eta|_{[t'_i, t''_i]}\big] = c_i.$$

Let us consider a minimizer $\gamma(t)$ of $L-\eta$, and the associated trajectory of the Euler-Lagrange flow $X(t) = (d\gamma(t), t \bmod 1)$. By Lemma 2.9, there exists a sequence $t_i \in]t'_i, t''_i[$ of times such that

$$d\big(X(t_i), \tilde{\mathcal{M}}(c_i)\big) \leqslant \epsilon_i.$$

If the cohomology classes c_i are equal to a fixed one c_∞ for $i \geqslant i_0$, then one can take η such that $\eta|_{[t'_{i_0}, \infty)}$ is closed and periodic. The trajectory $X|_{[t'_{i_0}, \infty)}$ is then a minimizer of $L - c_\infty$, hence it is asymptotic to $\tilde{\mathcal{L}}(c_\infty)$ by definition. The same holds for α-limits.

The proof of (B) is similar. We use a 1-form η on $M \times \mathbb{R}$ such that the minimizers of $L-\eta$ are extremals of L, and such that, for each i, the form $\eta_i = \eta|_{[t_i - \mathcal{T}_{\epsilon_i}(c_i), t_i + \mathcal{T}_{\epsilon_i}(c_i)]}$ is closed and periodic and satisfies $\big[\eta_i\big] = c_i$. We then conclude that the minimizing trajectories of $L - \eta$ have the desired property using the second part of Lemma 2.9. □

2.11 corollary (see also 4.10) : If there exist two C-equivalent classes c_0 and c_1 such that $\tilde{\mathcal{M}}(c_0)$ and $\tilde{\mathcal{M}}(c_1)$ are disjoint, then the time one map of the Euler Lagrange flow has positive topological entropy.

Proof Notice first that $\tilde{\mathcal{G}}(c_0)$ and $\tilde{\mathcal{G}}(c_1)$ are disjoint if $\tilde{\mathcal{M}}(c_0)$ and $\tilde{\mathcal{M}}(c_1)$ are. Else the intersection $\tilde{\mathcal{G}}(c_0) \cap \tilde{\mathcal{G}}(c_1)$ would carry an invariant measure, the support of which would belong both to $\tilde{\mathcal{M}}(c_0)$ and $\tilde{\mathcal{M}}(c_1)$. Let us now chose $\epsilon < d(\tilde{\mathcal{G}}(c_0), \tilde{\mathcal{G}}(c_1))/2$. Let T be an integer greater than $T(c_0, c_1) + T(c_1, c_0) + \mathcal{T}_\epsilon(c_0) + \mathcal{T}_\epsilon(c_1)$. Let $\phi_T : TM \longrightarrow TM$ be the time T flow, we want to prove that there exists a compact invariant set on which the dynamics of ϕ_T is semi-conjugated to a Bernoulli shift. To do so, we consider the disjoint neighborhoods $U_i = \{d(X, \tilde{\mathcal{G}}_0(c_i)) \leqslant \epsilon\}$, for $i = 0$ or 1, and the compact ϕ_T-invariant set

$$K = \bigcap_{k \in \mathbb{Z}} \phi_{kT}(U_0 \cup U_1).$$

Let f be the mapping from $U_0 \cup U_1$ to the set $\{0, 1\}$ which takes value 0 on U_0 and 1 on U_1. Let us endow the set $\Sigma = \{0,1\}^{\mathbb{Z}}$ with the product of discrete topologies. Define the continuous map $h : K \longrightarrow \Sigma$ by $(h(x))_i = f(\phi_{iT}(x))$. By definition, we have $h \circ \phi_T = \sigma \circ h$, where σ is the shift $\sigma(a)_i = a_{i+1}$. The point here is that the map h is surjective, in view of Theorem (B). More precisely, let us fix a sequence $a = (a_i) \in \Sigma$. Applying the theorem with $c_i = c_{a_i}$, $\epsilon_i = \epsilon$ and $t_i = iT$, we obtain the existence of a trajectory of ϕ_T in $h^{-1}(a_i)$. □

3 Globally minimizing orbits

We have achieved our main goal, proving Theorem 0.6. However, the hypothesis of this theorem is rather abstract, and some additional work is required in order to understand this hypothesis. In the present section, we will describe the various sets of globally minimizing orbits which have been defined in the literature. Since most of the proofs have been written only in the autonomous case, we prove most of the results we state, except the graph properties, mostly due to Mather, and for which we send the reader to [10], [5] and [3].

3.1 The Lagrangian L is called critical if the infimum of the actions of all closed curves is 0. It is equivalent to require that the minimum of the actions of all invariant probability measures is 0. Any Lagrangian satisfying the hypotheses of 0.1 can be made critical by the addition of a real constant. See section 4 below for more details.

3.2 Let L be a critical Lagrangian. For all $t' \geqslant t$ we define the function

$$\begin{aligned} F_{t,t'} : M \times M &\longrightarrow \mathbb{R} \\ (x, x') &\longmapsto \min_{\gamma \in \Gamma} \int_t^{t'} L(d\gamma(u), u)\, du \end{aligned}$$

where the minimum is taken on the set Γ of curves $\gamma \in C^1([t, t'], M)$ satisfying $\gamma(t) = x$ and $\gamma(t') = x'$. We also define, for each $(s, s') \in S^2$ the function

$$\begin{aligned} \Phi_{s,s'} : M \times M &\longrightarrow \mathbb{R} \\ (x, x') &\longmapsto \inf F_{t,t'}(x, x') \end{aligned}$$

where the infimum is taken on the set of $(t, t') \in \mathbb{R}^2$ such that $s = t \bmod 1$, $s' = t' \bmod 1$, and $t' \geqslant t + 1$. Following Mather, we introduce one more function

$$\begin{aligned} h_{s,s'} : M \times M &\longrightarrow \mathbb{R} \\ (x, x') &\longmapsto \liminf_{t'-t \longrightarrow \infty} F_{t,t'}(x, x') \end{aligned}$$

where the liminf is restricted to the set of $(t, t') \in \mathbb{R}^2$ such that $s = t \bmod 1$ and $s' = t' \bmod 1$. These functions have symmetric counterparts

$$d_{s,s'}(x, x') = h_{s,s'}(x, x') + h_{s',s}(x', x) \quad \text{and} \quad \tilde{d}_{s,s'}(x, x') = \Phi_{s,s'}(x, x') + \Phi_{s',s}(x', x)$$

If L is critical, then $d \geqslant \tilde{d} \geqslant 0$.

3.3 LEMMA The function

$$\begin{aligned} F : \mathbb{R} \times \mathbb{R} \times M \times M &\longrightarrow \mathbb{R} \\ (t, t', x, x') &\longmapsto F_{t,t'}(x, x') \end{aligned}$$

is Lipschitz and bounded on $\{t' \geqslant t + 1\}$.

PROOF We first prove the Lipschitz property. The proof of boundedness will be given in 3.8. In order to study the dependence in the space variables x and x', let us fix a number $\Delta \geqslant 1$ greater than the diameter of M. In view of Lemma 1.4 and 2.3, there exists a constant K such that, if $t' \geqslant t + 1$ and if $\gamma \in C^1([t, t'], M)$ is a minimizer, then $\|d\gamma\| \leqslant K$. Let us set

$$B = \max_{(z,t) \in TM \times \mathbb{R}, \|z\| \leqslant K + 3\Delta} |L(z, t)|.$$

Consider $t' \geqslant t + 1$ and four points x_0, x_0', x_1, x_1' in M. There is a minimizing curve $\gamma \in C^1([t, t'], M)$ such that $A(\gamma) = F_{t,t'}(x_0, x_0')$. Let us set

$$\delta = \min\{1/3, d(x_0, x_1)\} \quad \text{and} \quad \delta' = \min\{1/3, d(x_0', x_1')\}.$$

The geodesic $x \in C^1([t, t+\delta], M)$ between x_1 and $\gamma(t+\delta)$ satisfies

$$\|dx\| \leqslant d(x_1, \gamma(t+\delta))/\delta \leqslant \big(d(x_0, \gamma(t+\delta)) + d(x_0, x_1)\big)/\delta \leqslant K + d(x_0, x_1)/\delta \leqslant K + 3\Delta,$$

hence $A(x) \leqslant B\delta$. The same estimate is true with the geodesic $x' \in C^1([t'-\delta', t'], M)$ connecting $\gamma(t'-\delta')$ and x'_1. We have

$$\begin{aligned}
F_{t,t'}(x_1, x'_1) &\leqslant F_{t,t+\delta}(x_1, \gamma(t+\delta)) + F_{t+\delta, t'-\delta'}(\gamma(t+\delta), \gamma(t'+\delta')) + F_{t'-\delta', t'}(\gamma(t'-\delta'), x'_1)\\
&\leqslant F_{t+\delta, t'-\delta'}(\gamma(t+\delta), \gamma(t'+\delta')) + B\delta + B\delta'\\
&\leqslant F_{t,t'}(x_0, x'_0) - A(\gamma|_{[t,t+\delta]}) - A(\gamma|_{[t'-\delta, t']}) + B\delta + B\delta'\\
&\leqslant F_{t,t'}(x_0, x'_0) + 2B\delta + 2B\delta'.\\
&\leqslant F_{t,t'}(x_0, x'_0) + 2Bd(x_0, x_1) + 2Bd(x'_0, x'_1).
\end{aligned}$$

This proves that $2B$ is a Lipschitz constant for all the functions $F_{t,t'}$ with $t' \geqslant t+1$. There remains to study the dependence in the time variable t', the dependence in t is similar. Let us consider three times t, t', t'' such that $t+1 \leqslant t' \leqslant t''$ and two points x and x' in M. Let $\gamma : [0, t''] \longrightarrow M$ be a minimizing curve between x and x'. Recall that $\|d\gamma\| \leqslant K$. We have

$$F_{t,t''}(x, x') = F_{t,t'}(x, \gamma(t')) + F_{t',t''}(\gamma(t), x').$$

Observing that

$$|F_{t',t''}(\gamma(t), x')| \leqslant B(t'' - t')$$

in view of the definition of B and that

$$|F_{t,t'}(x, \gamma(t')) - F_{t,t'}(x, x')| \leqslant 2Bd(\gamma(t), x') \leqslant 2BK(t'' - t')$$

in view of the Lipschitz property just proved for $F_{t,t'}$, we conclude that

$$|F_{t,t''}(x, x') - F_{t,t'}(x, x')| \leqslant B(2K+1)(t'' - t)$$

hence F is Lipschitz. We need to introduce some definitions before we prove that this function is bounded. The proof will be given in 3.8.

3.4 It is useful to define distinguished classes of minimizers. Recall that L is a critical Lagrangian. A curve $\gamma \in C^1(I, M)$ is called semi-static if

$$A\big(\gamma|_{[a,b]}\big) = \Phi_{a \bmod 1, b \bmod 1}\big(\gamma(a), \gamma(b)\big)$$

for all $[a, b] \subset I$. An orbit $X(t) = (d\gamma(t), t \bmod 1)$ is called semi-static if γ is a semi-static curve. It is clear that semi-static orbits are minimizing. A curve $\gamma \in C^1(I, M)$ is called static if

$$A\big(\gamma|_{[a,b]}\big) = -\Phi_{b \bmod 1, a \bmod 1}\big(\gamma(b), \gamma(a)\big)$$

for all $[a, b] \subset I$. If γ is not semi-static, then there exists $[a, b]$ such that

$$A\big(\gamma|_{[a,b]}\big) > \Phi_{a \bmod 1, b \bmod 1}\big(\gamma(a), \gamma(b)\big)$$

hence

$$A\big(\gamma|_{[a,b]}\big) + \Phi_{b \bmod 1, a \bmod 1}\big(\gamma(b), \gamma(a)\big) > \tilde{d}_{a \bmod 1, b \bmod 1}\big(\gamma(a), \gamma(b)\big) \geqslant 0$$

hence γ is not static. It follows that static curves are semi-static. We call $\tilde{\mathcal{N}}$ the union in $TM \times S$ of the images of global semi-static orbits (semi-static orbits with $I = \mathbb{R}$) and $\tilde{\mathcal{A}}$ the union of global static orbits. Clearly,

$$\tilde{\mathcal{A}} \subset \tilde{\mathcal{N}} \subset \tilde{\mathcal{G}}.$$

It has became usual to call $\tilde{\mathcal{A}}$ the Aubry set, and $\tilde{\mathcal{N}}$ the Mañe set.

3.5 Let $\gamma \in C^1(\mathbb{R}, M)$ be a static curve, and $I \subset \mathbb{R}$ be a compact interval. There exists a sequence $T_k \longrightarrow \infty$ of integers and a sequence $\gamma_k : [-T_k/2, T_k/2] \longrightarrow M$ of piecewise C^1 curves which satisfies $A(\gamma_k) \longrightarrow 0$ and such that $\gamma_k|_I = \gamma|_I$, and $\gamma_k(T_k/2) = \gamma_k(-T_k/2)$.

In order to prove this result, let us consider a sequence $t_k \longrightarrow \infty$ of integer times. Since the curve γ is static,

$$\Phi_{0,0}(\gamma(-t_k), \gamma(t_k)) = -A(\gamma|_{[-t_k,t_k]}).$$

As a consequence, there exist a sequence $T_k \geqslant 2t_k + 1$ of integers and a sequence $x_k : [t_k, T_k - t_k] \longrightarrow M$ of C^1 curves such that $x_k(t_k) = \gamma(t_k)$, $x_k(T_k - t_k) = \gamma(-t_k)$, and $A(x_k) \leqslant A(\gamma|_{[-t_k,t_k]}) + 1/k$. Let $\tilde{\gamma}_k : \mathbb{R} \longrightarrow M$ be the periodic curve of period T_k which coincides with γ on $[-t_k, t_k]$ and with x_k on $[t_k, T_k - t_k]$. Setting $\gamma_k = \tilde{\gamma}_k|_{[-T_k/2,T_k/2]}$, we have $A(\gamma_k) = A(x_k) + A(\gamma|_{[-t_k,t_k]}) \leqslant 1/k$. □

3.6 Conversely, let us consider an absolutely continuous curve $\gamma : I \longrightarrow M$, where $I \subset \mathbb{R}$ is a compact interval of times. Assume that there exists a sequence $\gamma_k : \mathbb{R} \longrightarrow M$ of absolutely continuous periodic curves of period $T_k \in \mathbb{N}$ which is converging uniformly on I to γ and such that $A\big(\gamma_k|_{[0,T_k]}\big) \longrightarrow 0$. Then the curve γ is static.

We can assume without loss of generality that T_k is greater than the length of I (T_k is not supposed to be the smallest period of γ_k). Let us take an interval $[t, t'] \subset I$. We have

$$A\big(\gamma_k|_{[0,T_k]}\big) = A\big(\gamma_k|_{[t,t']}\big) + A\big(\gamma_k|_{[t',t+T_k]}\big) \longrightarrow 0.$$

On the other hand, we see from 1.2 that

$$A\big(\gamma|_{[t,t']}\big) \leqslant \liminf A\big(\gamma_k|_{[t,t']}\big).$$

We also have, by continuity of Φ,

$$\begin{aligned}\Phi_{t' \bmod 1, t \bmod 1}(\gamma(t'), \gamma(t)) &= \lim \Phi_{t' \bmod 1, t \bmod 1}(\gamma_k(t'), \gamma_k(t + T_k)) \\ &\leqslant \liminf A\big(\gamma_k|_{[t',t+T_k]}\big),\end{aligned}$$

so that

$$A\big(\gamma|_{[t,t']}\big) + \Phi_{t' \bmod 1, t \bmod 1}(\gamma(t'), \gamma(t) \leqslant 0$$

and γ is static. □

3.7 Lemma We have the equivalence

$$d_{s,s}(x, x) = 0 \Longleftrightarrow \tilde{d}_{s,s}(x, x) = 0 \Longleftrightarrow x \in \mathcal{A}_s,$$

and the set $\tilde{\mathcal{A}}$ is a non empty compact invariant set.

Proof Since $d \geqslant \tilde{d} \geqslant 0$, it is enough to prove that $d_{s,s}(x, x) = 0$ if $\tilde{d}_{s,s}(x, x) = 0$ to prove the first equivalence. Assume that $\tilde{d}_{s,s}(x, x) = 0$. Recall that $\tilde{d}_{s,s}(x, x) = 2\Phi_{s,s}(x, x)$. Either the infimum in the definition of Φ is a minimum, or it is a liminf. If it is a liminf, the proof is over. If it is reached, there is a curve $\gamma : [t, t'] \longrightarrow M$ such that $\gamma(t) = \gamma(t') = x$ and $t \bmod 1 = s = t' \bmod 1$, satisfying $A(\gamma) = 0$. In this case, we can paste γ with itself several times and build a curve $\gamma_k : [t, t + k(t' - t)]$ such that $\gamma_k(t) = \gamma_k(t + k(t' - t)) = x$ and such that $A(\gamma_k) = 0$. It follows that $h_{s,s}(x, x) = 0$, hence $d_{s,s}(x, x) = 0$. This ends the proof of the first equivalence.

Let us suppose that $d_{s,s}(x,x) = 0$, and prove that $x \in \mathcal{A}_s$. By definition of d, there is a sequence $\gamma_k \in C^1([t_k, t'_k], M)$ of minimizing curves such that $A(\gamma_k) \longrightarrow 0$, $\gamma_k(t_k) = x$, $\gamma_k(t'_k) = x$ and such that $t_k \bmod 1 = s = t'_k \bmod 1$ and $t'_k - t_k \longrightarrow \infty$. We can see γ_k as a sequence of periodic curves in $C^1(\mathbb{R}, M)$ of period $t'_k - t_k$. By Lemma 1.6 we can suppose, taking a subsequence, that the curves γ_k are converging uniformly on compact sets to a minimizer $\gamma \in C^1(\mathbb{R}, M)$. It follows from 3.6 that γ is static, hence $x \in \mathcal{A}_s$.

Conversely, if $x \in \mathcal{A}_s$, there exists a static curve $\gamma : \mathbb{R} \longrightarrow M$ such that $\gamma(s) = x$, where we also note s the real number such that $s \bmod 1 = s$. It is then a direct consequence of 3.5 that $d_{s,s}(x,x) = 0$. The set $\mathcal{A}$ is not empty because it is clear that the minimum of the continuous function $x \longmapsto d_{s,s}(x,x)$ has to be 0 for each s if L is critical. Finally, $\tilde{\mathcal{A}}$ is clearly invariant since it is defined as a union of orbits. $\square$

3.8 We are now in a position to prove that the function F is bounded. Let

$$A = \sup_{t,x,x'} F_{t,t+1/3}(x,x') \quad \text{and} \quad B = \sup_{s,s',x,x'} \Phi_{s,s'}(x,x'),$$

both A and B are finite. let $\gamma \in C^1(\mathbb{R}, M)$ be a semi-static curve. There exist semi-static curves since we just proved the existence of static curves. Let us chose $t' \geqslant t+1$ and $x, x' \in M$. We have

$$\begin{aligned} F_{t,t'}(x,x') \;&\leqslant\; F_{t,t+1/3}\big(x, \gamma(t+1/3)\big) + \\ &\quad\; F_{t+1/3,t'-1/3}\big(\gamma(t+1/3), \gamma(t'-1/3)\big) + F_{t'-1/3,t'}\big(\gamma(t'-1/3), x'\big) \\ &\leqslant\; A + B + A, \end{aligned}$$

where we have used that, since γ is semi-static,

$$F_{t+1/3,t'-1/3}\big(\gamma(t+1/3), \gamma(t'-1/3)\big) = \Phi_{(t+1/3) \bmod 1, (t'-1/3) \bmod 1}\big(\gamma(t+1/3), \gamma(t'-1/3)\big).$$

Recalling that the functions $F_{t,t'}$ are equilipschitz, we obtain the existence of a constant C such that

$$F_{t,t'}(x,x') \leqslant C$$

for all $t' \geqslant t+1$ and all $(x,x') \in M^2$. In order to end the proof, notice that, when k is an integer such that $t+k \geqslant t'+1$,

$$F_{t,t'}(x,x') + F_{t',t+k}(x',x) \geqslant 0,$$

hence $F_{t,t'} \geqslant -C$. $\square$

3.9 Lemma We have the inclusions

$$\tilde{\mathcal{M}} \subset \tilde{\mathcal{L}} \subset \tilde{\mathcal{A}} \subset \tilde{\mathcal{N}} \subset \tilde{\mathcal{G}}.$$

Proof It is enough to prove that $\tilde{\mathcal{L}} \subset \tilde{\mathcal{A}}$. Let $X : [0,\infty) \longrightarrow TM \times S$ be a minimizing orbit and let $\gamma = \pi \circ X$ be its projection on M. There exists a sequence $t_k \longrightarrow \infty$ of times such that $t_k \bmod 1 = s$ and $\gamma(t_k) \longrightarrow \omega$. We can assume in addition that $t_{k+1} - t_k \longrightarrow \infty$. Let us set $X_k(t) = X(t + [t_k])$. Taking a subsequence if necessary, we can suppose that the curves

X_k are converging uniformly on compact sets to a curve $Y(t) = (dx(t), t \bmod 1)$. In order to prove that x is a static curve, we write, for $t' \geqslant t+1$,

$$\begin{aligned}
& A(x|_{[t,t']}) + \Phi_{t' \bmod 1, t \bmod 1}(x(t'), x(t)) \\
= \; & \lim A\big(\gamma|_{[t+[t_k], t'+[t_k]]}\big) + \Phi_{t' \bmod 1, t \bmod 1}(x(t'), x(t)) \\
= \; & \lim \Big(A\big(\gamma|_{[t_{k-1}, t_{k+1}]}\big) - A\big(\gamma|_{[t_{k-1}, t+[t_k]]}\big) - A\big(\gamma|_{[t'+[t_k], t_{k+1}]}\big) \Big) \\
+ \; & \Phi_{t' \bmod 1, t \bmod 1}(x(t'), x(t)) \\
\leqslant \; & \liminf \Big(A\big(\gamma|_{[t_{k-1}, t_{k+1}]}\big) \Big) \\
- \; & \Big(\Phi_{s, t \bmod 1}(\omega, x(t)) + \Phi_{t' \bmod 1, s}(x(t'), \omega) - \Phi_{t' \bmod 1, t \bmod 1}(x(t'), x(t)) \Big) \\
\leqslant \; & \liminf \Big(A\big(\gamma|_{[t_{k-1}, t_{k+1}]}\big) \Big) \leqslant 0.
\end{aligned}$$

In this calculations, we have used Lemma 1.5 between the first line and the second, and we have used Lemma 3.3 to obtain the last inequality. More precisely, it follows from this lemma that the sum

$$\sum_{k=1}^{n} A\big(\gamma|_{[t_{2k-1}, t_{2k+1}]}\big) = A\big(\gamma|_{[t_1, t_{2n+1}]}\big) = F_{t_1, t_{2n+1}}\big(\gamma(t_1), \gamma(t_{2n+1})\big)$$

is bounded, which implies that the liminf is not positive. □

3.10 First Graph property Let us call $\Pi : TM \times S \longrightarrow M \times S$ the natural projection. Then $\Pi|_{\tilde{\mathcal{A}}}$ is a bilipschitz homeomorphism onto its image $\mathcal{A}$. In addition, we have

$$\tilde{\mathcal{N}} \cap \Pi^{-1}(\mathcal{A}) = \tilde{\mathcal{A}}.$$

In other words, there is a Lipschitz section $v : \mathcal{A} \longrightarrow TM \times S$ of Π with the property that, for each $(x, s) \in \mathcal{A}$, there is one and only one semi-static orbit $X(t)$ satisfying $\Pi(X(0)) = (x, s)$, this orbit is static and is given by $X(t) = \phi_t(v(x, s), s)$.

3.11 It is not hard to see that

$$\tilde{d}_{s,s'}(x, x') = d_{s,s'}(x, x')$$

if $(x, s) \in \mathcal{A}$ or $(x', s') \in \mathcal{A}$. We define an equivalence relation on $\mathcal{A}$ by saying that (x, s) and (x', s') are equivalent if and only if $d_{s,s'}(x, x') = 0$. We call static class an equivalence class of this relation. We also call static class the image by the Lipschitz vector field v of a static class in $M \times S$. Static classes are compact invariant subsets of $\tilde{\mathcal{A}}$.

Remark If $\gamma : [0, \infty) \longrightarrow M$ is minimizing, then the omega-limit set of the orbit $X(t) = (d\gamma, t \bmod 1)$ is contained in a static class.

Proof Let us consider sequences t_k and t'_k such that $t_k \bmod 1 = s$ and $t'_k \bmod 1 = s'$, and such that $X(t_k) \longrightarrow \tilde{\omega}$ and $X(t'_k) \longrightarrow \tilde{\omega}'$. We can assume in addition that $t_k - t'_k \longrightarrow \infty$ and that $t'_k - t_{k-1} \longrightarrow \infty$. If ω and ω' are the projections on M of $\tilde{\omega}$ and $\tilde{\omega}'$, then

$$d_{s,s'}(\omega, \omega') \leqslant \liminf A\big(\gamma|_{[t_k, t_{k+1}]}\big) \leqslant \liminf F_{t_k, t_{k+1}}(\gamma(t_k), \gamma(t_{k+1})) \leqslant 0,$$

where the last liminf is not positive in view of Lemma 3.3 since $\gamma(t_k)$ is convergent. $\square$
A semi-static curve thus has its alpha-limit contained in a static class, and its omega-limit contained in a static class. Each static class intersects $\tilde{\mathcal{M}}$.

Lemma A semi-static curve is static if and only if its alpha and omega-limit belong to the same static class. If $\tilde{\mathcal{A}}$ contains only one static class, then $\tilde{\mathcal{N}} = \tilde{\mathcal{A}}$. It is the case for example if $\tilde{\mathcal{M}}$ is transitive *i.e.* if it has a dense orbit.

Proof It is quite clear that if $\gamma(t)$ is a static curve, then

$$d_{t \bmod 1, t' \bmod 1}(\gamma(t), \gamma(t')) = \tilde{d}_{t \bmod 1, t' \bmod 1}(\gamma(t), \gamma(t')) = 0$$

for all $t \leqslant t'$. Taking the limit $t \longrightarrow -\infty$ and $t' \longrightarrow \infty$ we obtain that the alpha and omega limit belong to the same static class. On the other hand, let $\gamma(t)$ be a semi-static curve such that the alpha and omega-limit belong to the same static class. Let us consider sequences $t_k \longrightarrow -\infty$ and $t'_k \longrightarrow \infty$ of integers such that $\gamma(t_k)$ has a limit $\alpha \in M$ and $\gamma(t'_k) \longrightarrow \omega$. The hypothesis is that $d_{0,0}(\alpha, \omega) = 0$. For each $t' \geqslant t$, we have

$$d_{t \bmod 1, t' \bmod 1}(\gamma(t), \gamma(t')) + d_{0, t \bmod 1}(\alpha, \gamma(t)) + d_{t' \bmod 1, 0}(\gamma(t'), \omega) \leqslant d_{0,0}(\alpha, \omega) = 0,$$

hence $d_{t \bmod 1, t' \bmod 1}(\gamma(t), \gamma(t')) \leqslant 0$ and γ is static. $\square$

3.12 If $\tilde{\mathcal{S}} \subset TM \times S$ is a static class, we call $\tilde{\mathcal{S}}^+$ the set of points $(z, s) \in TM \times \mathbb{R}$ such that the orbit $\phi_t(z, s)$ is semi-static on an open neighborhood of $[0, \infty)$, and omega-asymptotic to $\tilde{\mathcal{S}}$. We define $\tilde{\mathcal{S}}^-$ in the same way with alpha-limits.

Second graph property For each static class $\tilde{\mathcal{S}}$, the restriction of Π to $\tilde{\mathcal{S}}^+$ is a bilipschitz homeomorphism onto its image, as well as the restriction of Π to $\tilde{\mathcal{S}}^-$. The set $\tilde{\mathcal{N}}$ is the union of the graphs $\tilde{\mathcal{N}} \cap \tilde{\mathcal{S}}^+$, as well as the union of the graphs $\tilde{\mathcal{N}} \cap \tilde{\mathcal{S}}^-$.

4 The averaged energy

We will now explain the method introduced by Mather to associate to each Lagrangian L satisfying the hypotheses 0.1 a family of invariant sets. We will also define the averaged energy α of Mather, and state some results of Massart [6] which establish a link between the averaged energy and the topology of the Aubry set. These results will be useful later to study the twist map case, and they show that no connecting orbit can be obtained from our results in the autonomous case.

4.1 Let us identify $H^1(S, \mathbb{R})$ with $\mathbb{R}$ in the standard way. To a closed one-form η on $M \times S$, we associates the cohomology $\lambda(\eta) \in \mathbb{R}$ of its restriction to $\{x\} \times S$, this cohomology does not depend on $x \in M$, and depends only of the cohomology of η. Recall that we have defined in 0.3 the class $[\eta] \in H^1(M, \mathbb{R})$ of any closed one form η on $M \times S$. The function

$$\eta \longmapsto ([\eta], \lambda(\eta))$$

induces an isomorphism between $H^1(M \times S, \mathbb{R})$ and $H^1(M, \mathbb{R}) \times \mathbb{R}$.

4.2 Let us fix a Lagrangian L, not necessarily critical. We say that a closed one-form η on $M \times S$ is critical if $L - \eta$ is critical.

Theorem (Mather [10]) There exists a convex and superlinear function

$$\alpha : H^1(M, \mathbb{R}) \longrightarrow \mathbb{R}$$

with the property that a closed one-form η is critical if and only if

$$\lambda(\eta) = -\alpha([\eta]).$$

We call the function α the averaged energy.

4.3 Given a critical form η, we can associate all the sets $\tilde{\mathcal{M}}, \tilde{\mathcal{A}}, \dots$ to the critical Lagrangian $L - \eta$. It is not hard to see that these sets depend only on the class $[\eta] \in H^1(M, \mathbb{R})$. In view of Mather's Theorem above, the function $\eta \longmapsto [\eta]$ restricted to critical forms is surjective, and induces an isomorphism in cohomology *i.e.* the cohomology in $H^1(M \times S, \mathbb{R})$ of a critical form η is determined by its cohomology $[\eta] \in H^1(M, \mathbb{R})$. We note

$$\tilde{\mathcal{M}}(c) \subset \tilde{\mathcal{L}}(c) \subset \tilde{\mathcal{A}}(c) \subset \tilde{\mathcal{N}}(c) \subset \tilde{\mathcal{G}}(c)$$

the sets $\tilde{\mathcal{M}}, \tilde{\mathcal{L}}, \dots$ associated to the critical Lagrangian $L - \eta$, where η is any critical form satisfying $[\eta] = c$. They are non empty compact sets invariant under the Euler-Lagrange flow of L.

4.4 We note $\partial\alpha(c)$ the subderivative of α at c, which is a compact and convex subset of $H_1(M, \mathbb{R})$. This is the set of rotation vectors of invariant measures of the Euler-Lagrange flow supported in $\tilde{\mathcal{A}}(c)$. These measures are the minimizing measures defined by Mather in [10], see [5].

4.5 Following Mather, we note

$$\beta : H_1(M, \mathbb{R}) \longrightarrow \mathbb{R}$$

the Fenchel conjugate of α. We call it the averaged action. For each $\omega \in H_1(M, \mathbb{R})$, the number $\beta(\omega)$ is the minimal action of invariant probability measures of rotation vector ω. There are interesting connections between the size of the flats of the averaged energy α and the topology of the invariant set $\mathcal{A}(c)$. In the following, we adapt to our needs some results of Massart [6].

4.6 A flat of α is a closed convex $K \subset H^1(M, \mathbb{R})$ such that $\alpha|_K$ is an affine function. To any closed convex set K, we associate the vector subspace $VK = \mathrm{Vect}(K - K)$. A point c is said to be in the interior of K if there exists a neighborhood U of 0 in VK such that $d + U \subset K$. The interior of a flat is not empty. Given $c \in H^1(M, \mathbb{R})$, we note $F(c)$ the union of all flats containing c in their interior. It is clear that $F(c)$ is a flat, we note $VF(c)$ the associated vector space. It is easy to see that $V\partial\alpha(c) \subset VF(c)^{\perp}$, although the equality does not always hold (for example, if α is differentiable at c, and strictly convex, then $VF(c) = V\partial\alpha(c) = \{0\}$).

4.7 Proposition (Massart, [6]) : If F is a flat of α, there exists an Aubry set $\tilde{\mathcal{A}}(F)$ which is the Aubry set $\tilde{\mathcal{A}}(c)$ for all cohomology class c in the interior of F, and is contained in the Aubry set of any cohomology class $c \in F$.

Conversely, if $\tilde{\mathcal{G}}(c) \cap \tilde{\mathcal{G}}(c') \neq \emptyset$, then the segment $[c, c']$ between c and c' is contained in a flat, or equivalently $\alpha|_{[c,c']}$ is affine. Here $[c, c'] \subset H^1(M, \mathbb{R})$ is the compact segment between c and c' (the convex envelope of $\{c, c'\}$).

Proof : Let us consider a flat F of α. Let η be a critical form such that $[\eta] = c$ is in the interior of F, and let μ be a closed one-form such that $\eta + \lambda\mu$ is critical for $\lambda \in [\lambda_0, \lambda_1]$, where $\lambda_0 < 0 < \lambda_1$. This is to say that $e = [\mu] \in VF$. Let us prove that $\tilde{\mathcal{A}}(c) \subset \tilde{\mathcal{A}}(c + \lambda e)$ for all $\lambda \in [\lambda_0, \lambda_1]$. Recall that $\tilde{\mathcal{A}}(c)$ is the union of orbits which are static for $L - \eta$, so that it is enough to prove that each curve $\gamma \in C^1(\mathbb{R}, M)$ which is static for $L - \eta$ is also static for $L - \eta - \lambda\mu$. Let $I \subset \mathbb{R}$ be a compact interval, and let γ_k be the sequence of periodic curves given by 3.5 applied with γ and I for $L - \eta$, so that $A_\eta(\gamma_k) \longrightarrow 0$. Since the form $\eta + \lambda\mu$ is critical and the curve γ_k is closed, we have

$$0 \leqslant A_{\eta+\lambda\mu}(\gamma_k) = A_\eta(\gamma_k) - t\langle e, [\gamma_k]\rangle,$$

where $[\gamma_k] \in H_1(M, \mathbb{R})$ is the homology of γ_k. We obtain that

$$\langle e, [\gamma_k]\rangle \leqslant \frac{A_\eta(\gamma_k)}{\lambda_1} \longrightarrow 0,$$

when $k \longrightarrow \infty$, hence $A_{\eta+\lambda\mu}(\gamma_k) \longrightarrow 0$. This implies, in view of 3.6, that $\gamma|_I$ is static for $L - \eta - \lambda\mu$. Since this holds for all I, the curve γ is static for the Lagrangian $L - \eta - \lambda\mu$ when $\lambda \in [\lambda_0, \lambda_1]$.

In order to prove the converse, let us consider two cohomology classes c and c' such that $\tilde{\mathcal{G}}(c) \cap \tilde{\mathcal{G}}(c') \neq \emptyset$. It is clear that in this case $\tilde{\mathcal{M}}(c) \cap \tilde{\mathcal{M}}(c') \neq \emptyset$. Choose a critical form η such that $[\eta] = c$ and a closed form μ such that $\eta + \mu$ is critical and satisfies $[\eta + \mu] = c'$. We are going to prove that the form $\eta + \lambda\mu$ is critical for each $\lambda \in [0, 1]$. Let us note $l(\lambda)$ the infimum of the actions of closed curves for $L - \eta - \lambda\mu$, we have to prove that $l(\lambda) = 0$ on $[0, 1]$. The function $\lambda \longrightarrow l(\lambda)$ is concave as an infimum of linear functions, and $l(0) = l(1) = 0$. As a consequence, $l(\lambda) \geqslant 0$ on $[0, 1]$. It remains to prove that $l(\lambda) \leqslant 0$. Let us consider a curve $\gamma \in C^1(\mathbb{R}, M)$ whose lifting $(d\gamma(t), t \bmod 1)$ is a recurrent orbit of the Euler-Lagrange flow contained in $\tilde{\mathcal{M}}(c) \cap \tilde{\mathcal{M}}(c')$. Let $t_k \longrightarrow \infty$ be a sequence of integer times such that $\gamma(t_k) \longrightarrow \gamma(0)$. Let us set $\delta_k = dist(\gamma_k(t_k), \gamma(0))$. Let $\gamma_k : [0, t_k] \longrightarrow M$ be the closed continuous curve such that $\gamma_k|_{[0,t_k-\delta_k]} = \gamma|_{[0,t_k-\delta_k]}$ and $\gamma_k|_{[t_k-\delta_k,t_k]}$ is a minimizing geodesic between its endpoints. Since the curve γ is static for $L-\eta$, we have, using the action associated to $L - \eta$

$$A\big(\gamma|_{[0,t_k]}\big) = -\Phi_{0,0}(\gamma(t_k), \gamma(0)) \longrightarrow -\Phi_{0,0}(\gamma(0), \gamma(0)) = 0$$

and a simple calculation shows that $A(\gamma_k) \longrightarrow 0$. The same holds for $L-\eta-\mu$, since the curve γ is also static for this Lagrangian. On the other hand, the action $A_\lambda(\gamma_k)$ of γ_k associated to $L - \eta - \lambda\mu$ is a linear function of λ for each k. As a consequence, we have $A_\lambda(\gamma_k) \longrightarrow 0$ uniformly on $[0, 1]$, hence $l(\lambda) \leqslant 0$ on $[0, 1]$. □

4.8 Following Massart, let us define two subspaces of $H^1(M, \mathbb{R})$ associated to the topology of $\tilde{\mathcal{A}}(c)$. By cohomology class of a closed one form of $M \times S$, we mean the cohomology class in $H^1(M, \mathbb{R})$ defined in 0.3. The subspace $E(c) \subset H^1(M, \mathbb{R})$ is the set of cohomology classes in $H^1(M, \mathbb{R})$ of closed one forms of $M \times S$ which have a support disjoint from $\mathcal{A}(c)$. The subspace $G(c) \subset H^1(M, \mathbb{R})$ is the set of cohomology classes in $H^1(M, \mathbb{R})$ of continuous closed one forms of $M \times S$ which vanish on $T_{(x,s)}(M \times S)$ for each $(x, s) \in \mathcal{A}(c)$. In the above definition, we call a continuous one-form closed if it is locally the differential of a C^1 function. It is not hard to define the cohomology of such a closed form, for example by considering its action on closed curves.

4.9 Theorem (Massart, [6]) We have the inclusions

$$E(c) \subset VF(c) \subset G(c).$$

The proof of this result in [6] is based on some regularity properties of the function h discovered by Fathi, see [3]. The generalization from the autonomous setting of [6] to the non autonomous setting here does not present any difficulty.

4.10 The orbits constructed in section 2 are non-trivial if they connect disjoint invariant sets. Hence interesting applications of these results are possible if and only if there exist C-equivalence classes which are not contained in any flat of α. For example, we have the following restatement of Corollary 2.11.

Corollary If there exist two C-equivalent classes c and c' such that $\alpha_{|[c,c']}$ is not affine, or equivalently such that no flat F contains c and c', then the time one map of the Euler-Lagrange flow has positive entropy.

4.11 In the autonomous case, the sets $\mathcal{A}_t(c)$ and $\mathcal{G}_t(c) \subset M$ do not depend on t, and we have, using the notations of 0.5,

$$R(c) = V\big(\mathcal{G}_t(c)\big)^{\perp} \subset V\big(\mathcal{A}_t(c)\big)^{\perp} = E(c) \subset VF(c)$$

for each t. Hence each C-equivalence class is contained in a flat of α, so that our results are of no interest in the autonomous case.

5 Convergence of the Lax-Oleinik semigroup

The Graph properties provide a good description of the Mañe set $\tilde{\mathcal{N}}$. However, the set involved in the hypothesis of Theorem 0.6 is the *a priori* larger set $\tilde{\mathcal{G}}$. The relations between the sets $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{N}}$ are related to the asymptotic behavior of the so called Lax-Oleinik semi-group. In all this section, we will consider a critical Lagrangian L as defined in 3.1. Results similar to the ones of this section have been obtained from the point of view of Hamilton-Jacobi equations in [14] by J. M. Roquejoffre.

5.1 We say that L is regular if the liminf in the definition of the functions $h_{s,s'}$ given in 3.2 is a limit for all s, s', x, x'. In this case, since the functions $F_{t,t'}$ are equilipschitz, we have uniform convergence of the sequence $F_{t,t'}$, $t \bmod 1 = s$, $t' \bmod 1 = s'$ to $h_{s,s'}$ for all s, s'. If L is regular and if η is an exact one-form on $M \times S$, then $L - \eta$ is regular.

5.2 It is usual to define the mapping $T_t : C(M, \mathbb{R}) \longrightarrow C(M, \mathbb{R})$ by the expression

$$T_t u(x) = \min_{y \in M} \big(u(y) + F_{0,t}(y, x)\big).$$

The sequence $(T_n)_{n \in \mathbb{N}}$ is a semi-group called the Lax-Oleinik semi-group, see [2],[3] and [4]. We say that the Lax-Oleinik semi-group is convergent if, for each function $u \in C(M, \mathbb{R})$, there exists a function $U \in C(M \times S, \mathbb{R})$ such that

$$\lim_{t \bmod 1 = s, t \to \infty} T_t u(x) = U(x, s).$$

It is known that the Lax-Oleinik semi-group is convergent if and only if L is regular, see [2] and [4]. We recall the argument. If L is regular, then the Lax-Oleinik semi-group is clearly convergent with limit

$$U(x, s) = \min_{y \in M} \big(u(y) + h_{0,s}(y, x)\big).$$

On the other hand, Assume that the Lax-Oleinik semi-group is convergent. Let us fix $t \in \mathbb{R}$ and $z \in M$, and set $u(x) = F_{t,k}(z, x)$, where $k \in \mathbb{N}$ is chosen such that $k \geqslant 1 + t$. For each $t' \geqslant k$, we have $F_{t,t'}(z, x) = T_{t'-k}u(x)$. If we fix $t' \bmod 1 = s'$ and let t' go to infinity, this is converging to $U(x, s')$, which has to be equal to $h_{s,s'}(z, x)$. It follows that L is regular.

5.3 Proposition If L is regular, then $\tilde{\mathcal{G}} = \tilde{\mathcal{N}}$.

Proof Let $\gamma \in C^1(\mathbb{R}, M)$ be a minimizing orbit. We have to prove that this orbit is semi-static. Let us consider a sequence $t_k \longrightarrow -\infty$ such that $s = t_k \bmod 1$ does not depend on k and such that $\alpha = \lim \gamma(t_k)$ exists. In the same way, we consider a sequence $t'_k \longrightarrow \infty$ and set $s' = t'_k \bmod 1$ and $\omega = \lim \gamma(t'_k)$. Since L is regular,we have

$$A\big(\gamma|_{[t_k,t'_k]}\big) = F_{t_k,t'_k}\big(\gamma(t_k), \gamma(t'_k)\big) \longrightarrow h_{s,s'}\big(\alpha, \omega\big).$$

Let us consider a compact interval of times $[a, b]$, and assume to make things simpler that $s' = a \bmod 1$ and $s = b \bmod 1$. For k large enough, we have

$$\begin{aligned} A\big(\gamma|_{[a,b]}\big) &= A\big(\gamma|_{[t_k,t'_k]}\big) - A\big(\gamma|_{[t_k,a]}\big) - A\big(\gamma|_{[b,t'_k]}\big) \\ &= F_{t_k,t'_k}(\gamma(t_k), \gamma(t'_k)) - F_{t_k,a}(\gamma(t_k), \gamma(a)) - F_{b,t'_k}(\gamma(b), \gamma(t'_k)). \end{aligned}$$

Taking the limit, we get

$$A\big(\gamma|_{[a,b]}\big) = h_{s,s'}(\alpha, \omega) - h_{s,s}(\alpha, \gamma(a)) - h_{s',s'}(\gamma(b), \omega).$$

On the other hand, we observe if L is regular that

$$h_{s,s'}(\alpha, \omega) \leqslant h_{s,s}(\alpha, \gamma(a)) + \Phi_{s,s'}(\gamma(a), \gamma(b)) + h_{s',s'}(\gamma(b), \omega).$$

As a consequence, we obtain

$$A\big(\gamma|_{[a,b]}\big) \leqslant \Phi_{s,s'}(\gamma(a), \gamma(b))$$

hence γ is semi-static. □

5.4 Lemma If for each $(x, s) \in \mathcal{M}$, the liminf in the definition of $h_{s,s}(x, x)$ is a limit, *i.e.* if

$$F_{t,t+n}(x, x) \underset{n \to \infty}{\longrightarrow} 0$$

for each $(x, s) \in \mathcal{M}$ and each t satisfying $t \bmod 1 = s$, then L is regular.

Corollary If $\tilde{\mathcal{M}}$ is a union of 1-periodic orbits, then L is regular.

PROOF Let us fix (x, s) and (x', s') in $M \times S$, and $\epsilon > 0$. We want to prove that there exists T such that, if t and t' satisfy $t \bmod 1 = s$, $t' \bmod 1 = s'$ and $t' \geqslant t + T$, then

$$F_{t,t'}(x, x') \leqslant h_{s,s'}(x, x') + \epsilon.$$

Let K be a common Lipschitz constant of all functions $F_{t,t'}$ with $t' \geqslant t + 1$. Such a constant exists by Lemma 3.3. Let $\gamma : [t, t'] \longrightarrow M$ be a minimizing curve such that $A(\gamma) = F_{t,t'}(x, x')$ and such that $\gamma(t) = x$ and $\gamma(t') = x'$. By Lemma 2.9, it is possible to chose $t_0 \leqslant t_1 \leqslant t'_0$ such that $t_0 \bmod 1 = s$ and $t'_0 \bmod 1 = s'$, and a minimizing curve $\gamma \in C^1([t_0, t'_0], M)$ such that $A(\gamma) = F_{t_0,t'_0}(x, x')$ and such that $\gamma(t_0) = x$, $\gamma(t'_0) = x'$ and $d(\gamma(t_1), \mathcal{M}_{t_1}) \leqslant \epsilon/4K$. Since $h_{s,s'}(x, x') = \liminf F_{t,t'}(x, x')$, we can suppose in addition that

$$F_{t_0,t'_0}(x, x') \leqslant h_{s,s'}(x, x') + \epsilon/2.$$

Let $x_1 = \gamma(t_1)$, we have

$$F_{t_0,t'_0}(x, x') = F_{t_0,t_1}(x, x_1) + F_{t_1,t'_0}(x_1, x'),$$

and there exists a point $y \in \mathcal{M}_{t_1}$ such that $d(x_1, y) \leqslant \epsilon/4K$. It follows that

$$\left|F_{t_0,t'_0}(x, x') - F_{t_0,t_1}(x, y) - F_{t_1,t'_0}(y, x')\right| \leqslant \epsilon/2,$$

hence

$$F_{t_0,t_1}(x, y) + F_{t_1,t'_0}(y, x') \leqslant h_{s,s'}(x, x') + \epsilon.$$

Writing and $t' - t = t'_0 - t_0 + n$ with $n \in \mathbb{N}$, we have

$$F_{t,t'}(x, x') = F_{t_0,t'_0+n}(x, x') \leqslant F_{t_0,t_1}(x, y) + F_{t_1,t_1+n}(y, y) + F_{t_1+n,t'_0+n}(y, x').$$

Taking the limsup yields

$$\limsup F_{t,t'}(x, x') \leqslant F_{t_0,t_1}(x, y) + 0 + F_{t_1,t'_0}(y, x') \leqslant h_{s,s'}(x, x') + \epsilon.$$

Since this holds for all $\epsilon > 0$, the lemma is proved. Let us now prove the corollary. If $\gamma \in C^1(\mathbb{R}, M)$ is 1-periodic and minimizing, then for each t the sequence

$$F_{t,t+n}(\gamma(t), \gamma(t + n)) = nF_{t,t+1}(\gamma(t), \gamma(t + 1))$$

is bounded, hence $F_{t,t+n}(\gamma(t), \gamma(t)) = 0$ for each n. As a consequence, if $\tilde{\mathcal{M}}$ is a union of 1-periodic orbits, then the hypothesis of the lemma is satisfied and L is regular. □

5.5 One may wish to consider the given Lagrangian L, which is 1-periodic in time, as a k-periodic function of time only. This is best done in our setting by considering the mapping

$$\begin{aligned} P_k : TM \times S &\longrightarrow TM \times S \\ (x, v, t) &\longmapsto (x, v/k, kt) \end{aligned}$$

and the new 1-periodic Lagrangian $L^k = L \circ P_k$. This Lagrangian has the property that a curve $\gamma \in C^1(I, M)$ is an extremal of L^k if and only if the curve $\gamma^k : t \longmapsto \gamma(kt)$ is an extremal of L. We call $\mathcal{M}^k$, $\mathcal{A}^k$,... the various sets associated to L^k. It is clear that

$$P_k(\tilde{\mathcal{G}}^k) = \tilde{\mathcal{G}}.$$

On the other hand, we have

$$\tilde{\mathcal{N}} \subset P_k(\tilde{\mathcal{N}}^k)$$

but it is not hard to build examples where $\tilde{\mathcal{N}} \neq P_k(\tilde{\mathcal{N}}^k)$ (see [4]). Since $P_k(\tilde{\mathcal{N}}^k) \subset \tilde{\mathcal{G}}$, this provides examples where

$$\tilde{\mathcal{G}} \neq \tilde{\mathcal{N}}.$$

A direct consequence of Corollary 5.4 and Proposition 5.3 is
LEMMA If $\mathcal{M}$ is a union of k-periodic orbits, then L^k is regular, hence $\tilde{\mathcal{G}} = P_k(\tilde{\mathcal{N}}^k)$.

5.6 LEMMA If $\tilde{\mathcal{M}}$ is minimal in the sense of topological dynamics and if there exists a sequence γ_n of n-periodic curves such that $A(\gamma_n) \longrightarrow 0$, then L is regular, hence $\tilde{\mathcal{A}} = \tilde{\mathcal{N}} = \tilde{\mathcal{G}}$.

PROOF We can suppose that the curves γ_n are minimizers. Let us consider the n-periodic orbits $X_n(t) = (d\gamma_n(t), t \bmod 1)$. Let us also note X_n the image of X_n, which is a compact subset of $TM \times S$. Each subsequence of X_n has a convergent subsequence (for the Haußdorff topology). The limit of such a subsequence is an invariant subset of $\tilde{\mathcal{M}}$. Since $\tilde{\mathcal{M}}$ is minimal, this limit has to be $\tilde{\mathcal{M}}$, hence X_n is converging to $\tilde{\mathcal{M}}$ for the Haußdorff topology. It follows that each point $(x,s) \in \mathcal{M}$ is the limit of a sequence $(\gamma_n(t_n), s)$ with $t_n \bmod 1 = s$ for each n. Using Lemma 3.3, we get that

$$\limsup F_{t,t+n}(x,x) = \limsup F_{t,t+n}(\gamma_n(t_n), \gamma_n(t_n)) = \limsup A(\gamma_n) = 0$$

for each $(x,s) \in \mathcal{M}$ and each t satisfying $t \bmod 1 = s$. By Lemma 5.4, L is regular. $\square$

5.7 THEOREM (FATHI, [2]) If L does not depend on t, then it is regular.
As a consequence, in the autonomous case, the sets $\tilde{\mathcal{G}}$ and $\tilde{\mathcal{N}}$ are the same, hence our result is precisely the result of Mather in this case. See however 4.11.

6 TWIST MAPS

We are now going to specify our results in the case $M = S$. As we shall see, unlike Mather's theorem of [11], our result in high dimension is optimal when restricted to this case, in the sense that two cohomology classes c and c' are C-equivalent if and only if the sets $\tilde{\mathcal{G}}(c)$ and $\tilde{\mathcal{G}}(c')$ are not separated by a rotational invariant curve.

6.1 Let $f : TS \longrightarrow TS$ be the Poincaré return map associated to the section $TS \times \{0\}$. Moser has proved that any twist map of the annulus TS can be realized as the Poincaré map of a Lagrangian flow satisfying our hypotheses ([13]).

6.2 We identify $H^1(S,\mathbb{R})$ and $H_1(S,\mathbb{R})$ with $\mathbb{R}$ in the standard way. We shall use the term Lipschitz graph for a set which is the image of a subset of the basis S or $S \times S$ by a Lipschitz section of the tangent bundle $TS \longrightarrow S$ or $TS \times S \longrightarrow S \times S$. We shall use the term full Lipschitz graph for a set which is the image of a Lipschitz section of the tangent bundle. A rotational invariant curve for f is a closed curve of TS which is invariant by f and is not homotopic to a constant curve or equivalently an invariant curve the complement of which has two unbounded connected components. The Euler-Lagrange vectorfield gives a canonical homotopy between the identity and f, hence each rotational invariant circle has a well defined real rotation number. For each $c \in H^1(S, \mathbb{R})$, the set $\tilde{\mathcal{A}}_0(c)$ is a Lipschitz graph which is invariant by f. By the theory of homeomorphisms of the circle, the map f restricted to $\tilde{\mathcal{A}}_0(c)$ has a rotation number, which is the only subderivative of α at point c. Hence α is differentiable, and $\alpha'(c)$ is the rotation number of $f|_{\tilde{\mathcal{A}}_0(c)}$.

6.3 Theorem If $R(c) = 0$, then the set $\tilde{\mathcal{G}}(c)$ contains a rotational invariant curve of rotation number $\alpha'(c)$. This curve is a full Lipschitz graph.

Note that $R(c) = 0$ if and only if $\mathcal{G}(c) = S \times S$. We have to prove that $\tilde{\mathcal{G}}(c)$ contains a full Lipschitz graph if $\mathcal{G}(c) = S \times S$. Let us first mention a corollary.

6.4 Let $C \subset \mathbb{R}$ be the set of cohomology classes $c \in \mathbb{R}$ such that $R(c) = 0$, or equivalently such that $\mathcal{G}(c) = S \times S$. Since the mapping $c \longrightarrow \mathcal{G}(c)$ is upper semi-continuous (see 2.1), the set C is closed. Let $\Omega = \alpha'(C)$ be the set of rotation numbers of the sets $\tilde{\mathcal{G}}(c)$ which contain full Lipschitz graphs. Since the function α is convex and superlinear, the set Ω is closed. Assume that $\Omega \neq \mathbb{R}$, then the complement of Ω contains an open interval I. It follows from the theorem above that $R(c) \neq 0$ if $\alpha'(c) \in I$, hence the set $\alpha'^{-1}(I)$ is contained in a C-equivalence class. On the other hand, this set is not contained in a face of α, so that we have the following consequence of Corollary 4.10:

Corollary If $\Omega \neq \mathbb{R}$, then the diffeomorphism f has positive topological entropy.

Corollary If there exists $\omega \in \mathbb{R}$ such that no rotational invariant curve of rotation number ω exist, then the diffeomorphism f has positive topological entropy.

Let us now prove Theorem 6.3. We need the following result.

6.5 Proposition (Mather [8]) The function β is differentiable at irrational points. Equivalently, all the flats of α of dimension 1 have rational slope.

The original proof in [8],[7] of this result is rather complicated. We shall obtain it as a consequence of the inclusion $VF(c) \subset G(c)$ of 4.9. It is enough to prove that any continuous closed form η of $S \times S$ which vanishes on $\mathcal{M}(c)$ has trivial cohomology in $H^1(M, \mathbb{R})$. Let us consider the universal cover $\mathbb{R}^2$ of $S \times S$. The closed one form lifts to an exact form dg, where $g \in C^1(\mathbb{R}^2, \mathbb{R})$. We still call $\mathcal{M}$ the lifting of $\mathcal{M}(c)$. It is a union of embedded lines, which will be called the trajectories of $\mathcal{M}$. Since $\eta = 0$ on $\mathcal{M}(c)$, the function g is constant on each trajectory of $\mathcal{M}$. Let us call g_0 the restriction of g to $t = 0$, *i.e.* $g_0(x) = g(x, 0)$, and $\mathcal{M}_0 = \mathcal{M} \cap \{t = 0\} \subset \mathbb{R}$. Proving that η has trivial cohomology is equivalent to proving that the constant $g_0(x+1) - g_0(x)$ is null. It is enough to prove that the function g_0 is constant on $\mathcal{M}_0$. Let $]x^-, x^+[$ be a connected component of the complement of $\mathcal{M}_0$. Let $\gamma^\pm$ be the trajectory of $\mathcal{M}$ which contains $(x^\pm, 0)$. Since the function g is Lipschitz and constant on γ^- and γ^+, and since the distance between γ^- and γ^+ in $\mathbb{R}^2$ is zero, we have $g_0(x^-) = g_0(x^+)$. Since this holds for all connected component of the complement of $\mathcal{M}_0$, there exists a continuous function $\tilde{g}_0 \in C(\mathbb{R}, \mathbb{R})$ which is equal to g_0 on $\mathcal{M}_0$ and is constant on each connected component of the complement. This function is differentiable at each point, with zero derivative. Hence it is constant, so that the restriction of g_0 to $\mathcal{M}_0$ is constant. $\square$

6.6 Let us consider a rational number $\omega = p/q$ in lowest terms. Let us choose $c \in \partial\beta(\omega)$. We see from the theory of homeomorphisms of the circle that the Mather set $\tilde{\mathcal{M}}(c)$ is a union of orbits of period q, whose lifting $\bar{\gamma}$ to the universal cover $\mathbb{R}$ satisfy $\bar{\gamma}(t+q) = \bar{\gamma}(t)+p$. Among the curves $\xi \in C^1(\mathbb{R}, S)$ whose liftings $\bar{\xi}$ to the universal cover $\mathbb{R}$ satisfy $\bar{\xi}(t+q)-\bar{\xi}(t) = p$, those which are orbits of $\tilde{\mathcal{M}}(c)$ are precisely those which minimize the action. As a consequence, these orbits all have the same action $A(\gamma) = q\beta(p/q)$.

6.7 RATIONAL ROTATION NUMBER Let us assume that $\alpha'(c)$ is a rational number p/q in lowest terms. By Lemma 5.5, we have $\tilde{\mathcal{G}}(c) = P_q(\tilde{\mathcal{N}}^q(c))$. Let $\mathcal{H}$ be the closure of a connected component of the complement of $\mathcal{M}(c)$ in $M \times S$. The boundary of $\mathcal{H}$ is made of two periodic curves γ^+ and γ^-. We see from the second graph property that $\tilde{\mathcal{G}}(c) \cap \Pi^{-1}\mathcal{H}$ is the union of two graphs $\tilde{\mathcal{G}}^+$ and $\tilde{\mathcal{G}}^-$, where the orbits $\tilde{\mathcal{G}}^+$ are heteroclinic from γ^- to γ^+, as well as γ^- and γ^+ themselves, and the orbits of $\tilde{\mathcal{G}}^-$, are heteroclinic from γ^+ to γ^- as well as γ^- and γ^+. If none of the projected sets $\mathcal{G}^+ = \Pi(\tilde{\mathcal{G}}^+)$ and $\mathcal{G}^- = \Pi(\tilde{\mathcal{G}}^-)$ is $\mathcal{H}$, then their union is also properly contained in $\mathcal{H}$ *i.e.* $\mathcal{H} \cap \mathcal{G}(c) \neq \mathcal{H}$, hence $\mathcal{G}(c)$ is properly contained in $S \times S$ so that $R(c) = \mathbb{R}$. Else, $\tilde{\mathcal{G}}(c) \cap \Pi^{-1}\mathcal{H}$ contains a Lipschitz graph. If for all possible choice of $\mathcal{H}$ the second option holds, then clearly all the Lipschitz graphs can be glued together, and $\tilde{\mathcal{G}}(c)$ contains a full Lipschitz graph.

6.8 IRRATIONAL ROTATION NUMBER Let us assume that $\alpha'(c)$ is an irrational number ω. The Mather set $\tilde{\mathcal{M}}(c)$ is minimal in the sense of topological dynamics, and we have

$$\tilde{\mathcal{A}}(c) = \tilde{\mathcal{N}}(c) = \tilde{\mathcal{G}}(c).$$

As a consequence $\tilde{\mathcal{G}}(c)$ is a Lipschitz graph.

PROOF That the Mather set is minimal is a consequence of the theory of homeomorphisms of the circle. We can assume by subtracting a critical form η satisfying $[\eta] = c$ that $\beta(\omega) = 0 = \beta'(\omega)$. In view of Lemma 5.6, it is enough to prove the existence of a sequence γ_n of n-periodic orbits such that $A(\gamma_n) \longrightarrow 0$. For each integer n, let us consider a real number $c_n \in \partial\beta([n\omega]/n)$, so that $\tilde{\mathcal{M}}(c_n)$ contains a periodic orbit γ_n of period n and rotation number $[n\omega]/n$. In view of 6.6, the orbit γ_n has action $A(\gamma_n) = n\beta([n\omega]/n)$ which is converging to 0 because $\beta(\omega) = 0 = \beta'(\omega)$. □

6.9 In terms of the Lax-Oleinik semi-group, we have proved the following. Let L be a critical Lagrangian, and let ω be the rotation number of $\tilde{\mathcal{A}}$. Let us consider the integer k defined by $k = 1$ if ω is irrational, and $k = q$ if $\omega = p/q$ in lowest terms. Then the semi-group $T_n^k, n \in \mathbb{N}$ is converging. Here we may view equivalently T_n^k as T_{kn}, or as the Lax-Oleinik semi-group associated to L^k. In other words, the semi-group T_n has k-periodic asymptotic orbits. Part of this result was obtained by J. M. Roquejoffre in [14]. In the paper of Roquejoffre, the convergence is proved in the case of an irrational rotation number only under the additional assumption that the Mather set is the full circle.

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