Université Paris Dauphine — Année 2002-2003
U.F.R. Mathématiques de la Décision
MD2 - Statistique

Correction examen de contrôle continu du 07/05/2003

Exercice 1

1

$$\begin{aligned}
\mathbb{E}(XY) &= \int_{-a}^{0}\left(\int_{0}^{a+x}\frac{3y^2x}{a^3}dy\right)dx + \int_{0}^{a}\left(\int_{0}^{a-x}\frac{3y^2x}{a^3}dy\right)dx \\
\mathbb{E}(XY) &= \int_{-a}^{0}\frac{(a+x)^3x}{a^3}dx + \int_{0}^{a}\frac{(a-x)^3x}{a^3}dx \\
\mathbb{E}(XY) &= 0
\end{aligned}$$

2

$$\begin{aligned}
f_X(x) &= \left(\int_{0}^{a+x}\frac{3y}{a^3}dy\right)1_{]-a,0]}(x) + \left(\int_{0}^{a-x}\frac{3y}{a^3}dy\right)1_{]0,a[}(x) \\
f_X(x) &= \frac{3(a+x)^2}{2a^3}1_{]-a,0]}(x) + \frac{3(a-x)^2}{2a^3}1_{]0,a[}(x)
\end{aligned}$$

$$\begin{aligned}
f_Y(y) &= \left(\int_{y-a}^{a-y}\frac{3y}{a^3}dx\right)1_{]0,a[}(y) \\
f_Y(y) &= \frac{6y(a-y)}{a^3}1_{]0,a[}(y)
\end{aligned}$$

3

$$\begin{aligned}
\mathbb{E}(X) &= \int_{-a}^{0}\frac{3x(a+x)^2}{2a^3}dx + \int_{0}^{a}\frac{3x(a-x)^2}{2a^3}dx \\
\mathbb{E}(X) &= 0
\end{aligned}$$

Ainsi, $\text{cov}(X,Y) = 0$.

4

$$\begin{aligned} f_{Y|X=x}(y) &= \frac{f_{(X,Y)}(x,y)}{f_X(x)} \\ f_{Y|X=x}(y) &= \frac{2y}{(a+x)^2}1_{]-a,0]}(x)1_{]0,a+x[}(y) + \frac{2y}{(a-x)^2}1_{]0,a[}(x)1_{]0,a-x[}(y) \end{aligned}$$

5

$$\mathbb{E}(Y|X) = \left(\frac{2(a+X)}{3}\right)1_{]-a,0]}(X) + \left(\frac{2(a-X)}{3}\right)1_{]0,a[}(X)$$

Exercice 2

1

$$\begin{aligned} I(n,a) &= \int_0^{+\infty} x^n \exp(-ax)dx \\ I(n,a) &= \int_0^{+\infty} \frac{u^n}{a^{n+1}} \exp(-u)du \\ I(n,a) &= \frac{\Gamma(n+1)}{a^{n+1}} \\ I(n,a) &= \frac{n!}{a^{n+1}} \end{aligned}$$

2

$$\int_0^{+\infty} \lambda x^2 \exp(-ax)dx = 1$$

$$\lambda = \frac{a^3}{2}$$

$$\begin{aligned} \mathbb{E}(X) &= \left(\frac{a^3}{2}\right)\int_0^{+\infty} x^3 \exp(-ax)dx \\ \mathbb{E}(X) &= \frac{3}{a} \end{aligned}$$

Ainsi, si $\mathbb{E}(X) = 200$ alors $a = 0.015$.

3

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\frac{\alpha}{(1+x^2+y^2)^{3/2}}\right)dxdy=1$$
$$\iff \alpha\int_0^{+\infty}\left(\int_0^{2\pi}\frac{\rho}{(1+\rho^2)^{3/2}}d\theta\right)d\rho=1$$
$$\iff 2\pi\alpha\int_0^{+\infty}\rho\left(1+\rho^2\right)^{-3/2}d\rho=1$$
$$\iff \alpha=\frac{1}{2\pi}$$

Exercice 3

1

$$X\sim\mathcal{N}(0,4/3)$$

Par symétrie, $Y\sim\mathcal{N}(0,4/3)$.

2

$$Y|X=x\sim\mathcal{N}(x/2,1)$$
$$\mathbb{E}(Y|X)=\left[\frac{X}{2}\right]$$

3

$$\alpha\sqrt{2\pi}=\frac{\sqrt{3}}{\sqrt{2\pi}2}$$
$$\alpha=\frac{\sqrt{3}}{4\pi}$$

Nous avons $x^2-xy+y^2=(xy)\begin{bmatrix}4/3 & 2/3\\ 2/3 & 4/3\end{bmatrix}^{-1}\begin{pmatrix}x\\ y\end{pmatrix}$. Dans la mesure où $\mathbb{E}(X)=0$, nous avons $\mathbb{E}(XY)=2/3$.
Ainsi, $\mathrm{corr}(X,Y)=1/2$.