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BAYESIAN SEMI-PARAMETRIC ESTIMATION OF THE LONG-MEMORY PARAMETER UNDER FEXP-PRIORS: SUPPLEMENTARY MATERIAL *

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This supplementary material contains technical lemmas and inequalities used in the main text of *Bayesian semi-parametric estimation of the long-memory parameter under FEXP-priors*. It starts with a series of (in)equalities on integrals (Lemmas 1.1 and 1.2) and a deviation bound for quadratic forms (Lemma 1.3). In Lemmas 2.1-2.6 (section 2) we prove results on the asymptotic behavior of Toeplitz matrices. These results require the Hölder constants of various functions, which are given in section 3. Finally, the proof of Lemma B.2 is given in section 4.

1. Technical results. Let $\eta_j = -1_{j>0}2/j$ and recall that $\bar{\theta}_{d,k} = \theta_{o[k]} + (d_o - d)\eta_{[k]}$. Let the sequence $\{a_j\}$ be defined as $a_j = \theta_{o,j} + (d_o - d)\eta_j$ when $j > k$ and $a_j = 0$ when $j \leq k$. In addition, define

$$H_k(x) = \sum_{j=k+1}^{\infty} \eta_j \cos(jx), \qquad G_k(x) = \sum_{j=1}^{k} \eta_j \cos(jx), \tag{1.1}$$

$$\Delta_{d,k}(x) = \sum_{j=k+1}^{\infty} (\theta_{o,j} + (d_o - d)\eta_j) \cos(jx) = \sum_{j=k+1}^{\infty} a_j \cos(jx). \tag{1.2}$$

Using this notation we can write

$$-2 \log |1 - e^{ix}| = -\log(2 - 2\cos(x)) = G_k(x) + H_k(x), \tag{1.3}$$

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$$
\begin{aligned}
f_{d,k}(x) = f_{d,k,\bar\theta_{d,k}}(x) &= f_o(x)\exp\left\{-\sum_{j=k+1}^{\infty} a_j\cos(jx)\right\} \\
&= f_o(x)e^{-\Delta_{d,k}(x)} = f_o(x)e^{(d-d_o)H_k(x)-\Delta_{d_o,k}(x)}.
\end{aligned} \tag{1.4}
$$

Given d,k and θ_o, the sequence $\{a_j\}$ represents the closest possible distance between f_o and $f_{d,k,\theta}$, since

$$
l(f_o, f_{d,k}) = l(f_o, f_{d,k,\bar\theta_{d,k}}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\Delta^2_{d,k}(x)dx = \sum_{j>k} a_j^2. \tag{1.5}
$$

From (1.4) it also follows that for all d,

$$
\frac{\partial}{\partial d} f_{d,k} = H_k f_{d,k}. \tag{1.6}
$$

LEMMA 1.1. *When $\theta_o \in \Theta(\beta, L_o)$, there exist constants such that for any positive integer k,*

$$
\begin{aligned}
k^{-1} \lesssim \int_{-\pi}^{\pi} H_k^2(x)dx \quad &\lesssim \quad k^{-1}, \qquad (1.7)\\
\sum_{l>k}|\theta_{o,l}| \quad &= \quad O(k^{-\beta+\frac{1}{2}}), \quad \sum_{l\geq 0}|\theta_{o,l}| = O(1), \qquad (1.8)\\
\int_{-\pi}^{\pi}\Delta_{d_o,k}(x)H_k(x)dx \quad &= \quad \sum_{j>k}\eta_j\theta_{o,j} = O\left(k^{-\frac{1+2\beta}{2}}\right), \qquad (1.9)\\
\int_{-\pi}^{\pi}\Delta^2_{d_o,k}(x)dx \quad &= \quad \sum_{l>k}\theta^2_{o,l} = O\left(k^{-2\beta}\right), \qquad (1.10)\\
\int_{-\pi}^{\pi}\Delta^2_{d_o,k}(x)H_k(x)dx \quad &= \quad O\left(k^{-2\beta-1}\right), \qquad (1.11)\\
\int_{-\pi}^{\pi} H_k^4(x)dx \quad &\lesssim \quad \frac{\log k}{k}. \qquad (1.12)
\end{aligned}
$$

When $k\to\infty$, the big-O in (1.8)-(1.11) may be replaced by a small-o, since $\sum_{l>k}\theta^2_{o,l}l^{2\beta}$ then tends to zero.

PROOF. The result for $\int H_k^2(x)dx$ follows directly from the definition of H_k. The assumption that $\theta_o \in \Theta(\beta, L_o)$ and the Cauchy-Schwarz inequality imply that

$$
\sum_{l>k}|\theta_{o,l}| \leq \sqrt{\sum_{l>k}\theta^2_{o,l}l^{2\beta}}\sqrt{\sum_{l>k}l^{-2\beta}} = O(k^{-\beta+\frac{1}{2}}),
$$

proving the first result in (1.8). Similarly, one can prove (1.9). For (1.10), note that $\sum_{l>k}\theta_{o,l}^2 \le k^{-2\beta}\sum_{l>k}\theta_{o,l}^2 l^{2\beta}$. For the other bounds we omit the details of the proof. They follow from the fact that for all sequences a, b and c,

$$\begin{aligned}
&2\sum_{l,m,n>k} a_l b_m c_n \int_{-\pi}^{\pi} \cos(lx)\cos(mx)\cos(nx)dx \\
&\qquad = \sum_{m,n>k} b_m c_n \sum_{l>k} a_l \int_{-\pi}^{\pi} \cos(lx)\left(\cos((m+n)x)+\cos((m-n)x)\right)dx \\
&\qquad = \sum_{m,n>k} a_{m+n} b_m c_n + \sum_{m,n>k;m-n>k} a_{m-n} b_m c_n.
\end{aligned}$$

□

Before stating the next lemma we give bounds for the functions H_k and G_k. Since $-2\log|1-e^{ix}| = -\log(x^2+O(x^4))$, there exist positive constants c, B_0, B_1 and B_2 such that

$$|H_k(x)| \ge B_0|\log x|, \quad |x| \le ck^{-1}, \tag{1.13}$$

$$|H_k(x)| \le B_1|\log x| + B_2\log k, \quad x\in[-\pi,\pi]. \tag{1.14}$$

LEMMA 1.2. *Let $a_j = (\theta_{o,j}-(d-d_o)\eta_j)1_{j>k}$, as in (1.2). Then for $p\ge 1$ and $q=2,3,4$ there exist constants $c(p,q)$ such that for all $d\in(-\frac{1}{2},\frac{1}{2})$ and $k\le \exp(|d-d_o|^{-1})$,*

$$\begin{aligned}
\int_{-\pi}^{\pi}\left(\frac{f_o(x)}{f_{d,k}(x)}\right)^p |H_k|^q(x)dx &= O\left(\frac{(\log k)^{c(p,q)}}{k}\right) \\
&\quad + O((\log k)^{q+pB_2|d-d_o|}|d-d_o|^{-\frac{q}{2}}e^{-|d-d_o|^{-1}}),
\end{aligned} \tag{1.15}$$

(1.16)
$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{f_o}{f_{d,k}}(x)-1\right)\cos(ix)\cos(jx)dx = \frac{1}{2}a_{i+j}1_{i+j>k} + O\left(\sum_{j>k}a_j^2\right),$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{f_o}{f_{d,k}}(x)-1\right)H_k^2(x)dx = O(|d-d_o|k^{-1}\log k), \tag{1.17}$$

where the constant B_2 in (1.15) is as in (1.14), and the constants in (1.16) and (1.17) are uniform in d. The constant $c(p,q)$ in (1.15) equals $0,\frac{1}{2},1$ when respectively $q=2,3,4$.

Proof. When $d = d_o$, (1.15) directly follows from (1.7) and (1.12), because of the boundedness of $(f_o/f_{d_o,k})^p = \exp\{p\Delta_{d_o,k}\}$. Now suppose $d \neq d_o$. Let $C_k = \max_{x\in[-\pi,\pi]} \exp\{|\Delta_{d_o,k}(x)|\}$ and $b_m = \max_{x\in[m,\pi]} |(d-d_o)H_k(x)|$, for $m = e^{-\frac{1}{|d-d_o|}} < e^{-1}$. Since $\sum_{j=0}^{\infty} |\theta_{o,j}| < \infty$, the sequence C_k is bounded by some constant C. To prove (1.15) we write

$$\begin{aligned}
&\frac{1}{2}\int_{-\pi}^{\pi} \left(\frac{f_o(x)}{f_{d,k}(x)}\right)^p |H_k|^q(x)dx \\
&\quad= \int_0^m \left(\frac{f_o(x)}{f_{d,k}(x)}\right)^p |H_k|^q(x)dx + \int_m^{\pi} \left(\frac{f_o(x)}{f_{d,k}(x)}\right)^p |H_k|^q(x)dx.
\end{aligned} \tag{1.18}$$

We first bound the last integral in the preceding display, by substitution of $(f_o/f_{d,k})^p = \exp\{p\Delta_{d,k}\} = \exp\{-p(d-d_o)H_k + p\Delta_{d_o,k}\}$. From (1.14) it follows that

$$b_m \leq |d-d_o|(B_1|d-d_o|^{-1} + B_2\log k) \leq B_1 + B_2,$$

as $k \leq \exp(|d-d_o|^{-1})$. Hence we obtain $(f_o/f_{d,k})^p \leq Ce^{b_m}$ on (m,π). For $q = 2$ and $q = 4$ the bound on the last integral in (1.18) therefore follows from (1.7) and (1.12); for $q = 3$ the bound follows from the Cauchy-Schwarz inequality.

Next we bound the first integral in (1.18). Because the function $x^{|d-d_o|}(\log x)^2$ has a local maximum of $4|d-d_o|^{-2}e^{-2}$ at $x = e^{-2/|d-d_o|}$, $(\log x)^2 \leq 4x^{-|d-d_o|}|d-d_o|^{-2}e^{-2}$ for all $x \in [0,m]$. Again using (1.14) we find that

$$\begin{aligned}
\int_0^m \left(\frac{f_o(x)}{f_{d,k}(x)}\right)^p |H_k|^q(x)dx &\lesssim \sum_{j=0}^{q}\binom{q}{j}\int_0^m (B_1|\log x|)^j\,(B_2\log k)^{q-j}\,e^{-p(d-d_o)H_k(x)}dx \\
&\lesssim \sum_{j=0}^{q}\binom{q}{j}(\log k)^{q-j+pB_2|d-d_o|}\int_0^m (B_1|\log x|)^j\,x^{-pB_1|d-d_o|}dx \\
&\leq \sum_{j=0}^{q}\binom{q}{j}(\log k)^{q-j+pB_2|d-d_o|}\left(\frac{2B_1^2}{e|d-d_o|}\right)^{\frac{j}{2}}\int_0^m x^{-(j/2+pB_1)|d-d_o|}dx \\
&\lesssim (\log k)^{q+pB_2|d-d_o|}|d-d_o|^{-\frac{q}{2}}e^{-1/|d-d_o|}.
\end{aligned}$$

We now prove (1.16).

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{f_o}{f_{d,k}}(x) - 1\right)\cos(ix)\cos(jx)dx \\
&\quad= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(e^{\Delta_{d,k}(x)} - 1\right)\cos(ix)\cos(jx)dx \\
&\quad\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\Delta_{d,k}(x) + \frac{1}{2}\Delta_{d,k}^2(x)e^{(\Delta_{d,k}(x))_+}\right)\cos(ix)\cos(jx)dx.
\end{aligned}$$

The linear term equals

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Delta_{d,k}(x)\cos(ix)\cos(jx)dx$$
$$=\frac{1}{4\pi}\int_{-\pi}^{\pi}\left(\sum_{l>k}a_l\cos(lx)\right)(\cos((i+j)x)+\cos((i-j)x))\,dx=\frac{1}{2}a_{i+j}1_{i+j>k}.$$

For the quadratic term we have

(1.19)
$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\Delta_{d,k}^2(x)e^{(\Delta_{d,k}(x))_+}\cos(ix)\cos(jx)dx\right|\leq\frac{1}{2\pi}\int_{-\pi}^{\pi}\Delta_{d,k}^2(x)e^{(\Delta_{d,k}(x))_+}dx$$
$$\leq\frac{1}{2\pi}\int_0^m\Delta_{d,k}^2(x)e^{-\Delta_{d,k}(x)}dx+\frac{(1+Ce^{b_m})}{2\pi}\int_{-\pi}^{\pi}\Delta_{d,k}^2(x)dx.$$

This is $O(\sum_{j>k}a_j^2)$, which follows from (1.5) and integration over $(0,e^{-\frac{1}{\bar{v}_n}})$ and $(e^{-\frac{1}{\bar{v}_n}},\pi)$ as above.

To prove (1.17), write $\exp(\Delta_{d,k})-1=\Delta_{d,k}+\Delta_{d,k}^2e^{\xi}$ with $\Delta_{d,k}=-(d-d_o)H_k(x)+\Delta_{d_o,k}(x)$ and $|d-d_o|\leq\bar{v}_n$, substitute (1.14) and proceed as in the proof of (1.15) above. The biggest term is a multiple of $|d-d_o|\int_{-\pi}^{\pi}|H_k(x)|^3dx$, which is $O(\bar{v}_nk^{-1})$. This is larger than the approximation error when $\beta>\frac{1}{2}(1+\sqrt{2})$. □

Lemma 1.3. *Let A be a symmetric matrix matrix such that $|A|=1$ and let $Y=(Y_1,\dots,Y_n)$ be a vector of independent standard normal random variables. Then for any $\alpha>0$,*

$$P\left(Y^tAY-tr(A)>n^{\alpha}\right)\leq\exp\{-n^{\alpha}/8\}.$$

Proof. Note that $\|A\|\leq|A|=1$ so that for all $s\leq 1/4$, $sy^tAy\leq s_0y^ty\|A\|\leq y^ty/4$ and $\exp\{sY^tAY\}$ has finite expectation. Choose $s=1/4$, then by Markov's inequality,

$$\begin{aligned}P\left(Y^tAY-\mathrm{tr}(A)>n^{\alpha}\right)&\leq e^{-n^{\alpha}/4}Ee^{(Y^tAY-\mathrm{tr}(A))/4}\\&=\exp\left\{-n^{\alpha}/4-\frac{1}{2}\log\det[I_n-A/2]-\mathrm{tr}(A)/4\right\}\\&\leq\exp\left\{-n^{\alpha}/4+\mathrm{tr}(A^2)/4\right\}.\end{aligned}$$

The last inequality follows from the fact that $A(I_n-\tau A/2)^{-1}$ has eigenvalues $\lambda_j(1-\tau\lambda_j/2)^{-1}$, where λ_j are the eigenvalues of A for all $\tau\in(0,1)$. Hence,

$\mathrm{tr}(A^2(I_n - \tau A/2)^{-2})$ is bounded by $4\mathrm{tr}(A^2)$. The result follows from the fact that when n is large enough $n^\alpha > 2\mathrm{tr}(A^2) = 2$. $\square$

2. Convergence of the trace of a product of Toeplitz matrices.

Suppose $T_n(f_j)$ $(j = 1, \ldots, p)$ are covariance matrices associated with spectral densities f_j. According to a classical result by Grenander and Szégö (Grenander and Szegö (1958)),

$$\frac{1}{n}\mathrm{tr}\left[\prod_{j=1}^{p} T_n(f_j)\right] \to (2\pi)^{2p-1}\int_{-\pi}^{\pi}\prod_{j=1}^{p} f_j(x)dx.$$

In this section we give a series of related results. We first recall a result from Rousseau et al. (2010).

Lemma 2.1. *Let $1/2 > t > 0$ and $L^{(i)}, M^{(i)} > 0$, $\rho_i \in (0,1]$, $d_i \in [-1/2+t, 1/2-t]$ for all $i = 1, ..., 2p$ and let f_i, $(i \leq 2p)$ be functions on $[-\pi, \pi]$ satisfying*

(2.1)
$$|f_i(x)| = |x|^{-2d_i} g_i(x),\ |g_i(x)| \leq M^{(i)},\ |g_i(x) - g_i(y)| \leq \frac{M^{(i)}|x-y|}{|x| \wedge |y|} + L^{(i)}|x-y|^{\rho_i}$$

and assume that $\sum_{i=1}^{p}(d_{2i-1} + d_{2i}) < \frac{1}{2}$. Then for all $\epsilon > 0$ there exists a constant K depending only on ϵ, t and $q = \sum_{j=1}^{p}(d_{2j-1} + d_{2j})_+$ such that

$$\left|\frac{1}{n}tr\left[\prod_{j=1}^{p} T_n(f_{2j-1})T_n(f_{2j})\right] - (2\pi)^{2p-1}\int_{-\pi}^{\pi}\prod_{j=1}^{2p} f_j(x)dx\right|$$
$$\leq K\sum_{j=2}^{2p}\left(\prod_{i\neq j} M^{(i)}\right) L^{(j)} n^{-\rho_j+\epsilon+2q} + K\prod_{i=1}^{2p} M^{(i)} n^{-1+q+\epsilon}.$$

To prove a similar result involving also inverses of matrices, we need the following two lemmas. They can be found elsewhere, but as we make frequent use of them they are included for easy reference and are formulated in a way better suited to our purpose. The first lemma can be found on p.19 of Rousseau et al. (2010), and is an extension of Lemma 5.2 in Dahlhaus (1989).

Lemma 2.2. *Suppose that for $0 < t < 1/2$ and $d \in [-1/2+t, 1/2-t]$*

$$|f(x)| = |x|^{-2d} g(x), \quad m \leq |g(x)| \leq M, \quad |g(x) - g(y)| \leq L|x-y|^\rho \tag{2.2}$$

and assume that $0 < m \leq 1 \leq M < +\infty$ *and* $L \geq 1$*. Then, for all* $\epsilon > 0$*, there exists a constant* K *depending on* t *and* ϵ *only such that*

$$|I_n - T_n^{\frac{1}{2}}(f) T_n\left(\frac{1}{4\pi^2 f}\right) T_n^{\frac{1}{2}}(f)|^2 \leq KL\frac{M^2}{m^2} n^{1-\rho+\epsilon}.$$

PROOF. By Lemma 2.1,

$$|I_n - T_n^{\frac{1}{2}}(f) T_n\left(\frac{1}{4\pi^2 f}\right) T_n^{\frac{1}{2}}(f)|^2 = \operatorname{tr}\left\{ I_n - 2T_n^{\frac{1}{2}}(f) T_n\left(\frac{1}{4\pi^2 f}\right) T_n^{\frac{1}{2}}(f) \right.$$
$$\left. + T_n^{\frac{1}{2}}(f) T_n\left(\frac{1}{4\pi^2 f}\right) T_n(f) T_n\left(\frac{1}{4\pi^2 f}\right) T_n^{\frac{1}{2}}(f) \right\}$$

converges to zero, the approximation error being bounded by $K[L(1 + M^2/m^2) + M^2/m^2]$. □

The next result can be found as Lemma 3 in Lieberman et al. (2011), and is an extension of Lemma 5.3 in Dahlhaus (1989).

LEMMA 2.3. *Suppose that* f_1 *and* f_2 *are such that* $|f_1(x)| \geq m|x|^{-2d_1}$ *and* $|f_2(x)| \leq M|x|^{-2d_2}$ *for constants* $d_1, d_2 \in (-\frac{1}{2}, \frac{1}{2})$ *and* $m, M > 0$*. Then*

$$\|T_n^{-\frac{1}{2}}(f_1) T_n^{\frac{1}{2}}(f_2)\| \leq C\frac{M}{m} n^{(d_2 - d_1)_+ + \epsilon}.$$

PROOF. In the proof of Lemma 5.3 on p. 1761 in Dahlhaus (1989), the first inequality only depends on the upper and lower bounds m and M. □

Using the preceding lemmas, the approximation result given in Lemma 2.1 for traces of matrix products can be extended to include matrix inverses.

LEMMA 2.4. *Suppose that* f *satisfies* (2.2) *with constants* d*,* ρ*,* L*,* m *and* M*. For* f_{2j}*,* $j = 1, \dots, p$*, assume that* (2.1) *holds with constants* d_{2j}*,* ρ_{2j}*,* $L^{(2j)}$ *and* $M^{(2j)}$ *(*$j = 1, \dots, p$*). For convenience, we denote* $M^{(2j-1)} = m^{-1}$ *,* $\rho_{2j-1} = \rho$ *and* $L^{(2j-1)} = L$ *(*$j = 1, \dots, p$*). Suppose in addition that* $d, d_{2j} \in [-\frac{1}{2} + t, \frac{1}{2} - t]$ *satisfy* $\sum_{j=1}^{p}(d_{2j} - d)_+ < \frac{1}{2}(\rho - \frac{1}{2})$*, and let* $q = \sum_{j=1}^{p}(d_{2j} - d)_+$*.*

Then for all $\epsilon > 0$ there exists a constant K such that

$$\begin{aligned}
&\left| \frac{1}{n} tr \left\{ \prod_{j=1}^{p} T_n^{-1}(f) T_n(f_{2j}) \right\} - \frac{1}{2\pi} \int_{-\pi}^{\pi} \prod_{j=1}^{p} \frac{f_{2j}(x)}{f(x)} dx \right| \\
&\quad \leq K \left[\sum_{j=2}^{2p} \left(\prod_{i \neq j}^{2p} M^{(i)} \right) L^{(j)} n^{-\rho_j} + n^{-1} \prod_{i \leq 2p} M^{(i)} \right] n^{\epsilon + 2q} \\
&\quad + \left(\prod_{j=1}^{p} M^{(2j)} \right) \left(L \frac{M}{m} \right)^{\frac{(p+1)}{2}} n^{(1-\rho)\frac{(p+1)}{2} - 1 + \epsilon + 2q},
\end{aligned} \tag{2.3}$$

and setting $\tilde{f} = 1/(4\pi^2 f)$,

$$\begin{aligned}
&\frac{1}{n} \left| tr \left\{ \prod_{j=1}^{p} T_n^{-1}(f) T_n(f_{2j}) \right\} - tr \left\{ \prod_{j=1}^{p} T_n(\tilde{f}) T_n(f_j) \right\} \right| \\
&\leq \left(\prod_{j=1}^{p} M^{(2j)} \right) \left(L \frac{M}{m} \right)^{\frac{(p+1)}{2}} n^{(1-\rho)\frac{(p+1)}{2} - 1 + \epsilon + 2q}.
\end{aligned} \tag{2.4}$$

PROOF. Without loss of generality, we consider the f_{2j}'s to be nonnegative When this is not the case, we write $f_{2j} = f_{2j}^{+} - f_{2j}^{-}$ and treat the positive and negative part separately; see also Dahlhaus (1989) , p. 1755-56. To prove (2.4), we use the construction of Lemma 5 from Lieberman et al. (2011), who treat the case $\rho = 1$ and $d_{2j} = d'$. Inspection of their proof shows that this extends to $\rho \neq 1$ and d_{2j} that differ with j. To prove (2.3), we use the construction of Dahlhaus' Theorem 5.1 (see also the remark on p. 744 of Lieberman and Phillips (2004), after (28)), and apply Lemma 2.1 with $f_{2j-1} = \tilde{f} = \frac{1}{4\pi^2 f}$, $j = 1, \ldots, p$. This gives the first term on the right in (2.3). The last term in (2.3) follows from (2.4). □

Although the bound provided by Lemma 2.4 is sufficiently tight for most purposes, certain applications require sharper bounds. These can only be obtained if we exploit specific properties of f and f_{2j}. In Lemma 2.5 below we improve on the first term on the right in (2.3). This is useful when for example $b_i(x) = cos(jx)$; the Lipschitz constant L is then of order $O(k)$, but the boundedness of b_i actually allows a better result. In Lemma 2.6 we improve on the last term of (2.3).

LEMMA 2.5. *Let $f(x) = |x|^{-2d}g(x)$ with $-1/2 < d < 1/2$ and g a bounded Lipschitz function satisfying $m < g < M$, with Lipschitz constant L.*

- *Let $b_1, \dots, b_p$ be bounded functions and let $\|b\|_\infty$ denote a common upper bound for these functions. Then for all $\epsilon > 0$,*

$$\begin{aligned}(2.5)\qquad &\left| tr\left[\prod_{i=1}^{p} T_n(b_i f)T_n(f^{-1})\right] - (2\pi)^p tr\left[\prod_{i=1}^{p} T_n(b_i)\right]\right| \\ &\qquad \le Cn^{\epsilon}\left(\frac{M}{m}\right)^p \|b\|_\infty^{p-1}\left(\|b\|_\infty + L\sum_{j=1}^{p}\|b_j\|_2\right).\end{aligned}$$

- *Let b_j $(j \ge 2)$ be bounded functions. Let b_1 be such that $\|b_1\|_2 < +\infty$, and assume that for all $a > 0$ there exists $M'(a) > 0$ such that*

$$\int_{-\pi}^{\pi} |b_1(x)||x|^{-1+a}dx \le M'(a).$$

Then for all $a > 0$

$$\begin{aligned}(2.6)\qquad &\left| tr\left[\prod_{i=1}^{p} T_n(b_i f)T_n(f^{-1})\right] - (2\pi)^p tr\left[\prod_{i=1}^{p} T_n(b_i)\right]\right| \\ &\le C\left(\frac{M}{m}\right)^p \prod_{i\ge 2}\|b_i\|_\infty \left(n^{3pa}M'(a) + L(\log n)^{2p-1}\|b_1\|_2\right).\end{aligned}$$

PROOF. We prove (2.5); the proof of (2.6) follows exactly the same lines. We define $\Delta_n(x) = e^{ix}$ and $L_n(x) = n \wedge |x|^{-1}$ where the latter is an upper bound of the former. Using the decomposition as on p. 1761 in Dahlhaus

(1989) or as in the proof of we find that

$$
\begin{aligned}
&\left|\mathrm{tr}\left[\prod_{i=1}^{p} T_n(b_i f)T_n(f^{-1})\right] - (2\pi)^p \mathrm{tr}\left[\prod_{i=1}^{p} T_n(b_i)\right]\right| \\
&\leq C\left|\int_{[-\pi,\pi]^{2p}} \prod_{i=1}^{p} b_i(x_{2i-1})\left(\prod_{i=1}^{p}\frac{f(x_{2i-1})}{f(x_{2i})}-1\right)\Delta_n(x_1-x_2)\dots\Delta_n(x_{2p}-x_1)dx\right| \\
&\leq C\left|\int_{[-\pi,\pi]^{2p}} \prod_{i=1}^{p} b_i(x_{2i-1})\frac{g(x_{2i-1})}{g(x_{2i})}\left(\prod_{i=1}^{p}\frac{|x_{2i-1}|^{-2d}}{|x_{2i}|^{-2d}}-1\right)\Delta_n(x_1-x_2)\dots\Delta_n(x_{2p}-x_1)dx\right| \\
&+C\left|\int_{[-\pi,\pi]^{2p}} \prod_{i=1}^{p} b_i(x_{2i-1})\left(\prod_{i=1}^{p}\frac{g(x_{2i-1})}{g(x_{2i})}-1\right)\Delta_n(x_1-x_2)\dots\Delta_n(x_{2p}-x_1)dx\right| \\
&\leq C\left(\frac{M\|b\|_\infty}{m}\right)^p \sum_{j=1}^{p}\int_{[-\pi,\pi]^{2p}}\prod_{i=1}^{j}\frac{|x_{2i-1}-x_{2i}|^{1-3a}}{(|x_{2i}|\wedge|x_{2i-1}|)^{1-a}}L_n(x_1-x_2)\dots L_n(x_{2p}-x_1)dx \\
&+CL\left(\frac{M\|b\|_\infty}{m}\right)^{p-1}\sum_{j=1}^{p}\int_{[-\pi,\pi]^{2p}}|b_j(x_{2j-1})||x_{2j-1}-x_{2j}|L_n(x_1-x_2)\dots L_n(x_{2p}-x_1)dx \\
&\leq C\left(\frac{M\|b\|_\infty}{m}\right)^p n^{3pa}\left(\int_{[-\pi,\pi]}|x|^{-1+a}dx\right)^p + CL\left(\frac{M\|b\|_\infty}{m}\log n\right)^{p-1}(\log n)^{2p-1}\sum_{j=1}^{p}\|b_j\|_2.
\end{aligned}
$$

□

LEMMA 2.6. *Let $\tilde{f} = 1/(4\pi^2 f)$, and let $\rho > 1/2$ and $L > 1$, then under the conditions of Lemma 2.4 we have the following alternative bound for (2.4):*

(2.7)
$$
\begin{aligned}
&\left| tr\left\{\prod_{j=1}^{p} T_n^{-1}(f)T_n(f_{2j})\right\} - tr\left\{\prod_{j=1}^{p} T_n(\tilde{f})T_n(f_{2j})\right\}\right| \\
&\lesssim \sqrt{L}n^{(1-\rho/2)+2q+\epsilon}\left\{\sum_{j=1}^{p-1}\left(\sqrt{M_{2p}}\prod_{l=j+1}^{p-1}M^{(2l)}\right)\times\left(\int_{-\pi}^{\pi}\frac{|f_{2p}(x)|}{f(x)}\prod_{l=1}^{j}\frac{f_{2l}^2}{f^2}(x)dx\right)^{\frac{1}{2}}\right. \\
&\left.+\prod_{l=2}^{p}M^{(2l)}\left(\int_{-\pi}^{\pi}\frac{f_2^2}{f^2}(x)dx\right)^{\frac{1}{2}} + error\right\}
\end{aligned}
$$

where

$$
error \leq L^{3/4}n^{(1-3\rho)/4}\prod_{l=1}^{p}M^{(2l)} + \sum_{j=1}^{p}\sqrt{L^{(2j)}n^{-\rho_{2j}}M^{(2j)}}\prod_{l\neq j}M^{(2l)}
$$

REMARK 2.1. *The constant appearing on the right hand side of (2.7) depends on M and m, but in all our applications of Lemma 2.6, the constants M and m will be bounded and of no consequence.*

PROOF. Following the construction of Dahlhaus (1989), equation (13), we write $|\mathrm{tr}\{\prod_{j=1}^p T_n^{-1}(f)T_n(f_{2j})\} - \mathrm{tr}\{\prod_{j=1}^p T_n(\tilde{f})T_n(f_{2j})\}|$ as

$$\begin{aligned}(2.8)\qquad &\left|\mathrm{tr}\left\{\prod_{j=1}^p A_j - \prod_{j=1}^p B_j\right\}\right| \\ &= \left|\mathrm{tr}\left\{(A_1 - B_1)\prod_{l=2}^p A_l + \sum_{j=2}^p \left(\prod_{l=1}^{j-1} B_l\right)(A_j - B_j)\prod_{l=j+1}^p A_l\right\}\right|,\end{aligned}$$

where $A_j = T_n^{\frac{1}{2}}(f_{2j-2})T_n^{-1}(f)T_n^{\frac{1}{2}}(f_{2j})$, $B_j = T_n^{\frac{1}{2}}(f_{2j-2})T_n(\tilde{f})T_n^{\frac{1}{2}}(f_{2j})$ and $f_0 := f_{2p}$ (similarly for ρ_0, $L^{(0)}$ and $M^{(0)}$). When $j = p$, the factor $\prod_{l=j+1}^p A_l$ is understood to be the identity. Without loss of generality, the functions f_{2j} are assumed to be positive (it suffices to write $f_{2j} = f_{2j+} - f_{2j-}$). Lemma 2.3 implies that for each j,

$$(2.9)\qquad \|T_n^{-\frac{1}{2}}(f)T_n^{\frac{1}{2}}(f_{2j})\| \lesssim \frac{M^{(2j)}}{m} n^{(d_{2j}-d)_+ + \epsilon}.$$

Using the relations in (1.6) (main paper) it then follows that

$$\begin{aligned}(2.10)\qquad \left\|\prod_{l=j+1}^p A_l\right\| &\le \prod_{l=j+1}^p \|T_n^{\frac{1}{2}}(f_{2l})T_n^{-\frac{1}{2}}(f)\|^2 \\ &\lesssim \left(\prod_{l=j+1}^{p-1} M^{(2l)}\right) n^{2\sum_{l=j+1}^{p-1}(d_{2l}-d)_+ + (d_{2j}-d)_+ + (d_{2p}-d)_+} \sqrt{M^{(2p)}M^{(2j)}}.\end{aligned}$$

First we treat the term $(A_1 - B_1)\prod_{l=2}^p A_l$ on the right in (2.8). Writing

$R = I_n - T_n^{\frac{1}{2}}(f)T_n(\tilde{f})T_n^{\frac{1}{2}}(f)$, it follows that

(2.11)
$$\begin{aligned}
&\left|\operatorname{tr}\left[(A_1 - B_1)\prod_{l=2}^{p} A_l\right]\right| \\
&\quad = \left|\operatorname{tr}\left[T_n^{\frac{1}{2}}(f_{2p})T_n^{-\frac{1}{2}}(f)RT_n^{-\frac{1}{2}}(f)T_n(f_2)T_n^{-\frac{1}{2}}(f)T_n^{-\frac{1}{2}}(f)T_n^{\frac{1}{2}}(f_4)\prod_{l=3}^{p} A_l\right]\right| \\
&\quad \leq |R||T_n^{-\frac{1}{2}}(f)T_n(f_2)T_n^{-\frac{1}{2}}(f)|\|T_n^{\frac{1}{2}}(f_{2p})T_n^{-\frac{1}{2}}(f)\|\|T_n^{-\frac{1}{2}}(f)T_n^{\frac{1}{2}}(f_4)\|\left\|\prod_{l=3}^{p} A_l\right\| \\
&\quad \lesssim L^{\frac{1}{2}} n^{(1-\rho)/2+\epsilon+\frac{1}{2}+2q}\prod_{l=2}^{p} M^{(2l)}\left(\int_{-\pi}^{\pi}\frac{f_2^2(x)}{f^2(x)}dx + \text{error}\right)^{\frac{1}{2}} \\
&\quad \lesssim L^{\frac{1}{2}} n^{(1-\rho)/2+\epsilon+\frac{1}{2}+2q}\prod_{l=2}^{p} M^{(2l)}\times \\
&\qquad \left(\int_{-\pi}^{\pi}\frac{f_2^2(x)}{f^2(x)}dx + n^{\epsilon+2q}\left(L^{3/2}\left(M^{(2)}\right)^2 n^{(1-3\rho)/2} + M^{(2)}L^{(2)}n^{-\rho_2}\right)\right)^{\frac{1}{2}}.
\end{aligned}$$

The first inequality follows from the relations in (1.6) (main paper). The second inequality follows after writing $|T_n^{-\frac{1}{2}}(f)T_n(f_2)T_n^{-\frac{1}{2}}(f)|$ as the sum of a limiting integral and an approximation error; in addition we use (2.9) and Lemma 2.2, by which

$$|R|^2 \leq KL(M/m)^2 n^{1-\rho+\epsilon} \lesssim Ln^{1-\rho+\epsilon}. \tag{2.12}$$

This follows from Lemma 2.4, which we use to bound the approximation error. The second term within the brackets in (2.11) constitutes part of the term *error*.

Next we bound the term $(\prod_{l=1}^{j-1} B_l)(A_j - B_j)\prod_{l=j+1}^{p} A_l$ in (2.8) for $j = 2$. Similar to the preceding decomposition, we have

$$\begin{aligned}
\left|\operatorname{tr}\left[B_1(A_2 - B_2)\prod_{l=3}^{p} A_l\right]\right| &= \left|\operatorname{tr}\left[B_1T_n^{\frac{1}{2}}(f_2)T_n^{-\frac{1}{2}}(f)RT_n^{-\frac{1}{2}}(f)T_n^{\frac{1}{2}}(f_4)\prod_{l=3}^{p} A_l\right]\right| \\
&\leq |B_1T_n^{\frac{1}{2}}(f_2)T_n^{-\frac{1}{2}}(f)||R|\|T_n^{-\frac{1}{2}}(f)T_n^{\frac{1}{2}}(f_4)\|\left\|\prod_{l=3}^{p} A_l\right\|.
\end{aligned}$$

The terms $|R|$, $\|T_n^{-\frac{1}{2}}(f)T_n^{\frac{1}{2}}(f_4)\|$ and $\|\prod_{l=3}^{p} A_l\|$ are bounded as in (2.9), (2.10) and (2.12). For the term $|B_1T_n^{\frac{1}{2}}(f_2)T_n^{-\frac{1}{2}}(f)|$ we have the decomposition

$$
\begin{aligned}
&|B_1 T_n^{\frac12}(f_2) T_n^{-\frac12}(f)|^2 = \operatorname{tr}\left[T_n^{-\frac12}(f) T_n^{\frac12}(f_2) B_1^t B_1 T_n^{\frac12}(f_2) T_n^{-\frac12}(f)\right] \\
&= \operatorname{tr}\left[B_1^t B_1 T_n^{\frac12}(f_2) T_n^{-1}(f) T_n^{\frac12}(f_2)\right] = \operatorname{tr}\left[B_1^t B_1 T_n^{\frac12}(f_2) T_n(\tilde f) T_n^{\frac12}(f_2)\right] \\
&\qquad + \operatorname{tr}\left[B_1^t B_1 T_n^{\frac12}(f_2) T_n^{-\frac12}(f) R T_n^{-\frac12}(f) T_n^{\frac12}(f_2)\right] \\
&\le |B_1 T_n^{\frac12}(f_2) T_n^{\frac12}(\tilde f)|^2 + |B_1^t B_1||R|\|T_n^{\frac12}(f_2) T_n^{-\frac12}(f)\|^2.
\end{aligned}
$$

Using again Lemmas 2.1, 2.2 and 2.3, we find that the first term on the right is bounded by

$$
n\left\{\int_{-\pi}^{\pi} \frac{|f_{2p}(x)| f_2^2(x)}{f^3(x)} dx + M^{(2p)} M^{(2)} n^{4(d_2-d)_+ + 2(d_{2p}-d)_+ + \epsilon}\left(M^{(2)} L n^{-\rho} + L^{(2)} n^{-\rho_2}\right)\right\}
$$

and the second term by

$$
\begin{aligned}
&n^{\frac12 - \frac{\rho}{2} + (d_2-d)_+ + \epsilon}\sqrt{L} M^{(2)}\left[n\int_{-\pi}^{\pi} \frac{f_{2p}^2(x) f_2^2(x)}{f^4(x)} dx + M^{(2p)} M^{(2)} n^{4(d_2-d)_+ + 2(d_{2p}-d)_+ + \epsilon} \times\right. \\
&\qquad \left.\times\left((M^{(2)})^2 (M^{(2p)})^2 L n^{1-\rho} + (M^{(2p)})^2 M^{(2)} L^{(2)} n^{1-\rho_2} + (M^{(2)})^2 M^{(2p)} L^{(2p)} n^{1-\rho_{2p}}\right)\right]^{\frac12}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&\left|\operatorname{tr}\left[B_1(A_2 - B_2)\prod_{l=3}^{p} A_l\right]\right| \\
&\quad \lesssim L^{\frac12} n^{\frac12 + (1-\rho)/2 + \epsilon + 2q}\sqrt{M^{(2p)}}\prod_{l=2}^{p-1} M^{(2l)}\left[\int_{-\pi}^{\pi} \frac{f_2^2(x)|f_{2p}(x)|}{f^3(x)} dx + M^{(2p)} (M^{(2)})^2 L n^{-\rho}\right. \\
&\quad + n^{-\rho/2}(L)^{1/2}\sqrt{M^{(2)}}\left(\int_{-\pi}^{\pi} \frac{f_2^2(x) f_{2p}^2(x)}{f^4(x)} dx\right)^{\frac12} + L^{(2)} M^{(2)} M^{(2p)} n^{-\rho_2} \\
&\quad \left. + (LL^{(2)})^{1/2} M^{(2p)} (M^{(2)})^{3/2} n^{-(\rho+\rho_2)/2} + (LL^{(2p)} M^{(2p)})^{1/2} (M^{(2)})^2 n^{-(\rho+\rho_{2p})/2}\right]^{\frac12}.
\end{aligned}
$$

Note that

$$
\left(\int_{-\pi}^{\pi} \frac{f_2^2(x) f_{2p}(x)^2}{f^4(x)} dx\right)^{\frac12} \lesssim M^{(2)} M^{(2p)},
$$

$$
(LL^{(2)} M^{(2p)})^{1/2} (M^{(2)})^{3/2} n^{-(\rho+\rho_2)/2} \le L (M^{(2)})^2 n^{-\rho} + L^{(2)} M^{(2)} M^{(2p)} n^{-\rho_2}
$$

and $Ln^{-\rho} \lesssim L^{3/2} n^{(1-3\rho)/2}$. Therefore the terms on the right are of the same order as the right hand side of (2.11). A similar argument applies to the term $(LL^{(2p)} M^{(2p)})^{1/2} (M^{(2)})^2 n^{-(\rho+\rho_{2p})/2}$.

Finally, we bound the term $(\prod_{l=1}^{j-1} B_l)(A_j - B_j) \prod_{l=j+1}^{p} A_l$ in (2.8) for $j \geq 3$. For $j \geq 3$, Lemma 2.1 implies that

$$|\prod_{l=1}^{j-1} B_l|^2 = n \left(\int_{-\pi}^{\pi} \frac{f_{2p}(x) f_{2j-2}(x)}{f^2(x)} \prod_{l=1}^{j-2} \frac{f_{2l}^2(x)}{f^2(x)} dx + error_j \right),$$

where

$$error_j \lesssim n^{\epsilon + 2\sum_{l=1}^{j-1}(d_{2l}-d)_+} \prod_{l=1}^{j-1} M^{(2l)} M^{(2l-2)} \left(Ln^{-\rho} + \sum_{l=1}^{j-1} \frac{L^{(2l)}}{M^{(2l)}} n^{-\rho_{2l}}) \right).$$

Consequently, we have for all $j \geq 2$

$$\begin{aligned} &\left| \mathrm{tr} \left[\left(\prod_{l=1}^{j-1} B_l \right) (A_j - B_j) \prod_{l=j+1}^{p} A_l \right] \right| \\ &\leq |\prod_{l=1}^{j-1} B_l||R| \prod_{l=j+1}^{p} \|A_l\| \|T_n^{\frac{1}{2}}(f_{2j}) T_n^{-\frac{1}{2}}(f)\| \|T_n^{\frac{1}{2}}(f_{(2j-2)}) T_n^{-\frac{1}{2}}(f)\| \\ &\lesssim L^{\frac{1}{2}} n^{\frac{1}{2}+(1-\rho)/2+2q+\epsilon} \sqrt{M^{(2p)}} \prod_{l=j+1}^{p-1} M^{(2l)} \left(\int_{-\pi}^{\pi} \frac{f_{2p}(x) f_{2j}(x)}{f^2(x)} \prod_{l=1}^{j-1} \frac{f_{2l}^2(x)}{f^2(x)} dx + error_j \right)^{\frac{1}{2}}. \end{aligned}$$

for all $j \geq 3$, which finishes the proof of Lemma 2.6. □

3. Hölder constants of various functions.

LEMMA 3.1. *Let $\theta_o \in \Theta(\beta, L_o)$. Then f_o satisfies condition (2.2) with $\rho = 1$ when $\beta > \frac{3}{2}$, and with any $\rho < \beta - \frac{1}{2}$ when $\beta \leq \frac{3}{2}$. The Hölder-constant only depends on L_o. When $\theta \in \Theta_k(\beta, L)$, $f_{d,k,\theta}$ satisfies (2.2) with $\rho = 1$, regardless of β. The Hölder-constant is of order $k^{\frac{3}{2}-\beta}$. The function $-\log(2 - 2\cos(x)) f_{d,k,\theta}$ satisfies condition (2.1) with $\rho = 1$ and Hölder-constant of order $k^{\frac{3}{2}-\beta}$. The functions $G_k f_{d,k,\theta}$ and $H_k f_{d,k,\theta}$, with G_k and H_k as in (1.1), satisfy (2.1) with $\rho = 1$ and Hölder-constant of order k.*

PROOF. The function $\sum_{j=0}^{\infty} \theta_{o,j} \cos(jx)$ (i.e. the logarithm of the short-memory part of f_o), has smoothness $\rho < \beta - \frac{1}{2}$, since

$$\sum_{j=0}^{\infty} |\theta_{o,j}||\cos(jx) - \cos(jy)| \leq \left(\sum_{j=0}^{\infty} |\theta_{o,j}|j^{\rho}\right) |x-y|^{\rho}$$
$$\leq \left(\sum_{j=0}^{\infty} \theta_{o,j}^2 j^{2\beta}\right)^{\frac{1}{2}} \left(\sum_{j=0}^{\infty} j^{-2(\beta-\rho)}\right)^{\frac{1}{2}} |x-y|^{\rho},$$

which is finite only when $\rho < \beta - \frac{1}{2}$. Since $\sum_{j=0}^{\infty} |\theta_j| \lesssim \sqrt{L}$ when $\theta \in \Theta(\beta - 1/2, L)$ and $\beta > 1$, the functions $\sum_{j=0}^{\infty} \theta_j \cos(jx)$ and $\exp\{\sum_{j=0}^{\infty} \theta_j \cos(jx)\}$ have the same smoothness; only the values of L and M differ. The same calculation can be made when the FEXP-expansion is finite: when $\theta \in \Theta_k(\beta, L)$, then for all $x, y \in [-\pi, \pi]$,

$$\left|\sum_{j=0}^{k} \theta_j(\cos(jx) - \cos(jy))\right| \leq |x-y| \sum_{j=0}^{k} j|\theta_j| \lesssim \sqrt{L} k^{\frac{3}{2}-\beta} |x-y|. \tag{3.1}$$

Since

$$|G_k(x) - G_k(y)| \leq 2 \sum_{j=1}^{k} \eta_j |\cos(jx) - \cos(jy)| = O(k)|x-y|, \tag{3.2}$$

$G_k f_{d,k,\theta}$ has Hölder-smoothness $\rho = 1$, its Hölder-constant being $O(k)$. The same result holds for $H_k f_{d,k,\theta}$, since $H_k(x) = -\log(2 - 2\cos(x)) - G_k(x)$ (see (1.3)) and $k^{\frac{3}{2}-\beta} = o(k)$ for all $\beta > 1$. □

4. Proof of Lemma B.2. For easy reference we first restate the result. Let $W_\sigma(d)$ denote any of the quadratic forms

$$X^t T_n^{-1}(f_{d,k}) B_\sigma(d, \bar{\theta}_{d,k}) X - \mathrm{tr}\left[T_n(f_o) T_n^{-1}(f_{d,k}) B_\sigma(d, \bar{\theta}_{d,k})\right]$$

in (B.2) (in the main paper). Then for any $j \leq J$, $(l_1, \ldots, l_j) \in \{0, \ldots, k\}^j$ and $\sigma \in \mathcal{S}(l_1, \ldots, l_j)$, we have

$$|W_\sigma(d) - W_\sigma(d_o)| = \mathbf{o}_{\mathbf{P_o}}(|d-d_o| n^{\frac{1}{2}+\epsilon} k^{-\frac{1}{2}}), \tag{4.1}$$

$$\begin{aligned} &\mathrm{tr}\left[B_\sigma(d, \bar{\theta}_{d,k})\right] - \mathrm{tr}\left[B_\sigma(d_o, \bar{\theta}_{d_o})\right] \\ &= (d-d_o)\mathrm{tr}[T_{1,\sigma}(d_o,k)] + (d-d_o)^2 \mathbf{o}(n^{\epsilon+\frac{1}{2}} k^{-\frac{1}{2}+(1-\beta/2)_+}) \\ &= (d-d_o)\mathrm{tr}[T_{1,\sigma}(d_o,k)] + (d-d_o)^2 \mathbf{o}(n^{1-\delta}/k), \end{aligned} \tag{4.2}$$

$$\begin{aligned}\text{tr}\left[(T_n(f_o)T_n^{-1}(f_{d,k})-I_n)B_\sigma(d,\bar\theta_{d,k})\right]-\text{tr}\left[(T_n(f_o)T_n^{-1}(f_{d_o,k})-I_n)B_\sigma(d_o,\bar\theta_{d_o,k})\right]\\ =(d-d_o)\text{tr}[T_{2,\sigma}(d_o,k)]+(d-d_o)^2\mathbf{o}(n/k)+(d-d_o)\mathbf{o}(n^{\epsilon+\frac12}k^{-\frac12}).\end{aligned} \tag{4.3}$$

Proof of Lemma B.2. We first prove (4.2). Developing the left-hand side in d we obtain, for all j, $(l_1,\dots,l_j)\in\{0,\dots,k\}^j$ and $\sigma\in\mathcal{S}_j$,

$$\begin{aligned}&\text{tr}\left[B_\sigma(d,\bar\theta_{d,k})\right]-\text{tr}\left[B_\sigma(d_o,\bar\theta_{d_o})\right]\\ &\quad=(d-d_o)\text{tr}\left[B'_\sigma(d_o,\bar\theta_{d_o,k})\right]+\frac{(d-d_o)^2}{2}\text{tr}\left[B''_\sigma(\bar d,\bar\theta_{\bar d})\right],\end{aligned} \tag{4.4}$$

where $\bar d\in(d,d_o)$, and B' and B'' denote the first and second derivative with respect to d, respectively. Writing

$$\begin{aligned}\tilde B_{\sigma(i)}(d,k)&=T_n(H_k\nabla_{\sigma(i)}f_{d,k})T_n^{-1}(f_{d,k})\\ &\quad-T_n(\nabla_{\sigma(i)}f_{d,k})T_n^{-1}(f_{d,k})T_n(H_kf_{d,k})T_n^{-1}(f_{d,k}),\end{aligned}$$

it follows that $B'_\sigma(d,\bar\theta_{d,k})$ equals

$$B'_\sigma(d,\bar\theta_{d,k})=\sum_{i=1}^{|\sigma|}\prod_{j<i}T_n(\nabla_{\sigma(j)}f_{d,k})T_n^{-1}(f_{d,k})\tilde B_{\sigma(i)}(d,k)\prod_{j>i}T_n(\nabla_{\sigma(j)}f_{d,k})T_n^{-1}(f_{d,k}).$$

We recall the definition of $T_{1,\sigma}$ in Lemma B.1 (main paper), and conclude that $B'_\sigma(d,\bar\theta_{d,k})=T_{1,\sigma}(d,k)$. Consequently, the first term on the right in (4.4) equals $(d-d_o)\text{tr}[T_{1,\sigma}(d_o,k)]$.

The second derivative $B''_\sigma(d,\bar\theta_{d,k})$ equals

$$\begin{aligned}&2\sum_{i_1<i_2}^{|\sigma|}\prod_{j<i_1}T_n(\nabla_{\sigma(j)}f_{d,k})T_n^{-1}(f_{d,k})\tilde B_{\sigma(i_1)}(d,k)\prod_{i_1<j<i_2}T_n(\nabla_{\sigma(j)}f_{d,k})T_n^{-1}(f_{d,k})\\ &\qquad\times\tilde B_{\sigma(i_2)}(d,k)\prod_{i_2<j}T_n(\nabla_{\sigma(j)}f_{d,k})T_n^{-1}(f_{d,k})\\ &\quad+\sum_{i=1}^{|\sigma|}\prod_{j<i}T_n(\nabla_{\sigma(j)}f_{d,k})T_n^{-1}(f_{d,k})\tilde B'_{\sigma(i)}(d,k)\prod_{i<j}T_n(\nabla_{\sigma(j)}f_{d,k})T_n^{-1}(f_{d,k}).\end{aligned}$$

We now show that $\text{tr}\left[B''_\sigma(d,\bar\theta_{d,k})\right]=\mathbf{o}(n^{\epsilon+\frac12}k^{-\frac12+(1-\beta/2)_+})$. From Lemma 2.4 and the above expression for $B''_\sigma(d,\bar\theta_{d,k})$, it can be seen that $\text{tr}\left[B''_\sigma(d,\bar\theta_{d,k})\right]$ converges to zero. To bound the approximation error, we cannot use directly Lemma 2.4 because the bound in (2.4) becomes too large when $\beta<2$ and

$|\sigma|$ is larger than 1. We therefore use Lemmas 2.1 and 2.6. Let $A''_\sigma(d,\bar\theta_{d,k})$ be the matrix obtained after replacing every factor $T_n^{-1}(f_{d,k})$ in $B''_\sigma(d,\bar\theta_{d,k})$ by $T_n(\tilde f_{d,k})$, for $\tilde f_{d,k} = f_{d,k}^{-1}/(4\pi^2)$. We recall from Lemma 3.1 that the Lipschitz constant of $f_{d,k}$ is $O(k^{(2-\beta)_+})$, and for $H_k^j f_{d,k}$ and $H_k^j \nabla_{\sigma(m)} f_{d,k}$ ($m \le |\sigma|$, $j = 1,2$) it is $O(k \log k)$. Consequently, Lemma 2.1 implies that

$$\left|\mathrm{tr}\left[A''_\sigma(d,\bar\theta_{d,k})\right]\right| \quad = \quad O(kn^\epsilon) = \mathbf{o}(n^{\epsilon+\frac12}k^{-\frac12+(1-\beta/2)_+})$$

when $k \le k_n$ and $\beta > 1$. It follows from Lemma 2.6 that

$$\begin{aligned}\left|\mathrm{tr}\left[A''_\sigma(d,\bar\theta_{d,k})\right] - \mathrm{tr}\left[B''_\sigma(d,\bar\theta_{d,k})\right]\right| &= O(n^{1/2+\epsilon}k^{(1-\beta/2)_+})\left(\int_{-\pi}^{\pi} H_k^2(x)dx\right)^{\frac12} \\ &= \mathbf{o}(n^{\epsilon+\frac12}k^{-\frac12+(1-\beta/2)_+}).\end{aligned}$$

Note that in the case where $B''_\sigma(d,\bar\theta_{d,k})$ contains a Toeplitz matrix of the form $T_n(H_k^2 f_{d,k})$ or $T_n(H_k^2\nabla_{\sigma(m)}f_{d,k})$ then it contains no other Toeplitz matrix involving H_k and we can set $f_2 = H_k^2 f_{d,k}$ or $f_2 = H_k^2\nabla_{\sigma(m)}f_{d,k}$ and use Remark 2.1; this leads to the above error rate. Combining the preceding results for $|\mathrm{tr}[A''_\sigma(d,\bar\theta_{d,k})]|$ and $|\mathrm{tr}[A''_\sigma(d,\bar\theta_{d,k})] - \mathrm{tr}[B''_\sigma(d,\bar\theta_{d,k})]|$ we obtain that

$$\left|\mathrm{tr}\left[B''_\sigma(d,\bar\theta_{d,k})\right]\right| \quad = \quad \mathbf{o}(n^{\epsilon+\frac12}k^{-\frac12+(1-\beta/2)_+}) = \mathbf{o}(n^{1-\delta}/k),$$

which completes the proof of (4.2).

Next, we prove (4.3). Writing $f_o - f_{d,k} = f_o - f_{d_o,k} + f_{d_o,k} - f_{d,k}$, it follows that the left-hand side of (4.3) equals

$$\begin{aligned}&\mathrm{tr}\left[T_n(f_o - f_{d,k})T_n^{-1}(f_{d,k})B_\sigma(d,\bar\theta_{d,k})\right] - \mathrm{tr}\left[T_n(f_o - f_{d_o,k})T_n^{-1}(f_{d_o,k})B_\sigma(d_o,\bar\theta_{d_o,k})\right] \\ &= \mathrm{tr}\left[T_n(f_o - f_{d_o,k})\left\{T_n^{-1}(f_{d,k})B_\sigma(d,\bar\theta_{d,k}) - T_n^{-1}(f_{d_o,k})B_\sigma(d_o,\bar\theta_{d_o,k})\right\}\right] \\ &+ \mathrm{tr}\left[T_n(f_{d_o,k} - f_{d,k})T_n^{-1}(f_{d,k})B_\sigma(d,\bar\theta_{d,k})\right] \\ &:= C_1 + C_2.\end{aligned}$$

Using (1.4) we write $f_{d,k} = f_{d_o,k}e^{(d-d_o)H_k}$ and $f_o = f_{d_o,k}e^{\Delta_{d_o,k}}$, and we develop $C_\sigma(d,\bar\theta_{d,k}) = T_n^{-1}(f_{d,k})B_\sigma(d,\bar\theta_{d,k})$ around $d = d_o$. It follows that

$$\begin{aligned}C_1 &= \mathrm{tr}\left[T_n(f_o - f_{d_o,k})\left\{T_n^{-1}(f_{d,k})B_\sigma(d,\bar\theta_{d,k}) - T_n^{-1}(f_{d_o,k})B_\sigma(d_o,\bar\theta_{d_o,k})\right\}\right] \\ &= (d-d_o)\mathrm{tr}\left[T_n(f_o - f_{d_o,k})C'_\sigma(d_o,\bar\theta_{d_o,k})\right] \\ &\quad + (d-d_o)^2\int_0^1 (1-u)\mathrm{tr}\left[T_n(f_o - f_{d_o,k})C''_\sigma(d_u,\bar\theta_{d_u,k})\right]du,\end{aligned}$$

with $d_u = ud + (1-u)d_o$. For the first term on the right, we write, using Lemmas 2.1 and 2.6,

$$\begin{aligned}
&\mathrm{tr}\left[T_n(f_o - f_{d_o,k})C_\sigma^{'}(d_o, \bar{\theta}_{d_o,k})\right] \\
&\quad = \frac{n}{2\pi}\int_{-\pi}^{\pi} \frac{f_o - d_{d_o,k}}{f_{d_o,k}} H_k(x)\cos(l_1 x)\ldots\cos(l_{|\sigma|}x)dx + \text{error},
\end{aligned}$$

where σ is a partition of $\{1, ..., j\}$ and the error term is

$$O\left(\|\Delta_{d_o,k}\|_\infty n^\epsilon (k + k^{0.5(3/2-\beta)_+}\frac{\sqrt{n}}{\sqrt{k}} + k^{0.5(3/2-\beta)_+}\left(\frac{n}{k^{2\beta}} + k\|\Delta_{d_o,k}\|_\infty\right)^{\frac{1}{2}}\right),$$

which is $o(k^{-1/2}n^{1/2-\delta})$. Similarly, Lemmas 2.1 and 2.6 imply that there exists $c \in \mathbb{R}$ such that for al $d \in (d_o - \bar{v}_n, d_o + \bar{v}_n)$

$$\begin{aligned}
&\mathrm{tr}\left[T_n(f_o - f_{d_o,k})C_\sigma^{''}(d, \bar{\theta}_{d,k})\right] \\
&\quad = \frac{cn}{2\pi}\int_{-\pi}^{\pi} \frac{f_o - d_{d_o,k}}{f_{d_o,k}} H_k^2(x)\cos(l_1 x)\ldots\cos(l_{|\sigma|}x)dx + \text{error},
\end{aligned}$$

where the error term is of order

$$O\left(\|\Delta_{d_o,k}\|_\infty n^\epsilon (k + k^{\frac{1}{2}(2-\beta)_+}\frac{\sqrt{n}}{\sqrt{k}} + k^{\frac{1}{2}(2-\beta)_+}(\frac{n}{k^{2\beta}} + k\|\Delta_{d_o,k}\|_\infty)^{\frac{1}{2}}\right) = o\left(\frac{n^{1-\delta}}{k}\right).$$

This implies that $C_1 = O(S_n(d))$.

Using a Taylor expansion of $C_\sigma(d, \bar{\theta}_{d,k})$ and of $e^{-(d-d_o)H_k}$ around d_o, it follows that

$$\begin{aligned}
C_2 &= -(d-d_o)\mathrm{tr}\left[T_n(f_{d_o,k}H_k)T_n^{-1}(f_{d_o,k})B_\sigma(d_o, \bar{\theta}_{d_o,k})\right] \\
&\quad - \frac{1}{2}(d-d_o)^2\mathrm{tr}\left[T_n(f_{d_o,k}H_k^2 e^{-t(d^{'}-d_o)H_k})C_\sigma(d', \bar{\theta}_{d',k}) + 2T_n(f_{d_o,k}H_k)C_\sigma^{'}(d', \bar{\theta}_{d',k})\right],
\end{aligned}$$

for some $d^{'}$ between d and d_o. The first term equals $\mathrm{tr}[T_{2,\sigma}]$. The second equals

$$\begin{aligned}
&-\frac{1}{2}(d-d_o)^2\mathrm{tr}\left[T_n(f_{d',k}H_k^2)C_\sigma(d', \bar{\theta}_{d',k}) + 2T_n(f_{d_o,k}H_k)C_\sigma^{'}(d^{'}, \bar{\theta}_{d',k})\right] \\
&\quad = -\frac{n(d-d_o)^2}{2\pi}\int_{-\pi}^{\pi} H_k^2(x)\cos(l_1 x)\ldots\cos(l_{|\sigma|}x)dx + \text{error},
\end{aligned}$$

where the error term is $O\left(n^\epsilon\left(k + k^{0.5(2-\beta)_+}(nk^{-1} + kn^\epsilon)^{1/2}\right)\right) = o(k^{-1}n^{1-\delta})$. Therefore

$$C_2 = (d-d_o)\mathrm{tr}[T_{2,\sigma}] + O(n/k).$$

Finally, to prove (4.1), let $Z = T_n^{-\frac{1}{2}}(f_o)X$ and let $A_d = T_n^{\frac{1}{2}}(f_o)T_n^{-1}(f_{d,k})B_\sigma(d,\bar\theta_{d,k})T_n^{\frac{1}{2}}(f_o)$. Then for any $|d-d_o| \leq \bar v_n$, we have

$$W_\sigma(d) - W_\sigma(d_o) = Z^t(A_d - A_{d_o})Z - \mathrm{tr}\left(A_d - A_{d_o}\right).$$

Writing A'_d for the derivative of A_d with respect to d, it follows that

$$A_d - A_{d_o} = (d-d_o)A'_{\bar d}, \tag{4.5}$$

for some $\bar d$ between d and d_o. Using (1.6), we find that

$$\begin{aligned} A'_d &= T_n^{\frac{1}{2}}(f_o)T_n^{-1}(f_{d,k})T_n(H_k f_{d,k})T_n^{-1}(f_{d,k})T_n(B_\sigma(d,\bar\theta_{d,k}))T_n^{\frac{1}{2}}(f_o) \\ &+ T_n^{\frac{1}{2}}(f_o)T_n^{-1}(f_{d,k})B'_\sigma(d,\bar\theta_{d,k})T_n^{\frac{1}{2}}(f_o). \end{aligned}$$

Therefore, Lemma 2 of Lieberman et al. (2011) and the inequalities in (1.6) (main paper) imply that

(4.6)

$$\begin{aligned} &|A_d - A_{d_o}| \leq |d-d_o||A'_{\bar d}| \\ &\quad\leq C|d-d_o|\|T_n^{\frac{1}{2}}(f_o)T_n^{-\frac{1}{2}}(f_{\bar d,k})\|^2 \prod_{i=1}^{|\sigma|}\|T_n^{-\frac{1}{2}}(f_{\bar d,k})B_{\sigma(i)}(\bar d,\bar\theta_{\bar d,k})T_n^{\frac{1}{2}}(f_{\bar d,k})\| \\ &+ |T_n^{-\frac{1}{2}}(f_{\bar d,k})T_n(H_k\nabla_{\sigma(i)}f_{\bar d,k})T_n^{-\frac{1}{2}}(f_{\bar d,k})| \\ &= |d-d_o|n^\epsilon O\left(|T_n^{-\frac{1}{2}}(f_{\bar d,k})T_n(H_k f_{\bar d,k})T_n^{-\frac{1}{2}}(f_{\bar d,k})| + |T_n^{-\frac{1}{2}}(f_{\bar d,k})T_n(H_k\nabla_{\sigma(i)}f_{\bar d,k})T_n^{-\frac{1}{2}}(f_{\bar d,k})|\right), \end{aligned}$$

where $\sigma(i)$ can also be the empty set, in which case $\nabla_{\sigma(i)}f_{\bar d,k} = f_{\bar d,k}$. We bound the terms between brackets using Lemma 2.4, with $p=2$, $f = f_{\bar d,k}$ and $g_1 = g_2$ equalling either $H_k f_{\bar d,k}$ or $H_k\nabla_{\sigma(i)}f_{\bar d,k}$. The Hölder constants of these functions are given by Lemma 3.1. Hence we find that

(4.7)

$$\begin{aligned} &|T_n^{-\frac{1}{2}}(f_{\bar d,k})T_n(H_k f_{\bar d,k})T_n^{-\frac{1}{2}}(f_{\bar d,k})|^2 = \mathrm{tr}\left[\left(T_n^{-1}(f_{\bar d,k})T_n(H_k f_{\bar d,k})\right)^2\right] \\ &\quad= \frac{n}{2\pi}\int_{-\pi}^{\pi} H_k^2(x)dx + O(n^\epsilon(k+k^{2-\beta})) = O(n^{1-1/(2\beta)}(\log n)^{1/(2\beta)}). \end{aligned}$$

The last inequality follows from equation (1.7) in Lemma 1.1 and the fact that $k = k_n$ and $\beta > 1$. Similarly, it follows that

$$|T_n^{-\frac{1}{2}}(f_{\bar d,k})T_n(H_k\nabla_{\sigma(i)}f_{\bar d,k})T_n^{-\frac{1}{2}}(f_{\bar d,k})|^2 = O(n^{1-1/(2\beta)}(\log n)^{1/(2\beta)}). \tag{4.8}$$

Inserting (4.6), (4.7) and (4.8) in (4.5), we find that $|A_d - A_{d_o}| \leq |d - d_o| n^{1/2-1/(4\beta)+\epsilon}$, for all $|d - d_o| \leq \bar{v}_n$ and all $\epsilon > 0$, when n is large enough. Consequently, we can apply Lemma 1.3 with $A = (A_d - A_{d_o})/|A_d - A_{d_o}|$, so that when n is large enough

$$\sup_{|d-d_o|\leq\bar{v}_n} P_o\left(|W_\sigma(d) - W_\sigma(d_o)| > |d - d_o| n^{2\epsilon+\frac{1}{2}-\frac{1}{4\beta}}\right) \leq e^{-n^\epsilon/8}. \tag{4.9}$$

Using the above computations with $|d - d'| \leq n^{-2}$, we obtain

$$|W_\sigma(d) - W_\sigma(d')| \leq n^{-2+\epsilon}\left(n + Z^t Z\right).$$

Hence, for all $\epsilon < \frac{1}{2}$ and $c > 0$,

$$P_o\left(\sup_{|d'-d|\leq n^{-2}} |W_\sigma(d) - W_\sigma(d')| > n^{-\epsilon}\right) \leq P_o\left(Z^t Z > n^{2-2\epsilon}\right) \leq e^{-cn}, \tag{4.10}$$

provided n is large enough. Hence, we obtain (4.1) by combining (4.9) and (4.10) in a simple chaining argument over the interval $(d_o - \bar{v}_n, d_o + \bar{v}_n)$. $\square$

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